Asset Management
Lecture 3. Smart Beta, Factor Investing and Alternative Risk Premia

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General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `http://www.thierry-roncalli.com/RiskBasedAM.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 8 (6 hours, AM+PM)
- January 15 (6 hours, AM+PM)
- January 22 (6 hours, AM+PM)
- January 29 (6 hours, AM+PM)

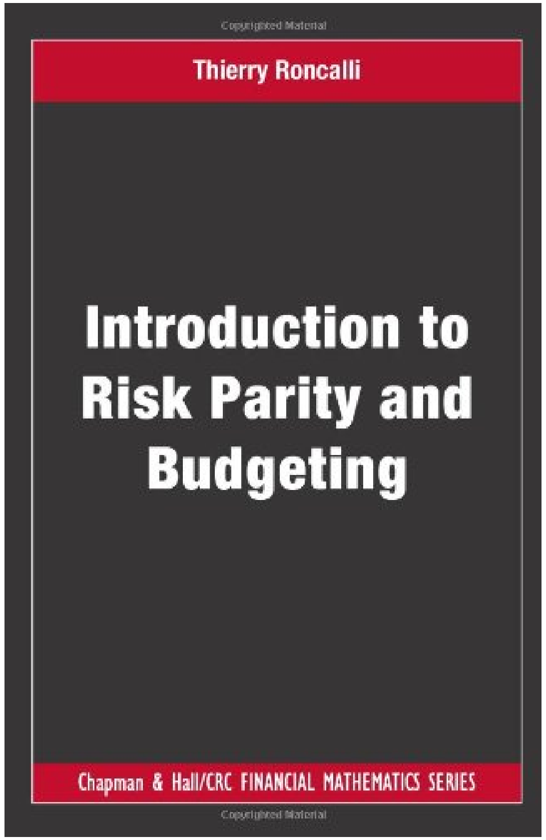
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Textbook

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskBasedAM.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- **Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia**
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Cap-weighted indexation and modern portfolio theory

Rationale of market-cap indexation

- **Separation Theorem**: there is one unique risky portfolio owned by investors called the tangency portfolio (Tobin, 1958)
- **CAPM**: the tangency portfolio is the Market portfolio, best represented by the capitalization-weighted index (Sharpe, 1964)
- **Performance of active management**: negative alpha in equity mutual funds on average (Jensen, 1968)
- **EMH**: markets are efficient (Fama, 1970)
- **Passive management**: launch of the first index fund (John McQuown, Wells Fargo Investment Advisors, Samsonite Luggage Corporation, 1971)
- **First S&P 500 index fund** by Wells Fargo and American National Bank in Chicago (1973)
- The **first listed ETF** was the SPDRs (Ticker: SPY) in 1993

Index funds

Mutual Fund (MF)

A mutual fund is a **collective investment fund** that are regulated and sold to the general public

Exchange Traded Fund (ETF)

It is a mutual fund which trades **intra-day** on a securities exchange (thanks to market makers)

Exchange Traded Product (ETP)

It is a security that is **derivatively-priced** and that trades intra-day on an exchange. ETPs includes exchange traded funds (ETFs), exchange traded vehicles (ETVs), exchange traded notes (ETNs) and certificates.

Pros of market-cap indexation

- A convenient and **recognized approach** to participate to broad equity markets
- **Management simplicity**: low turnover & transaction costs

Construction of an equity index

- We consider an index universe composed of n stocks
- Let $P_{i,t}$ be the price of the i^{th} stock and $R_{i,t}$ be the corresponding return between times $t-1$ and t:

$$R_{i,t} = \frac{P_{i,t}}{P_{i,t-1}} - 1$$

- The value of the index B_t at time t is defined by:

$$B_t = \varphi \sum_{i=1}^{n} N_i P_{i,t}$$

where φ is a scaling factor and N_i is the total number of shares issued by the company i

Construction of an equity index

- Another expression of B_t is[1]:

$$\begin{aligned} B_t &= \varphi \sum_{i=1}^{n} N_i P_{i,t-1} \left(1 + R_{i,t}\right) \\ &= B_{t-1} \frac{\sum_{i=1}^{n} N_i P_{i,t-1} \left(1 + R_{i,t}\right)}{\sum_{i=1}^{n} N_i P_{i,t-1}} \\ &= B_{t-1} \sum_{i=1}^{n} w_{i,t-1} \left(1 + R_{i,t}\right) \end{aligned}$$

  where $w_{i,t-1}$ is the weight of the i^{th} stock in the index:

$$w_{i,t-1} = \frac{N_i P_{i,t-1}}{\sum_{i=1}^{n} N_i P_{i,t-1}}$$

- The computation of the index value B_t can be done at the closing time t and also in an intra-day basis

[1] B_0 can be set to an arbitrary value (e.g. 100, 500, 1000 or 5000)

Construction of an equity index

Remark

The previous computation is purely theoretical because the portfolio corresponds to all the shares outstanding of the n stocks $\Rightarrow$ impossible to hold this portfolio

Remark

Most of equity indices use floating shares[a] instead of shares outstanding

[a] They indicate the number of shares available for trading

Replication of an equity index

- In order to replicate this index, we must build a hedging strategy that consists in investing in stocks
- Let S_t be the value of the strategy (or the index fund):

$$S_t = \sum_{i=1}^{n} n_{i,t} P_{i,t}$$

  where $n_{i,t}$ is the number of stock i held between $t-1$ and t

- The tracking error is the difference between the return of the strategy and the return of the index:

$$e_t (S \mid B) = R_{S,t} - R_{B,t}$$

Replication of an equity index

The quality of the replication process is measured by the volatility $\sigma(e_t(S \mid B))$ of the tracking error. We may distinguish several cases:

1. Index funds with low tracking error volatility (less than 10 bps) $\Rightarrow$ physical replication or synthetic replication
2. Index funds with moderate tracking error volatility (between 10 bps and 50 bps) $\Rightarrow$ sampling replication
3. Index funds with higher tracking error volatility (larger than 50 bps) $\Rightarrow$ equity universes with liquidity problems and enhanced/tilted index funds

Replication of an equity index

- In a capitalization-weighted index, the weights are given by:

$$w_{i,t} = \frac{C_{i,t}}{\sum_{j=1}^{n} C_{j,t}} = \frac{N_{i,t} P_{i,t}}{\sum_{j=1}^{n} N_{j,t} P_{j,t}}$$

  where $N_{i,t}$ and $C_{i,t} = N_{i,t} P_{i,t}$ are the number of shares outstanding and the market capitalization of the i^{th} stock

- If we have a perfect match at time $t-1$:

$$\frac{n_{i,t-1} P_{i,t-1}}{\sum_{i=1}^{n} n_{i,t-1} P_{i,t-1}} = w_{i,t-1}$$

  we have a perfect match at time t:

$$n_{i,t} = n_{i,t-1}$$

Replication of an equity index

- We do not need to rebalance the hedging portfolio because of the relationship:

$$n_{i,t}P_{i,t} \propto w_{i,t}P_{i,t}$$

- Therefore, it is not necessarily to adjust the portfolio of the strategy (except if there are subscriptions or redemptions)

A CW index fund remains the most efficient investment in terms of management simplicity, turnover and transaction costs

Cons of market-cap indexation

- Trend-following strategy: momentum bias leads to bubble risk exposure as weight of best performers ever increases
  $\Rightarrow$ Mid 2007, financial stocks represent 40% of the Eurostoxx 50 index
- Growth bias as high valuation multiples stocks weight more than low-multiple stocks with equivalent realized earnings.
  $\Rightarrow$ Mid 2000, the 8 stocks of the technology/telecom sectors represent 35% of the Eurostoxx 50 index
  $\Rightarrow$ 2½ years later after the dot.com bubble, these two sectors represent 12%
- Concentrated portfolios
  $\Rightarrow$ The top 100 market caps of the S&P 500 account for around 70%
- Lack of risk diversification and high drawdown risk: no portfolio construction rules leads to concentration issues (e.g. sectors, stocks).

Cons of market-cap indexation

Some illustrations

- Mid 2000: 8 Technology/Telecom stocks represent 35% of the Eurostoxx 50 index
- In 2002: 7.5% of the Eurostoxx 50 index is invested into Nokia with a volatility of 70%
- Dec. 2006: 26.5% of the MSCI World index is invested in financial stocks
- June 2007: 40% of the Eurostoxx 50 is invested into Financials
- January 2013: 20% of the S&P 500 stocks represent 68% of the S&P 500 risk
- Between 2002 and 2012, two stocks contribute on average to more than 20% of the monthly performance of the Eurostoxx 50 index

Cons of market-cap indexation

Table 1: Weight and risk concentration of several equity indices (June 29, 2012)

Ticker	Weights				Risk contributions			
	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%
SX5P	30.8	24.1	48.1	71.3	26.3	19.0	40.4	68.6
SX5E	31.2	23.0	46.5	72.1	31.2	20.5	44.7	73.3
INDU	33.2	23.0	45.0	73.5	35.8	25.0	49.6	75.9
BEL20	39.1	25.8	49.4	79.1	45.1	25.6	56.8	82.5
DAX	44.0	27.5	56.0	81.8	47.3	27.2	59.8	84.8
CAC	47.4	34.3	58.3	82.4	44.1	31.9	57.3	79.7
AEX	52.2	37.2	61.3	86.0	51.4	35.3	62.0	84.7
HSCEI	54.8	39.7	69.3	85.9	53.8	36.5	67.2	85.9
NKY	60.2	47.9	70.4	87.7	61.4	49.6	70.9	88.1
UKX	60.8	47.5	73.1	88.6	60.4	46.1	72.8	88.7
SXXE	61.7	49.2	73.5	88.7	63.9	51.6	75.3	90.1
SPX	61.8	52.1	72.0	87.8	59.3	48.7	69.9	86.7
MEXBOL	64.6	48.2	75.1	91.8	65.9	45.7	78.6	92.9
IBEX	64.9	51.7	77.3	90.2	68.3	58.2	80.3	91.4
SXXP	65.6	55.0	76.4	90.1	64.2	52.0	75.5	90.0
NDX	66.3	58.6	77.0	89.2	64.6	56.9	74.9	88.6
TWSE	79.7	73.4	86.8	95.2	79.7	72.6	87.3	95.7
TPX	80.8	72.8	88.8	96.3	83.9	77.1	91.0	97.3
KOSPI	86.5	80.6	93.9	98.0	89.3	85.1	95.8	98.8

$\mathcal{G}(x)$ = Gini coefficient, $\mathbb{L}(x)$ = Lorenz curve

Cons of market-cap indexation

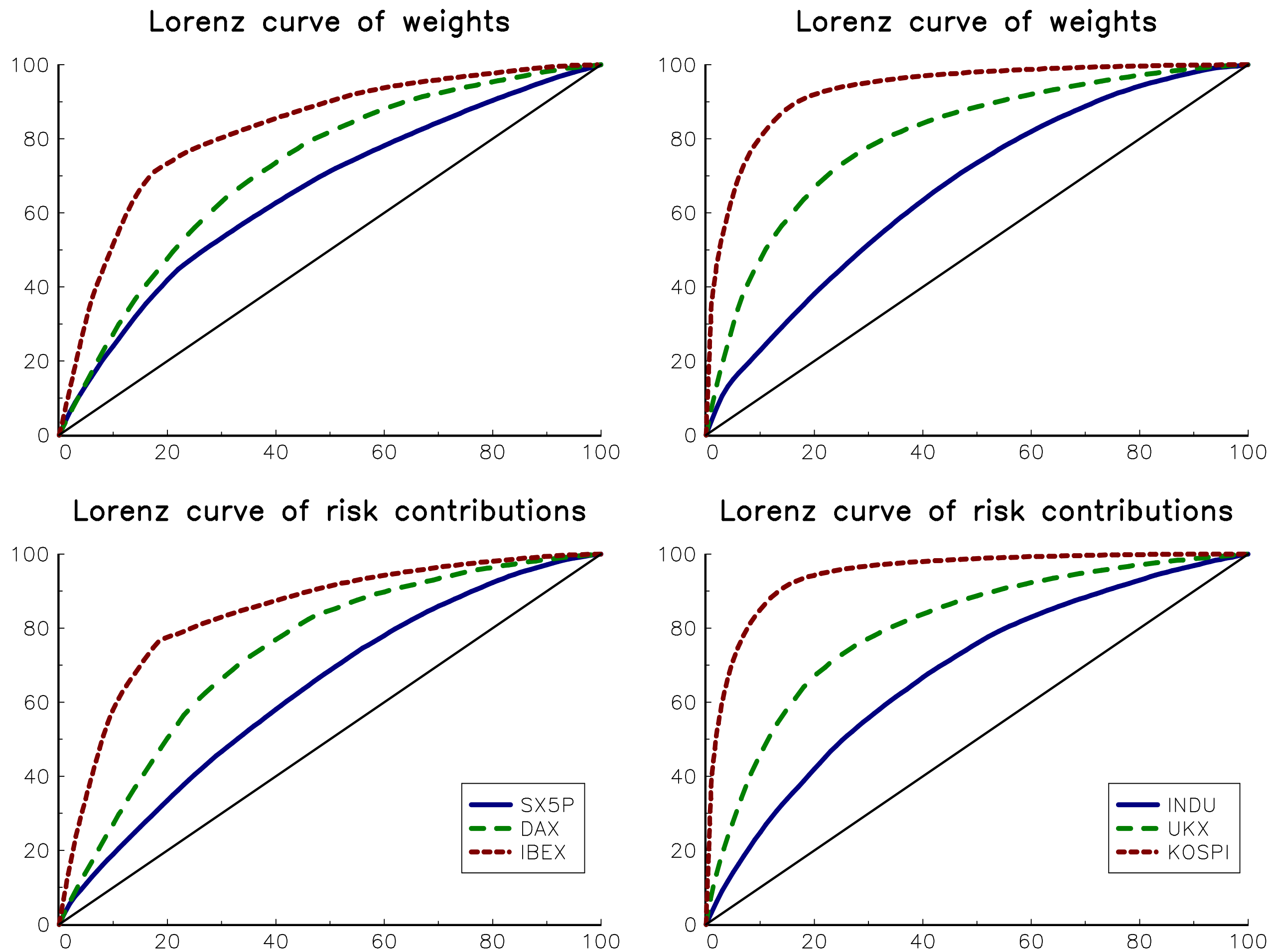


Figure 1: Lorenz curve of several equity indices (June 29, 2012)

Capturing the equity risk premium

	APPLE	EXXON	MSFT	J&J	IBM	PFIZER	CITI	McDO
	Cap-weighted allocation (in %)							
Dec. 1999	1.05	12.40	38.10	7.94	12.20	12.97	11.89	3.46
Dec. 2004	1.74	22.16	19.47	12.61	11.00	13.57	16.76	2.70
Dec. 2008	6.54	35.03	14.92	14.32	9.75	10.30	3.15	5.98
Dec. 2010	18.33	22.84	14.79	10.52	11.29	8.69	8.51	5.02
Dec. 2012	26.07	20.55	11.71	10.12	11.27	9.62	6.04	4.61
Jun. 2013	20.78	19.80	14.35	11.64	11.36	9.51	7.79	4.77
	Implied risk premium (in %)							
Dec. 1999	5.96	2.14	8.51	3.61	5.81	5.91	6.19	2.66
Dec. 2004	3.88	2.66	2.79	2.03	2.32	3.90	3.02	1.86
Dec. 2008	9.83	11.97	10.48	6.24	7.28	8.06	17.15	6.28
Dec. 2010	5.38	3.85	4.42	2.29	3.66	3.76	6.52	2.54
Dec. 2012	5.87	2.85	3.58	1.44	2.80	1.77	5.91	1.88
Jun. 2013	5.59	2.79	3.60	1.55	2.92	1.91	5.24	1.82
	Expected performance contribution (in %)							
Dec. 1999	1.01	4.31	52.63	4.66	11.52	12.43	11.94	1.49
Dec. 2004	2.41	21.04	19.44	9.15	9.12	18.93	18.11	1.79
Dec. 2008	6.60	43.00	16.04	9.17	7.28	8.52	5.55	3.85
Dec. 2010	23.58	21.01	15.62	5.77	9.89	7.81	13.27	3.05
Dec. 2012	42.41	16.23	11.61	4.04	8.73	4.71	9.88	2.40
Jun. 2013	33.96	16.18	15.10	5.28	9.69	5.32	11.93	2.53

Alternative-weighted indexation

Definition

Alternative-weighted indexation aims at building passive indexes where the weights are not based on market capitalization

Alternative-weighted indexation

Three kinds of responses:

1. Fundamental indexation (capturing *alpha*?)
    1. Dividend yield indexation
    2. RAFI indexation
2. Risk-based indexation (capturing *diversification*?)
    1. Equally weighted portfolio
    2. Minimum variance portfolio
    3. Equal risk contribution portfolio
    4. Most diversified portfolio
3. Factor investing (capturing *normal returns or beta*? *abnormal returns or alpha*?)
    1. The market risk factor is not the only systematic risk factor
    2. Other factors: size, value, momentum, low beta, quality, etc.

Alternative-weighted indexation

2008

Smart Beta
=
Fundamental Indexation
+
Risk-Based Indexation

Today

Smart Beta
=
Risk-Based Indexation
+
Factor Investing

Equally-weighted portfolio

- The underlying idea of the equally weighted or '$1/n$' portfolio is to define a portfolio independently from the estimated statistics and properties of stocks
- If we assume that it is impossible to predict return and risk, then attributing an equal weight to all of the portfolio components constitutes a natural choice
- We have:

$$x_i = x_j = \frac{1}{n}$$

Equally-weighted portfolio

The portfolio volatility is equal to:

$$\begin{aligned}\sigma^2(x) &= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i>j} x_i x_j \rho_{i,j} \sigma_i \sigma_j \\ &= \frac{1}{n^2} \left(\sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i>j} \rho_{i,j} \sigma_i \sigma_j \right)\end{aligned}$$

If we assume that $\sigma_i \leq \sigma_{\max}$ and $0 \leq \rho_{i,j} \leq \rho_{\max}$, we obtain:

$$\begin{aligned}\sigma^2(x) &\leq \frac{1}{n^2} \left(\sum_{i=1}^{n} \sigma_{\max}^2 + 2 \sum_{i>j} \rho_{\max} \sigma_{\max}^2 \right) \\ &\leq \frac{1}{n^2} \left(n \sigma_{\max}^2 + 2 \frac{n(n-1)}{2} \rho_{\max} \sigma_{\max}^2 \right) \\ &\leq \left(\frac{1 + (n-1)\rho_{\max}}{n} \right) \sigma_{\max}^2\end{aligned}$$

Equally-weighted portfolio

We deduce that:

$$\lim_{n \to \infty} \sigma(x) \leq \sigma_{\max}(x) = \sigma_{\max}\sqrt{\rho_{\max}}$$

Table 2: Value of $\sigma_{\max}(x)$ (in %)

		$\sigma_{\max}$ (in %)					
		5.00	10.00	15.00	20.00	25.00	30.00
$\rho_{\max}$ (in %)	10.00	1.58	3.16	4.74	6.32	7.91	9.49
	20.00	2.24	4.47	6.71	8.94	11.18	13.42
	30.00	2.74	5.48	8.22	10.95	13.69	16.43
	40.00	3.16	6.32	9.49	12.65	15.81	18.97
	50.00	3.54	7.07	10.61	14.14	17.68	21.21
	75.00	4.33	8.66	12.99	17.32	21.65	25.98
	90.00	4.74	9.49	14.23	18.97	23.72	28.46
	99.00	4.97	9.95	14.92	19.90	24.87	29.85

Equally-weighted portfolio

If the volatilities are the same ($\sigma_i = \sigma$) and the correlation matrix is constant ($\rho_{i,j} = \rho$), we deduce that:

$$\sigma(x) = \sigma \sqrt{\frac{1 + (n-1)\rho}{n}}$$

Correlations are more important than volatilities to benefit from diversification (= risk reduction)

Equally-weighted portfolio

Result

The main interest of the EW portfolio is the volatility reduction

It is called "naive diversification"

Equally-weighted portfolio

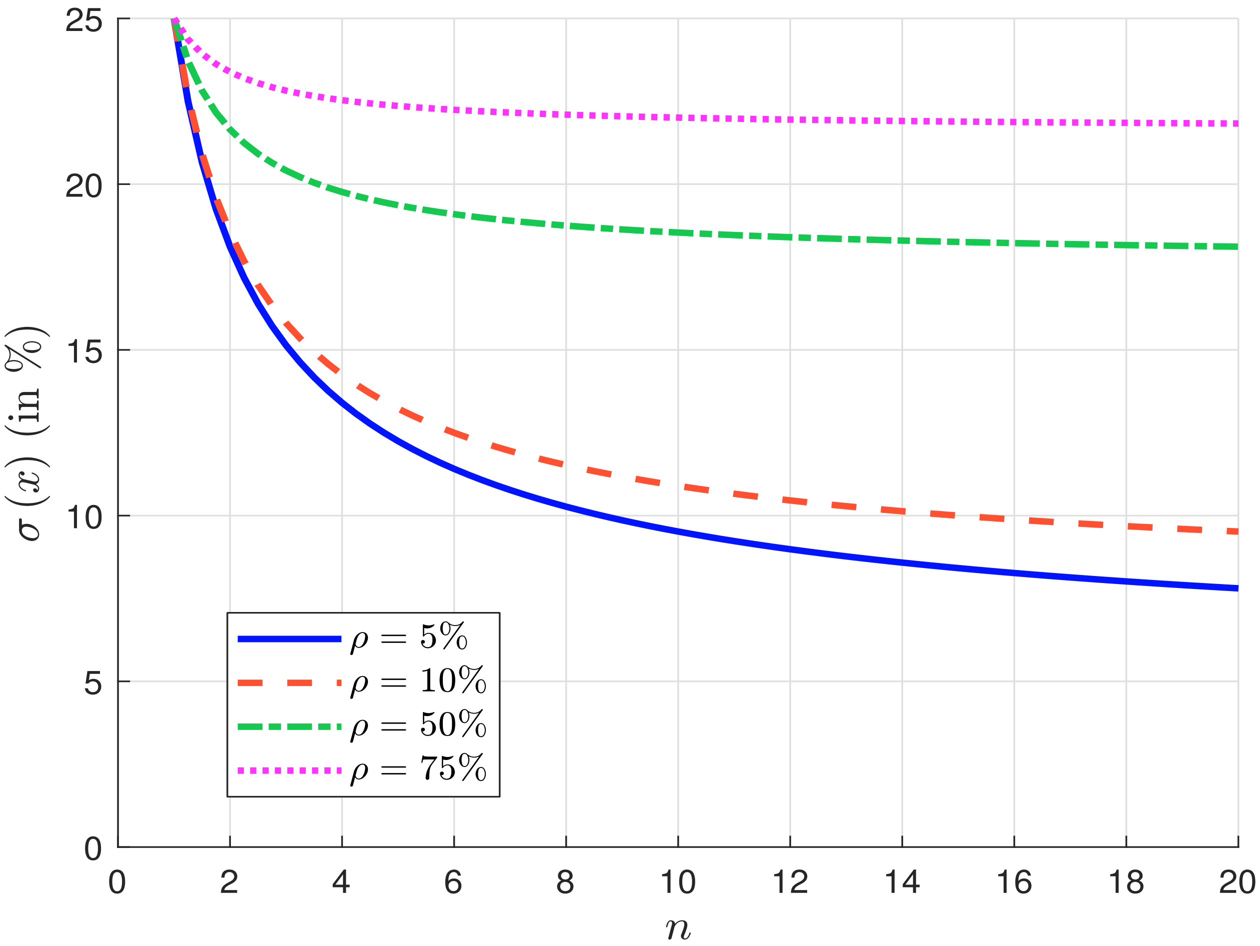


Figure 2: Illustration of the diversification effect ($\sigma = 25\%$)

Equally-weighted portfolio

Another interest of the EW portfolio is its good out-of-sample performance:

> *"We evaluate the out-of-sample performance of the sample-based mean-variance model, and its extensions designed to reduce estimation error, relative to the naive $1/n$ portfolio. Of the 14 models we evaluate across seven empirical datasets, none is consistently better than the $1/n$ rule in terms of Sharpe ratio, certainty-equivalent return, or turnover, which indicates that, out of sample, the gain from optimal diversification is more than offset by estimation error" (DeMiguel et al., 2009)*

Minimum variance portfolio

The global minimum variance (GMV) portfolio corresponds to the following optimization program:

$$
\begin{aligned}
x_{\mathrm{gmv}} & = & \arg\min \frac{1}{2} x^\top \Sigma x \\
\text{u.c.} & & \mathbf{1}_n^\top x = 1
\end{aligned}
$$

Minimum variance portfolio

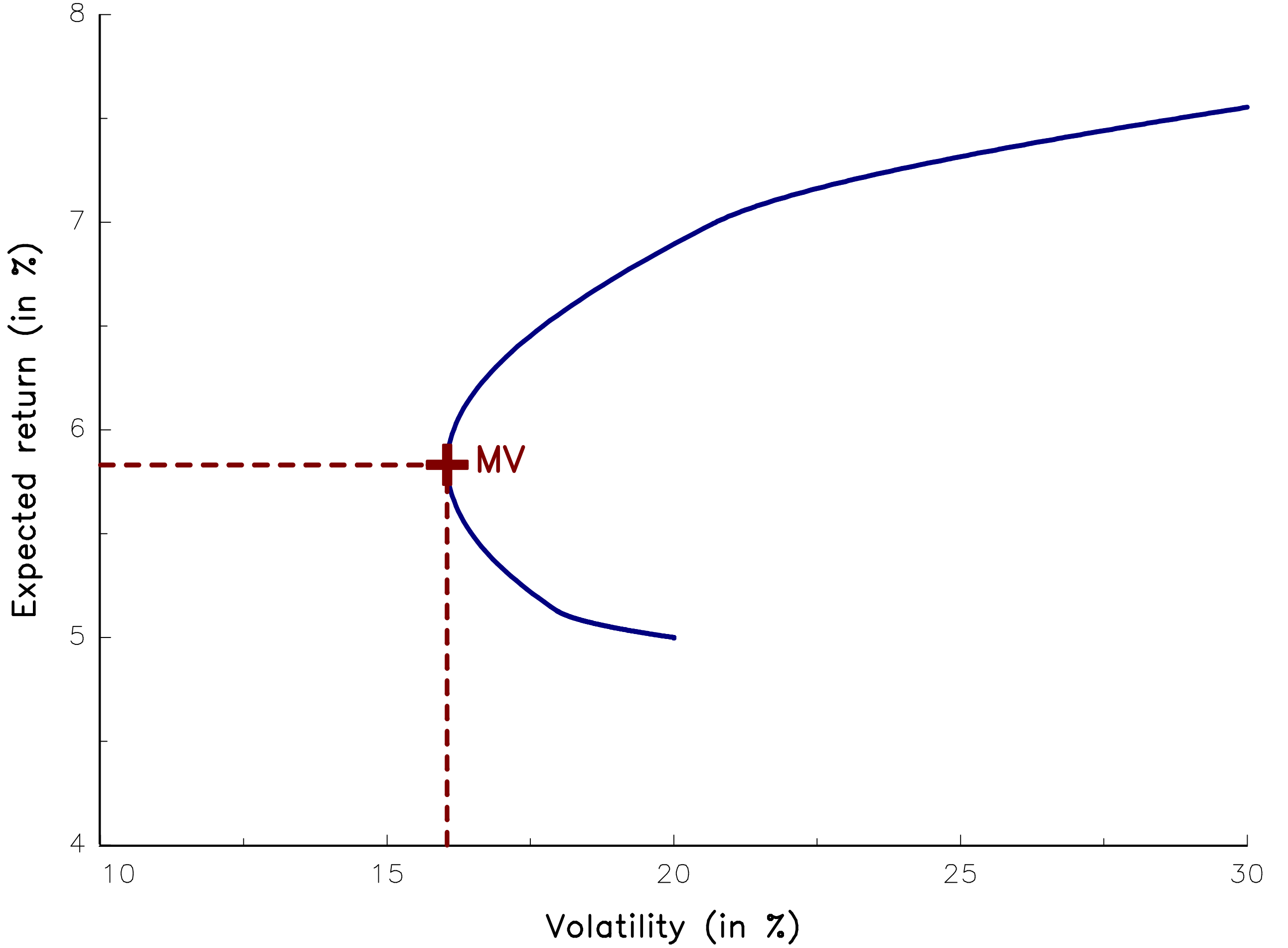


Figure 3: Location of the minimum variance portfolio in the efficient frontier

Minimum variance portfolio

The Lagrange function is equal to:

$$\mathcal{L}(x; \lambda_0) = \frac{1}{2} x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x - 1 \right)$$

The first-order condition is:

$$\frac{\partial \, \mathcal{L}(x; \lambda_0)}{\partial \, x} = \Sigma x - \lambda_0 \mathbf{1}_n = \mathbf{0}_n$$

We deduce that:

$$x = \lambda_0 \Sigma^{-1} \mathbf{1}_n$$

Since we have $\mathbf{1}_n^\top x = 1$, the Lagrange multiplier satisfies:

$$\mathbf{1}_n^\top \left(\lambda_0 \Sigma^{-1} \mathbf{1}_n \right) = 1$$

or:

$$\lambda_0 = \frac{1}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

Minimum variance portfolio

Theorem

The GMV portfolio is given by the following formula:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1}\mathbf{1}_n}$$

Minimum variance portfolio

The volatility of the GMV portfolio is equal to:

$$
\begin{aligned}
\sigma^2(x_{\mathrm{gmv}}) &= x_{\mathrm{gmv}}^\top \Sigma x_{\mathrm{gmv}} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \Sigma \frac{\Sigma^{-1} \mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1} \Sigma \Sigma^{-1} \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2} \\
&= \frac{1}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}
\end{aligned}
$$

Another expression of the GMV portfolio is:

$$
x_{\mathrm{gmv}} = \sigma^2(x_{\mathrm{gmv}}) \Sigma^{-1} \mathbf{1}_n
$$

Minimum variance portfolio

Example 1

The investment universe is made up of 4 assets. The volatility of these assets is respectively equal to 20%, 18%, 16% and 14%, whereas the correlation matrix is given by:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.40 & 0.20 & 1.00 & \\ 0.10 & 0.40 & 0.70 & 1.00 \end{pmatrix}$$

Minimum variance portfolio

We have:

$$\Sigma = \begin{pmatrix} 400.00 & 180.00 & 128.00 & 28.00 \\ 180.00 & 324.00 & 57.60 & 100.80 \\ 128.00 & 57.60 & 256.00 & 156.80 \\ 28.00 & 100.80 & 156.80 & 196.00 \end{pmatrix} \times 10^4$$

It follows that:

$$\Sigma^{-1} = \begin{pmatrix} 54.35 & -37.35 & -50.55 & 51.89 \\ -37.35 & 62.97 & 41.32 & -60.11 \\ -50.55 & 41.32 & 124.77 & -113.85 \\ 51.89 & -60.11 & -113.85 & 165.60 \end{pmatrix}$$

Minimum variance portfolio

We deduce that:

$$\Sigma^{-1}\mathbf{1}_4 = \begin{pmatrix} 18.34 \\ 6.83 \\ 1.69 \\ 43.53 \end{pmatrix}$$

We also have $\mathbf{1}_4^\top \Sigma^{-1}\mathbf{1}_4 = 70.39$, $\sigma^2(x_{\mathrm{gmv}}) = 1/70.39 = 1.4206\%$ and $\sigma(x_{\mathrm{gmv}}) = \sqrt{1.4206\%} = 11.92\%$. Finally, we obtain:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_4}{\mathbf{1}_4^\top \Sigma^{-1}\mathbf{1}_4} = \begin{pmatrix} 26.05\% \\ 9.71\% \\ 2.41\% \\ 61.84\% \end{pmatrix}$$

We verify that $\sum_{i=1}^{4} x_{\mathrm{gmv},i} = 100\%$ and $\sqrt{x_{\mathrm{gmv}}^\top \Sigma x_{\mathrm{gmv}}} = 11.92\%$

Minimum variance portfolio

- If we assume that the correlation matrix is constant – $C = \mathcal{C}_n(\rho)$, the covariance matrix is $\Sigma = \sigma\sigma^\top \circ \mathcal{C}_n(\rho)$ with $\mathcal{C}_n(\rho)$ the constant correlation matrix. We deduce that:

$$\Sigma^{-1} = \Gamma \circ \mathcal{C}_n^{-1}(\rho)$$

with $\Gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}$ and:

$$\mathcal{C}_n^{-1}(\rho) = \frac{\rho \mathbf{1}_n \mathbf{1}_n^\top - ((n-1)\rho + 1) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

- By using the trace property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we can show that:

$$x_{\mathrm{gmv},i} = \frac{-((n-1)\rho+1)\sigma_i^{-2} + \rho\sum_{j=1}^n (\sigma_i\sigma_j)^{-1}}{\sum_{k=1}^n \left(-((n-1)\rho+1)\sigma_k^{-2} + \rho\sum_{j=1}^n (\sigma_k\sigma_j)^{-1}\right)}$$

Minimum variance portfolio

- The denominator is the scaling factor such that $\mathbf{1}_n^\top x_{\mathrm{gmv}} = 1$. We deduce that the optimal weights are given by the following relationship:

$$x_{\mathrm{gmv},i} \propto \frac{((n-1)\rho + 1)}{\sigma_i^2} - \frac{\rho}{\sigma_i} \sum_{j=1}^{n} \frac{1}{\sigma_j}$$

Minimum variance portfolio

Here are some special cases:

1. The lower bound of $\mathcal{C}_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$ and we have:

$$\begin{aligned} x_{\mathrm{gmv},i} &\propto -\frac{\rho}{\sigma_i}\sum_{j=1}^{n}\frac{1}{\sigma_j} \\ &\propto \frac{1}{\sigma_i} \end{aligned}$$

   The weights are proportional to the inverse volatilities (GMV = ERC)

2. If the assets are uncorrelated ($\rho = 0$), we obtain:

$$x_i \propto \frac{1}{\sigma_i^2}$$

   The weights are proportional to the inverse variances

Minimum variance portfolio

3. If the assets are perfectly correlated ($\rho = 1$), we have:

$$x_{\mathrm{gmv},i} \propto \frac{1}{\sigma_i}\left(\frac{n}{\sigma_i} - \sum_{j=1}^{n}\frac{1}{\sigma_j}\right)$$

We deduce that:

$$\begin{aligned} x_{\mathrm{gmv},i} \geq 0 \quad &\Leftrightarrow \quad \frac{n}{\sigma_i} - \sum_{j=1}^{n}\frac{1}{\sigma_j} \geq 0 \\ &\Leftrightarrow \quad \sigma_i \leq \left(\frac{1}{n}\sum_{j=1}^{n}\sigma_j^{-1}\right)^{-1} \\ &\Leftrightarrow \quad \sigma_i \leq \bar{H}\left(\sigma_1, \ldots, \sigma_n\right) \end{aligned}$$

where $\bar{H}\left(\sigma_1, \ldots, \sigma_n\right)$ is the harmonic mean of volatilities

Minimum variance portfolio

Example 2

We consider a universe of four assets. Their volatilities are respectively equal to 4%, 6%, 8% and 10%. We assume also that the correlation matrix C is uniform and is equal to $\mathcal{C}_n(\rho)$.

Minimum variance portfolio

Table 3: Global minimum variance portfolios

Asset	ρ −20%	0%	20%	50%	70%	90%	99%
1	44.35	53.92	65.88	90.65	114.60	149.07	170.07
2	25.25	23.97	22.36	19.04	15.83	11.20	8.38
3	17.32	13.48	8.69	−1.24	−10.84	−24.67	−33.09
4	13.08	8.63	3.07	−8.44	−19.58	−35.61	−45.37
$\sigma(x^\star)$	1.93	2.94	3.52	3.86	3.62	2.52	0.87

Table 4: Long-only minimum variance portfolios

Asset	ρ −20%	0%	20%	50%	70%	90%	99%
1	44.35	53.92	65.88	85.71	100.00	100.00	100.00
2	25.25	23.97	22.36	14.29	0.00	0.00	0.00
3	17.32	13.48	8.69	0.00	0.00	0.00	0.00
4	13.08	8.63	3.07	0.00	0.00	0.00	0.00
$\sigma(x^\star)$	1.93	2.94	3.52	3.93	4.00	4.00	4.00

Minimum variance portfolio

In practice, we impose no short selling constraints

$\Downarrow$

Smart beta products (funds and indices) corresponds to long-only minimum variance portfolios

Minimum variance portfolio

Remark

The minimum variance strategy is related to the low beta effect (Black, 1972; Frazzini and Pedersen, 2014) or the low volatility anomaly (Haugen and Baker, 1991).

Minimum variance portfolio

We consider the single-factor model of the CAPM:

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$$

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where:

- $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of betas
- σ_m^2 is the variance of the market portfolio
- $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of specific variances

Minimum variance portfolio

Sherman-Morrison-Woodbury formula

Suppose u and v are two $n \times 1$ vectors and A is an invertible $n \times n$ matrix. We can show that:

$$\left(A + uv^\top\right)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1} uv^\top A^{-1}$$

Minimum variance portfolio

We have:

$$\Sigma = D + (\sigma_m \beta)(\sigma_m \beta)^\top$$

We apply the Sherman-Morrison-Woodbury with $A = D$ and $u = v = \sigma_m \beta$:

$$\begin{aligned}
\Sigma^{-1} &= \left(D + (\sigma_m \beta)(\sigma_m \beta)^\top\right)^{-1} \\
&= D^{-1} - \frac{1}{1 + (\sigma_m \beta)^\top D^{-1} (\sigma_m \beta)} D^{-1} (\sigma_m \beta)(\sigma_m \beta)^\top D^{-1} \\
&= D^{-1} - \frac{\sigma_m^2}{1 + \sigma_m^2 \left(\beta^\top D^{-1} \beta\right)} \left(D^{-1}\beta\right)\left(D^{-1}\beta\right)^\top
\end{aligned}$$

Minimum variance portfolio

We have:

$$D^{-1}\beta = \tilde{\beta}$$

with $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$ and:

$$\begin{aligned} \varphi &= \beta^\top D^{-1}\beta \\ &= \tilde{\beta}^\top \beta \\ &= \sum_{i=1}^n \frac{\beta_i^2}{\tilde{\sigma}_i^2} \end{aligned}$$

We obtain:

$$\Sigma^{-1} = D^{-1} - \frac{\sigma_m^2}{1+\varphi\sigma_m^2}\tilde{\beta}\tilde{\beta}^\top$$

The GMV portfolio is equal to:

$$\begin{aligned} x_{\mathrm{gmv}} &= \sigma^2\left(x_{\mathrm{gmv}}\right)\Sigma^{-1}\mathbf{1}_n \\ &= \sigma^2\left(x_{\mathrm{gmv}}\right)\left(D^{-1}\mathbf{1}_n - \frac{\sigma_m^2}{1+\varphi\sigma_m^2}\tilde{\beta}\tilde{\beta}^\top\mathbf{1}_n\right) \end{aligned}$$

Minimum variance portfolio

It follows that:

$$
\begin{aligned}
x_{\mathrm{gmv},i} &= \sigma^2\left(x_{\mathrm{gmv}}\right)\left(\frac{1}{\tilde{\sigma}_i^2}-\frac{\sigma_m^2\left(\tilde{\beta}^\top \mathbf{1}_n\right)}{1+\varphi\sigma_m^2}\frac{\beta_i}{\tilde{\sigma}_i^2}\right) \\
&= \frac{\sigma^2\left(x_{\mathrm{gmv}}\right)}{\tilde{\sigma}_i^2}\left(1-\frac{\beta_i}{\beta^\star}\right)
\end{aligned}
$$

where:

$$
\beta^\star = \frac{1+\varphi\sigma_m^2}{\sigma_m^2\left(\tilde{\beta}^\top \mathbf{1}_n\right)}
$$

The minimum variance portfolio is positively exposed to stocks with low beta:

$$
\left\{
\begin{array}{l}
\beta_i < \beta^\star \Rightarrow x_{\mathrm{gmv},i} > 0 \\
\beta_i > \beta^\star \Rightarrow x_{\mathrm{gmv},i} < 0
\end{array}
\right.
$$

Moreover, the absolute weight is a decreasing function of the idiosyncratic volatility: $\tilde{\sigma}_i \searrow \Rightarrow \left|x_{\mathrm{gmv},i}\right| \nearrow$

Minimum variance portfolio

The previous formula has been found by Scherer (2011). Clarke et al. (2011) have extended it to the long-only minimum variance:

$$x_{\mathrm{mv},i} = \frac{\sigma^2 \left(x_{\mathrm{gmv}}\right)}{\tilde{\sigma}_i^2} \left(1 - \frac{\beta_i}{\beta^\star}\right)$$

where the threshold $\beta^\star$ is defined as follows:

$$\beta^\star = \frac{1 + \sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i \beta_i}{\sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i}$$

In this case, if $\beta_i > \beta^\star$, $x_i^\star = 0$

Minimum variance portfolio

Example 3

We consider an investment universe of five assets. Their beta is respectively equal to 0.9, 0.8, 1.2, 0.7 and 1.3 whereas their specific volatility is 4%, 12%, 5%, 8% and 5%. We also assume that the market portfolio volatility is equal to 25%.

Minimum variance portfolio

- In the case of the GMV portfolio, we have $\varphi = 1879.26$ and $\beta^{\star} = 1.0972$
- In the case of the long-only MV portfolio, we have $\varphi = 121.01$ and $\beta^{\star} = 0.8307$

Table 5: Composition of the MV portfolio

Asset	β_i	$\tilde{\beta}_i$	x_i	
			Unconstrained	Long-only
1	0.90	562.50	147.33	0.00
2	0.80	55.56	24.67	9.45
3	1.20	480.00	−49.19	0.00
4	0.70	109.37	74.20	90.55
5	1.30	520.00	−97.01	0.00
Volatility			11.45	19.19

Minimum variance portfolio

In practice, we use a constrained long-only optimization program:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\
\text{u.c.} & \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
x \in \mathcal{DC}
\end{array}
\right.
\end{aligned}
$$

$\Rightarrow$ we need to impose some diversification constraints ($x \in \mathcal{DC}$) because Markowitz optimization leads to corner solutions that are not diversified

Minimum variance portfolio

Three main approaches:

1. In order to reduce the concentration of a few number of assets, we can use upper bound on the weights:

$$x_i \leq x_i^+$$

For instance, we can set $x_i \leq 5\%$, meaning that the weight of an asset cannot be larger than 5%. We can also impose lower and upper bounds by sector:

$$s_j^- \leq \sum_{i \in \mathcal{S}_j} x_i \leq s_j^+$$

For instance, if we impose that $3\% \leq \sum_{i \in \mathcal{S}_j} x_i \leq 20\%$, this implied that the weight of each sector must be between 3% and 20%.

Minimum variance portfolio

2. We can impose some constraints with respect to the benchmark composition:

$$\frac{b_i}{m} \leq x_i \leq m \cdot b_i$$

where b_i is the weight of asset i in the benchmark (or index) b. For instance, if $m = 2$, the weight of asset i cannot be lower than 50% of its weight in the benchmark. It cannot also be greater than twice of its weight in the benchmark.

3. The third approach consists of imposing a weight diversification based on the Herfindahl index:

$$\mathcal{H}(x) = \sum_{i=1}^{n} x_i$$

Minimum variance portfolio

- The inverse of the Herfindahl index is called the effective number of bets (ENB):

$$\mathcal{N}(x) = \mathcal{H}^{-1}(x)$$

- $\mathcal{N}(x)$ represents the equivalent number of equally-weighted assets. We can impose a sufficient number of effective bets:

$$\mathcal{N}(x) \geq \mathcal{N}_{\text{min}}$$

- During the period 2000-2020, the ENB of the S&P 500 index is between 90 and 130:

$$90 \leq \mathcal{N}(b) \leq 130$$

- During the same period, the ENB of the S&P 500 minimum variance portfolio is between 15 and 30:

$$15 \leq \mathcal{N}(x) \leq 30$$

- We conclude that the S&P 500 minimum variance portfolio is less diversified than the S&P 500 index

Minimum variance portfolio

We can impose:

$$\mathcal{N}(x) \geq m \cdot \mathcal{N}(b)$$

For instance, if $m = 1.5$, the ENB of the S&P 500 minimum variance portfolio will be 50% larger than the ENB of the S&P 500 index

We notice that:

$$\begin{aligned}\mathcal{N}(x) \geq \mathcal{N}_{\min} &\Leftrightarrow \mathcal{H}(x) \leq \mathcal{N}_{\min}^{-1} \\ &\Leftrightarrow x^\top x \leq \mathcal{N}_{\min}^{-1}\end{aligned}$$

The optimization problem becomes:

$$\begin{aligned}x^\star(\lambda) &= \arg\min \frac{1}{2} x^\top \Sigma x + \lambda \left(x^\top x - \mathcal{N}_{\min}^{-1}\right) \\ &\text{u.c.} \quad \left\{\begin{array}{l}\mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n\end{array}\right.\end{aligned}$$

Minimum variance portfolio

We can rewrite the objective function as follows:

$$\mathcal{L}(x;\lambda) = \frac{1}{2}x^\top \Sigma x + \lambda x^\top I_n x = \frac{1}{2}x^\top (\Sigma + 2\lambda I_n) x$$

We obtain a standard minimum variance optimization problem where the covariance matrix is shrunk

Remark

The optimal solution is found by applying the bisection algorithm to the QP problem in order to match the constraint:

$$\mathcal{N}(x^\star(\lambda)) = \mathcal{N}_{\min}$$

An alternative approach is to consider the ADMM algorithm (these numerical problems are studied in Lecture 5)

Most diversified portfolio

Definition

Choueifaty and Coignard (2008) introduce the concept of diversification ratio:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sigma(x)} = \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

$\mathcal{DR}(x)$ is the ratio between the weighted average volatility and the portfolio volatility

- The diversification ratio of a portfolio fully invested in one asset is equal to one:

$$\mathcal{DR}(e_i) = 1$$

- In the general case, it is larger than one:

$$\mathcal{DR}(x) \geq 1$$

Most diversified portfolio

The most diversified portfolio (or MDP) is defined as the portfolio which maximizes the diversification ratio:

$$\begin{aligned} x^{\star} &= \arg\max \ln \mathcal{DR}(x) \\ &\text{u.c. } \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Most diversified portfolio

The associated Lagrange function is equal to:

$$\begin{aligned}\mathcal{L}(x;\lambda_0,\lambda) &= \ln\left(\frac{x^\top\sigma}{\sqrt{x^\top\Sigma x}}\right)+\lambda_0\left(\mathbf{1}_n^\top x-1\right)+\lambda^\top\left(x-\mathbf{0}_n\right)\\ &= \ln\left(x^\top\sigma\right)-\frac{1}{2}\ln\left(x^\top\Sigma x\right)+\lambda_0\left(\mathbf{1}_n^\top x-1\right)+\lambda^\top x\end{aligned}$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0,\lambda)}{\partial\,x}=\frac{\sigma}{x^\top\sigma}-\frac{\Sigma x}{x^\top\Sigma x}+\lambda_0\mathbf{1}_n+\lambda=\mathbf{0}_n$$

whereas the Kuhn-Tucker conditions are:

$$\min\left(\lambda_i,x_i\right)=0\qquad\text{for } i=1,\ldots,n$$

Most diversified portfolio

The constraint $\mathbf{1}_n^\top x = 1$ can always be matched because:

$$\mathcal{DR}(\varphi \cdot x) = \mathcal{DR}(x)$$

We deduce that the MDP $x^\star$ satisfies:

$$\frac{\Sigma x^\star}{x^{\star\top} \Sigma x^\star} = \frac{\sigma}{x^{\star\top} \sigma} + \lambda$$

or:

$$\begin{aligned}
\Sigma x^\star &= \frac{\sigma^2(x^\star)}{x^{\star\top}\sigma}\sigma + \lambda \sigma^2(x^\star) \\
&= \frac{\sigma(x^\star)}{\mathcal{DR}(x^\star)}\sigma + \lambda \sigma^2(x^\star)
\end{aligned}$$

If the long-only constraint is not imposed, we have $\lambda = \mathbf{0}_n$

Most diversified portfolio

The correlation between a portfolio x and the MDP $x^{\star}$ is given by:

$$
\begin{aligned}
\rho\left(x, x^{\star}\right) &= \frac{x^{\top} \Sigma x^{\star}}{\sigma\left(x\right) \sigma\left(x^{\star}\right)} \\
&= \frac{1}{\sigma\left(x\right) \mathcal{DR}\left(x^{\star}\right)} x^{\top} \sigma + \frac{\sigma\left(x^{\star}\right)}{\sigma\left(x\right)} x^{\top} \lambda \\
&= \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)} + \frac{\sigma\left(x^{\star}\right)}{\sigma\left(x\right)} x^{\top} \lambda
\end{aligned}
$$

Most diversified portfolio

If $x^{\star}$ is the long-only MDP, we obtain (because $\lambda \geq \mathbf{0}_n$ and $x^{\top}\lambda \geq 0$):

$$\rho\left(x, x^{\star}\right) \geq \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)}$$

whereas we have for the unconstrained MDP:

$$\rho\left(x, x^{\star}\right) = \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)}$$

The 'core property' of the MDP

"The long-only MDP is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their diversification ratios" (Choueifaty et al., 2013)

Most diversified portfolio

The correlation between Asset i and the MDP is equal to:

$$\begin{aligned}\rho\left(e_i, x^\star\right) &= \frac{\mathcal{DR}\left(e_i\right)}{\mathcal{DR}\left(x^\star\right)} + \frac{\sigma\left(x^\star\right)}{\sigma\left(e_i\right)} e_i^\top \lambda \\ &= \frac{1}{\mathcal{DR}\left(x^\star\right)} + \frac{\sigma\left(x^\star\right)}{\sigma_i} \lambda_i\end{aligned}$$

Most diversified portfolio

Because $\lambda_i = 0$ if $x_i^\star > 0$ and $\lambda_i > 0$ if $x_i^\star = 0$, we deduce that:

$$\rho\left(e_i, x^\star\right) = \frac{1}{\mathcal{DR}\left(x^\star\right)} \quad \text{if} \quad x_i^\star > 0$$

and:

$$\rho\left(e_i, x^\star\right) \geq \frac{1}{\mathcal{DR}\left(x^\star\right)} \quad \text{if} \quad x_i^\star = 0$$

Most diversified portfolio

Another diversification concept

"Any stock not held by the MDP is more correlated to the MDP than any of the stocks that belong to it. Furthermore, all stocks belonging to the MDP have the same correlation to it. [...] This property illustrates that all assets in the universe are effectively represented in the MDP, even if the portfolio does not physically hold them. [...] This is consistent with the notion that the most diversified portfolio is the un-diversifiable portfolio" (Choueifaty et al., 2013)

Most diversified portfolio

Remark

In the case when the long-only constraint is omitted, we have $\rho(e_i, x^\star) = \rho(e_j, x^\star)$ meaning that the correlation with the MDP is the same for all the assets

Most diversified portfolio

Example 4

We consider an investment universe of four assets. Their volatilities are equal to 20%, 10%, 20% and 25%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.40 & 0.30 & 1.00 & \\ 0.50 & 0.10 & -0.10 & 1.00 \end{pmatrix}$$

Most diversified portfolio

Table 6: Composition of the MDP

Asset	Unconstrained		Long-only	
	$x_i^\star$	$\rho(e_i, x^\star)$	$x_i^\star$	$\rho(e_i, x^\star)$
1	-18.15	61.10	0.00	73.20
2	61.21	61.10	41.70	62.40
3	29.89	61.10	30.71	62.40
4	27.05	61.10	27.60	62.40
$\sigma(x^\star)$	9.31		10.74	
$\mathcal{DR}(x^\star)$	1.64		1.60	

Most diversified portfolio

Assumption $\mathcal{H}_0$: all the assets have the same Sharpe ratio

$$\frac{\mu_i - r}{\sigma_i} = s$$

Under $\mathcal{H}_0$, the diversification ratio of portfolio x is proportional to its Sharpe ratio:

$$\begin{aligned}
\mathcal{DR}(x) &= \frac{1}{s} \frac{\sum_{i=1}^n x_i (\mu_i - r)}{\sigma(x)} \\
&= \frac{1}{s} \frac{x^\top \mu - r}{\sigma(x)} \\
&= \frac{1}{s} \cdot \mathrm{SR}(x \mid r)
\end{aligned}$$

Most diversified portfolio

Optimality of the MDP

Under $\mathcal{H}_0$, maximizing the diversification ratio is then equivalent to maximizing the Sharpe ratio:

$$\text{MDP} = \text{MSR}$$

Most diversified portfolio

In the CAPM framework, Clarke *et al.* (2013) showed that:

$$x_i^\star = \mathcal{DR}\left(x^\star\right) \frac{\sigma_i \sigma\left(x^\star\right)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star}\right)$$

where $\sigma_i = \sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2}$ is the volatility of asset i, $\rho_{i,m} = \beta_i \sigma_m / \sigma_i$ is the correlation between asset i and the market portfolio and $\rho^\star$ is the threshold correlation given by this formula:

$$\rho^\star = \left(1 + \sum_{i=1}^n \frac{\rho_{i,m}^2}{1 - \rho_{i,m}^2}\right) \Bigg/ \left(\sum_{i=1}^n \frac{\rho_{i,m}}{1 - \rho_{i,m}^2}\right)$$

The weights are then strictly positive if $\rho_{i,m} < \rho^\star$

Most diversified portfolio

The MDP tends to be less concentrated than the MV portfolio because:

$$
\begin{aligned}
x_{\mathrm{mv},i} &= \frac{1}{\tilde{\sigma}_i^2} \times \cdots \\
x_{\mathrm{mdp},i} &= \frac{\sigma_i}{\tilde{\sigma}_i^2} \times \cdots \approx \frac{1}{\tilde{\sigma}_i} \times \cdots + \cdots
\end{aligned}
$$

ERC portfolio

In Lecture 2, we have seen that the ERC portfolio corresponds to the portfolio such that the risk contribution from each stock is made equal

The main advantages of the ERC allocation are the following:

1. It defines a portfolio that is well diversified in terms of risk and weights
2. Like the three previous risk-based methods, it does not depend on any expected returns hypothesis
3. It is less sensitive to small changes in the covariance matrix than MV or MDP portfolios (Demey *et al.*, 2010)

ERC portfolio

In the CAPM framework, Clarke *et al.* (2013) showed:

$$x_i^\star = \frac{\sigma^2(x^\star)}{\tilde{\sigma}_i^2}\left(\sqrt{\frac{\beta_i^2}{\beta^{\star 2}} + \frac{\tilde{\sigma}_i^2}{n\sigma^2(x^\star)}} - \frac{\beta_i}{\beta^\star}\right)$$

where:

$$\beta^\star = \frac{2\sigma^2(x^\star)}{\beta(x^\star)\,\sigma_m^2}$$

It follows that:

$$\lim_{n\to\infty} x_{\mathrm{erc}} = x_{\mathrm{ew}}$$

Comparison of the 4 Methods

Equally-weighted (EW)

- Weights are equal
- Easy to understand
- Contrarian strategy with a take-profit scheme
- The least concentrated in terms of weights
- Do not depend on risks

Most Diversified Portfolio (MDP)

- Also known as the Max Sharpe Ratio (MSR) portfolio of EDHEC
- Based on the assumption that sharpe ratio is equal for all stocks
- It is the tangency portfolio if the previous assumption is verified
- Sensitive to the covariance matrix

Minimum variance (MV)

- Low volatility portfolio
- The only optimal portfolio not depending on expected returns assumptions
- Good out of sample performance
- Concentrated portfolios
- Sensitive to the covariance matrix

Equal Risk Contribution (ERC)

- Risk contributions are equal
- Highly diversified portfolios
- Less sensitive to the covariance matrix (than the MV and MDP portfolios)
- Not efficient for universe with a large number of stocks (equivalent to the EW portfolio)

Some properties

In terms of bets

$$\begin{array}{ll} \exists i : w_i = 0 & \text{(MV - MDP)} \\ \forall i : w_i \neq 0 & \text{(EW - ERC)} \end{array}$$

In terms of risk factors

$$\begin{array}{rcll} x_i & = & x_j & \text{(EW)} \\ \frac{\partial\, \sigma(x)}{\partial\, x_i} & = & \frac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(MV)} \\ x_i \cdot \frac{\partial\, \sigma(x)}{\partial\, x_i} & = & x_j \cdot \frac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(ERC)} \\ \frac{1}{\sigma_i} \cdot \frac{\partial\, \sigma(x)}{\partial\, x_i} & = & \frac{1}{\sigma_j} \cdot \frac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(MDP)} \end{array}$$

Some properties

Proof for the MDP portfolio

For the unconstrained MDP portfolio, we recall that the first-order condition is given by:

$$\frac{\partial \mathcal{L}(x;\lambda_0,\lambda)}{\partial x_i} = \frac{\sigma_i}{x^\top \sigma} - \frac{(\Sigma x)_i}{x^\top \Sigma x} = 0$$

The scaled marginal volatility is then equal to the inverse of the diversification ratio of the MDP:

$$\begin{aligned}
\frac{1}{\sigma_i} \cdot \frac{\partial \sigma(x)}{\partial x_i} &= \frac{1}{\sigma_i} \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
&= \frac{\sigma(x)}{\sigma_i} \cdot \frac{(\Sigma x)_i}{x^\top \Sigma x} \\
&= \frac{\sigma(x)}{x^\top \sigma} = \frac{1}{\mathcal{DR}(x)}
\end{aligned}$$

Application to the Eurostoxx 50 index

Table 7: Composition in % (January 2010)

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
TOTAL	6.1		2.1		2			5.0	
BANCO SANTANDER	5.8		1.3		2				
TELEFONICA SA	5.0	31.2	3.5		2	10.0		5.0	5.0
SANOFI-AVENTIS	3.6	12.1	4.5	15.5	2	10.0	10.0	5.0	5.0
E.ON AG	3.6		2.1		2				1.4
BNP PARIBAS	3.4		1.1		2				
SIEMENS AG	3.2		1.5		2				
BBVA(BILB-VIZ-ARG)	2.9		1.4		2				
BAYER AG	2.9		2.6	3.7	2	2.2	5.0	5.0	5.0
ENI	2.7		2.1		2				
GDF SUEZ	2.5		2.6	4.5	2		5.4	5.0	5.0
BASF SE	2.5		1.5		2				
ALLIANZ SE	2.4		1.4		2				
UNICREDIT SPA	2.3		1.1		2				
SOC GENERALE	2.2		1.2	3.9	2		3.7		5.0
UNILEVER NV	2.2	11.4	3.7	10.8	2	10.0	10.0	5.0	5.0
FRANCE TELECOM	2.1	14.9	4.1	10.2	2	10.0	10.0	5.0	5.0
NOKIA OYJ	2.1		1.8	4.5	2		4.8		5.0
DAIMLER AG	2.1		1.3		2				
DEUTSCHE BANK AG	1.9		1.0		2				
DEUTSCHE TELEKOM	1.9		3.2	2.6	2	5.7	3.7	5.0	5.0
INTESA SANPAOLO	1.9		1.3		2				
AXA	1.8		1.0		2				
ARCELORMITTAL	1.8		1.0		2				
SAP AG	1.8	21.0	3.4	11.2	2	10.0	10.0	5.0	5.0
RWE AG (NEU)	1.7	2.7	2.7		2	7.0		5.0	
ING GROEP NV	1.6		0.8	0.4	2				
DANONE	1.6	1.9	3.4	1.8	2	8.7	3.3	5.0	5.0
IBERDROLA SA	1.6		2.5		2	5.1		5.0	1.2
ENEL	1.6		2.1		2			5.0	2.9
VIVENDI SA	1.6	2.8	3.1	4.5	2	10.0	5.9	5.0	5.0
ANHEUSER-BUSCH INB	1.6	0.2	2.7	10.9	2	2.1	10.0	5.0	5.0
ASSIC GENERALI SPA	1.6		1.8		2				
AIR LIQUIDE(L')	1.4		2.1		2			5.0	
MUENCHENER RUECKVE	1.3		2.1	2.1	2		3.1	5.0	5.0
SCHNEIDER ELECTRIC	1.3		1.5		2				
CARREFOUR	1.3	1.0	2.7	1.3	2	3.7	2.5	5.0	5.0
VINCI	1.3		1.6		2				
LVMH MOET HENNESSY	1.2		1.8		2				
PHILIPS ELEC(KON)	1.2		1.4		2				
L'OREAL	1.1	0.8	2.8		2	5.5		5.0	5.0
CIE DE ST-GOBAIN	1.0		1.1		2				
REPSOL YPF SA	0.9		2.0		2			5.0	
CRH	0.8		1.7	5.1	2		5.2		5.0
CREDIT AGRICOLE SA	0.8		1.1		2				
DEUTSCHE BOERSE AG	0.7		1.5		2				1.9
TELECOM ITALIA SPA	0.7		2.0		2				2.5
ALSTOM	0.6		1.5		2				
AEGON NV	0.4		0.7		2				
VOLKSWAGEN AG	0.2		1.8	7.1	2		7.4		5.0
Total of components	50	11	50	17	50	14	16	20	23

Some examples

To compare the risk-based methods, we report:

- The weights x_i in %
- The relative risk contributions $\mathcal{RC}_i$ in %
- The weight concentration $\mathcal{H}^\star(x)$ in % and the risk concentration $\mathcal{H}^\star(\mathcal{RC})$ in % where $\mathcal{H}^\star$ is the modified Herfindahl index[2]
- The portfolio volatility $\sigma(x)$ in %
- The diversification ratio $\mathcal{DR}(x)$

[2] We have:

$$\mathcal{H}^\star(\pi) = \frac{n\mathcal{H}(\pi) - 1}{n - 1} \in [0, 1]$$

Some examples

Example 5

We consider an investment universe with four assets. We assume that the volatility σ_i is the same and equal to 20% for all four assets. The correlation matrix C is equal to:

$$C = \begin{pmatrix} 100\% & & & \\ 80\% & 100\% & & \\ 0\% & 0\% & 100\% & \\ 0\% & 0\% & -50\% & 100\% \end{pmatrix}$$

Some examples

Table 8: Weights and risk contributions (Example 5)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	4.20	10.87	0.96	10.87	0.96	17.26	2.32
2	25.00	4.20	10.87	0.96	10.87	0.96	17.26	2.32
3	25.00	1.17	39.13	3.46	39.13	3.46	32.74	2.32
4	25.00	1.17	39.13	3.46	39.13	3.46	32.74	2.32
$\mathcal{H}^\star(x)$	0.00		10.65		10.65		3.20	
$\sigma(x)$	10.72		8.85		8.85		9.26	
$\mathcal{DR}(x)$	1.87		2.26		2.26		2.16	
$\mathcal{H}^\star(\mathcal{RC})$	10.65		10.65		10.65		0.00	

Some examples

Example 6

We modify the previous example by introducing differences in volatilities. They are 10%, 20%, 30% and 40% respectively. The correlation matrix remains the same as in Example 5.

Some examples

Table 9: Weights and risk contributions (Example 6)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	1.41	74.48	6.43	27.78	1.23	38.36	2.57
2	25.00	3.04	0.00	0.00	13.89	1.23	19.18	2.57
3	25.00	1.63	15.17	1.31	33.33	4.42	24.26	2.57
4	25.00	5.43	10.34	0.89	25.00	4.42	18.20	2.57
$\mathcal{H}^\star(x)$	0.00		45.13		2.68		3.46	
$\sigma(x)$	11.51		8.63		11.30		10.29	
$\mathcal{DR}(x)$	2.17		1.87		2.26		2.16	
$\mathcal{H}^\star(\mathcal{RC})$	10.31		45.13		10.65		0.00	

Some examples

Example 7

We now reverse the volatilities of Example 6. They are now equal to 40%, 30%, 20% and 10%.

Some examples

Table 10: Weights and risk contributions (Example 7)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	9.32	0.00	0.00	4.18	0.74	7.29	1.96
2	25.00	6.77	4.55	0.29	5.57	0.74	9.72	1.96
3	25.00	1.09	27.27	1.74	30.08	2.66	27.66	1.96
4	25.00	0.00	68.18	4.36	60.17	2.66	55.33	1.96
$\mathcal{H}^\star(x)$	0.00		38.84		27.65		19.65	
$\sigma(x)$	17.18		6.40		6.80		7.82	
$\mathcal{DR}(x)$	1.46		2.13		2.26		2.16	
$\mathcal{H}^\star(\mathcal{RC})$	27.13		38.84		10.65		0.00	

Some examples

Example 8

We consider an investment universe of four assets. The volatility is respectively equal to 15%, 30%, 45% and 60% whereas the correlation matrix C is equal to:

$$C = \begin{pmatrix} 100\% & & & \\ 10\% & 100\% & & \\ 30\% & 30\% & 100\% & \\ 40\% & 20\% & -50\% & 100\% \end{pmatrix}$$

Some examples

Table 11: Weights and risk contributions (Example 8)

	EW		MV		MDP		ERC	
Asset	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	2.52	82.61	11.50	0.00	0.00	40.53	4.52
2	25.00	5.19	17.39	2.42	0.00	0.00	22.46	4.52
3	25.00	3.89	0.00	0.00	57.14	12.86	21.12	4.52
4	25.00	9.01	0.00	0.00	42.86	12.86	15.88	4.52
$\mathcal{H}^{\star}(x)$	0.00		61.69		34.69		4.61	
$\sigma(x)$	20.61		13.92		25.71		18.06	
$\mathcal{DR}(x)$	1.82		1.27		2.00		1.76	
$\mathcal{H}^{\star}(\mathcal{RC})$	7.33		61.69		33.33		0.00	

Some examples

Example 9

Now we consider an example with six assets. The volatilities are 25%, 20%, 15%, 18%, 30% and 20% respectively. We use the following correlation matrix:

$$C = \begin{pmatrix} 100\% & & & & & \\ 20\% & 100\% & & & & \\ 60\% & 60\% & 100\% & & & \\ 60\% & 60\% & 60\% & 100\% & & \\ 60\% & 60\% & 60\% & 60\% & 100\% & \\ 60\% & 60\% & 60\% & 60\% & 60\% & 100\% \end{pmatrix}$$

Some examples

Table 12: Weights and risk contributions (Example 9)

	EW		MV		MDP		ERC	
Asset	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	16.67	3.19	0.00	0.00	44.44	8.61	14.51	2.72
2	16.67	2.42	6.11	0.88	55.56	8.61	18.14	2.72
3	16.67	2.01	65.16	9.33	0.00	0.00	21.84	2.72
4	16.67	2.45	22.62	3.24	0.00	0.00	18.20	2.72
5	16.67	4.32	0.00	0.00	0.00	0.00	10.92	2.72
6	16.67	2.75	6.11	0.88	0.00	0.00	16.38	2.72
$\mathcal{H}^\star(x)$	0.00		37.99		40.74		0.83	
$\sigma(x)$	17.14		14.33		17.21		16.31	
$\mathcal{DR}(x)$	1.24		1.14		1.29		1.25	
$\mathcal{H}^\star(\mathcal{RC})$	1.36		37.99		40.00		0.00	

Some examples

Example 10

To illustrate how the MV and MDP portfolios are sensitive to specific risks, we consider a universe of n assets with volatility equal to 20%. The structure of the correlation matrix is the following:

$$C = \begin{pmatrix} 100\% & & & & \\ \rho_{1,2} & 100\% & & & \\ 0 & \rho & 100\% & & \\ \vdots & \vdots & \ddots & 100\% & \\ 0 & \rho & \cdots & \rho & 100\% \end{pmatrix}$$

Some examples

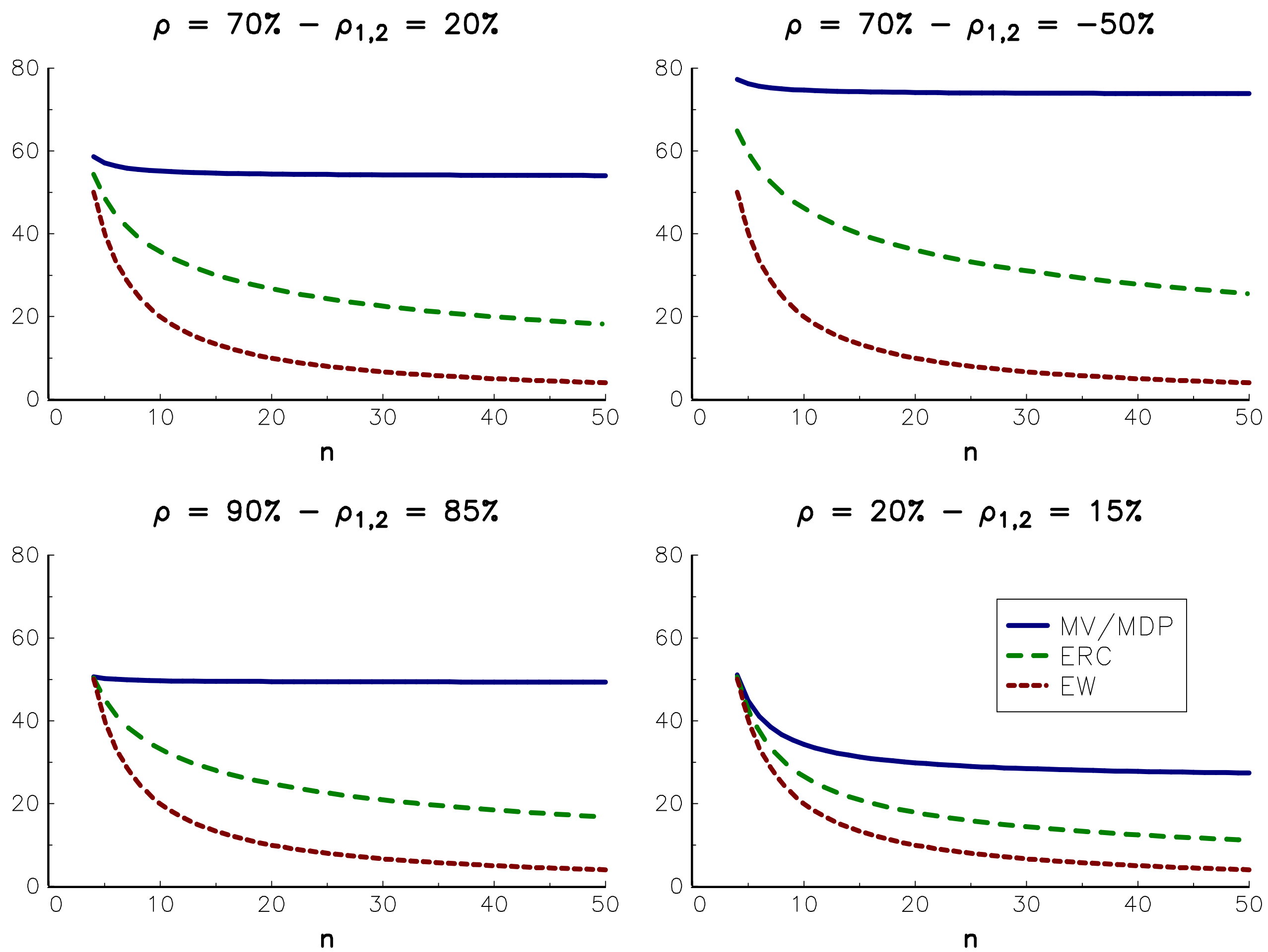


Figure 4: Weight of the first two assets in AW portfolios (Example 10)

Some examples

Example 11

We assume that asset returns follow the one-factor CAPM model. The idiosyncratic volatility $\tilde{\sigma}_i$ is set to 5% for all the assets whereas the volatility of the market portfolio σ_m is equal to 25%.

Some examples

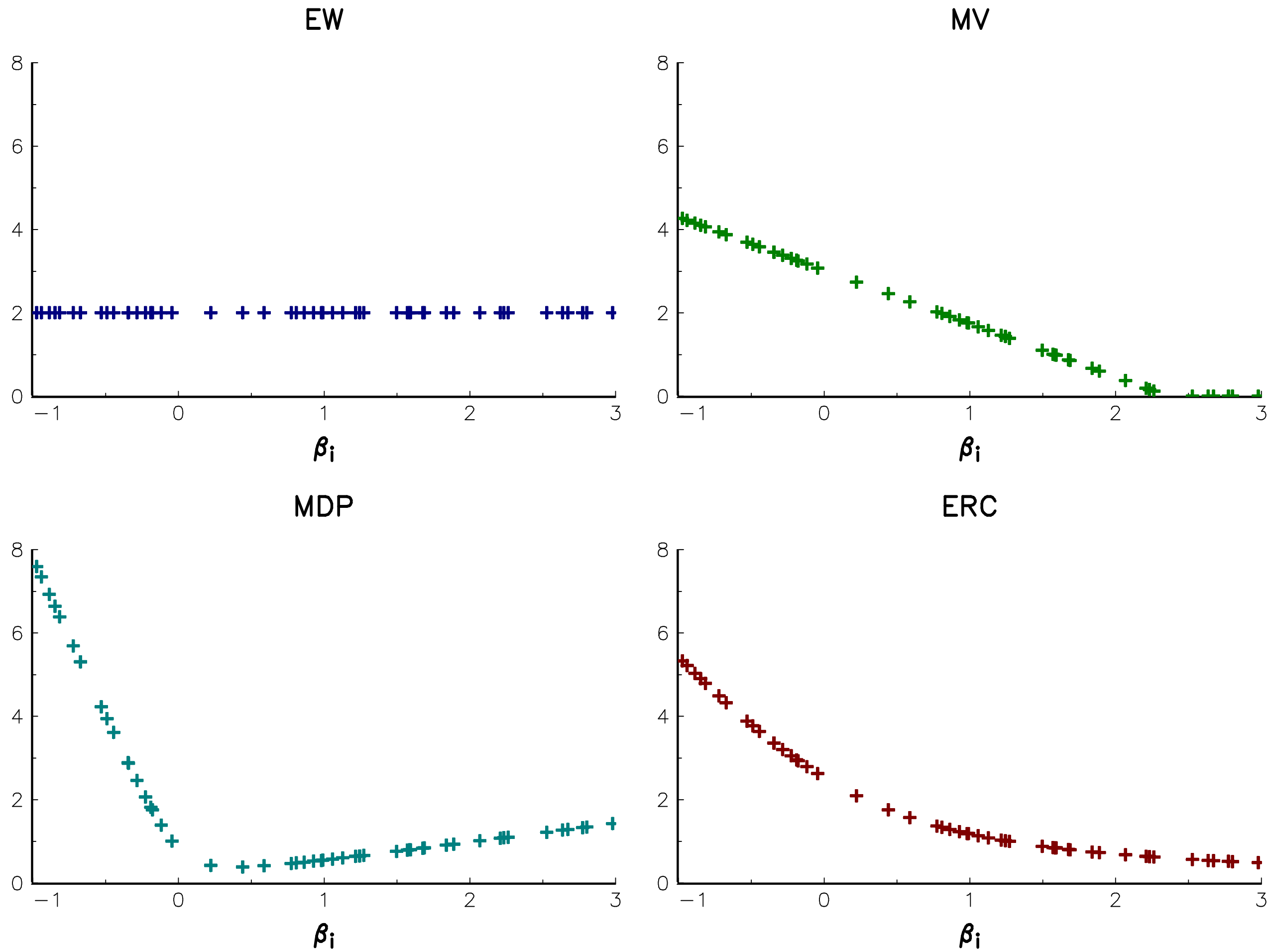


Figure 5: Weight with respect to the asset beta β_i (Example 11)

Smart beta products

- MSCI Equal Weighted Indexes (EW)
  www.msci.com/msci-equal-weighted-indexes
- S&P 500 Equal Weight Index (EW)
  www.spglobal.com/spdji/en/indices/equity/sp-500-equal-weight-index
- FTSE UK Equally Weighted Index Series (EW)
  www.ftserussell.com/products/indices/equally-weighted
- FTSE Global Minimum Variance Index Series (MV)
  www.ftserussell.com/products/indices/min-variance
- MSCI Minimum Volatility Indexes (MV)
  www.msci.com/msci-minimum-volatility-indexes
- S&P 500 Minimum Volatility Index (MV)
  www.spglobal.com/spdji/en/indices/strategy/sp-500-minimum-volatility-index
- FTSE Global Equal Risk Contribution Index Series (ERC)
  www.ftserussell.com/products/indices/erc
- TOBAM MaxDiv Index Series (MDP)
  www.tobam.fr/maximum-diversification-indexes

Smart beta products

Largest ETF issuers in Europe

1. iShares (BlackRock)
2. Xtrackers (DWS)
3. Lyxor ETF
4. UBS ETF
5. Amundi ETF

Largest ETF issuers in US

1. iShares (BlackRock)
2. SPDR (State Street)
3. Vanguard
4. Invesco PowerShares
5. First Trust

- Specialized smart beta ETF issuers: Wisdom Tree (US), Ossiam (Europe), Research affiliates (US), etc.
- Smart beta fund managers in Europe: Amundi, Ossiam, Quoniam, Robeco, Seeyond, Tobam, Unigestion, etc.
- ETFs, mutual funds, mandates

The case of bonds

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

The case of bonds

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

Bond indexation

Debt weighting

It is defined by:

$$x_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

GDP weighting

It is defined by:

$$x_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

Risk budgeting

It is defined by:

$$b_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

or:

$$b_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

⇒ The offering is very small compared to equity indices because of the liquidity issues (see Roncalli (2013), Chapter 4 for more details)

From CAPM to factor investing

How to define risk factors?

Risk factors are common factors that explain the cross-section variance of expected returns

- 1964: Market or MKT (or BETA) factor
- 1972: Low beta or BAB factor
- 1981: Size or SMB factor
- 1985: Value or HML factor
- 1991: Low volatility or VOL factor
- 1993: Momentum or WML factor
- 2000: Quality or QMJ factor

Systematic risk factors $\neq$ Idiosyncratic risk factors

Beta(s) $\neq$ Alpha(s)

Alpha or beta?

At the security level, there is a lot of idiosyncratic risk or alpha[3]:

	Common Risk	Idiosyncratic Risk
GOOGLE	47%	53%
NETFLIX	24%	76%
MASTERCARD	50%	50%
NOKIA	32%	68%
TOTAL	89%	11%
AIRBUS	56%	44%

Carhart's model with 4 factors, 2010-2014
Source: Roncalli (2017)

[3]The linear regression is:

$$R_i = \alpha_i + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \mathcal{F}_j + \varepsilon_i$$

In our case, we measure the alpha as $1 - \mathfrak{R}_i^2$ where:

$$\mathfrak{R}_i^2 = 1 - \frac{\sigma^2(\varepsilon_i)}{\sigma^2(R_i)}$$

The concept of alpha

- Jensen (1968) – **How to measure the performance of active fund managers?**

$$R_t^F = \alpha + \beta R_t^{MKT} + \varepsilon_t$$

Fund	Return	Rank	Beta	Alpha	Rank
A	12%	Best	1.0	−2%	Worst
B	11%	Worst	0.5	4%	Best

Market return = 14%

$$\Rightarrow \bar{\alpha} = -\text{fees}$$

- It is the beginning of passive management:
  - John McQuown (Wells Fargo Bank, 1971)
  - Rex Sinquefield (American National Bank, 1973)

Active management and performance persistence

- Hendricks *et al.* (1993) – **Hot Hands in Mutual Funds**

$$\operatorname{cov}\left(\alpha_t^{\mathsf{Jensen}}, \alpha_{t-1}^{\mathsf{Jensen}}\right) > 0$$

where:

$$\alpha_t^{\mathsf{Jensen}} = R_t^F - \beta^{\mathrm{MKT}} R_t^{MKT}$$

$\Rightarrow$ The persistence of the performance of active management is due to the **persistence of the alpha**

Risk factors and active management

- Grinblatt *et al.* (1995) – **Momentum investors versus Value investors**

  "*77% of mutual funds are momentum investors*"

- Carhart (1997):

$$\left\{ \begin{array}{l} \operatorname{cov}\left(\alpha_t^{\text{Jensen}}, \alpha_{t-1}^{\text{Jensen}}\right) > 0 \\ \operatorname{cov}\left(\alpha_t^{\text{Carhart}}, \alpha_{t-1}^{\text{Carhart}}\right) = 0 \end{array} \right.$$

  where:

$$\alpha_t^{\text{Carhart}} = R_t^F - \beta^{\text{MKT}} R_t^{\text{MKT}} - \beta^{\text{SMB}} R_t^{\text{SMB}} - \beta^{\text{HML}} R_t^{\text{HML}} - \beta^{\text{WML}} R_t^{\text{WML}}$$

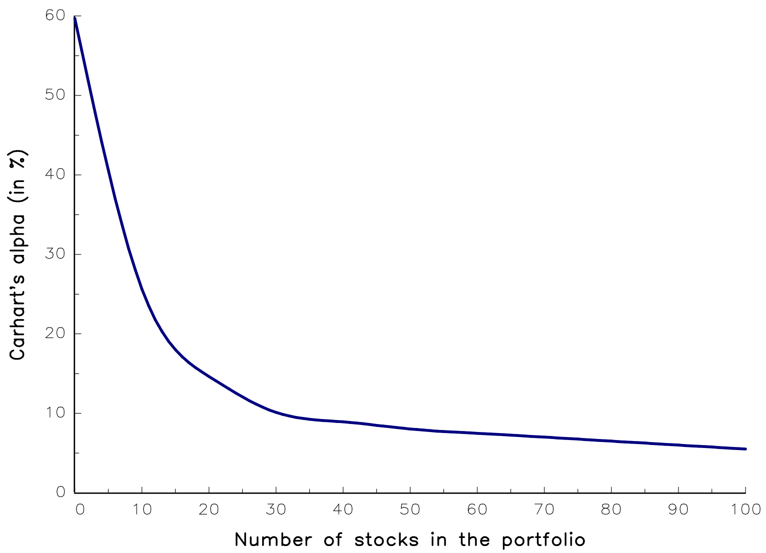
$\Rightarrow$ The (short-term) persistence of the performance of active management is due to the (short-term) **persistence of the performance of risk factors**

Diversification and alpha

David Swensen's rule for effective stock picking

Concentrated portfolio ⇒ No more than 20 bets?

Figure 6: Carhart's alpha decreases with the number of holding assets

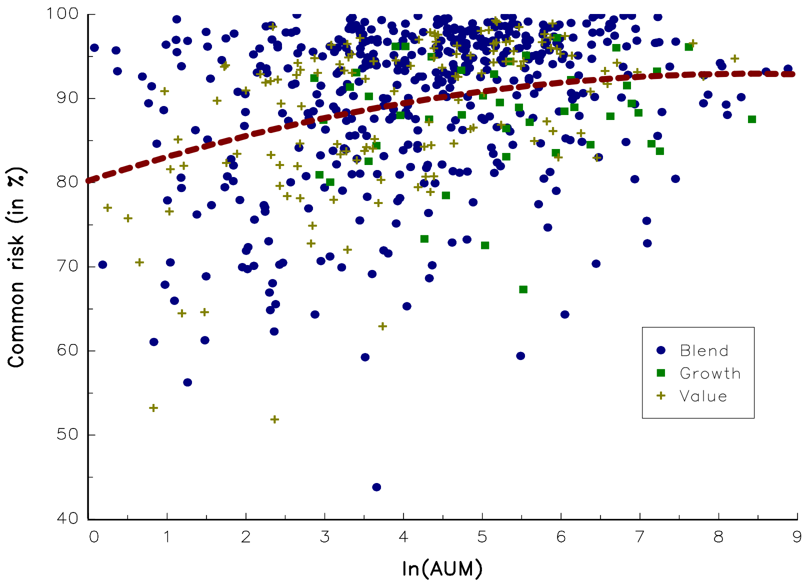


US equity markets, 2000-2014
Source: Roncalli (2017)

"*If you can identify six wonderful businesses, that is all the diversification you need. And you will make a lot of money. And I can guarantee that going into the seventh one instead of putting more money into your first one is going to be a terrible mistake. Very few people have gotten rich on their seventh best idea.*" (Warren Buffett, University of Florida, 1998).

Diversification and alpha

Figure 7: What proportion of return variance is explained by the 4-factor model?



Morningstar database, 880 mutual funds, European equities
Carhart's model with 4 factors, 2010-2014
Source: Roncalli (2017)

How many bets are there in large portfolios of institutional investors?

- **1986** Less than 10% of institutional portfolio return is explained by security picking and market timing (Brinson *et al.*, 1986)
- **2009** Professors' Report on the Norwegian GPFG: Risk factors represent 99.1% of the fund return variation (Ang *et al.*, 2009)

Risk factors versus alpha

What lessons can we draw from this?

Idiosyncratic risks and specific bets disappear in (large) diversified portfolios. Performance of institutional investors is then exposed to (common) risk factors.

Alpha is not scalable, but risk factors are scalable

$\Rightarrow$ Risk factors are the only bets that are compatible with diversification

Alpha

- Concentration
- Scarce?

$\neq$

Beta(s)

- Diversification
- Easy access?

Factor investing and active management

Misconception about active management

- Active management $= \boldsymbol{\alpha}$
- Passive management $= \boldsymbol{\beta}$

In this framework, passive management encompasses cap-weighted indexation, risk-based indexation and factor investing because these management styles do not pretend to create alpha

Factor investing and active management

"The question is when is active management good? The answer is never"

Eugene Fama, Morningstar ETF conference, September 2014

"So people say, 'I'm not going to try to beat the market. The market is all-knowing.' But how in the world can the market be all-knowing, if nobody is trying – well, not as many people – are trying to beat it?"

Robert Shiller, CNBC, November 2017

Factor investing and active management

- Discretionary active management ⇒ specific/idiosyncratic risks & rule-based management ⇒ factor investing and systematic risks?

- Are common risk factors exogenous or endogenous?
- Do risk factors exist without active management?

Risk factors first, active management second

or

Active management first, risk factors second

- Factor investing needs active investing
- Imagine a world without active managers, stock pickers, hedge funds, etc.

⇒ **Should active management be reduced to alpha management?**

Factor investing and active management

- Market risk factor = average of active management
- Low beta/low volatility strategies begin to be implemented in 2003-2004 (after the dot.com crisis)
- Quality strategies begin to be implemented in 2009-2010 (after the GFC crisis)

Alpha strategy $\Rightarrow$ **Risk Factor** (or a beta strategy)

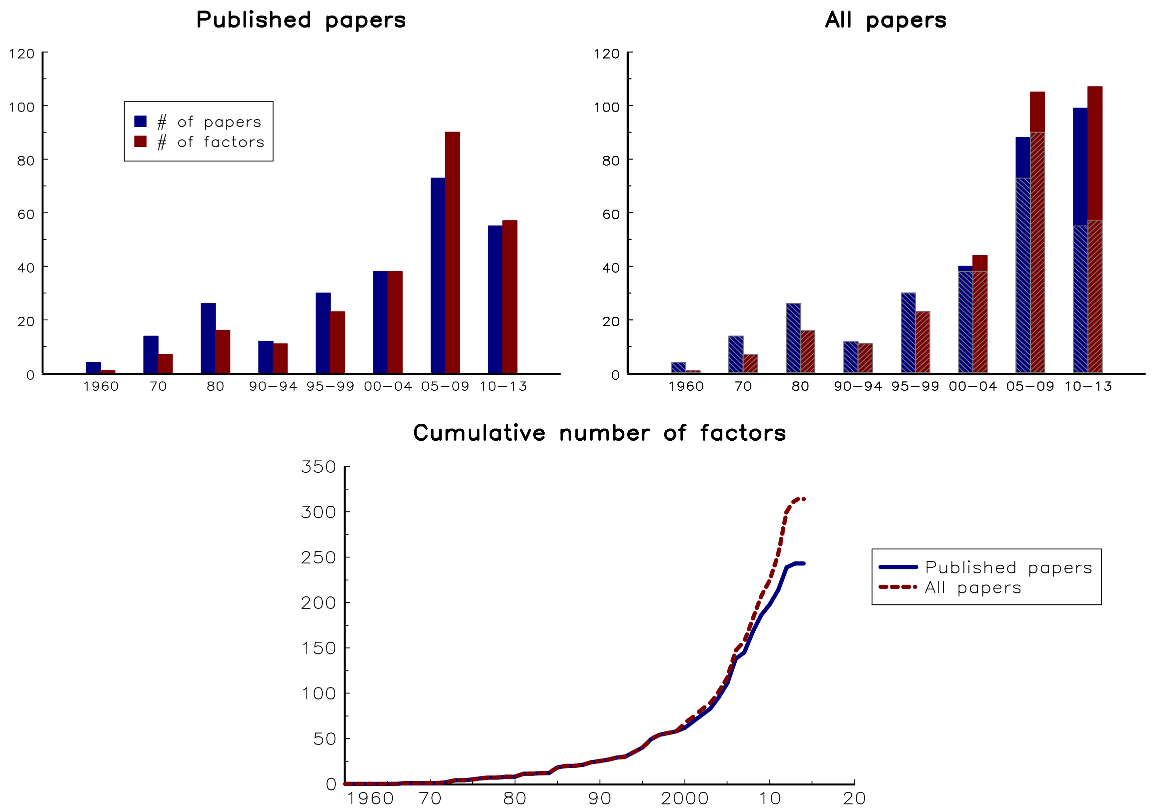
Factor investing and active management

α or β?

"[...] When an alpha strategy is massively invested, it has an enough impact on the structure of asset prices to become a risk factor.
[...] Indeed, an alpha strategy becomes a common market risk factor once it represents a significant part of investment portfolios and explains the cross-section dispersion of asset returns" (Roncalli, 2020)

The factor zoo

Figure 8: Harvey *et al.* (2016)



"*Now we have a zoo of new factors*" (Cochrane, 2011).

Factors, factors everywhere

"Standard predictive regressions fail to reject the hypothesis that the party of the U.S. President, the weather in Manhattan, global warming, El Niño, sunspots, or the conjunctions of the planets, are significantly related to anomaly performance. These results are striking, and quite surprising. In fact, some readers may be inclined to reject some of this paper's conclusions solely on the grounds of plausibility. I urge readers to consider this option carefully, however, as doing do so entails rejecting the standard methodology on which the return predictability literature is built."(Novy-Marx, 2014).

⇒ MKT, SMB, HML, WML, STR, LTR, VOL, IVOL, BAB, QMJ, LIQ, TERM, CARRY, DIV, JAN, CDS, GDP, INF, etc.

The alpha puzzle (Cochrane, 2011)

- Chaos

$$\mathbb{E}[R_i] - R_f = \boxed{\alpha_i}$$

- Sharpe (1964)

$$\mathbb{E}[R_i] - R_f = \beta_i^m (\mathbb{E}[R_m] - R_f)$$

- Chaos again

$$\mathbb{E}[R_i] - R_f = \boxed{\alpha_i} + \beta_i^m (\mathbb{E}[R_m] - R_f)$$

- Fama and French (1992)

$$\mathbb{E}[R_i] - R_f = \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \beta_i^{hml}\mathbb{E}[R_{hml}]$$

This is not the end of the story...

The alpha puzzle (Cochrane, 2011)

It's just the beginning!

- Chaos again

$$\mathbb{E}[R_i] - R_f = \boxed{\alpha_i} + \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \beta_i^{hml}\mathbb{E}[R_{hml}]$$

- Carhart (1997)

$$\mathbb{E}[R_i] - R_f = \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \beta_i^{hml}\mathbb{E}[R_{hml}] + \beta_i^{wml}\mathbb{E}[R_{wml}]$$

- Chaos again

$$\begin{aligned}\mathbb{E}[R_i] - R_f &= \boxed{\alpha_i} + \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \\ &\quad \beta_i^{hml}\mathbb{E}[R_{hml}] + \beta_i^{wml}\mathbb{E}[R_{wml}]\end{aligned}$$

- Etc.

How can alpha always come back?

The alpha puzzle (Cochrane, 2011)

1. **Because academic backtesting is not the real life**

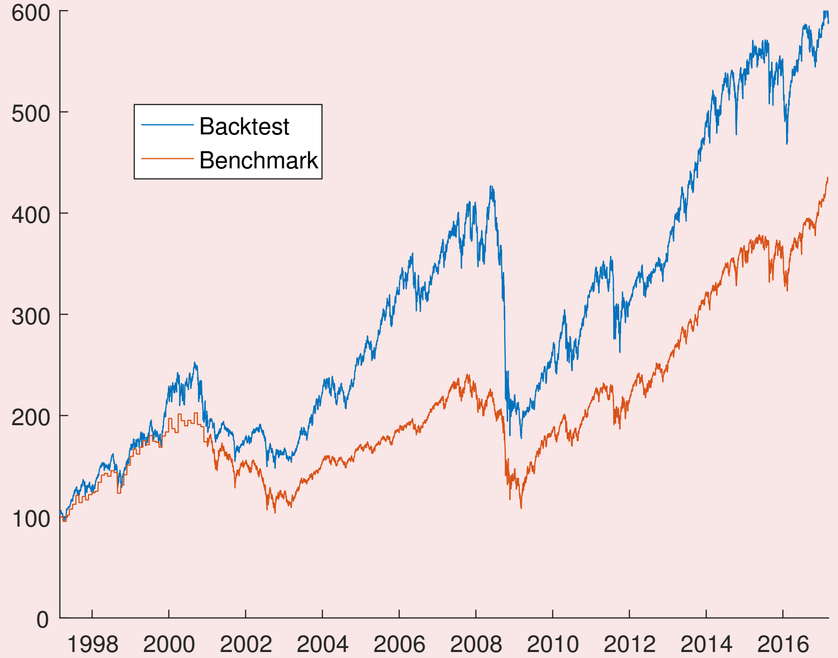
2. **Because risk factors are not independent in practice**

3. **Because the explanatory power of risk factors is time-varying**

4. **Because alpha and beta are highly related (beta strategy = successful alpha strategy)**

The issue of backtesting

Backtesting syndrome



The blue line is above the red line ⇒ it's OK!

⇒ Analytical models are important to understand a risk factor

The professional consensus

There is now a consensus among professionals that five factors are sufficient for the equity markets:

1. **Size**

   Small cap stocks $\neq$ Large cap stocks

2. **Value**

   Value stocks $\neq$ Non-value stocks (including growth stocks)

3. **Momentum**

   Past winners $\neq$ Past loosers

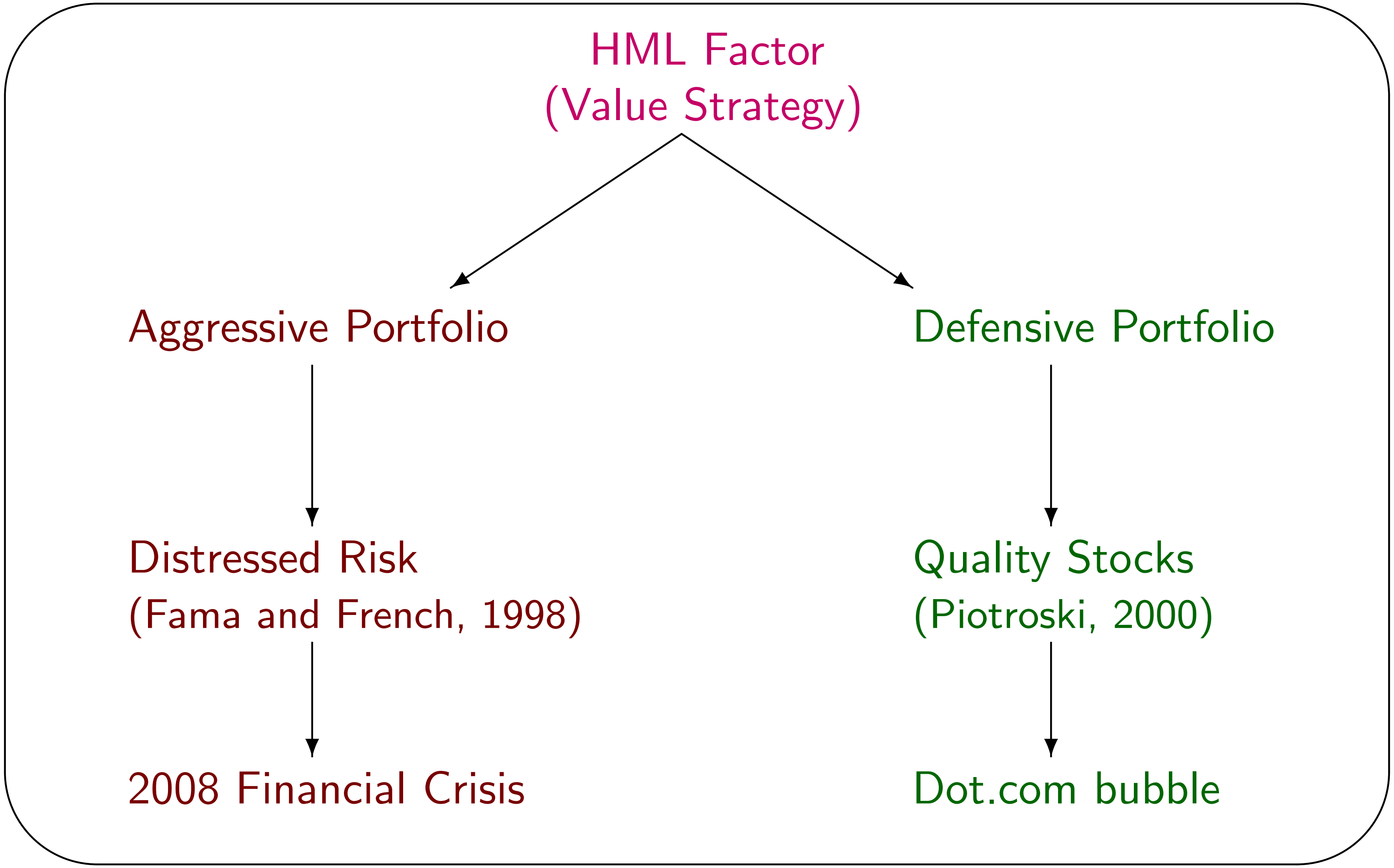
4. **Low-volatility**

   Low-vol (or low-beta) stocks $\neq$ High-vol (or high-beta stocks)

5. **Quality**

   Quality stocks $\neq$ Non-quality stocks (including junk stocks)

The example of the value risk factor



The example of the dividend yield risk factor

- Book-to-price (value risk factor):

$$\mathrm{B2P} = \frac{B}{P}$$

- Dividend yield (carry risk factor):

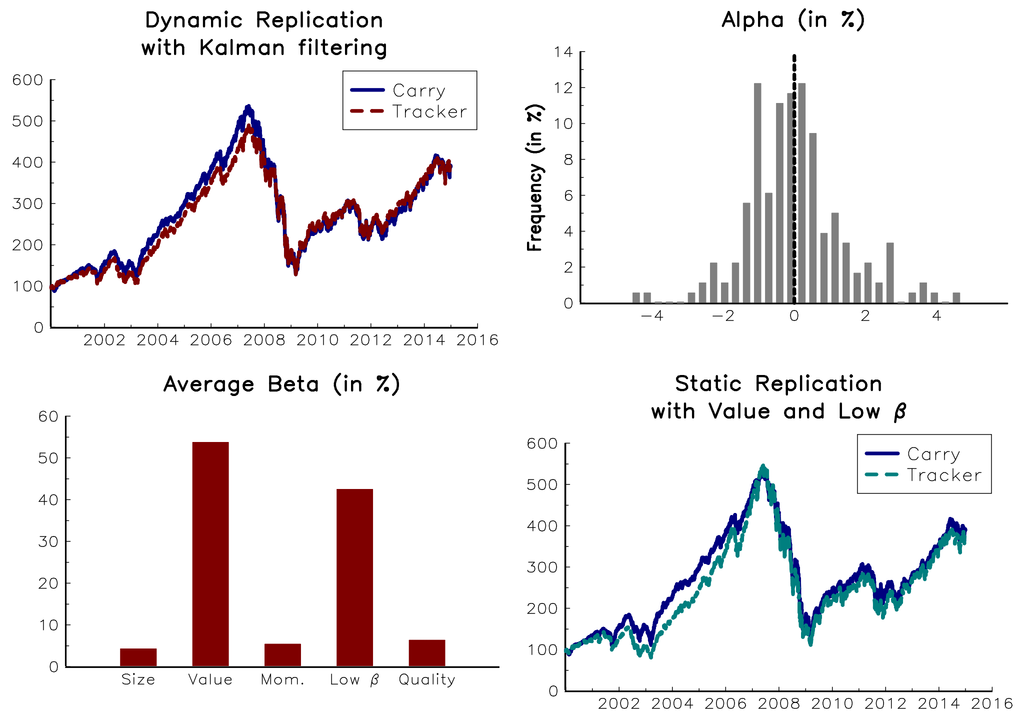
$$\begin{aligned} \mathrm{DY} &= \frac{D}{P} \\ &= \frac{D}{B} \times \frac{B}{P} \\ &= \mathrm{D2B} \times \mathrm{B2P} \end{aligned}$$

- Value component (book and dividend = low-frequency, price = high-frequency)
- Low-volatility component (bond-like stocks)

Risk factors are not orthogonal, they are correlated

The example of the dividend yield risk factor

Figure 9: Value, low beta and carry are not orthogonal risk factors



Source: Richard and Roncalli (2015)

$$\text{Carry} \simeq 60\% \text{ Value} + 40\% \text{ Low-volatility}$$

The example of the dividend yield risk factor

- Why Size + Value + Momentum + Low-volatility + Quality?
- Why not Size + Carry + Momentum + Low-volatility + Quality or Size + Carry + Momentum + Value + Quality?
- Because:

$$\begin{aligned}\text{Carry} &\simeq 60\%\ \text{Value} + 40\%\ \text{Low-volatility} \\ \text{Value} &\simeq 167\%\ \text{Carry} - 67\%\ \text{Low-volatility} \\ \text{Low-volatility} &\simeq 250\%\ \text{Carry} - 150\%\ \text{Value}\end{aligned}$$

Question

Why Value + Momentum + Low-volatility + Quality
and not
Size + Value + Momentum + Low-volatility + Quality?

General approach

- We consider a universe $\mathcal{U}$ of stocks (e.g. the MSCI World Index)
- We define a rebalancing period (e.g. every month, every quarter or every year)
- At each rebalancing date t_τ:
  - We define a score $\mathbb{S}_i(t_\tau)$ for each stock i
  - Stocks with high scores are selected to form the long exposure $\mathcal{L}(t_\tau)$ of the risk factor
  - Stocks with low scores are selected to form the short exposure $\mathcal{S}(t_\tau)$ of the risk factor
- We specify a weighting scheme $w_i(t_\tau)$ (e.g. value weighted or equally weighted)

General approach

- The performance of the risk factor between two rebalancing dates corresponds to the performance of the long/short portfolio:

$$\mathcal{F}(t) = \mathcal{F}(t_\tau) \cdot \left(\sum_{i \in \mathcal{L}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) - \sum_{i \in \mathcal{S}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) \right)$$

  where $t \in \left]t_\tau, t_{\tau+1}\right]$ and $\mathcal{F}(t_0) = 100$.

- In the case of a long-only risk factor, we only consider the long portfolio:

$$\mathcal{F}(t) = \mathcal{F}(t_\tau) \cdot \left(\sum_{i \in \mathcal{L}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) \right)$$

The Fama-French approach

The SMB and HML factors are defined as follows:

$$\mathrm{SMB}_t = \frac{1}{3}\left(R_t\left(\mathrm{SV}\right) + R_t\left(\mathrm{SN}\right) + R_t\left(\mathrm{SG}\right)\right) - \frac{1}{3}\left(R_t\left(\mathrm{BV}\right) + R_t\left(\mathrm{BN}\right) + R_t\left(\mathrm{BG}\right)\right)$$

and:

$$\mathrm{HML}_t = \frac{1}{2}\left(R_t\left(\mathrm{SV}\right) + R_t\left(\mathrm{BV}\right)\right) - \frac{1}{2}\left(R_t\left(\mathrm{SG}\right) + R_t\left(\mathrm{BG}\right)\right)$$

with the following 6 portfolios[4]:

	Value	Neutral	Growth
Small	SV	SN	SG
Big	BV	BN	BG

[4] We have:

- The scores are the market equity (ME) and the book equity to market equity (BE/ME)
- The size breakpoint is the median market equity (Small = 50% and Big = 50%)
- The value breakpoints are the 30th and 70th percentiles of BE/ME (Value = 30%, Neutral = 40% and Growth = 30%)

The Fama-French approach

Homepage of Kenneth R. French

You can download data at the following webpage:

https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

- Asia Pacific ex Japan
- Developed
- Developed ex US
- Europe
- Japan
- North American
- US

Quintile portfolios

In this approach, we form five quintile portfolios:

- Q_1 corresponds to the stocks with the highest scores (top 20%)
- Q_2, Q_3 and Q_4 are the second, third and fourth quintile portfolios
- Q_5 corresponds to the stocks with the lowest scores (bottom 20%)

$\Rightarrow$ The long/short risk factor is the performance of $Q_1 - Q_5$, whereas the long-only risk factor is the performance of Q_1

The construction of risk factors

Table 13: An illustrative example

Asset	Score	Rank	Quintile	Selected	L/S	Weight
A_1	1.1	3	Q_2			
A_2	0.5	4	Q_2			
A_3	-1.3	9	Q_5	✓	**Short**	-50%
A_4	1.5	2	Q_1	✓	**Long**	$+50\%$
A_5	-2.8	10	Q_5	✓	**Short**	-50%
A_6	0.3	5	Q_3			
A_7	0.1	6	Q_3			
A_8	2.3	1	Q_1	✓	**Long**	$+50\%$
A_9	-0.7	8	Q_4			
A_{10}	-0.3	7	Q_4			

The scoring system

Variable selection

- Size: market capitalization
- Value: Price to book, price to earnings, price to cash flow, dividend yield, etc.
- Momentum = one-year price return ex 1 month, 13-month price return minus one-month price return, etc.
- Low volatility = one-year rolling volatility, one-year rolling beta, etc.
- Quality: Profitability, leverage, ROE, Debt to Assets, etc.

The scoring system

Variable combination

- Z-score averaging
- Ranking system
- Bottom exclusion
- Etc.

⇒ Finally, we obtain one score for each stock (e.g. the value score, the quality score, etc.)

Single-factor exposure versus multi-factor portfolio

Single-factor

- Trading bet
- Tactical asset allocation (TAA)
- If the investor believe that value stocks will outperform growth stocks in the next six months, he will overweight value stocks or add an exposure on the value risk factor
- Active management

Multi-factor

- Long-term bet
- Strategic asset allocation (SAA)
- The investor believe that a factor investing portfolio allows to better capture the equity risk premium than a CW index
- Factor investing portfolio = diversified portfolio (across risk factors)

Multi-factor portfolio

- Long/short: Market + Size + Value + Momentum + Low-volatility + Quality
- Long-only: Size + Value + Momentum + Low-volatility + Quality (because the market risk factor is replicated by the other risk factors)

Risk factors in sovereign bonds

> *"Market participants have long recognized the importance of identifying the common factors that affect the returns on U.S. government bonds and related securities. To explain the variation in these returns, it is critical to distinguish the systematic risks that have a general impact on the returns of most securities from the specific risks that influence securities individually and hence a negligible effect on a diversified portfolio" (Litterman and Scheinkman, 1991, page 54).*

⇒ The 3-factor model of Litterman and Scheinkman (1991) is based on the PCA analysis:

- the level of the yield curve
- the steepness of the yield curve
- the curvature of the yield curve

Conventional bond model

- Let $B_i(t, D_i)$ be the zero-coupon bond price with maturity D_i:

$$B_i(t, D_i) = e^{-(R(t)+S_i(t))\,D_i}$$

where $R(t)$ is the risk-free interest rate and $S_i(t)$ is the credit spread

- L-CAPM of Acharya and Pedersen (2005):

$$R_i(t) = \underbrace{\underbrace{(R(t) + S_i(t))\,D_i}_{\text{Gross return}} - L_i(t)}_{\text{Net return}}$$

where $L_i(t)$ is the illiquidity cost of Bond i

Conventional bond model

We deduce that:

$$B_i(t, D_i) = e^{-((R(t)+S_i(t))\, D_i - L_i(t))}$$

and:

$$\begin{aligned} \mathrm{d} \ln B_i(t, D_i) &= -D_i \,\mathrm{d}R(t) - D_i \,\mathrm{d}S_i(t) + \mathrm{d}L_i(t) \\ &= -D_i \,\mathrm{d}R(t) - \mathrm{DTS}_i(t) \frac{\mathrm{d}S_i(t)}{S_i(t)} + \mathrm{d}L_i(t) \end{aligned}$$

where $\mathrm{DTS}_i(t) = D_i\, S_{i,t}$ is the duration-time-spread factor

Conventional bond model

Liquidity premia (Acharya and Pedersen, 2005)

The illiquidity premium $\mathrm{d}L_{i,t}$ can be decomposed into an illiquidity level component $\mathbb{E}\left[L_{i,t}\right]$ and three illiquidity covariance risks:

1. $\beta\left(L_i, L_M\right)$
   An asset that becomes illiquid when the market becomes illiquid should have a higher risk premium.
2. $\beta\left(R_i, L_M\right)$
   An asset that perform well in times of market illiquidity should have a lower risk premium.
3. $\beta\left(L_i, R_M\right)$
   Investors accept a lower risk premium on assets that are liquid in a bear market.

Conventional bond model

By assuming that:

$$\mathrm{d}L_{i,t} = \alpha_i(t) + \beta(L_i, L_M)\,\mathrm{d}L_M(t)$$

where α_i is the liquidity return that is not explained by the liquidity commonality, we obtain:

$$R_i(t) = \alpha_i(t) - D_i\,\mathrm{d}R(t) - \mathrm{DTS}_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} + \beta(L_i, L_M)\,\mathrm{d}L_M(t)$$

or:

$$R_i(t) = a(t) - D_i\,\mathrm{d}R(t) - \mathrm{DTS}_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} + \beta(L_i, L_M)\,\mathrm{d}L_M(t) + u_i(t)$$

Risk factors in corporate bonds

Conventional bond model (or the 'equivalent' CAPM for bonds)

The total return $R_i(t)$ of Bond i at time t is equal to:

$$R_i(t) = a(t) - \mathrm{MD}_i(t)\, R^I(t) - \mathrm{DTS}_i(t)\, R^S(t) + \mathrm{LTP}_i(t)\, R^L(t) + u_i(t)$$

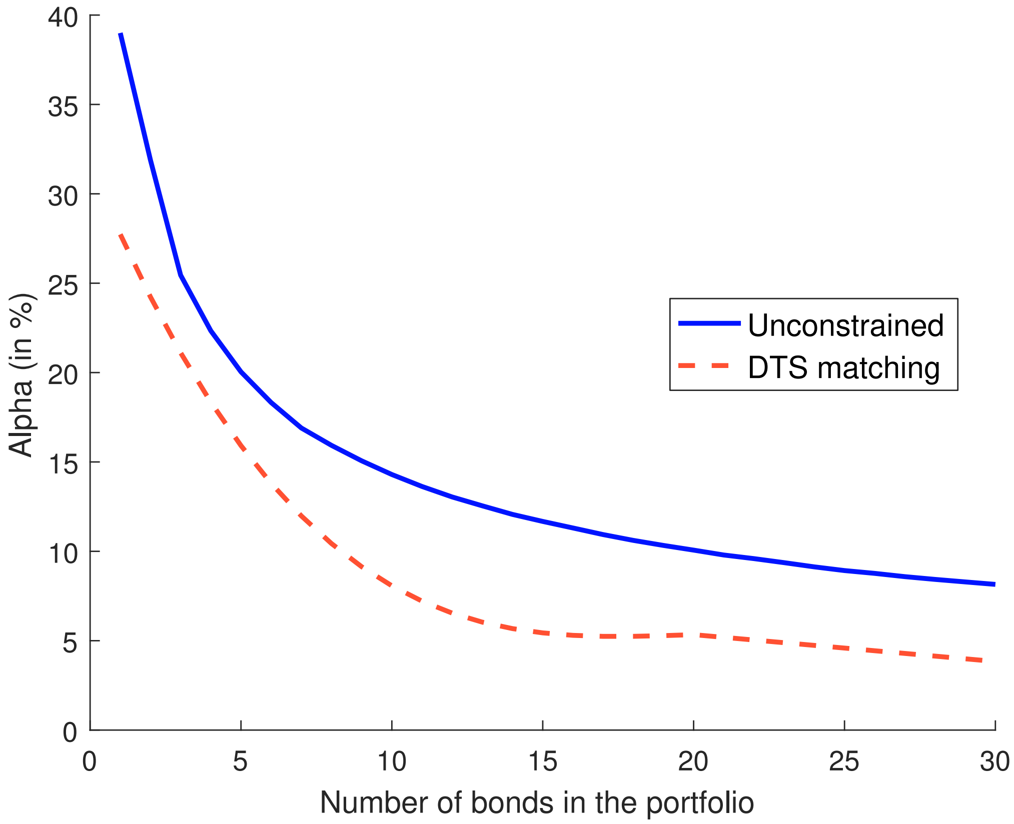
where:

- $a(t)$ is the constant/carry/zero intercept
- $\mathrm{MD}_i(t)$ is the modified duration
- $\mathrm{DTS}_i(t)$ is the duration-times-spread
- $\mathrm{LTP}_i(t)$ is the liquidity-time-price
- $u_i(t)$ is the residual

$\Rightarrow$ $R^I(t)$, $R^S(t)$ and $R^L(t)$ are the return components due to interest rate movements, credit spread variation and liquidity dynamics.

Risk factors in corporate bonds

Figure 10: Conventional alpha decreases with the number of holding assets



EURO IG corporate bonds, 2000-2015
Source: Amundi Research (2017)

- There is less traditional alpha in the bond market than in the stock market

Risk factors in corporate bonds

Since 2015

- Houweling and van Zundert (2017) — HZ
- Bektic, Neugebauer, Wegener and Wenzler (2017) — BNWW
- Israel, Palhares and Richardson (2017) — IPR
- Bektic, Wenzler, Wegener, Schiereck and Spielmann (2019) — BWWSS
- Ben Slimane, De Jong, Dumas, Fredj, Sekine and Srb (2019) — BDDFSS
- Etc.

Risk factors in corporate bonds

Study	HZ	BWWSS	IPR	BNWW
Period	1994-2015	1996-2016 (US) 2000-2016 (EU)	1997-2015	1999-2016
Universe	Bloomberg Barclays US IG & HY	BAML US IG & HY, EU IG	BAML US IG & HY	BAML US IG & HY
Investment		1Y variation in total assets		
Low risk	Short maturity + High rating		Leverage × Duration × Profitability	1Y equity beta
Momentum	6M bond return		6M bond return + 6M stock return	1Y stock return
Profitability		Earnings-to-book		
Size	Market value of issuer	Market capitalization		Market capitalization
Value	Comparing OAS to Maturity × Rating × 3M OAS variation	Price-to-book	Comparing OAS to Duration × Rating × Bond return volatility + Implied default probability	Price-to-book

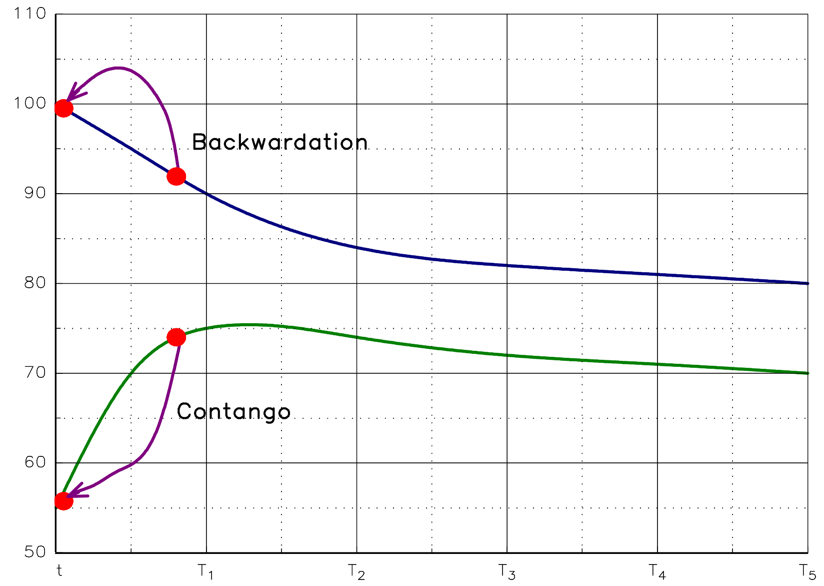
Risk factors in currency markets

- What are the main risk factors for explaining the cross-section of currency returns?
  1. Momentum (cross-section or time-series)
  2. Carry
  3. Value (short-term, medium-term or long-term)
- The dynamics of some currencies are mainly explained by:
  - Common risk factors (e.g. NZD or CAD)
  - Idiosyncratic risk factors (e.g. IDR or PEN)
- Carry-oriented currency? (e.g. JPY $\neq$ CHF)

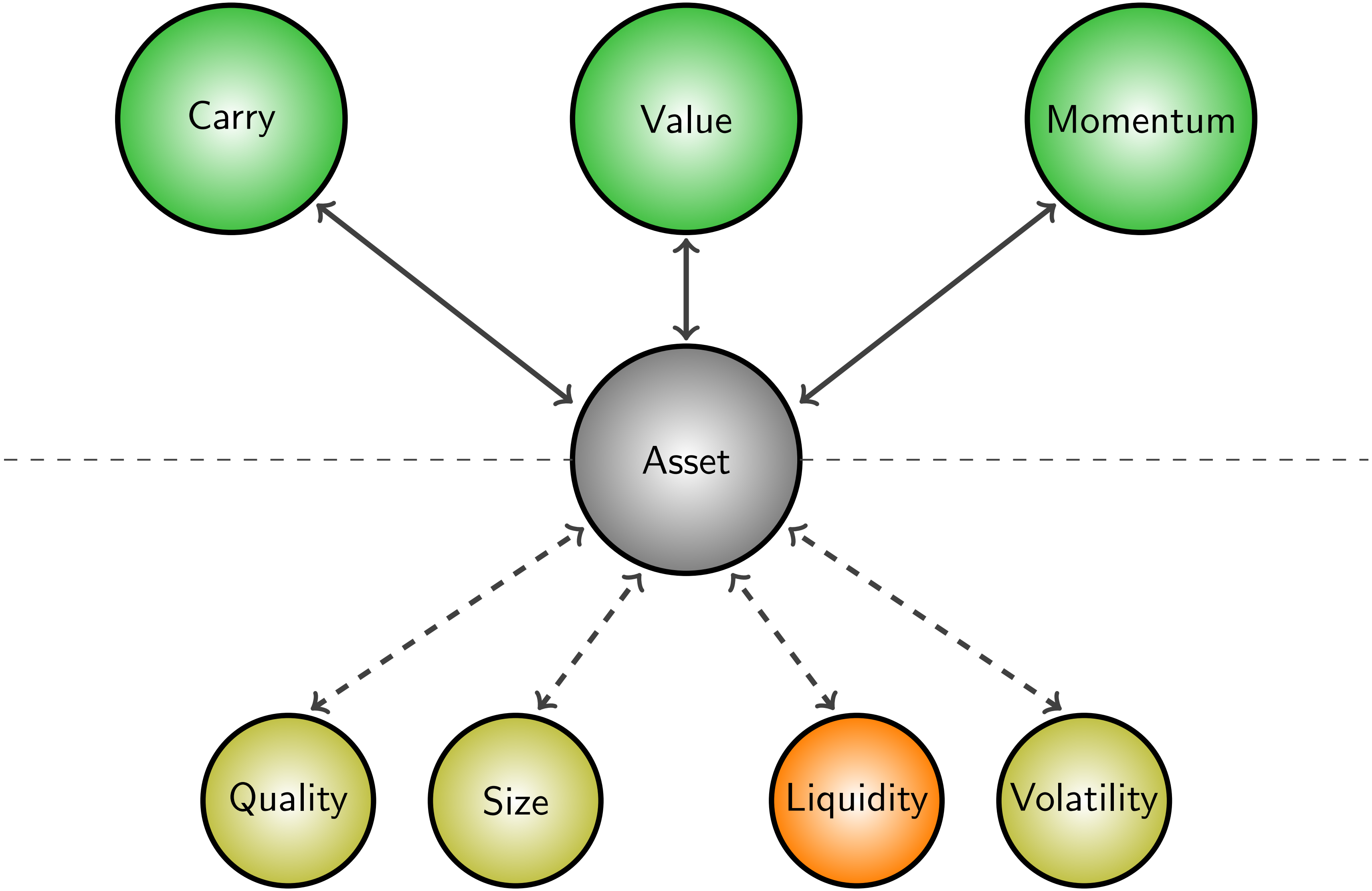
Risk factors in commodities

- Two universal strategies:
  - Contango/backwardation strategy
  - Trend-following strategy
- **CTA = Commodity Trading Advisor**
- Only two risk factors?
  - Carry
  - Momentum

Figure 11: Contango and backwardation movements in commodity futures contracts



Factor analysis of an asset



Factor analysis of an asset

Carry

- Yield
- Income generation
- Risk arbitrage

Value

- Fair price
- Overvalued / undervalued
- Fundamental

Momentum

- Price dynamics
- Trend-following
- Mean-reverting / Reversal

Liquidity

- Tradability property (transaction cost, execution time, scarcity)
- Supply/demande imbalance
- Bad times $\neq$ good times

The concept of alternative risk premia

There are many definitions of ARP:

- ARP $\approx$ factor investing (FI)
  (ARP = long/short portfolios, FI = long portfolios)
- ARP $\approx$ all the other risk premia (RP) than the equity and bond risk premia
- ARP $\approx$ quantitative investment strategies (QIS)

Sell-side

- CIBs & brokers
- ARP = QIS

Buy-side

- Asset managers & asset owners
- ARP = FI (for asset managers)
- ARP = RP (for asset owners)

The concept of alternative risk premia

Alternative Risk Premia

Alternative (or real) assets

- Private equity
- Private debt
- Real estate
- Infrastructure

Traditional financial assets

- Long/short risk factors in equities, rates, credit, currencies & commodities
- Risk premium strategy (e.g. carry, momentum, value, etc.)

The concept of alternative risk premia

- A risk premium is the expected excess return by the investor in order to accept the risk ⇒ any (risky) investment strategy has a risk premium!
- Generally, the term "*risk premium*" is associated to asset classes:
  - The equity risk premium
  - The risk premium of high yield bonds
- This means that a risk premium is the expected excess return by the investor in order to accept a future economic risk that cannot be diversifiable
  - For instance, the risk premium of a security does not integrate its specific risk

The concept of alternative risk premia

- What is the relationship between a risk factor and a risk premium?
  - A rewarded risk factor may correspond a to risk premium, while a non-rewarded risk factor is not a risk premium
  - A risk premium can be a risk factor if it helps to explain the cross-section of expected returns
  - The case of cat bonds:

Risk premium	✔
Risk factor	✘

Risk premia & non-diversifiable risk

Consumption-based model (Lucas, 1978; Cochrane, 2001)

A risk premium is a compensation for accepting (systematic) risk in **bad times**.

We have:

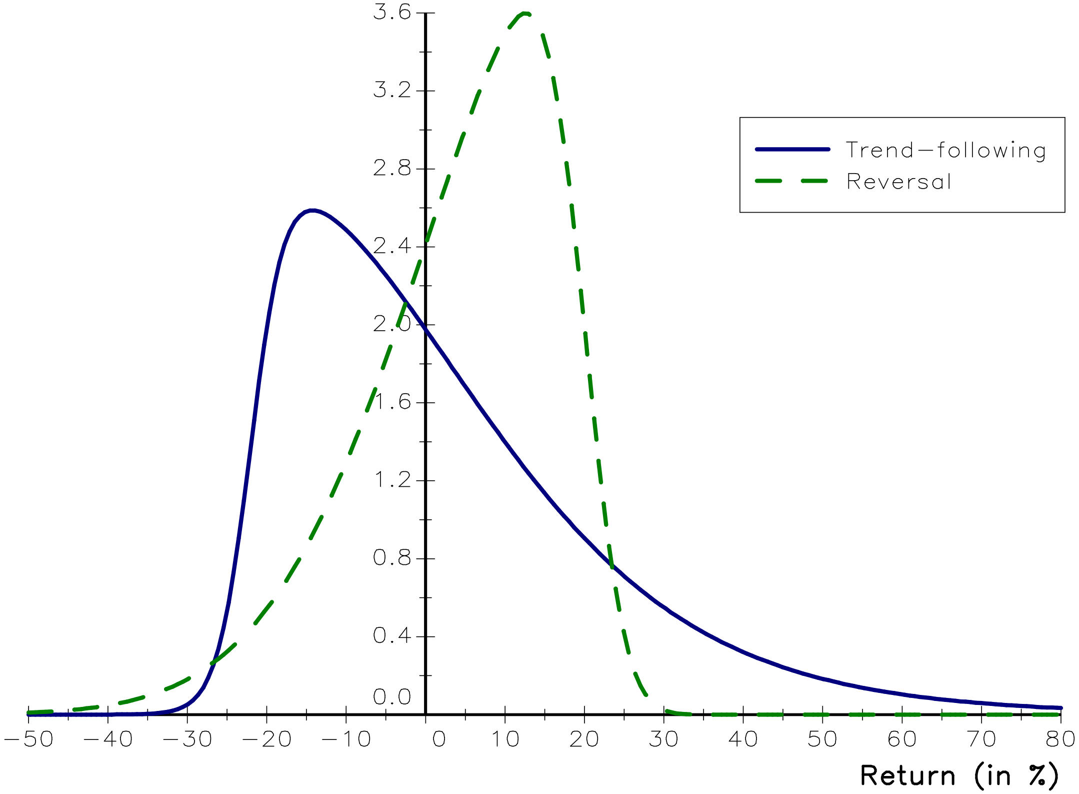
$$\underbrace{\mathbb{E}_t\left[R_{t+1} - R_{f,t}\right]}_{\text{Risk premium}} \propto -\underbrace{\rho\left(u'\left(C_{t+1}\right), R_{t+1}\right)}_{\text{Correlation term}} \times \underbrace{\sigma\left(u'\left(C_{t+1}\right)\right)}_{\text{Smoothing term}} \times \underbrace{\sigma\left(R_{t+1}\right)}_{\text{Volatility term}}$$

where R_{t+1} is the one-period return of the asset, $R_{f,t}$ is the risk-free rate, C_{t+1} is the future consumption and $u(C)$ is the utility function.

Main results

- Hedging assets help to smooth the consumption $\Rightarrow$ low or negative risk premium
- In bad times, risk premium strategies are correlated and have a negative performance ($\neq$ all-weather strategies)

Risk premia & bad times



The market must reward contrarian and value investors, not momentum investors

Behavioral finance and limits to arbitrage

Bounded rationality

Barberis and Thaler (2003), A Survey of Behavioral Finance.

Decisions of the other economic agents

$\Downarrow$

Feedback effects on our decisions!

Killing Homo Economicus

[...] "conventional economics assumes that people are highly rational, super rational and unemotional. They can calculate like a computer and have no self-control problems" (Richard Thaler, 2009).

"The people I study are humans that are closer to Homer Simpson" (Richard Thaler, 2017).

Behavioral finance and social preferences

- For example, momentum may be a rational behavior if the investor is not informed and his objective is to minimize the loss with respect to the 'average' investor.
- Absolute loss $\neq$ relative loss
- Loss aversion and performance asymmetry
- Imitations between institutional investors $\Rightarrow$ benchmarking
- Home bias

What does the theory become if utility maximization includes the performance of other economic agents?

$\Rightarrow$ **The crowning glory of tracking error and relative performance!**

Behavioral finance and market anomalies

Previously

Positive expected excess returns are explained by:

- risk premia

Today

Positive expected excess returns are explained by:

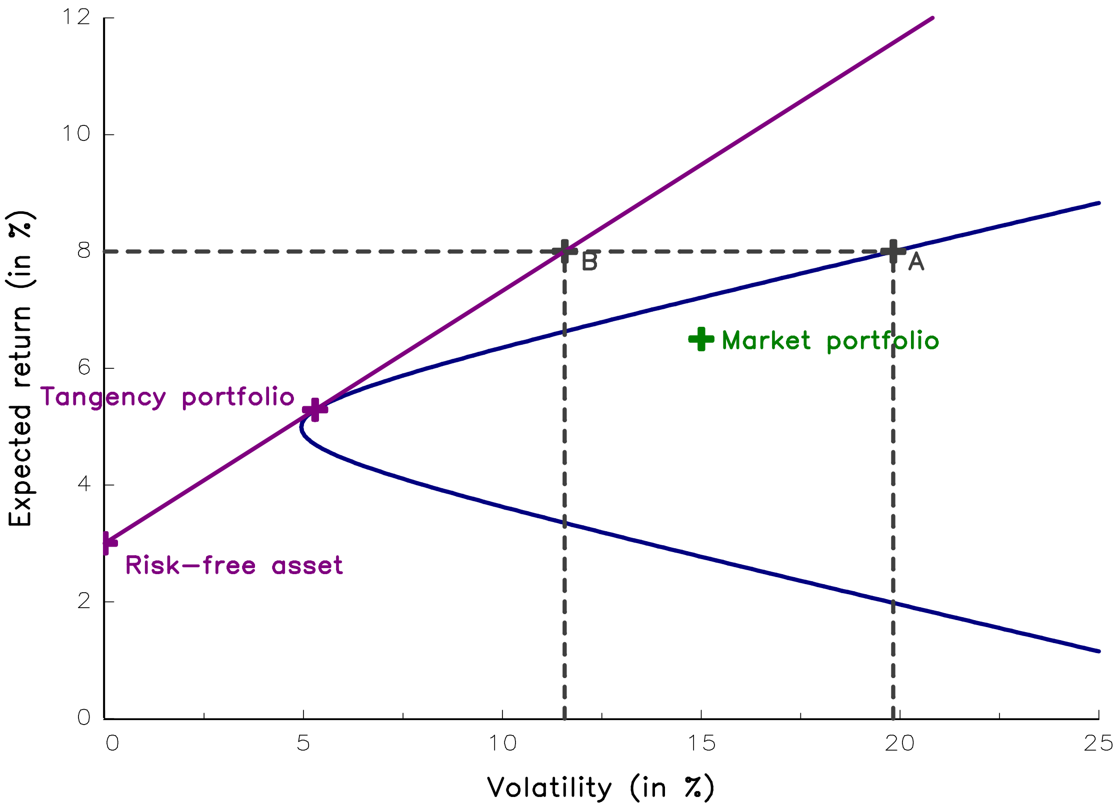
- risk premia
- or market anomalies

Market anomalies correspond to trading strategies that have delivered good performance in the past, but their performance cannot be explained by the existence of a systematic risk (in bad times). Their performance can only be explained by behavioral theories.

⇒ Momentum, low risk and quality risk factors are three market anomalies

The case of low risk assets

Figure 12: What is the impact of borrowing constraints on the market portfolio?



- The investor that targets a 8% expected return must choose Portfolio *B*
- The demand for high beta assets is higher than this predicted by CAPM
- This effect is called the low beta anomaly

Low risk assets have a higher Sharpe ratio than high risk assets

Skewness risk premia & market anomalies

Characterization of alternative risk premia

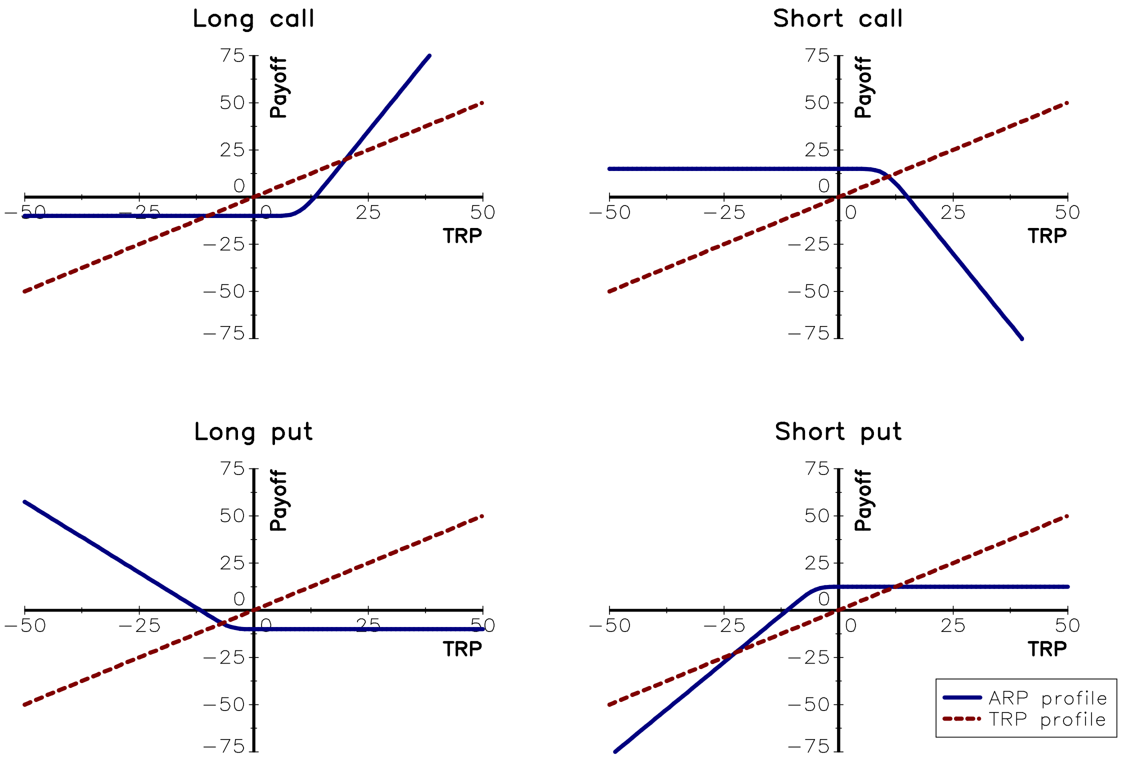
- An alternative risk premium (ARP) is a risk premium, which is not traditional
  - Traditional risk premia (TRP): equities, sovereign/corporate bonds
  - Currencies and some commodities are not TRP
- The drawdown of an ARP must be positively correlated to bad times
  - Risk premia $\neq$ insurance against bad times
  - (SMB, HML) $\neq$ WML
- Risk premia are an increasing function of the volatility and a decreasing function of the skewness

In the market practice, alternative risk premia recover:

1. Skewness risk premia (or pure risk premia), which present high negative skewness and potential large drawdown
2. Markets anomalies

Payoff function of alternative risk premia

Figure 13: Which option profile may be considered as a skewness risk premium?



- ~~Long call~~ (risk adverse)
- ~~Short call~~ (market anomaly)
- ~~Long put~~ (insurance)
- Short put

$\Rightarrow$ SMB, HML, ~~WML~~, ~~BAB~~, ~~QMJ~~

A myriad of alternative risk premia?

Figure 14: Mapping of risk premia strategies (based on existing products)

Strategy	Equities	Rates	Credit	Currencies	Commodities
Carry	Dividend futures High dividend yield	Forward rate bias Term structure slope Cross-term-structure	Forward rate bias	Forward rate bias	Forward rate bias Term structure slope Cross-term-structure
Event	Buyback Merger arbitrage				
Growth	Growth				
Liquidity	Amihud liquidity	Turn-of-the-month	Turn-of-the-month		Turn-of-the-month
Low beta	Low beta Low volatility				
Momentum	Cross-section Time-series	Cross-section Time-series	Time-series	Cross-section Time-series	Cross-section Time-series
Quality	Quality				
Reversal	Time-series Variance	Time-series		Time-series	Time-series
Size	Size				
Value	Value	Value	Value	PPP REER, BEER, FEER NATREX	Value
Volatility	Carry Term structure	Carry		Carry	Carry

Source: Roncalli (2017)

The carry risk premium

Underlying idea

Definition

- The investor takes an investment risk
- This investment risk is rewarded by a high and known yield
- Financial theory predicts a negative mark-to-market return that may reduce or write off the performance
- The investor hopes that the impact of the mark-to-market will be lower than the predicted value

$\Rightarrow$ Carry strategies are highly related to the concept of risk arbitrage[5]

- The carry risk premium is extensively studied by Koijen *et al.* (2018)
- The carry risk premium has a short put option profile

[5] An example is the carry strategy between pure money market instruments and commercial papers = not the same credit risk, not the same maturity risk, but the investor believes that the default will never occur!

The carry risk premium

Not one but several carry strategies

- Equity
  - Carry on dividend futures
  - Carry on stocks with high dividend yields (HDY)
- Rates (sovereign bonds)
  - Carry on the yield curve (term structure & roll-down)
- Credit (corporate bons)
  - Carry on bonds with high spreads
  - High yield strategy
- Currencies
  - Carry on interest rate differentials (uncovered interest rate parity)
- Commodities
  - Carry on contango & backwardation movements
- Volatility
  - Carry on option implied volatilities
  - Short volatility strategy

⇒ Many implementation methods: security-slope, cross-asset, long/short, long-only, basis arbitrage, etc.

The carry risk premium

Analytical model

- Let X_t be the capital allocated at time t to finance a futures position (or an unfunded forward exposure) on asset S_t
- By assuming that the futures price expires at the future spot price ($F_{t+1} = S_{t+1}$), Koijen *et al.* (2018) showed that:

$$\begin{aligned} R_{t+1}(X_t) - R_f &= \frac{F_{t+1} - F_t}{X_t} \\ &= \frac{S_{t+1} - F_t}{X_t} \\ &= \frac{S_t - F_t}{X_t} + \frac{\mathbb{E}_t[S_{t+1}] - S_t}{X_t} + \frac{S_{t+1} - \mathbb{E}_t[S_{t+1}]}{X_t} \end{aligned}$$

The carry risk premium

Analytical model

- At time $t+1$, the excess return of this investment is then equal to:

$$R_{t+1}(X_t) - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t[\Delta S_{t+1}]}{X_t} + \varepsilon_{t+1}$$

  where $\varepsilon_{t+1} = (S_{t+1} - \mathbb{E}_t[S_{t+1}])/X_t$ is the unexpected price change and $\mathcal{C}_t$ is the carry:

$$\mathcal{C}_t = \frac{S_t - F_t}{X_t}$$

- It follows that the expected excess return is the sum of the carry and the expected price change:

$$\mathbb{E}_t[R_{t+1}(X_t)] - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t[\Delta S_{t+1}]}{X_t}$$

- The nature of these two components is different:
  1. The carry is an ex-ante observable quantity (known value)
  2. The price change depends on the dynamic model of S_t (unknown value)

The carry risk premium

Analytical model

- If we assume that the spot price does not change (no-arbitrage assumption $\mathcal{H}$), the expected excess return is equal to the carry:

$$\frac{\mathbb{E}_t\left[\Delta S_{t+1}\right]}{X_t} = -\mathcal{C}_t$$

- The carry investor will prefer Asset i to Asset j if the carry of Asset i is higher:

$$\mathcal{C}_{i,t} \geq \mathcal{C}_{j,t} \Longrightarrow A_i \succ A_j$$

- The carry strategy would then be long on high carry assets and short on low carry assets.

Remark

In the case of a fully-collateralized position $X_t = F_t$, the value of the carry becomes:

$$\mathcal{C}_t = \frac{S_t}{F_t} - 1$$

The carry risk premium

Currency carry (or the carry trade strategy)

- Let S_t, i_t and r_t be the spot exchange rate, the domestic interest rate and the foreign interest rate for the period $[t, t+1]$
- The forward exchange rate F_t is equal to:

$$F_t = \frac{1 + i_t}{1 + r_t} S_t$$

- The carry is approximately equal to the interest rate differential:

$$\mathcal{C}_t = \frac{r_t - i_t}{1 + i_t} \simeq r_t - i_t$$

The carry risk premium

Currency carry (or the carry trade strategy)

- The carry strategy is long on currencies with high interest rates and short on currencies with low interest rates
- We can consider the following carry scoring (or ranking) system:

$$\mathcal{C}_t = r_t$$

Uncovered interest rate parity (UIP)

- An interest rate differential of 10% $\Rightarrow$ currency depreciation of 10% per year
- In 10 years, we must observe a depreciation of 65%!

The carry risk premium

Currency carry (or the carry trade strategy)

ARS	Argentine peso	KRW	Korean won
AUD	Australian dollar	LTL	Lithuanian litas
BGN	Bulgarian lev	LVL	Latvian lats
BHD	Bahraini dinar	MXN	Mexican peso
BRL	Brazilian real	MYR	Malaysian ringgit
CAD	Canadian dollar	NOK	Norwegian krone
CHF	Swiss franc	NZD	New Zealand dollar
CLP	Chilean peso	PEN	Peruvian new sol
CNY/RMB	Chinese yuan (Renminbi)	PHP	Philippine peso
COP	Colombian peso	PLN	Polish zloty
CZK	Czech koruna	RON	new Romanian leu
DKK	Danish krone	RUB	Russian rouble
EUR	Euro	SAR	Saudi riyal
GBP	Pound sterling	SEK	Swedish krona
HKD	Hong Kong dollar	SGD	Singapore dollar
HUF	Hungarian forint	THB	Thai baht
IDR	Indonesian rupiah	TRY	Turkish lira
ILS	Israeli new shekel	TWD	new Taiwan dollar
INR	Indian rupee	USD	US dollar
JPY	Japanese yen	ZAR	South African rand

The carry risk premium

Currency carry (or the carry trade strategy)

Baku *et al.* (2019, 2020) consider the most liquid currencies:

- **G10** AUD, CAD, CHF, EUR, GBP, JPY, NOK, NZD, SEK and USD
- **EM** BRL, CLP, CZK, HUF, IDR, ILS, INR, KRW, MXN, PLN, RUB, SGD, TRY, TWD and ZAR
- **G25** G10 + EM

They build currency risk factors using the following characteristics:

- The portfolio is equally-weighted and rebalanced every month
- The portfolio is notional-neutral (number of long exposures = number of short exposures)
- 3/3 for G10, 4/4 for EM and 7/7 for G25
- The long (resp. short) exposures correspond to the highest (resp. lowest) scores

The carry risk premium

Currency carry (or the carry trade strategy)

- Scoring system: $\mathbb{S}_{i,t} = \mathcal{C}_{i,t} = r_{i,t}$
- The carry strategy is long on currencies with high interest rates and short on currencies with low interest rates

Table 14: Risk/return statistics of the carry risk factor (2000-2018)

	G10	EM	G25
Excess return (in %)	3.75	11.21	7.22
Volatility (in %)	9.35	9.12	8.18
Sharpe ratio	0.40	1.23	0.88
Maximum drawdown (in %)	−31.60	−25.27	−17.89

Source: Baku *et al.* (2019, 2020)

The carry risk premium

Currency carry (or the carry trade strategy)

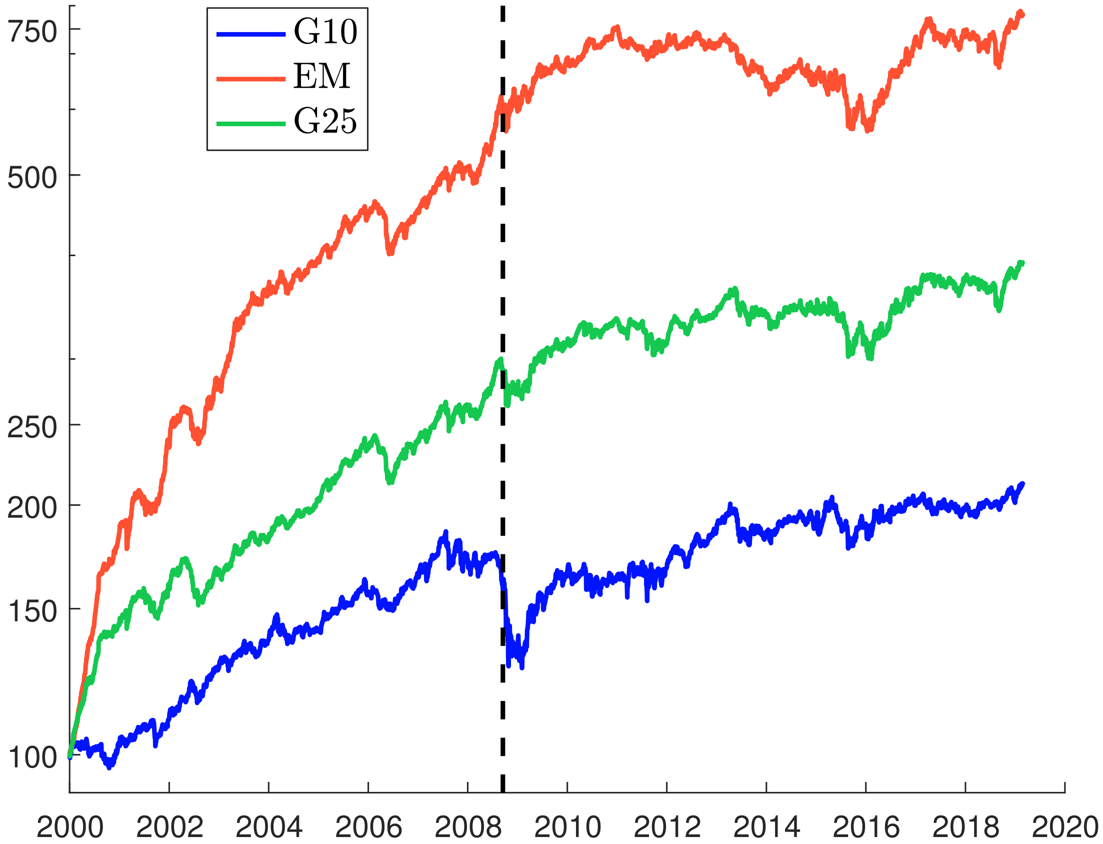


Figure 15: Cumulative performance of the carry risk factor

Source: Baku *et al.* (2019, 2020)

The carry risk premium

Equity carry

- We have:

$$\mathcal{C}_t \simeq \frac{\mathbb{E}_t [D_{t+1}]}{S_t} - r_t$$

where $\mathbb{E}_t [D_{t+1}]$ is the risk-neutral expected dividend for time $t+1$

- If we assume that dividends are constant, the carry is the difference between the dividend yield y_t and the risk-free rate r_t:

$$\mathcal{C}_t = y_t - r_t$$

- The carry strategy is long on stocks with high dividend yields and short on stocks with low dividend yields
- This strategy may be improved by considering forecasts of dividends. In this case, we have:

$$\mathcal{C}_t \simeq \frac{\mathbb{E}_t [D_{t+1}]}{S_t} - r_t = \frac{D_t + \mathbb{E}_t [\Delta D_{t+1}]}{S_t} - r_t = y_t + g_t - r_t$$

where g_t is the expected dividend growth

The carry risk premium

Equity carry

Carry strategy with dividend futures

Another carry strategy concerns dividend futures. The underlying idea is to take a long position on dividend futures where the dividend premium is the highest and a short position on dividend futures where the dividend premium is the lowest. Because dividend futures are on equity indices, the market beta exposure is generally hedged.

Why do we observe a premium on dividend futures?

⇒ Because of the business of structured products and options

The carry risk premium

Bond carry

- The price of a zero-coupon bond with maturity date T is equal to:

$$B_t(T) = e^{-(T-t)R_t(T)}$$

  where $R_t(T)$ is the corresponding zero-coupon rate

- Let $F_t(T, m)$ denote the forward interest rate for the period $[T, T+m]$, which is defined as follows:

$$B_t(T+m) = e^{-mF_t(T,m)} B_t(T)$$

  We deduce that:

$$F_t(T, m) = -\frac{1}{m} \ln \frac{B_t(T+m)}{B_t(T)}$$

  It follows that the instantaneous forward rate is given by this equation:

$$F_t(T) = F_t(T, 0) = \frac{-\partial \ln B_t(T)}{\partial T}$$

The carry risk premium
Bond carry

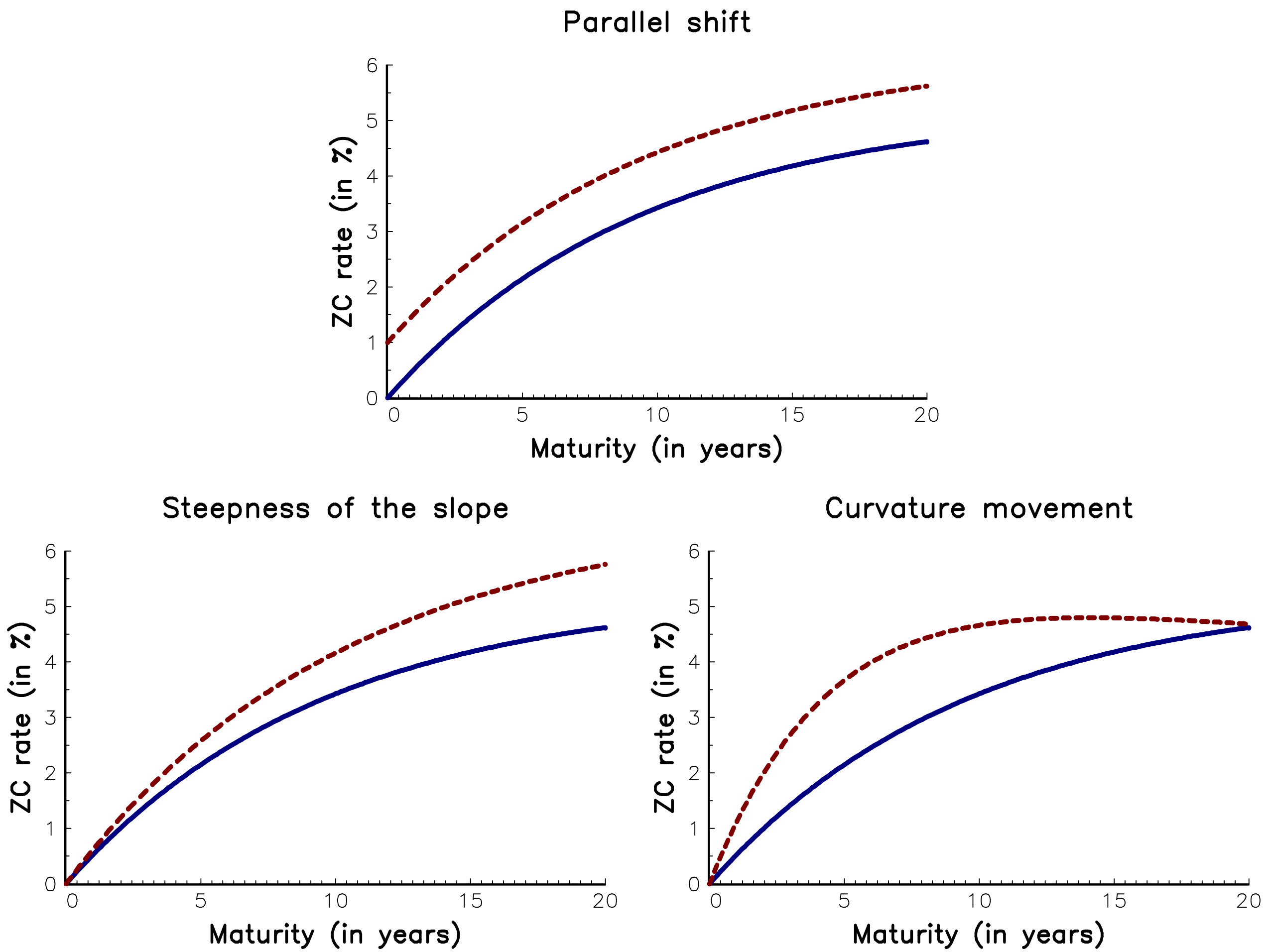


Figure 16: Movements of the yield curve

The carry risk premium

Bond carry

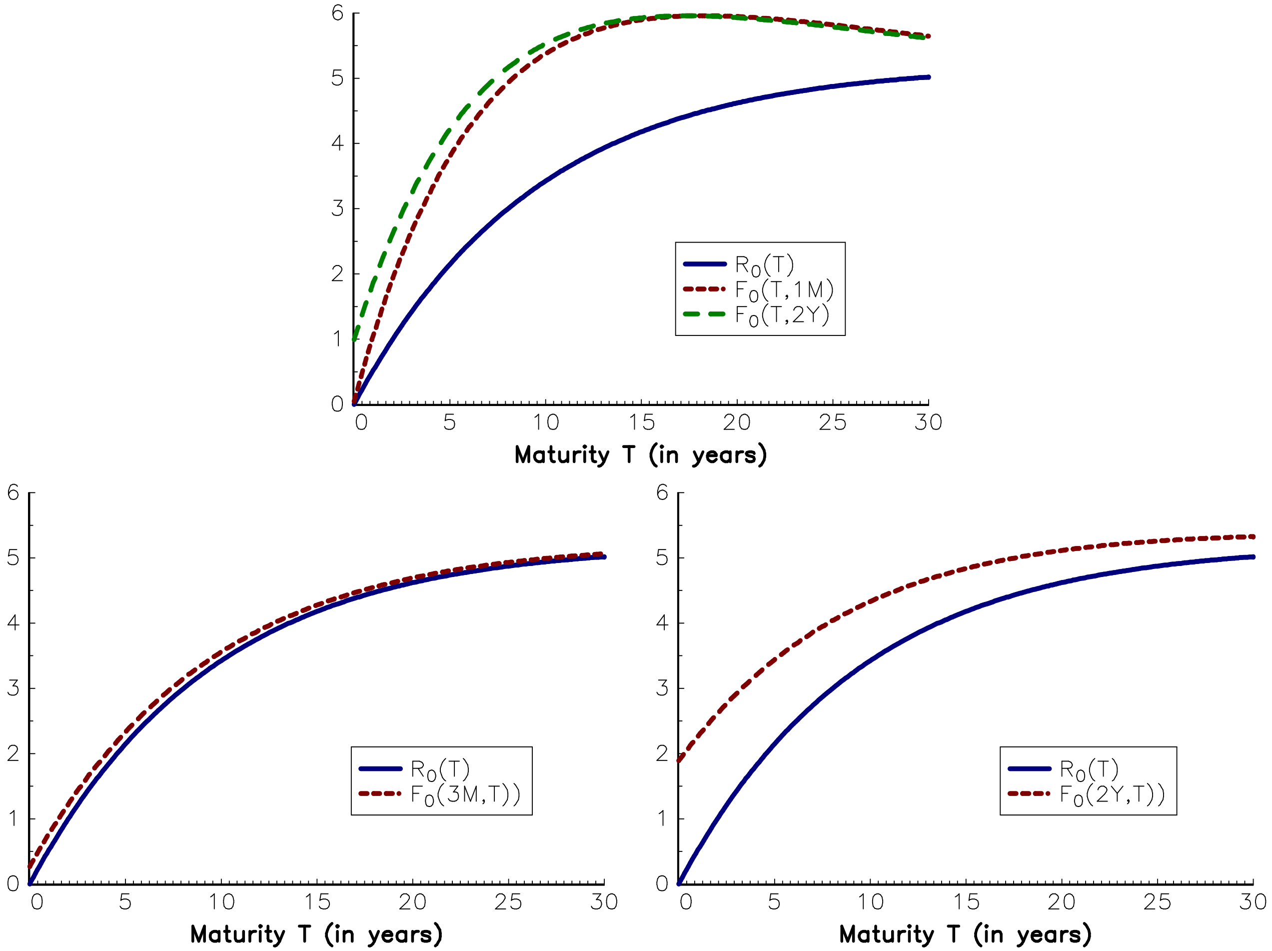


Figure 17: Sport and forward interest rates

The carry risk premium

Bond carry

1. The first carry strategy ("*forward rate bias*") consists in being long the forward contract on the forward rate $F_t(T, m)$ and selling it at time $t + \mathrm{d}t$ with $t + \mathrm{d}t \leq T$
   - Forward rates are generally higher than spot rates
   - Under the hypothesis $(\mathcal{H})$ that the yield curve does not change, rolling forward rate agreements can then capture the term premium and the roll down
   - The carry of this strategy is equal to:

$$\mathcal{C}_t = \underbrace{R_t(T) - r_t}_{\text{term premium}} + \underbrace{\partial_{\bar{T}} \bar{R}_t(\bar{T})}_{\text{roll down}}$$

   where $\bar{R}_t(\bar{T})$ is the zero-coupon rate with a constant time to maturity $\bar{T} = T - t$

The carry risk premium

Bond carry

Implementation

We notice that the difference is higher for long maturities. However, the risk associated with such a strategy is that of a rise in interest rates. This is why this carry strategy is generally implemented by using short-term maturities (less than two years)

The carry risk premium

Bond carry

2. The second carry strategy ("*carry slope*") corresponds to a long position in the bond with maturity T_2 and a short position in the bond with maturity T_1
   - The exposure of the two legs are adjusted in order to obtain a duration-neutral portfolio
   - This strategy is known as the slope carry trade
   - We have:

$$\mathcal{C}_t = \underbrace{\left(R_t\left(T_2\right) - r_t\right) - \frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\left(R_t\left(T_1\right) - r_t\right)}_{\text{duration neutral slope}} + \underbrace{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right) - \frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)}_{\text{duration neutral roll down}}$$

The carry risk premium

Bond carry

Implementation

The classical carry strategy is long 10Y/short 2Y

The carry risk premium

Bond carry

3. The third carry strategy ("*cross-carry slope*") is a variant of the second carry strategy when we consider the yield curves of several countries

Implementation

The portfolio is long on positive or higher slope carry and short on negative or lower slope carry

The carry risk premium

Credit carry

We consider a long position on a corporate bond and a short position on the government bond with the same duration

The carry is equal to:

$$\mathcal{C}_t = \underbrace{\mathcal{s}_t(T)}_{\text{spread}} + \underbrace{\partial_{\bar{T}} \bar{R}_t^{\star}(\bar{T}) - \partial_{\bar{T}} \bar{R}_t(\bar{T})}_{\text{roll down difference}}$$

where $\mathcal{s}_t(T) = R_t^{\star}(T) - R_t(T)$ is the credit spread, $R_t^{\star}(T)$ is the yield-to-maturity of the credit bond and $R_t^{\star}(T)$ is the yield-to-maturity of the government bond

The carry risk premium

Credit carry

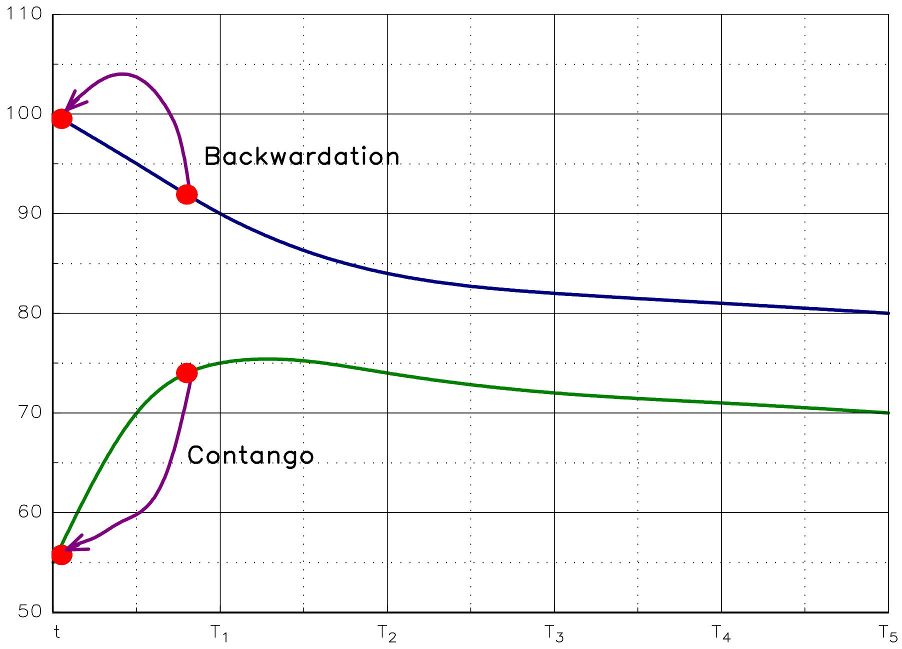
Two implementations

1. The first one is to build a long/short portfolio with corporate bond indices or baskets. The bond universe can be investment grade or high yield. In the case of HY bonds, the short exposure can be an IG bond index
2. The second approach consists in using credit default swaps (CDS). Typically, we sell credit protection on HY credit default indices (e.g. CDX.NA.HY) and buy protection on IG credit default indices (e.g. CDX.NA.IG)

The carry risk premium

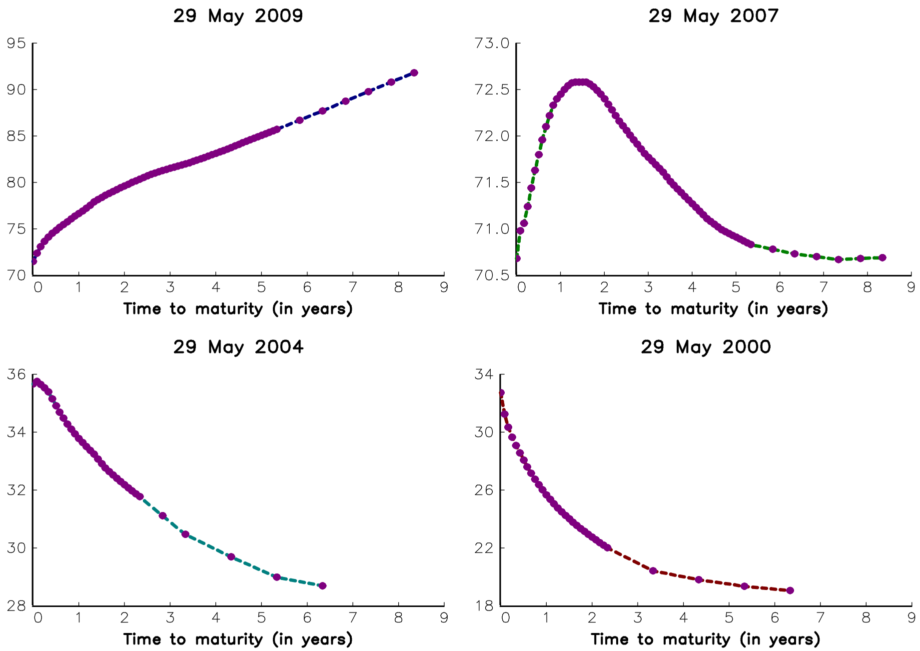
Commodity carry

Figure 18: Contango and backwardation movements in commodity futures contracts



Source: Roncalli (2013)

Figure 19: Term structure of crude oil futures contracts



Source: Roncalli (2013)

The carry risk premium

Volatility carry (or the short volatility strategy)

Volatility carry risk premium

- Long volatility $\Rightarrow$ negative carry ($\neq$ structural exposure)
- Short volatility $\Rightarrow$ positive carry, but the highest skewness risk
- The P&L of selling and delta-hedging an option is equal to:

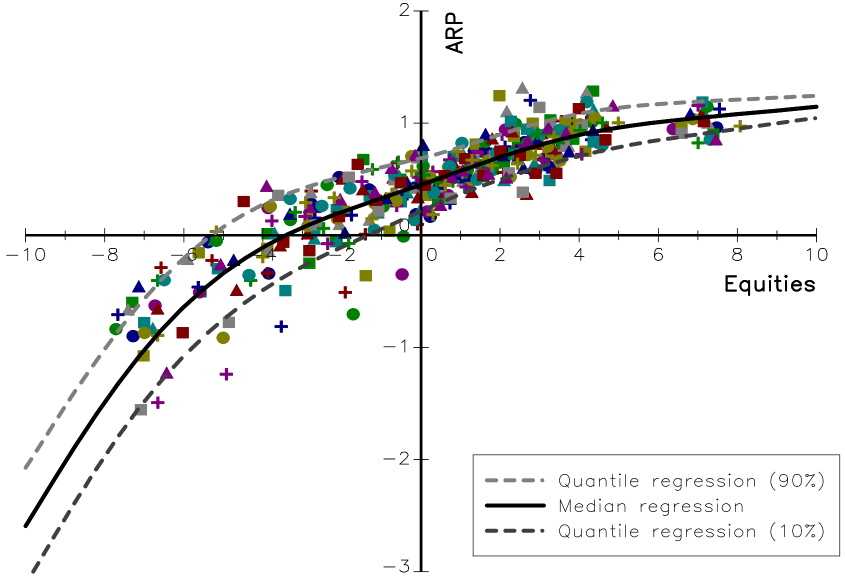
$$\Pi = \frac{1}{2}\int_0^T e^{r(T-t)} S_t^2 \Gamma_t \left(\Sigma_t^2 - \sigma_t^2\right) \,\mathrm{d}t$$

  where S_t is the price of the underlying asset, Γ_t is the gamma coefficient, Σ_t is the implied volatility and σ_t is the realized volatility
- $\Sigma_t \geq \sigma_t \Longrightarrow \Pi > 0$
- 3 main reasons:
  1. Asymmetric risk profile between the seller and the buyer
  2. Hedging demand imbalances
  3. Liquidity preferences

The carry risk premium

Volatility carry (or the short volatility strategy)

Figure 20: Non-parametric payoff of the US short volatility strategy



- Income generation
- Short put option profile
- Strategic asset allocation (≠ tactical asset allocation)
- Time horizon is crucial!

It is a skewness risk premium!

Carry strategies exhibit concave payoffs

The value risk premium

Definition

- Let $S_{i,t}$ be the market price of Asset i
- Let $S_i^\star$ be the fundamental price (or the fair value) of Asset i
- The value of Asset i is the relative difference between the two prices:

$$\mathcal{V}_{i,t} = \frac{S_i^\star - S_{i,t}}{S_{i,t}}$$

- The value investor will prefer Asset i to Asset j if the value of Asset i is higher:

$$\mathcal{V}_{i,t} \geq \mathcal{V}_{j,t} \Longrightarrow A_i \succ A_j$$

The value risk premium

The value strategy is an active management bet

- The price of Asset i is undervalued if and only if its value is negative:

$$\mathcal{V}_{i,t} \leq 0 \Leftrightarrow S_i^\star \leq S_{i,t}$$

  The value investor should sell securities with negative values

- The price of Asset i is overvalued if and only if its value is positive:

$$\mathcal{V}_{i,t} \geq 0 \Leftrightarrow S_i^\star \geq S_{i,t}$$

  The value investor should buy securities with positive values

Remark

While carry is an **objective** measure, value is a **subjective** measure, because the fair value is different from one investor to another (e.g. stock picking = value strategy)

The value risk premium

Computing the fair value

We need a model to estimate the fundamental price $S_i^\star$:

- Stocks: discounted cash flow (DCF) method, fundamental measures (B2P, PE, DY, EBITDA/EV, etc.), machine learning model, etc.
- Sovereign bonds: macroeconomic model, flows model, etc.
- Corporate bonds: Merton model, structural model, econometric model, etc.
- Foreign exchange rates: purchasing power parity (PPP), real effective exchange rate (REER), BEER, FEER, NATREX, etc.
- Commodities: statistical model (5-year average price), etc.

The value risk premium

Equity value

The equity strategy

If we assume that the weight of asset i is proportional to its book-to-price:

$$w_{i,t} \propto \frac{B_{i,t}}{P_{i,t}}$$

We obtain:

$$w_{i,t} = \underbrace{B_{i,t} \Big/ \sum_{j=1}^{n} B_{j,t}}_{\text{Fundamental component}} \times \underbrace{\sum_{j=1}^{n} P_{j,t} \Big/ P_{i,t}}_{\text{Reversal component}} \times \underbrace{\text{a cross-effect term}}_{\simeq \text{ constant}}$$

The value risk factor can be decomposed into two main components:

- a fundamental indexation pattern
- a reversal-based pattern

$\Rightarrow$ Reversal strategies $\approx$ value strategies

The value risk premium

Equity value

- In equities, the frequency of the reversal pattern is ≤ 1 month or $\geq$ 18 months
- In currencies and commodities, the frequency of the reversal pattern is very short (one or two weeks) or very long (≥ 3 years)

$\Rightarrow$ Value strategy in currencies and commodities?

The value risk premium

The payoff of the equity value risk premium

- We consider two Eurozone Value indices calculated by the same index sponsor
- The index sponsor uses the same stock selection process
- The index sponsor uses two different weighting schemes:
  - The first index considers a capitalization-weighted portfolio
  - The second index considers a minimum variance portfolio

$\Rightarrow$ We recall that the payoff of the low-volatility strategy is long put + short call

The value risk premium

The payoff of the equity value risk premium

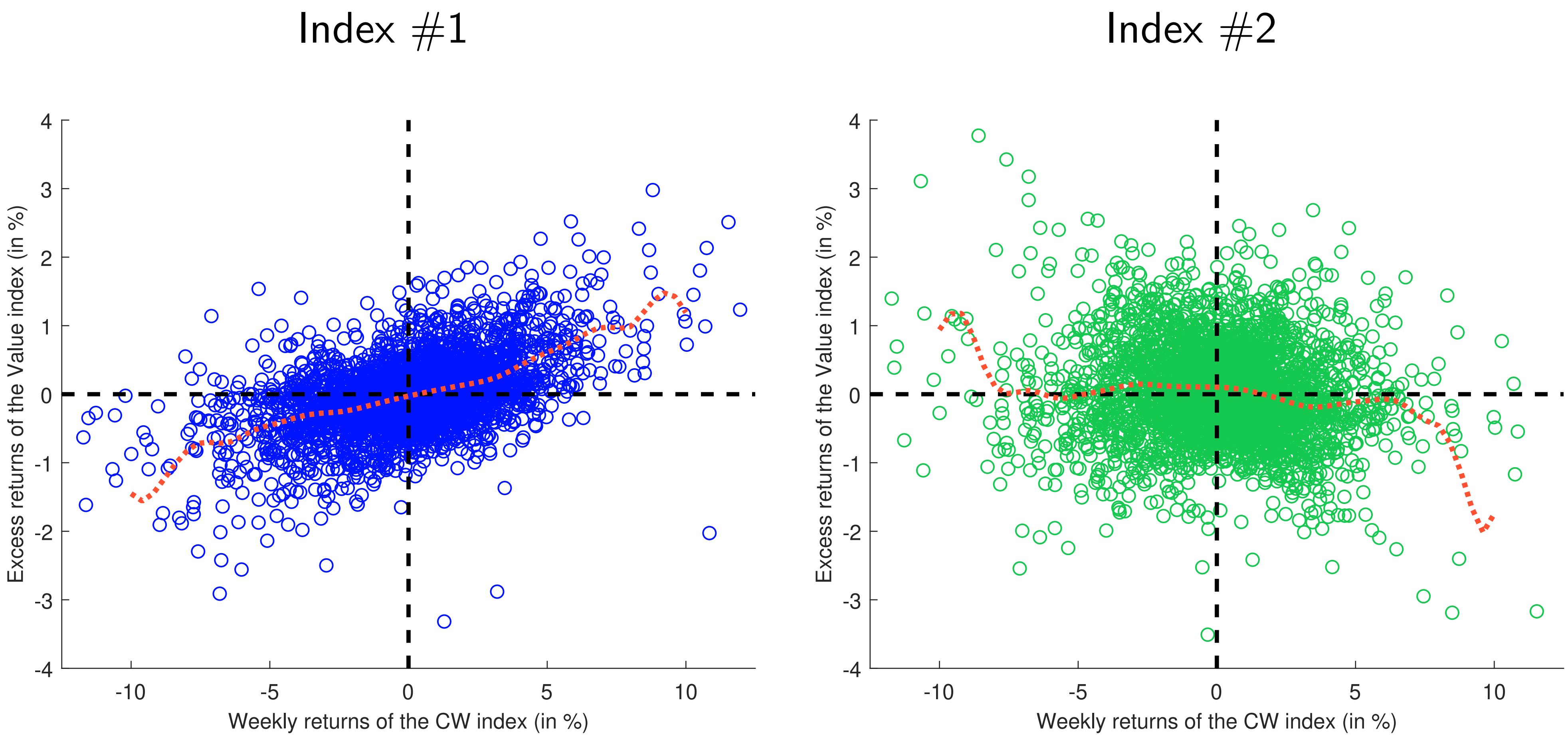


Figure 21: Which Eurozone value index has the right payoff?

The value risk premium

The payoff of the equity value risk premium

Answer

The payoff of the equity value risk premium is:

$$\textbf{Short Put} + \textbf{Long Call}$$

$\Rightarrow$ It is a skewness risk premium too!

- The design of the strategy is crucial (some weighting schemes may change or destroy the desired payoff!)
- Are the previous results valid for other asset classes, e.g. rates or currencies?

The value risk premium

Misunderstanding of the equity value risk premium

The dot-com crisis (2000-2003)

If we consider the S&P 500 index, we obtain:

- 55% of stocks post a negative performance

$$\approx 75\% \text{ of MC}$$

- 45% of stocks post a positive performance

$$\text{Maximum drawdown} = 49\ \%$$

Small caps stocks $\nearrow$
Value stocks $\nearrow$

The GFC crisis (2008)

If we consider the S&P 500 index, we obtain:

- 95% of stocks post a negative performance

$$\approx 97\% \text{ of MC}$$

- 5% of stocks post a positive performance

$$\text{Maximum drawdown} = 56\ \%$$

Small caps stocks $\searrow$
Value stocks $\searrow$

What is the impact of the liquidity risk premium?

The value risk premium
Extension to other asset classes

- Corporate bonds
  - Houweling and van Zundert (2017)
  - Ben Slimane *et al.* (2019)
  - Roncalli (2020)
- Currencies
  - MacDonald (1995)
  - Menkhoff *et al.* (2016)
  - Baku *et al.* (2019, 2020)

The momentum risk premium

Definition

- Let $S_{i,t}$ be the market price of Asset i
- We assume that:

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = \mu_{i,t}\,\mathrm{d}t + \sigma_{i,t}\,\mathrm{d}W_{i,t}$$

- The momentum of Asset i corresponds to its past trend:

$$\mathcal{M}_{i,t} = \hat{\mu}_{i,t}$$

- The momentum investor will prefer Asset i to Asset j if the momentum of Asset i is higher:

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \succ A_j$$

The momentum risk premium

Computing the momentum measure

- Past return (e.g. one-month, three-month, one-year, etc.)

$$\mathcal{M}_{i,t} = \frac{S_{i,t} - S_{i,t-h}}{S_{i,t-h}}$$

- Lagged past return[6]
- Econometric and statistical trend estimators (see Bruder *et al.* (2011) for a survey)

[6] For instance, the WML risk factor is generally implemented using the one-month lag of the twelve-month return:

$$\mathcal{M}_{i,t} = \frac{S_{i,t-1\mathrm{M}} - S_{i,t-13\mathrm{M}}}{S_{i,t-13\mathrm{M}}}$$

because the stock market is reversal within a one-month time horizon

The momentum risk premium

Three momentum strategies

1. Cross-section momentum (CSM)

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \succ A_j$$

2. Time-series momentum (TSM)

$$\mathcal{M}_{i,t} > 0 \Longrightarrow A_i \succ 0 \text{ and } \mathcal{M}_{i,t} < 0 \Longrightarrow A_i \prec 0$$

3. Reversal strategy:

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \prec A_j$$

Remark

Generally, the momentum risk premium corresponds to the CSM or TSM strategies. When we speak about momentum strategies, we can also include reversal strategies, which are more considered as trading strategies with high turnover ratios and very short holding periods (generally intra-day or daily frequency, less than one week most of the time)

The momentum risk premium

Cross-section versus time-series

Time-series momentum (TSM)

- The portfolio is long (resp. short) on the asset if it has a positive (resp. negative) momentum
- This strategy is also called "trend-following" or "trend-continuation"
- HF: CTA and managed futures
- Between asset classes

Cross-section momentum (CSM)

- The portfolio is long (resp. short) on assets that present a momentum higher (resp. lower) than the others
- This strategy is also called "winners minus losers" (or WML) by Carhart (1997)
- Within an asset class (equities, currencies)

⇒ These two momentum risk premia are very different and not well understood!

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- EWMA is the optimal trend estimator (Kalman-Bucy filtering)
- Two components
  - a short-term component given by the payoff (dynamics)
  - a long-term component given by the trading impact (performance)
- Main important parameters
  - The Sharpe ratio
  - The duration of the moving average
  - The correlation matrix
  - The term structure of the volatility
- Too much leverage kills momentum (high ruin probability)

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- The issue of diversification
  - Time-series momentum likes zero-correlated assets (e.g. multi-asset momentum premium)
  - Cross-section momentum likes highly correlated assets (e.g. equity momentum factor)
  - The number of assets decreases the P&L dispersion
  - The symmetry puzzle
  - The n/ρ trade-off
- Short-term versus long-term momentum
  - Short-term momentum is more risky than long-term momentum
  - The Sharpe ratio of long-term momentum is higher
  - The choice of the EWMA duration is more crucial for long-term momentum

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- The momentum strategy outperforms the buy-and-hold strategy when the Sharpe ratio is lower than 35%
- The specific nature of equities and bonds
  - Performance of equity momentum is explained by leverage patterns
  - Performance of bond momentum is explained by frequency patterns
- A lot of myths about the performance of CTAs (equity contribution, option profile, hedging properties)
- Momentum strategies are not alpha or absolute return strategies, but diversification strategies

The momentum risk premium

Trend-following strategies (or TSM) exhibit a convex payoff

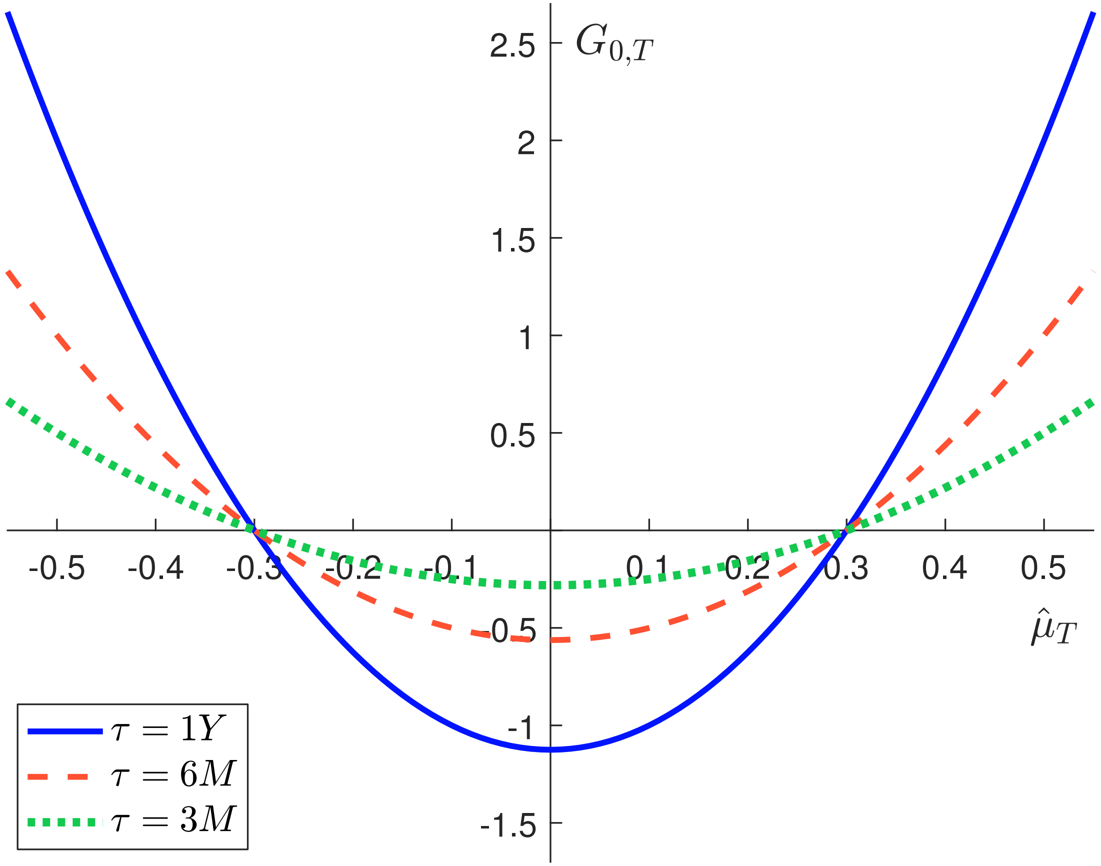


Figure 22: Option profile of the trend-following strategy

- λ is the parameter of the EWMA estimator
- $\tau = 1/\lambda$ is the duration of the EWMA estimator
- Market anomaly: the systematic risk is limited in bad times
- **Trend-following strategies exhibit a convex payoff**

The momentum risk premium

The loss of a trend-following strategy is bounded

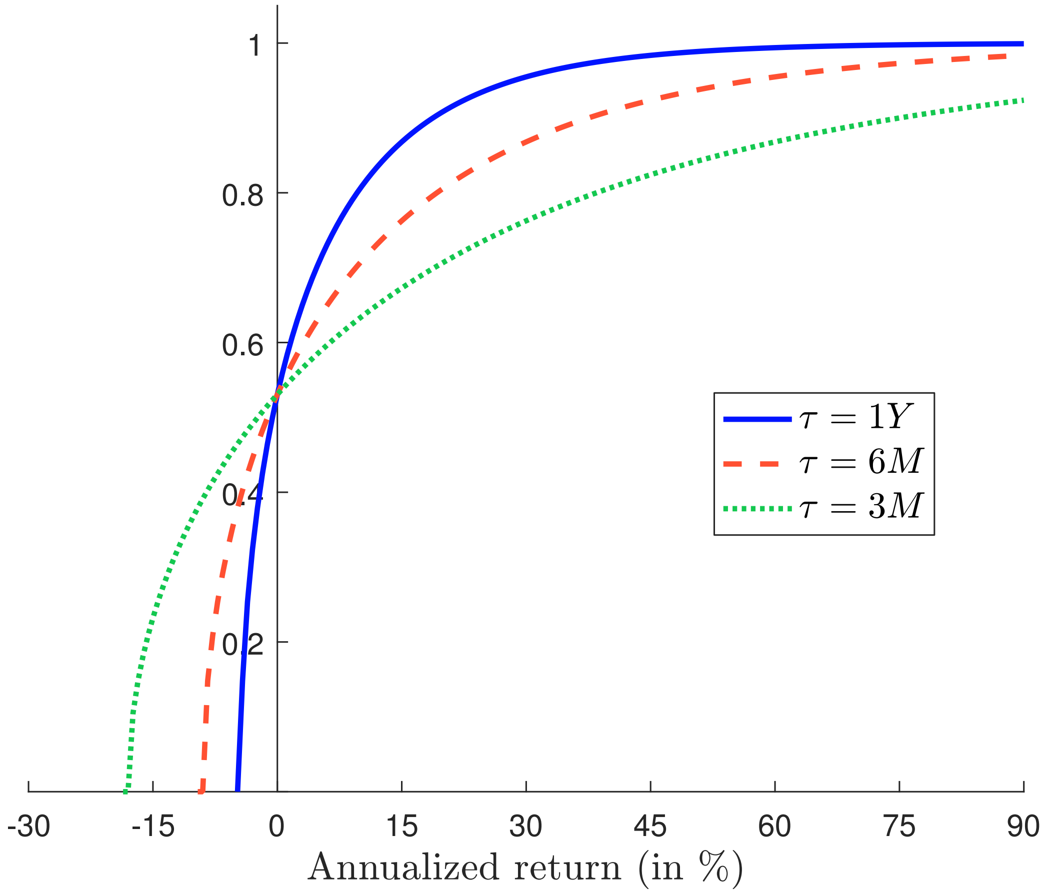


Figure 23: Cumulative distribution function of g_t ($s_t = 0$)

- s_t is the Sharpe ratio
- g_t is the trading impact
- **The loss is bounded**
- The gain may be infinite
- The return variance of short-term momentum strategies is larger than the return variance of long-term momentum strategies
- The skewness is positive

The momentum risk premium

Trend-following strategies exhibit positive skewness

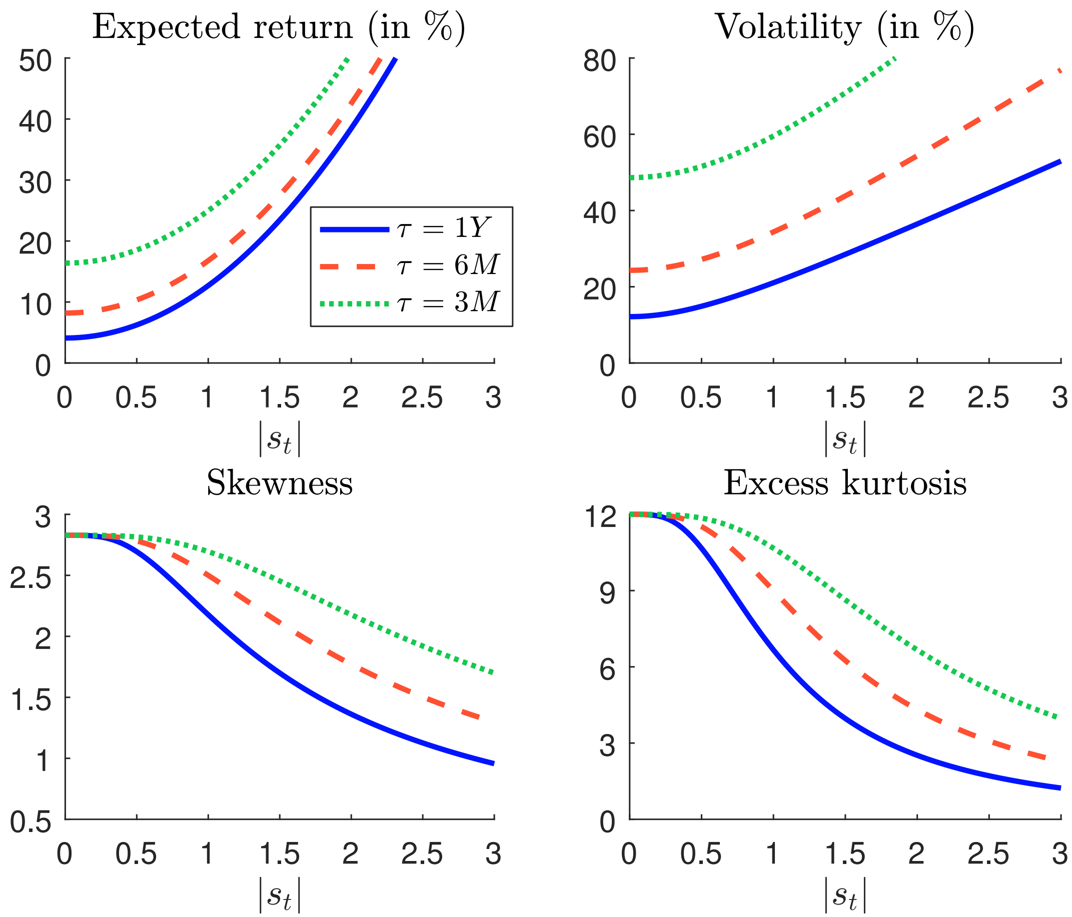


Figure 24: Statistical moments of the momentum strategy

- Short-term trend-following strategies are more risky than long-term trend-following strategies
- The skewness is positive
- It is a market anomaly, not a skewness risk premium

The momentum risk premium

Short-term versus long-term trend-following strategies

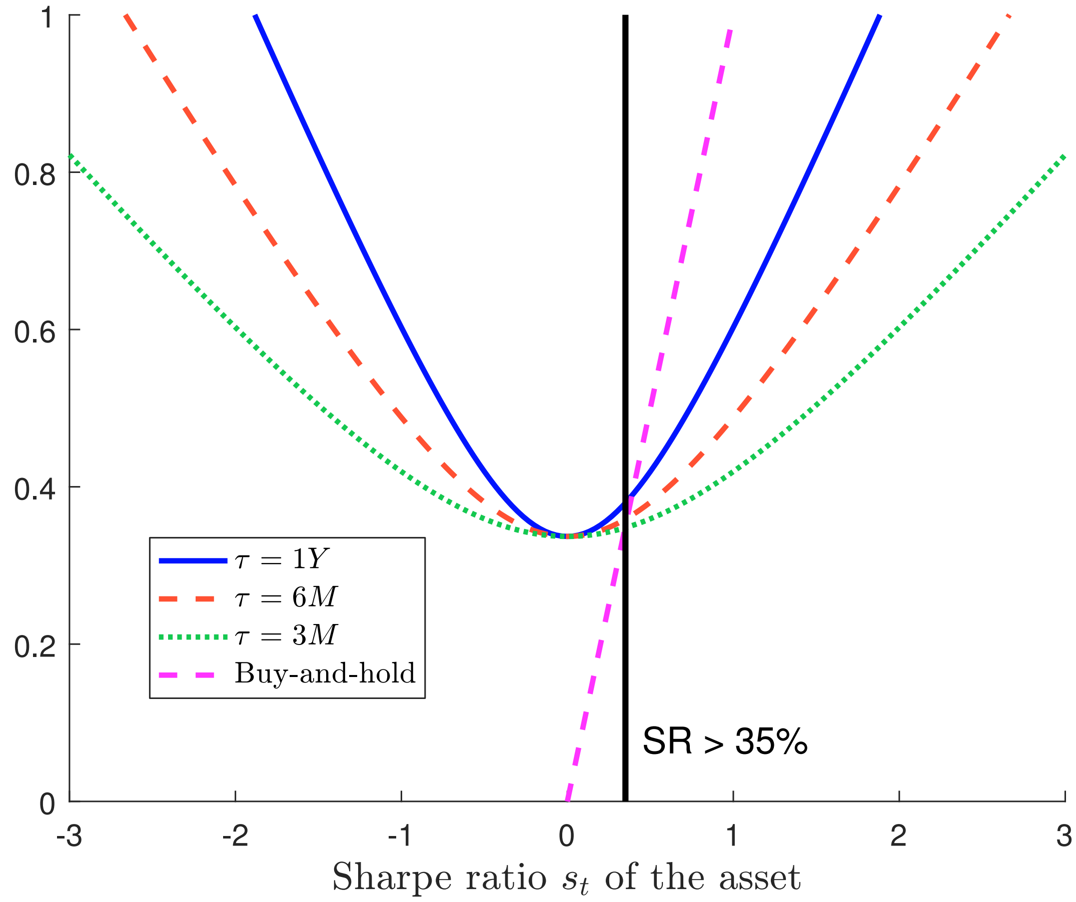


Figure 25: Sharpe ratio of the momentum strategy

- When the Sharpe ratio of the underlying is lower than 35%, the momentum strategy dominates the buy-and-hold strategy
- The Sharpe ratio of long-term momentum strategies is higher than the Sharpe ratio of short-term momentum strategies

The momentum risk premium

Relationship with the Black-Scholes robustness

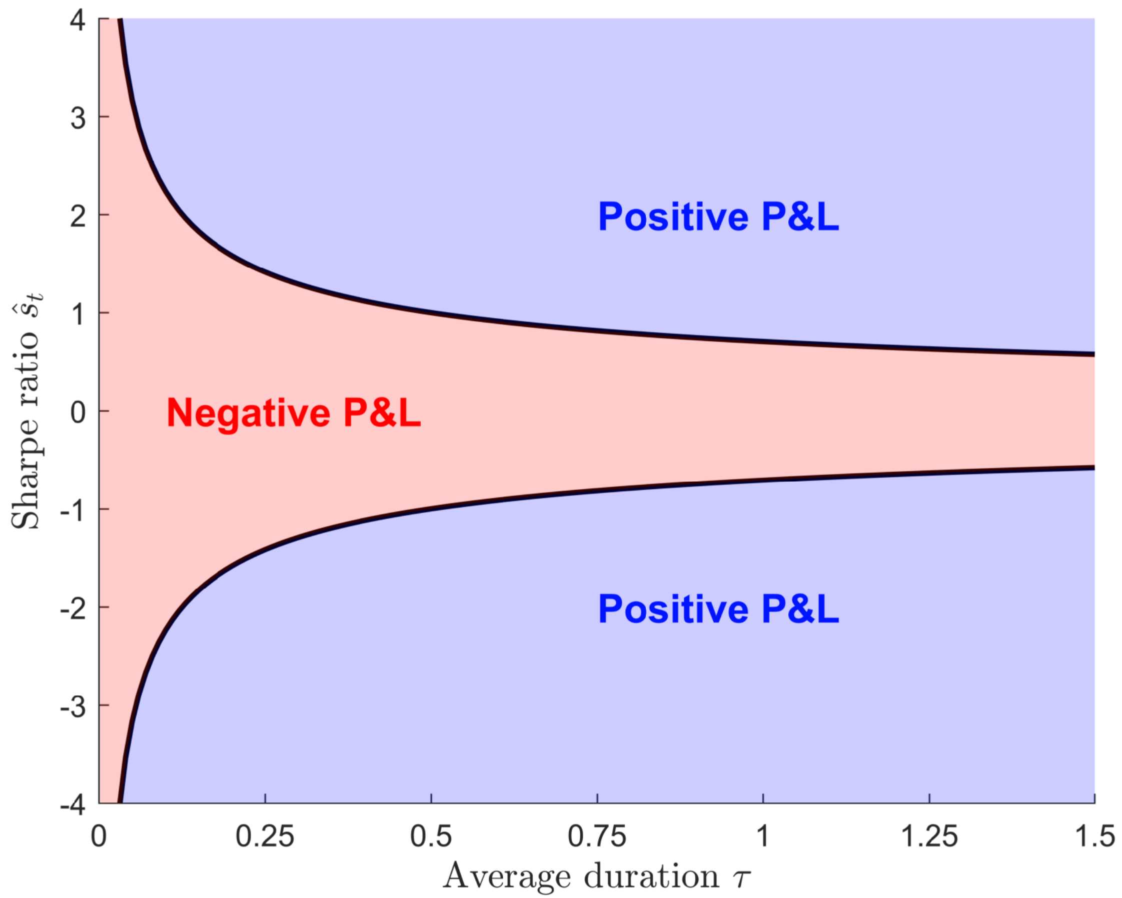


Figure 26: Admissible region for positive P&L

- Delta-hedging: implied volatility vs realized volatility
- Trend-following: duration vs realized Sharpe ratio
- The critical value for the Sharpe ratio is 1.41 for 3M and 0.71 for 1Y

The momentum risk premium

Impact of the correlation on trend-following strategies

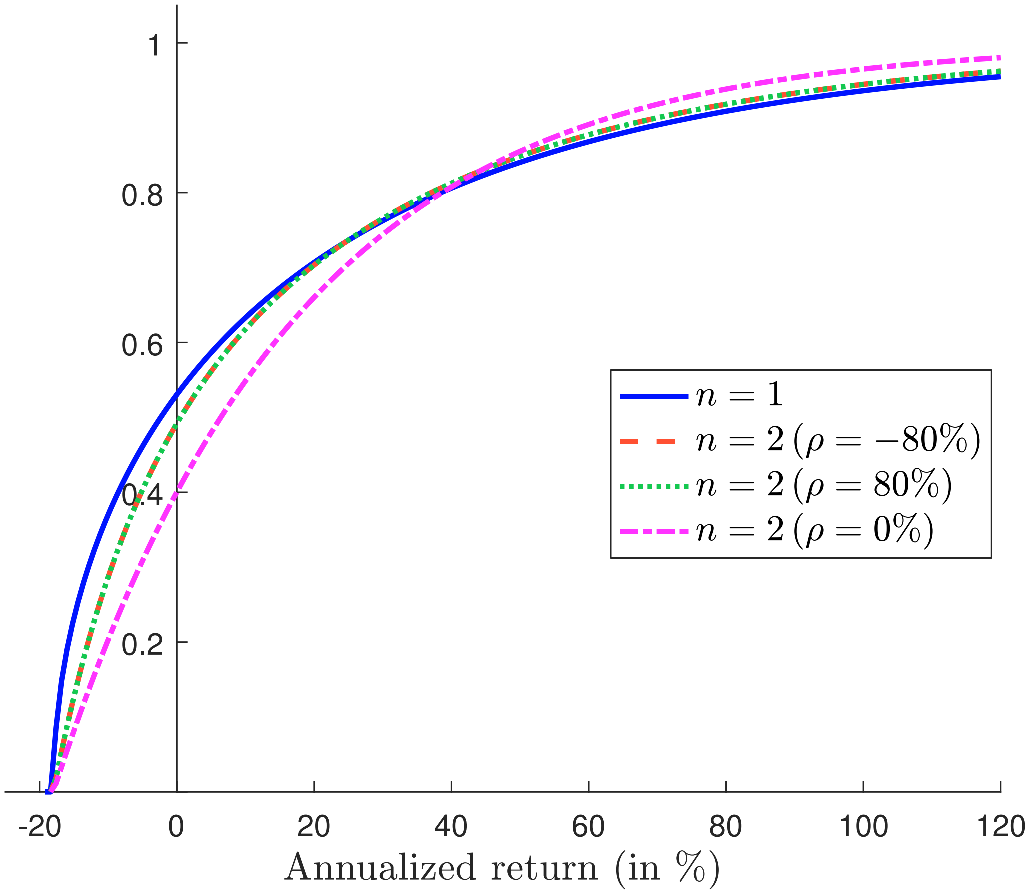


Figure 27: Cumulative distribution function of g_t ($s_t = 0$)

- Sign of correlation does not matter when the Sharpe ratio of assets is zero
- Symmetry puzzle

positive correlation
=
negative correlation

The momentum risk premium

Correlation and diversification

Long-only versus long/short diversification

We consider a portfolio (α_1, α_2) composed of two assets. We have:

$$\sigma(\rho) = \sqrt{\alpha_1^2 \sigma_1^2 + 2\rho\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2}$$

- In the case of a long-only portfolio, the best case for diversification is reached when the correlation is equal to -1:

$$|\alpha_1\sigma_1 - \alpha_2\sigma_2| = \sigma(-1) \leq \sigma(\rho) \leq \sigma(1) = \alpha_1\sigma_1 + \alpha_2\sigma_2$$

- In the case of a long/short portfolio, we generally have $\operatorname{sgn}(\alpha_1\alpha_2) = \operatorname{sgn}(\rho)$. Therefore, the best case for diversification is reached when the correlation is equal to zero: $\sigma(0) \leq \sigma(\rho)$. Indeed, when the correlation is -1, the investor is long on one asset and short on the other asset, implying that this is the same bet.

The momentum risk premium

The number of assets/correlation trade-off

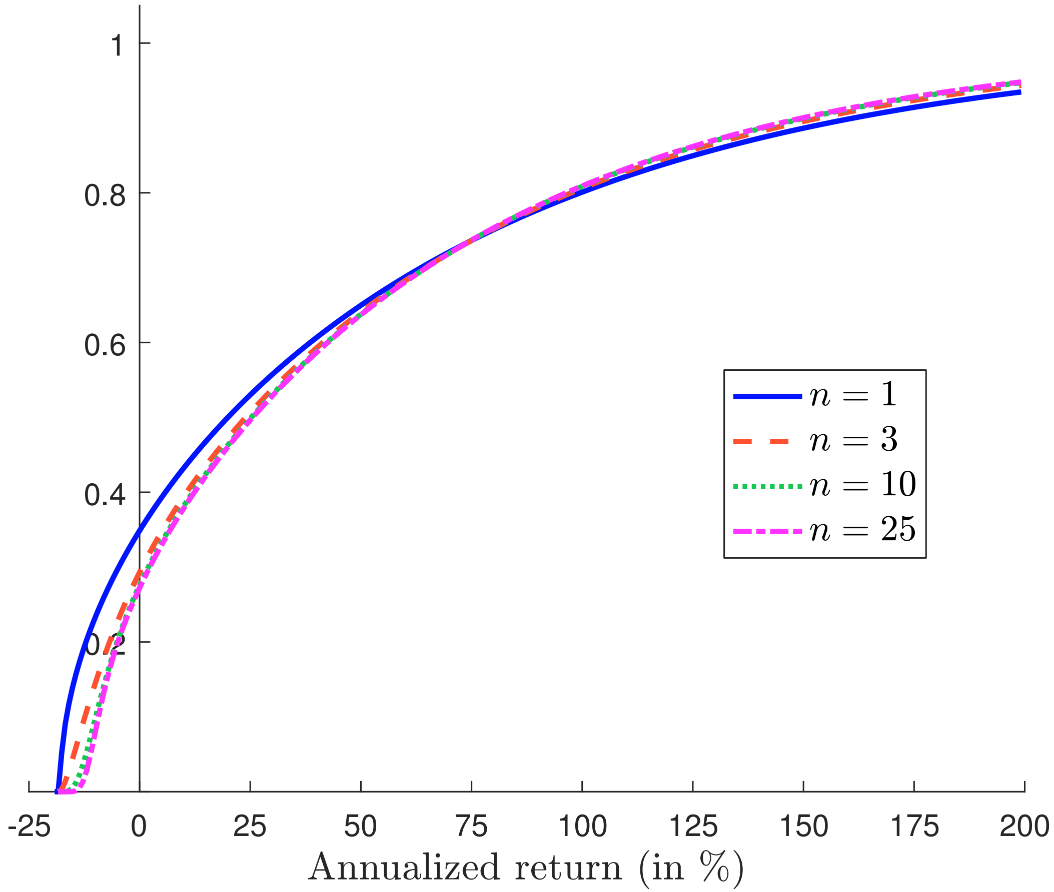


Figure 28: Impact of the number of assets on $\Pr\{g_t \leq g\}$ ($s_t = 2$, $\rho = 80\%$)

- **Correlation is not the friend of time-series momentum**
- A momentum strategy prefers a few number of assets with high Sharpe ratio absolute values than a large number of assets with low Sharpe ratio absolute values

The momentum risk premium

TSM versus CSM

Time-series momentum

- Absolute trends

$$\begin{cases} \hat{\mu}_{i,t} \geq 0 \Rightarrow e_{i,t} \geq 0 \\ \hat{\mu}_{i,t} < 0 \Rightarrow e_{i,t} < 0 \end{cases}$$

- CTA hedge funds
- Alternative risk premia in multi-asset portfolios

Cross-section momentum

- Relative trends

$$\begin{cases} \hat{\mu}_{i,t} \geq \bar{\mu}_t \Rightarrow e_{i,t} \geq 0 \\ \hat{\mu}_{i,t} < \bar{\mu}_t \Rightarrow e_{i,t} < 0 \end{cases}$$

where:

$$\bar{\mu}_t = \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}_{j,t}$$

- Statistical arbitrage / relative value
- Factor investing in equity portfolios

Beta strategy or Alpha strategy?

The momentum risk premium

Performance of cross-section momentum risk premium

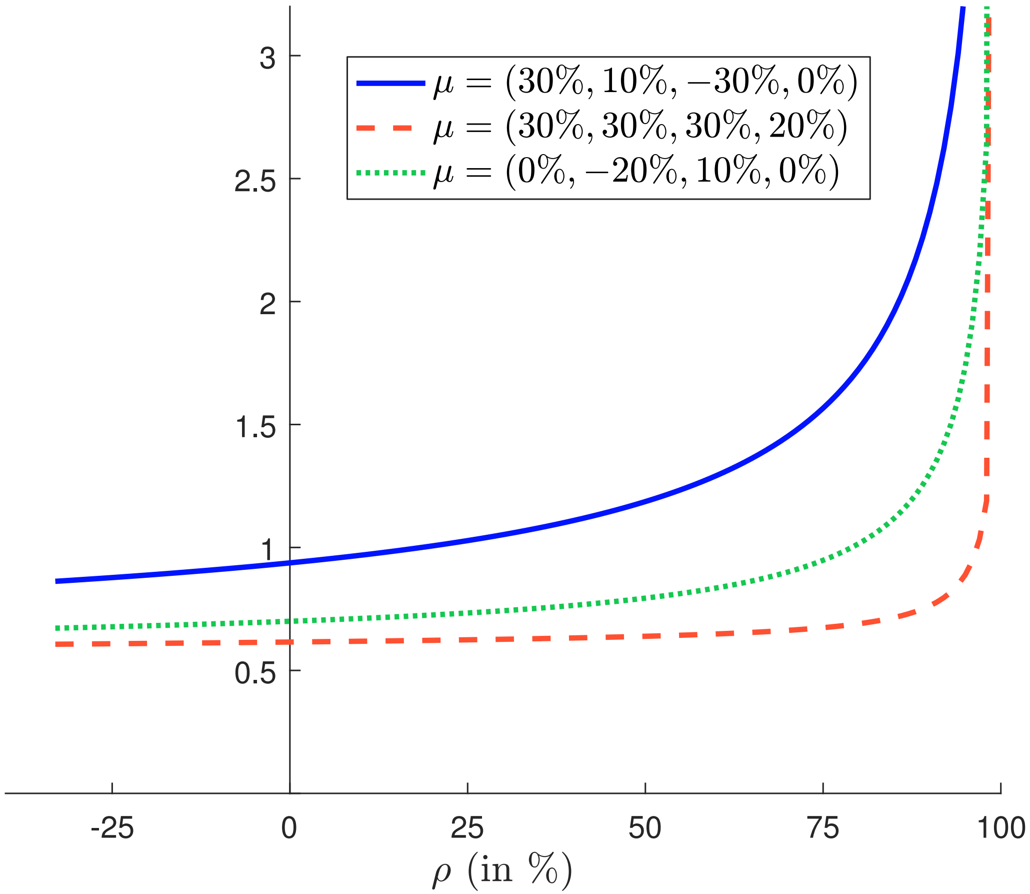


Figure 29: Sharpe ratio of the CSM strategy

- **Correlation is the friend of cross-section momentum!**
- Statistical arbitrage / relative value

The momentum risk premium

Naive replication of the SG CTA Index

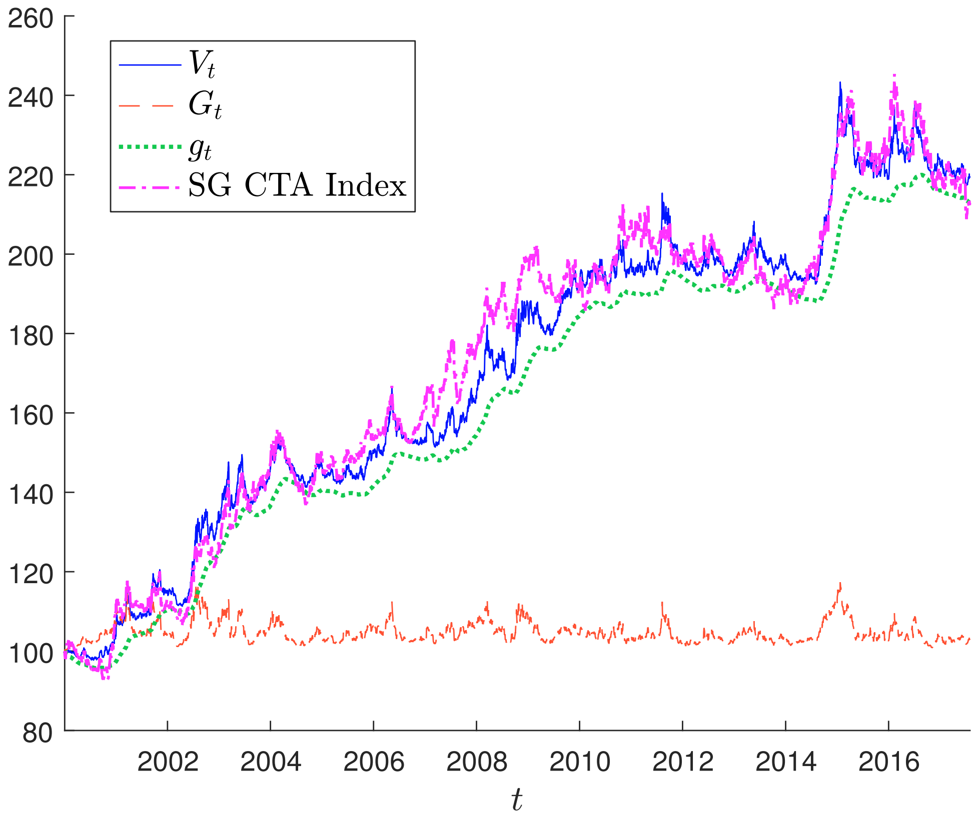


Figure 30: Comparison between the cumulative performance of the naive replication strategy and the SG CTA Index

- The performance of trend-followers comes from the trading impact
- Currencies and commodities are the main contributors!
- Mixing asset classes is the key point in order to capture the diversification premium

The momentum risk premium

Trend-following strategies benefit from traditional risk premia

Table 15: Exposure average of the trend-following strategy (in %)

Asset Class	Average Exposure	Short Exposure	Long Exposure	Short Frequency	Long Frequency
Bond	58%	−100%	122%	29%	**71%**
Equity	52%	−88%	**160%**	44%	56%
Currency	**18%**	−103%	115%	45%	55%
Commodity	**23%**	−108%	113%	41%	59%

- The specific nature of bonds: long exposure frequency $>$ short exposure frequency; long leverage $\approx$ short leverage
- The specific nature of equities: short exposure frequency $\approx$ long exposure frequency; long leverage $>$ short leverage

The momentum risk premium

The myth of short selling

- Equity and bond momentum strategies benefit from the existence of a risk premium
- Currency and commodity momentum strategies benefit from (positive / negative) trend patterns
- Leverage management $\succ$ short management
- The case of equities in the 2008 GFC, the stock-bond correlation and the symmetry puzzle

The good performance of CTAs in 2008 is not explained by their short exposure in equities, but by their long exposure in bonds

The momentum risk premium

The reversal strategy

- The reversal strategy may be defined as the opposite of the momentum strategy (CSM or TSM)
- It is also known as the mean-reverting strategy

How to reconciliate reversal and trend-following strategies?

Because they don't use the same trend windows and holding periods[7]

[7] Generally, reversal strategies use short-term or very long-term trends while trend-following strategies use medium-term trends

The momentum risk premium

The reversal strategy

The mean-reverting (or autocorrelation) strategy

- Let $R_{i,t} = \ln S_{i,t} - \ln S_{i,t-1}$ be the one-period return
- We note $\rho_i(h) = \rho(R_{i,t}, R_{i,t-h})$ the autocorrelation function
- Asset i exhibits a mean-reverting pattern if the short-term autocorrelation $\rho_i(1)$ is negative
- In this case, the short-term reversal is defined by the product of the autocorrelation and the current return:

$$\mathcal{R}_{i,t} = \rho_i(1) \cdot R_{i,t}$$

- The short-term reversal strategy is then defined by the following rule:

$$\mathcal{R}_{i,t} \geq \mathcal{R}_{j,t} \Longrightarrow i \succ j$$

The momentum risk premium

The reversal strategy

First implementation of the autocorrelation strategy

- If $\mathcal{R}_{i,t}$ is positive, meaning that the current return $R_{i,t}$ is negative, we should buy the asset, because a negative return is followed by a positive return on average
- If $\mathcal{R}_{i,t}$ is negative, meaning that the current return $R_{i,t}$ is positive, we should sell the asset, because a positive return is followed by a negative return on average

The momentum risk premium

The reversal strategy

The variance swap strategy

- We assume that the one-period asset return follows an AR(1) process:

$$R_{i,t} = \rho R_{i,t-1} + \varepsilon_t$$

where $|\rho| < 1$, $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ and $\operatorname{cov}\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$ for $j \geq 1$

- Let $\mathrm{RV}\left(h\right)$ be the annualized realized variance of the h-period asset return $R_{i,t}\left(h\right) = \ln S_{i,t} - \ln S_{i,t-h}$
- Hamdan *et al.* (2016) showed that:

$$\mathbb{E}\left[\mathrm{RV}\left(h\right)\right] = \phi\left(h\right)\mathbb{E}\left[\mathrm{RV}\left(1\right)\right]$$

where:

$$\phi\left(h\right) = 1 + 2\rho\frac{1-\rho^{h-1}}{1-\rho} - 2\sum_{j=1}^{h-1}\frac{j}{h}\rho^j$$

The momentum risk premium

The reversal strategy

The variance swap strategy

- We notice that:

$$\lim_{h \to \infty} \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] = \left(1 + \frac{2\rho}{1-\rho}\right) \cdot \mathbb{E}\left[\mathrm{RV}\left(1\right)\right]$$

- When the autocorrelation is negative, this implies that the long-term frequency variance is lower than the short-term frequency variance
- More generally, we have:

$$\begin{cases} \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] < \mathbb{E}\left[\mathrm{RV}\left(1\right)\right] & \text{if } \rho < 0 \\ \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] \geq \mathbb{E}\left[\mathrm{RV}\left(1\right)\right] & \text{otherwise} \end{cases}$$

The momentum risk premium

The reversal strategy

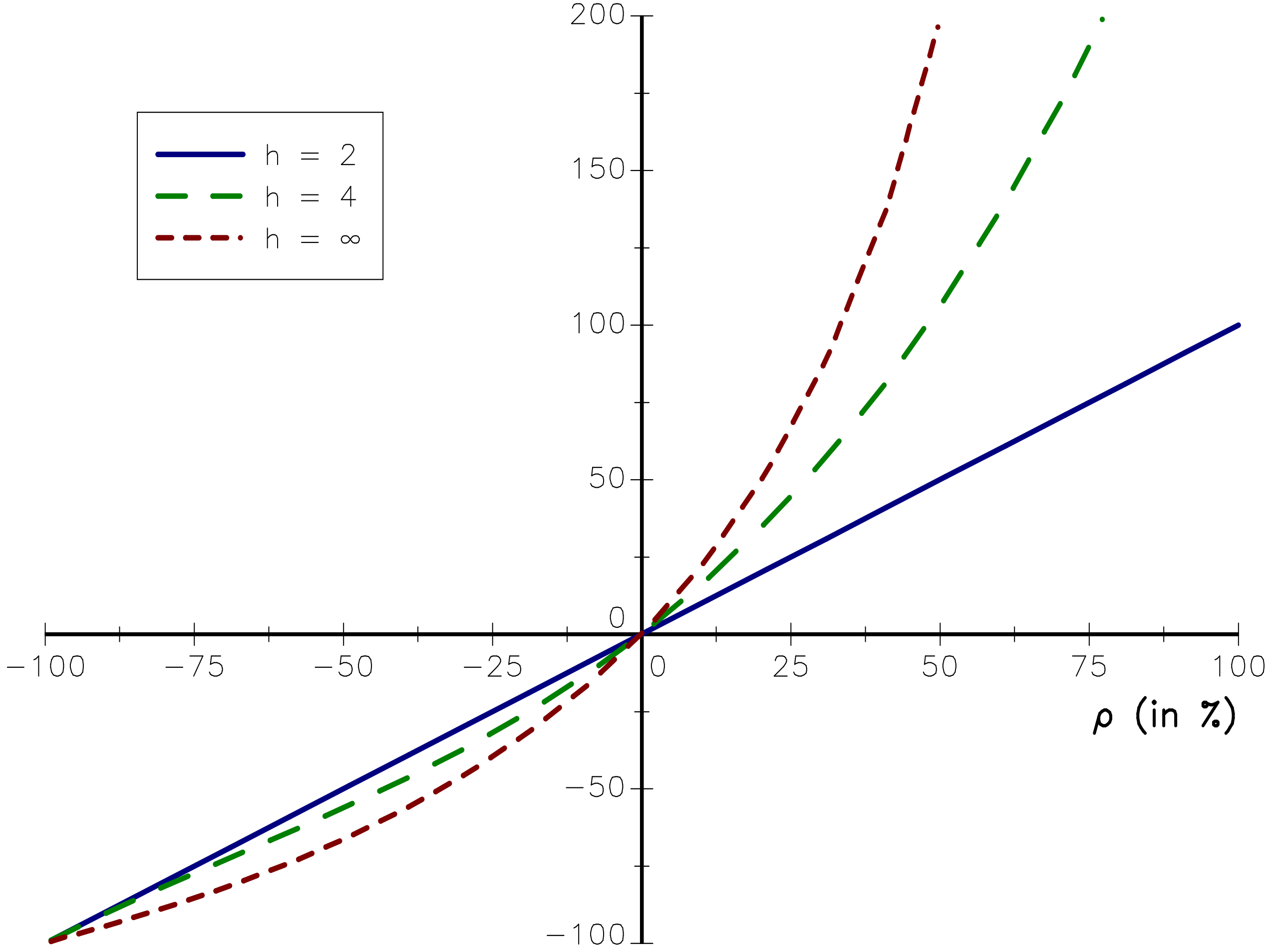


Figure 31: Variance ratio $(\mathrm{RV}(h) - \mathrm{RV}(1)) / \mathrm{RV}(1)$ (in %)

The momentum risk premium

The reversal strategy

Second implementation of the autocorrelation strategy

- The spread between daily/weekly and weekly/monthly variance swaps depends on the autocorrelation of daily returns
- The reversal strategy consists in being long on the daily/weekly variance swaps and short on the weekly/monthly variance swaps

The momentum risk premium

The reversal strategy

The long-term reversal strategy

- The long-term return reversal is defined by the difference between long-run and short-period average prices:

$$\mathcal{R}_{i,t} = \bar{S}_{i,t}^{LT} - \bar{S}_{i,t}^{ST}$$

- Typically, $\bar{S}_{i,t}^{ST}$ is the average price over the last year and $\bar{S}_{i,t}^{LT}$ is the average price over the last five years
- The long-term return reversal strategy follows the same rule as the short-term reversal strategy
- This reversal strategy is equivalent to a value strategy because the long-run average price can be viewed as an estimate of the fundamental price in some asset classes

The momentum risk premium

The reversal strategy

Implementation of the long-term reversal strategy

- If $\mathcal{R}_{i,t}$ is positive, the long-term mean of the asset price is above its short-term mean $\Rightarrow$ we should buy the asset
- If $\mathcal{R}_{i,t}$ is negative, the long-term mean of the asset price is below its short-term mean $\Rightarrow$ we should sell the asset

The liquidity risk premium

What means "*liquidity risk*"?

"[...] there is also broad belief among users of financial liquidity — traders, investors and central bankers — that the principal challenge is not the **average level** *of financial liquidity ... but its* **variability** *and uncertainty " (Persaud, 2003).*

The liquidity risk premium

The liquidity-adjusted CAPM

L-CAPM (Acharya and Pedersen, 2005)

We note L_i the relative (stochastic) illiquidity cost of Asset i. At the equilibrium, we have:

$$\mathbb{E}\left[R_i - L_i\right] - R_f = \tilde{\beta}_i \left(\mathbb{E}\left[R_M - L_M\right] - R_f\right)$$

where:

$$\tilde{\beta}_i = \frac{\operatorname{cov}\left(R_i - L_i, R_M - L_M\right)}{\operatorname{var}\left(R_M - L_M\right)}$$

CAPM in the frictionless economy

$\Downarrow$

CAPM in net returns (including illiquidity costs)

The liquidity risk premium

The liquidity-adjusted CAPM

- The liquidity-adjusted beta can be decomposed into four beta(s):

$$\tilde{\beta}_i = \beta_i + \beta\left(L_{i,}, L_M\right) - \beta\left(R_{i,}, L_M\right) - \beta\left(L_{i,}, R_M\right)$$

  where:

  - $\beta_i = \beta\left(R_{i,} R_M\right)$ is the standard market beta;
  - $\beta\left(L_{i,}, L_M\right)$ is the beta associated to the commonality in liquidity with the market liquidity;
  - $\beta\left(R_{i,}, L_M\right)$ is the beta associated to the return sensitivity to market liquidity;
  - $\beta\left(L_{i,}, R_M\right)$ is the beta associated to the liquidity sensitivity to market returns.

- The risk premium is equal to:

$$\begin{aligned}\pi_i &= \mathbb{E}\left[L_i\right] + \left(\beta_i + \beta\left(L_{i,}, L_M\right)\right)\pi_M - \\ &\quad \left(\tilde{\beta}_i \mathbb{E}\left[L_M\right] + \left(\beta\left(R_{i,}, L_M\right) + \beta\left(L_{i,}, R_M\right)\right)\pi_M\right)\end{aligned}$$

The liquidity risk premium

The liquidity-adjusted CAPM

Acharya and Pedersen (2005)

If assets face some liquidity costs, the relationship between the risk premium and the beta of asset i becomes:

$$\mathbb{E}[R_i] - R_f = \alpha_i + \beta_i \left(\mathbb{E}[R_M] - R_f\right)$$

where α_i is a function of the relative liquidity of Asset i with respect to the market portfolio and the liquidity beta(s):

$$\begin{aligned}\alpha_i &= \left(\mathbb{E}[L_i] - \tilde{\beta}_i \mathbb{E}[L_M]\right) + \\ &\quad \beta\left(L_{i,}, L_M\right)\pi_M - \beta\left(R_{i,}, L_M\right)\pi_M - \beta\left(L_{i,}, R_M\right)\pi_M\end{aligned}$$

The liquidity risk premium

Disentangling the liquidity alpha

- We deduce that:

$$\alpha_i \neq \mathbb{E}\left[L_i\right]$$

meaning that the risk premium of an illiquid asset is not the systematic risk premium plus a premium due the illiquidity level:

$$\mathbb{E}\left[R_i\right] - R_f \neq \mathbb{E}\left[L_i\right] + \beta_i \left(\mathbb{E}\left[R_M\right] - R_f\right)$$

- The 4 liquidity premia are highly correlated[8] ($\mathbb{E}\left[L_i\right]$, $\beta\left(L_{i,}, L_M\right)$, $\beta\left(R_{i,}, L_M\right)$ and $\beta\left(L_{i,}, R_M\right)$).
- Acharaya and Pedersen (2005) found that $\mathbb{E}\left[L_i\right]$ represents 75% of α_i on average. The 25% remaining are mainly explained by the liquidity sensitivity to market returns – $\beta\left(L_{i,}, R_M\right)$.

[8]For instance, we have $\rho\left(\beta\left(L_{i,}, L_M\right), \beta\left(R_{i,}, L_M\right)\right) = -57\%$, $\rho\left(\beta\left(L_{i,}, L_M\right), \beta\left(L_{i,}, R_M\right)\right) = -94\%$ and $\rho\left(\beta\left(R_{i,}, L_M\right), \beta\left(L_{i,}, R_M\right)\right) = 73\%$.

The liquidity risk premium

Three liquidity risks

In fact, we have:

$$\alpha_i = \text{illiquidity level} + \text{illiquidity covariance risks}$$

1. $\beta(L_{i,}, L_M)$
   - An asset that becomes illiquid when the market becomes illiquid should have a higher risk premium
   - Substitution effects when the market becomes illiquid
2. $\beta(R_{i,}, L_M)$
   - Assets that perform well in times of market illiquidity should have a lower risk premium
   - Relationship with solvency constraints
3. $\beta(L_{i,}, R_M)$
   - Investors accept a lower risk premium on assets that are liquid in a bear market
   - Selling markets $\neq$ buying markets

The liquidity risk premium

How does market liquidity impact risk premia?

Three main impacts

- Effect on the risk premium
- Effect on the price dynamics
  If liquidity is persistent, negative shock to liquidity implies low current returns and high predicted future returns:

$$\operatorname{cov}\left(L_{i,t}, R_{i,t}\right) < 0 \text{ and } \partial_{L_{i,t}} \mathbb{E}_t\left[R_{i,t+1}\right] > 0$$

- Effect on portfolio management

- Sovereign bonds
- Corporate bonds
- Stocks
- Small caps
- Private equities

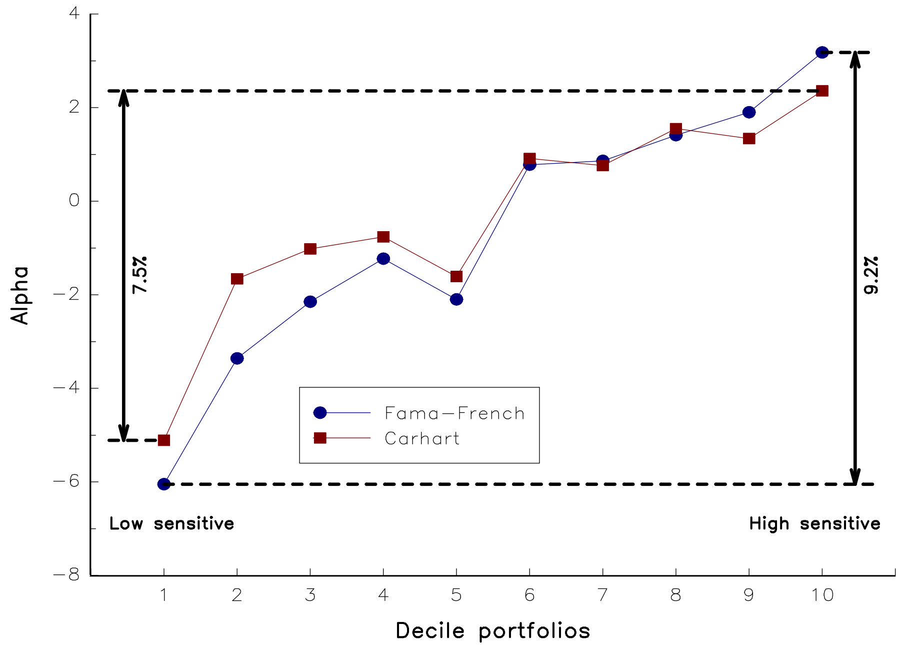
The liquidity risk premium

Application to stocks

Pastor and Stambaugh (2003) include a liquidity premium in the Fama-French-Carhart model:

$$\begin{aligned}\mathbb{E}\left[R_i\right] - R_f &= \beta_i^M \left(\mathbb{E}\left[R_M\right] - R_f\right) + \beta_i^{SMB} \mathbb{E}\left[R_{SMB}\right] + \beta_i^{HML} \mathbb{E}\left[R_{HML}\right] + \\ & \quad \beta_i^{WML} \mathbb{E}\left[R_{WML}\right] + \beta_i^{LIQ} \mathbb{E}\left[R_{LIQ}\right]\end{aligned}$$

where LIQ measures the shock or innovation of the aggregate liquidity.



Alphas of decile portfolios sorted on predicted liquidity beta(s)

Long Q10 / Short Q1:

- 9.2% wrt 3F Fama-French model
- 7.5% wrt 4F Carhart model

The liquidity risk premium

Impact of the liquidity on the stock market

The dot-com crisis (2000-2003)

If we consider the S&P 500 index, we obtain:

- 55% of stocks post a negative performance

$$\approx 75\% \text{ of MC}$$

- 45% of stocks post a positive performance

$$\text{Maximum drawdown} = 49\ \%$$

Small caps stocks $\nearrow$
Value stocks $\nearrow$

The GFC crisis (2008)

If we consider the S&P 500 index, we obtain:

- 95% of stocks post a negative performance

$$\approx 97\% \text{ of MC}$$

- 5% of stocks post a positive performance

$$\text{Maximum drawdown} = 56\ \%$$

Small caps stocks $\searrow$
Value stocks $\searrow$

The liquidity risk premium

The specific status of the stock market

The interconnectedness nature of illiquid assets and liquid assets: the example of the Global Financial Crisis

- Subprime crisis $\Leftrightarrow$ banks (credit risk)
- Banks $\Leftrightarrow$ asset management, e.g. hedge funds (funding & leverage risk)
- Asset management $\Leftrightarrow$ equity market (liquidity risk)
- Equity market $\Leftrightarrow$ banks (asset-price & collateral risk)

The equity market is the ultimate liquidity provider:
GFC $\gg$ internet bubble

Remark

$^1/_3$ *of the losses in the stock market is explained by the liquidity supply*

The liquidity risk premium

Relationship between diversification & liquidity

During good times

- Medium correlation between liquid assets
- Illiquid assets have low impact on liquid assets
- Low substitution effects

Main effect:

$$\mathbb{E}\left[L_i\right]$$

During bad times

- High correlation between liquid assets
- Illiquid assets have a high impact on liquid assets
- High substitution effects

Main effects:

$$\beta\left(L_i, R_M\right) \text{ and } \beta\left(R_i, L_M\right)$$

The skewness puzzle

Skewness aggregation $\neq$ volatility aggregation

When we accumulate long/short skewness risk premia in a portfolio, the volatility of this portfolio decreases dramatically, but its skewness risk generally increases!

- Skewness diversification $\neq$ volatility diversification

$$\begin{aligned} \sigma(X_1 + X_2) &\leq \sigma(X_1) + \sigma(X_2) \\ |\gamma_1(X_1 + X_2)| &\nleq |\gamma_1(X_1) + \gamma_1(X_2)| \end{aligned}$$

Skewness is not a convex risk measure

The skewness puzzle

Example 12

We assume that (X_1, X_2) follows a bivariate log-normal distribution $\mathcal{LN}\left(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho\right)$. This implies that $\ln X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $\ln X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the correlation between $\ln X_1$ and $\ln X_2$.

The skewness puzzle

We recall that the skewness of X_1 is equal to:

$$\gamma_1(X_1) = \frac{\mu_3(X_1)}{\mu_2^{3/2}(X_1)} = \frac{e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2}{\left(e^{\sigma_1^2} - 1\right)^{3/2}}$$

whereas the skewness of $X_1 + X_2$ is equal to:

$$\gamma_1(X_1 + X_2) = \frac{\mu_3(X_1 + X_2)}{\mu_2^{3/2}(X_1 + X_2)}$$

where $\mu_n(X)$ is the n^{th} central moment of X

The skewness puzzle

In order to find the skewness of the sum $X_1 + X_2$, we need a preliminary result. By denoting $X = \alpha_1 \ln X_1 + \alpha_2 \ln X_2$, we have[9]:

$$\mathbb{E}\left[e^X\right] = e^{\mu_X + \frac{1}{2}\sigma_X^2}$$

where:

$$\mu_X = \alpha_1 \mu_1 + \alpha_2 \mu_2$$

and:

$$\sigma_X^2 = \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1 \alpha_2 \rho \sigma_1 \sigma_2$$

It follows that:

$$\mathbb{E}\left[X_1^{\alpha_1} X_2^{\alpha_2}\right] = e^{\alpha_1 \mu_1 + \alpha_2 \mu_2 + \frac{1}{2}\left(\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1 \alpha_2 \rho \sigma_1 \sigma_2\right)}$$

[9] Because X is a Gaussian random variable

The skewness puzzle

We have:

$$\mu_2 (X_1 + X_2) = \mu_2 (X_1) + \mu_2 (X_2) + 2 \operatorname{cov} (X_1, X_2)$$

where:

$$\mu_2 (X_1) = e^{2\mu_1 + \sigma_1^2} \left(e^{\sigma_1^2} - 1 \right)$$

and:

$$\operatorname{cov} (X_1, X_2) = \left(e^{\rho \sigma_1 \sigma_2} - 1 \right) e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2}$$

The skewness puzzle

For the third moment of $X_1 + X_2$, we use the following formula:

$$\mu_3 (X_1 + X_2) = \mu_3 (X_1) + \mu_3 (X_2) + 3 (\operatorname{cov}(X_1, X_1, X_2) + \operatorname{cov}(X_1, X_2, X_2))$$

where:

$$\mu_3 (X_1) = e^{2\mu_1 + \frac{3}{2}\sigma_1^2} \left(e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2 \right)$$

and:

$$\operatorname{cov}(X_1, X_1, X_2) = \left(e^{\rho\sigma_1\sigma_2} - 1 \right) e^{2\mu_1 + \sigma_1^2 + \mu_2 + \frac{\sigma_2^2}{2}} \left(e^{\sigma_1^2 + \rho\sigma_1\sigma_2} + e^{\sigma_2^2} - 2 \right)$$

The skewness puzzle

We deduce that:

$$\gamma_1 (X_1 + X_2) = \frac{\mu_3 (X_1 + X_2)}{\mu_2^{3/2} (X_1 + X_2)}$$

where:

$$\begin{aligned} \mu_2 (X_1 + X_2) &= e^{2\mu_1 + \sigma_1^2} \left(e^{\sigma_1^2} - 1 \right) + e^{2\mu_2 + \sigma_2^2} \left(e^{\sigma_2^2} - 1 \right) + \\ & \quad 2 \left(e^{\rho \sigma_1 \sigma_2} - 1 \right) e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \end{aligned}$$

and:

$$\begin{aligned} \mu_3 (X_1 + X_2) &= e^{2\mu_1 + \frac{3}{2}\sigma_1^2} \left(e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2 \right) + e^{2\mu_2 + \frac{3}{2}\sigma_2^2} \left(e^{3\sigma_2^2} - 3e^{\sigma_2^2} + 2 \right) + \\ & \quad 3 \left(e^{\rho \sigma_1 \sigma_2} - 1 \right) e^{2\mu_1 + \sigma_1^2 + \mu_2 + \frac{\sigma_2^2}{2}} \left(e^{\sigma_1^2 + \rho \sigma_1 \sigma_2} + e^{\sigma_2^2} - 2 \right) + \\ & \quad 3 \left(e^{\rho \sigma_1 \sigma_2} - 1 \right) e^{\mu_1 + \frac{1}{2}\sigma_1^2 + 2\mu_2 + \sigma_2^2} \left(e^{\sigma_2^2 + \rho \sigma_1 \sigma_2} + e^{\sigma_1^2} - 2 \right) \end{aligned}$$

The skewness puzzle

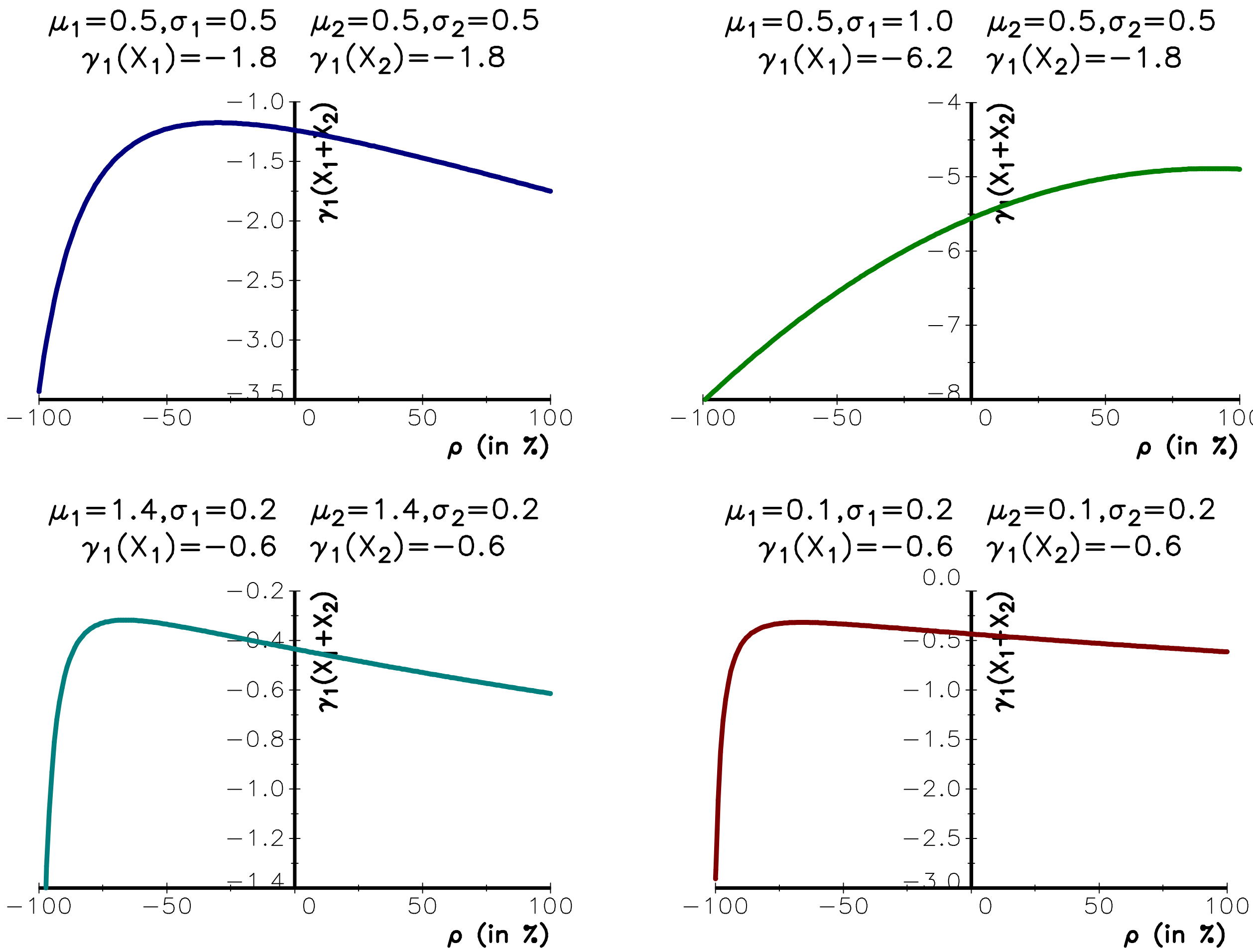


Figure 32: Skewness aggregation of the random vector $(-X_1, -X_3)$

The skewness puzzle

Why?

- Volatility diversification works very well with L/S risk premia:

$$\sigma(R(x)) \approx \frac{\bar{\sigma}}{\sqrt{n}}$$

- Drawdown diversification don't work very well because bad times are correlated and are difficult to hedge:

$$\mathrm{DD}(x) \approx \overline{\mathrm{DD}}$$

The skewness puzzle

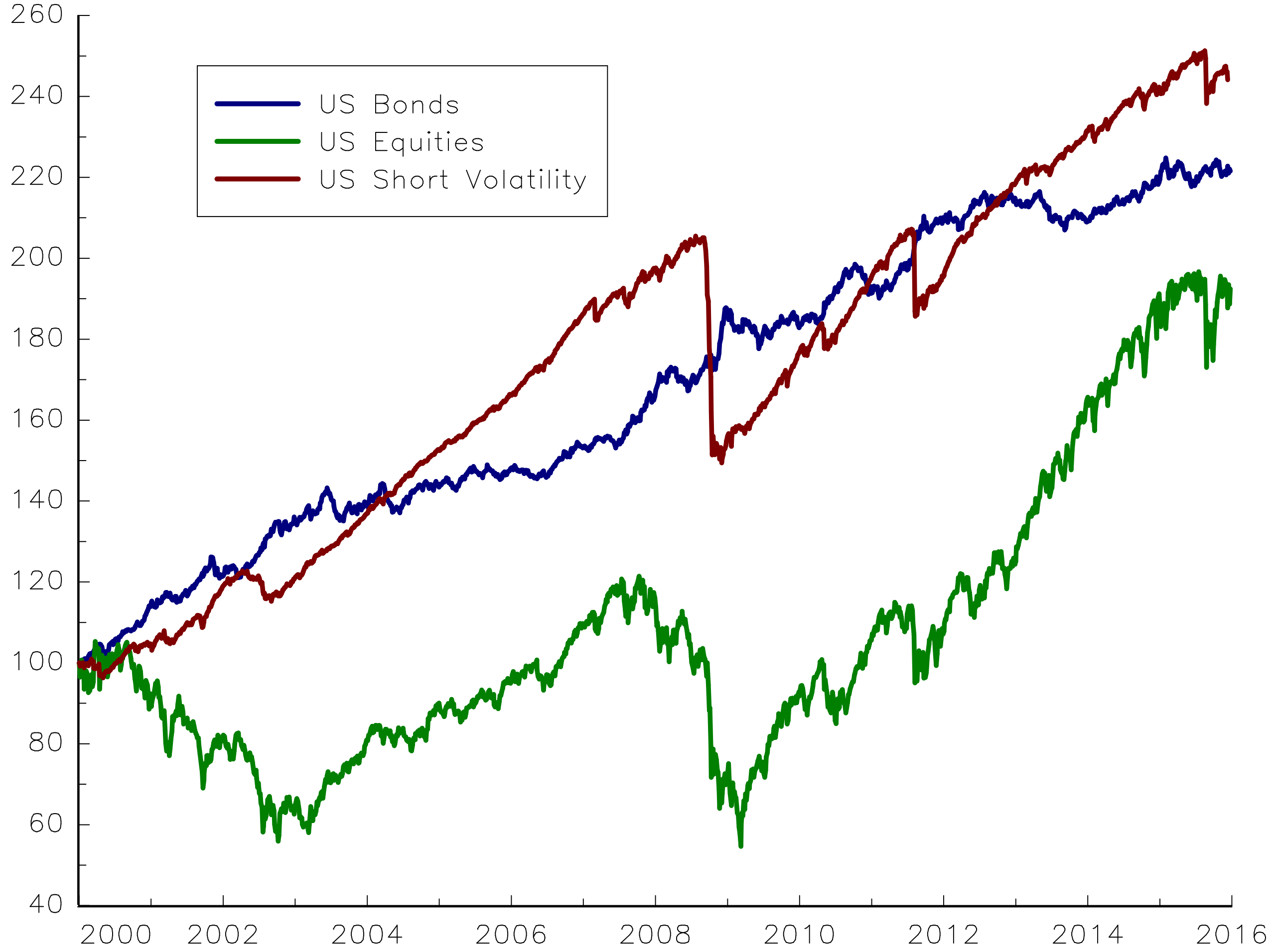


Figure 33: Cumulative performance of US 10Y bonds, US equities and US short volatility

Source: Bruder *et al.* (2016)

The correlation puzzle

We consider the Gaussian random vector (R_1, R_2, R_3), whose volatilities are equal to 25%, 12% and 9.76%. The correlation matrix is given by:

$$C = \begin{pmatrix} 100\% & & \\ -25.00\% & 100\% & \\ 55.31\% & 66.84\% & 100\% \end{pmatrix}$$

Good diversification? (correlation approach)

If R_i represents an asset return (or an excess return), we conclude that (R_1, R_2, R_3) is a well-diversified investment universe

Bad diversification? (payoff approach)

However, we have:

$$R_3 = 0.30 R_1 + 0.70 R_2$$

The correlation puzzle

Fantasies about correlations

- Negative correlations are good for diversification
- Positive correlations are bad for diversification

- If $\rho(R_1, R_2)$ is close to -1, can we hedge Asset 1 with Asset 2?
- If $\rho(R_1, R_2)$ is close to -1, can we diversify Asset 1 with Asset 2?
- If $\rho(R_1, R_2)$ is close to $+1$, can we hedge Asset 1 with a short position on Asset 2?
- If $\rho(R_1, R_2)$ is close to $+1$, can we diversify Asset 1 with a short position on Asset 2?
- Does $\rho(R_1, R_2) = -70\%$ correspond to a better diversification pattern than $\rho(R_1, R_2) = +70\%$?

There is a confusion between diversification and hedging!

The payoff approach

Table 16: Correlation matrix between asset classes (2000-2016)

		Equity				Bond			
		US	Euro	UK	Japan	US	Euro	UK	Japan
Equity	US	100%							
	Euro	78%	100%						
	UK	79%	87%	100%					
	Japan	53%	57%	55%	100%				
Bond	US	−35%	−39%	−32%	−29%	100%			
	Euro	−17%	−16%	−16%	−16%	58%	100%		
	UK	−31%	−37%	−30%	−31%	72%	63%	100%	
	Japan	−17%	−18%	−16%	−33%	37%	31%	36%	100%

Correlation = Pearson correlation = Linear correlation

The payoff approach

Let us consider a Gaussian random vector defined as follows:

$$\left(\begin{array}{c} Y \\ X \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_y \\ \mu_x \end{array}\right), \left(\begin{array}{cc} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{array}\right)\right)$$

The conditional distribution of Y given $X = x$ is a MN distribution:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{yy|x} = \sigma^2\left[Y \mid X = x\right] = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$$

We deduce that:

$$\begin{array}{rcl} Y & = & \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right) + u \\ & = & \underbrace{\left(\mu_y - \Sigma_{yx}\Sigma_{xx}^{-1}\mu_x\right)}_{\beta_0} + \underbrace{\Sigma_{yx}\Sigma_{xx}^{-1}}_{\beta^\top} x + u \end{array}$$

where u is a centered Gaussian random variable with variance $s^2 = \Sigma_{yy|x}$.

The payoff approach

Correlation = linear payoff

It follows that the payoff function is defined by the curve:

$$y = f(x)$$

where:

$$\begin{aligned} f(x) &= \mathbb{E}[R_2 \mid R_1 = x] \\ &= (\mu_2 - \beta_{2|1}\mu_1) + \beta_{2|1}x \end{aligned}$$

The payoff approach

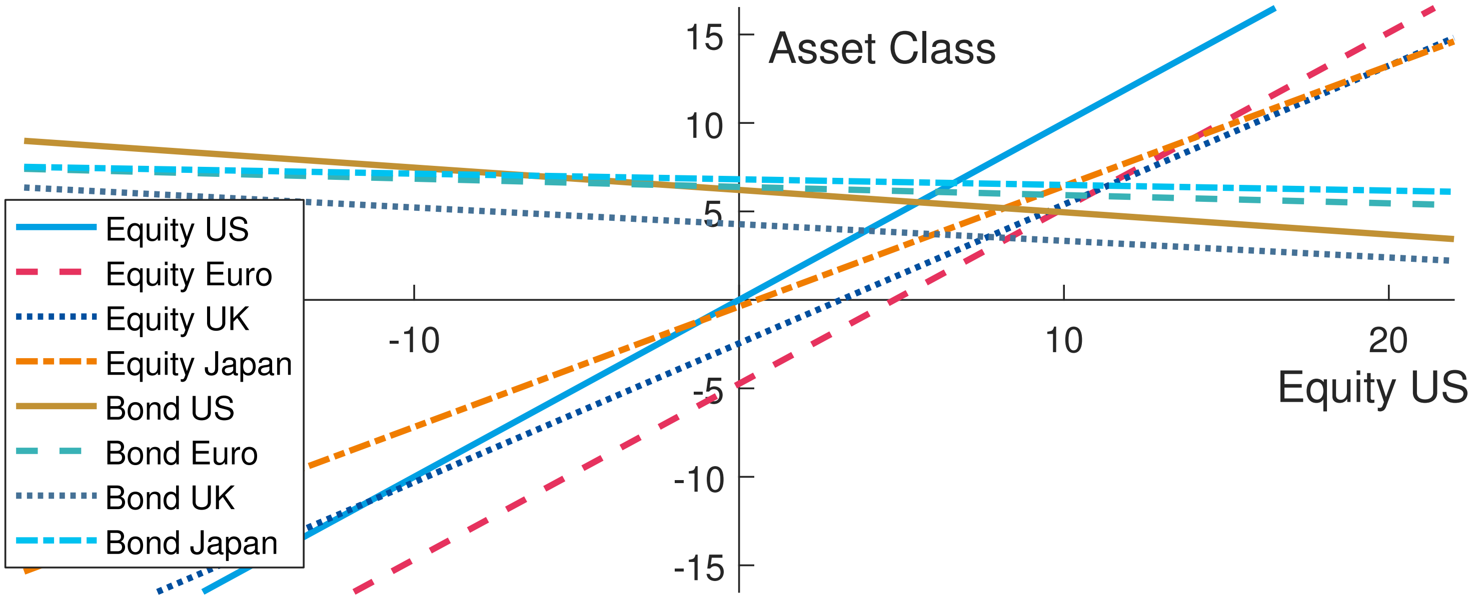


Figure 34: Linear payoff function with respect to the S&P 500 Index

A long-only diversified stock-bond portfolio makes sense!

The payoff approach

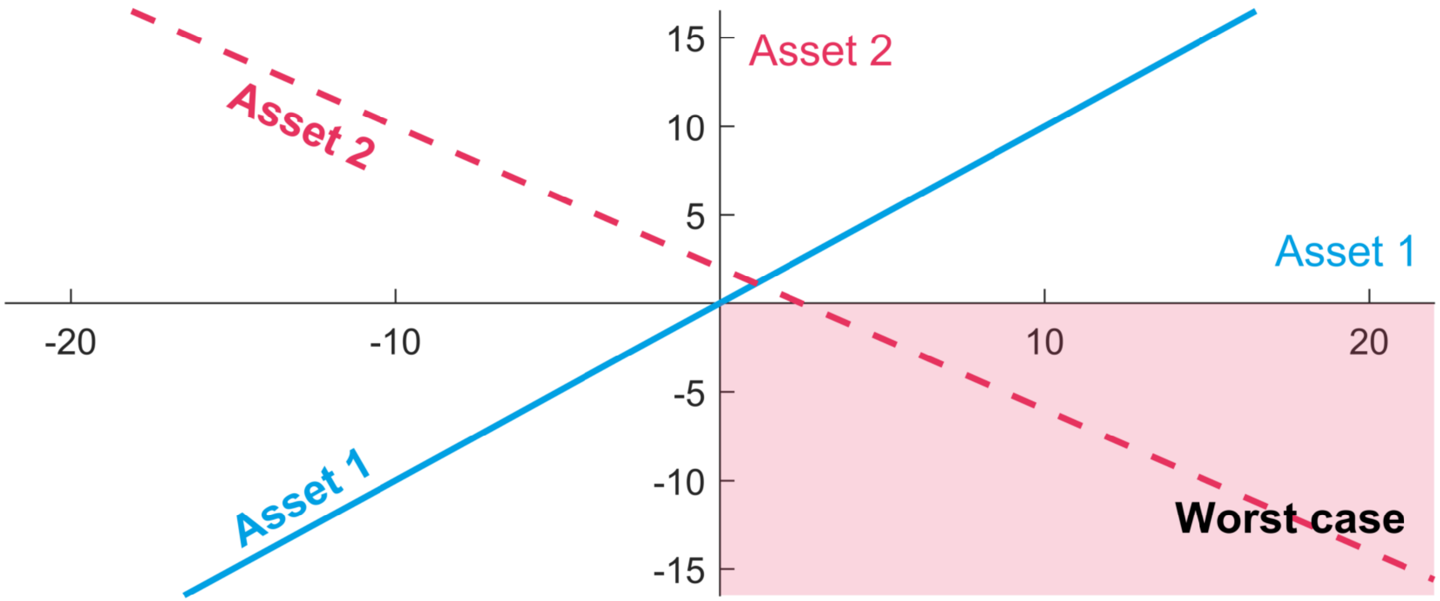


Figure 35: Worst diversification case

What is good diversification? What is bad diversification?

Negative correlation does not necessarily imply good diversification!

The payoff approach

Concave payoff

- Negative skewness
- Positive vega
- Hit ratio $\geq 50\%$
- Gain frequency $>$ loss frequency
- Average gain $<$ average loss
- Positively correlated with bad times

Convex payoff

- Positive skewness
- Negative vega
- Hit ratio $\leq 50\%$
- Gain frequency $<$ loss frequency
- Average gain $>$ average loss
- Negatively correlated with bad times?

Volatility Carry $\neq$ **Time-series Momentum**

The payoff approach

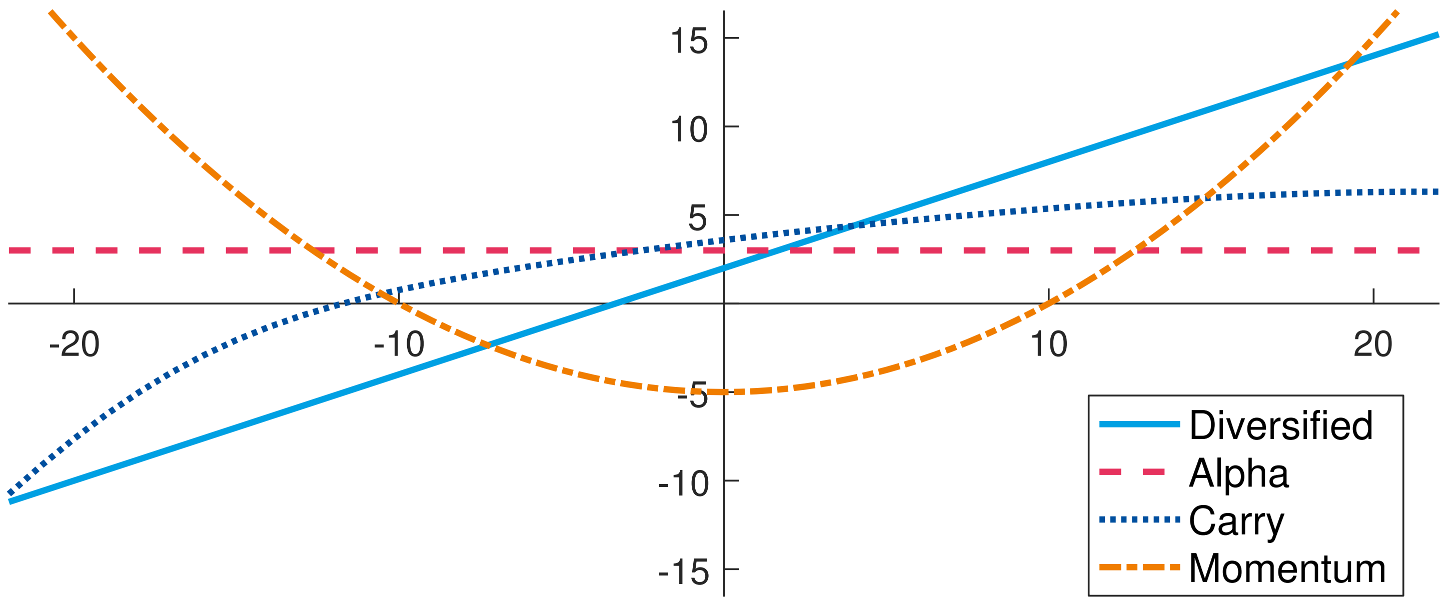


Figure 36: What does portfolio optimization produce with convex and concave strategies?

- Momentum = low allocation during good times and high allocation after bad times
- Carry = high allocation during good times and low allocation after bad times

The payoff approach

The magic formula

Long-run positive correlations, but...

...negative correlations is bad times

The payoff approach

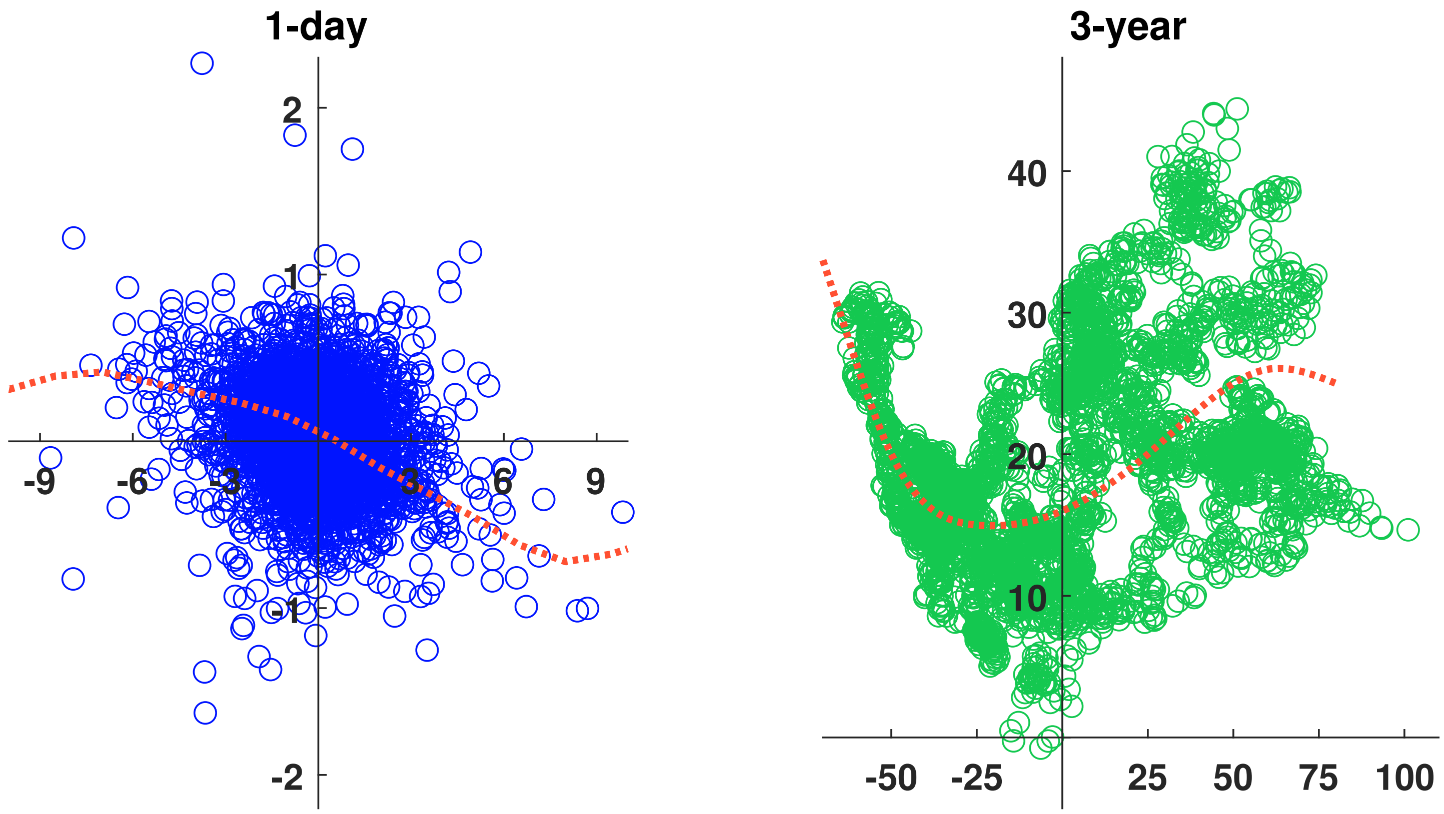


Figure 37: Stock/bond payoff (EUR)

Daily diversification is different than 3-year diversification

Equally-weighted portfolio

Exercise

We note Σ the covariance matrix of n asset returns. In what follows, we consider the equally weighted portfolio based on the universe of these n assets.

Equally-weighted portfolio

Question 1

Let $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$ be the elements of the covariance matrix Σ.

Equally-weighted portfolio

Question 1.a

Compute the volatility $\sigma(x)$ of the EW portfolio.

Equally-weighted portfolio

The elements of the covariance matrix are $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$. If we consider a portfolio $x = (x_1, \ldots, x_n)$, its volatility is:

$$\begin{aligned}\sigma(x) &= \sqrt{x^\top \Sigma x} \\ &= \sqrt{\sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i>j} x_i x_j \rho_{i,j} \sigma_i \sigma_j}\end{aligned}$$

For the equally weighted portfolio, we have $x_i = n^{-1}$ and:

$$\sigma(x) = \frac{1}{n} \sqrt{\sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i>j} \rho_{i,j} \sigma_i \sigma_j}$$

Equally-weighted portfolio

Question 1.b

Let $\sigma_0(x)$ and $\sigma_1(x)$ be the volatility of the EW portfolio when the asset returns are respectively independent and perfectly correlated. Calculate $\sigma_0(x)$ and $\sigma_1(x)$.

Equally-weighted portfolio

We have:

$$\sigma_0(x) = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2}$$

and:

$$\begin{aligned}
\sigma_1(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i\sum_{j=1}^{n}\sigma_j} \\
&= \frac{1}{n}\sqrt{\left(\sum_{i=1}^{n}\sigma_i\right)^2} = \frac{\sum_{i=1}^{n}\sigma_i}{n} \\
&= \bar{\sigma}
\end{aligned}$$

Equally-weighted portfolio

Question 1.c

We assume that the volatilities are the same. Find the expression of the portfolio volatility with respect to the mean correlation $\bar{\rho}$. What is the value of $\sigma(x)$ when $\bar{\rho}$ is equal to zero? What is the value of $\sigma(x)$ when n tends to $+\infty$?

Equally-weighted portfolio

If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\sigma(x) = \frac{\sigma}{n}\sqrt{n + 2\sum_{i>j}\rho_{i,j}}$$

Let $\bar{\rho}$ be the mean correlation. We have:

$$\bar{\rho} = \frac{2}{n^2 - n}\sum_{i>j}\rho_{i,j}$$

We deduce that:

$$\sum_{i>j}\rho_{i,j} = \frac{n(n-1)}{2}\bar{\rho}$$

Equally-weighted portfolio

We finally obtain:

$$\begin{aligned}\sigma(x) &= \frac{\sigma}{n}\sqrt{n + n(n-1)\bar{\rho}} \\ &= \sigma\sqrt{\frac{1+(n-1)\bar{\rho}}{n}}\end{aligned}$$

When $\bar{\rho}$ is equal to zero, the volatility $\sigma(x)$ is equal to $\sigma/\sqrt{n}$. When the number of assets tends to $+\infty$, it follows that:

$$\lim_{n\rightarrow\infty} \sigma(x) = \sigma\sqrt{\bar{\rho}}$$

Equally-weighted portfolio

Question 1.d

We assume that the correlations are uniform ($\rho_{i,j} = \rho$). Find the expression of the portfolio volatility as a function of $\sigma_0(x)$ and $\sigma_1(x)$. Comment on this result.

Equally-weighted portfolio

If $\rho_{i,j} = \rho$, we obtain:

$$\begin{aligned}
\sigma(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j} \\
&= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j - \rho\sum_{i=1}^{n}\sigma_i^2} \\
&= \frac{1}{n}\sqrt{(1-\rho)\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j}
\end{aligned}$$

Equally-weighted portfolio

We have:

$$\sum_{i=1}^{n} \sigma_i^2 = n^2 \sigma_0^2 (x)$$

and:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j = n^2 \sigma_1^2 (x)$$

It follows that:

$$\sigma (x) = \sqrt{(1 - \rho) \sigma_0^2 (x) + \rho \sigma_1^2 (x)}$$

When the correlation is uniform, the variance $\sigma^2 (x)$ is the weighted average between $\sigma_0^2 (x)$ and $\sigma_1^2 (x)$.

Equally-weighted portfolio

Question 2.a

Compute the normalized risk contributions $\mathcal{RC}_i^\star$ of the EW portfolio.

Equally-weighted portfolio

The risk contributions are equal to:

$$\mathcal{RC}_i^\star = \frac{x_i \cdot (\Sigma x)_i}{\sigma^2(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}\mathcal{RC}_i^\star &= \frac{\sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j}{n^2\sigma^2(x)} \\ &= \frac{\sigma_i^2 + \sigma_i \sum_{j\neq i} \rho_{i,j}\sigma_j}{n^2\sigma^2(x)}\end{aligned}$$

Equally-weighted portfolio

Question 2.b

Deduce the risk contributions $\mathcal{RC}_i^\star$ when the asset returns are respectively independent and perfectly correlated[a].

[a] We note them $\mathcal{RC}_{0,i}^\star$ and $\mathcal{RC}_{1,i}^\star$.

Equally-weighted portfolio

If asset returns are independent, we have:

$$\mathcal{RC}_{0,i}^{\star} = \frac{\sigma_i^2}{\sum_{i=1}^n \sigma_i^2}$$

In the case of perfect correlation, we obtain:

$$\begin{aligned} \mathcal{RC}_{1,i}^{\star} &= \frac{\sigma_i^2 + \sigma_i \sum_{j \neq i} \sigma_j}{n^2 \bar{\sigma}^2} \\ &= \frac{\sigma_i \sum_j \sigma_j}{n^2 \bar{\sigma}^2} \\ &= \frac{\sigma_i}{n \bar{\sigma}} \end{aligned}$$

Equally-weighted portfolio

Question 2.c

Show that the risk contribution $\mathcal{RC}_i$ is proportional to the ratio between the mean correlation of asset i and the mean correlation of the asset universe when the volatilities are the same.

Equally-weighted portfolio

If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\begin{aligned}\mathcal{RC}_i^\star &= \frac{\sigma^2 + \sigma^2 \sum_{j \neq i} \rho_{i,j}}{n^2 \sigma^2 (x)} \\ &= \frac{\sigma^2 + (n-1)\,\sigma^2 \bar{\rho}_i}{n^2 \sigma^2 (x)} \\ &= \frac{1 + (n-1)\,\bar{\rho}_i}{n\,(1 + (n-1)\,\bar{\rho})}\end{aligned}$$

It follows that:

$$\lim_{n \rightarrow \infty} \frac{1 + (n-1)\,\bar{\rho}_i}{1 + (n-1)\,\bar{\rho}} = \frac{\bar{\rho}_i}{\bar{\rho}}$$

We deduce that the risk contributions are proportional to the ratio between the mean correlation of asset i and the mean correlation of the asset universe.

Equally-weighted portfolio

Question 2.d

We assume that the correlations are uniform ($\rho_{i,j} = \rho$). Show that the risk contribution $\mathcal{RC}_i$ is a weighted average of $\mathcal{RC}^{\star}_{0,i}$ and $\mathcal{RC}^{\star}_{1,i}$.

Equally-weighted portfolio

We recall that we have:

$$\sigma(x) = \sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}$$

It follows that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \frac{(1-\rho)\,\sigma_0(x)\,\partial_{x_i}\sigma_0(x) + \rho\sigma_1(x)\,\partial_{x_i}\sigma_1(x)}{\sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}} \\ &= \frac{(1-\rho)\,\sigma_0(x)\,\mathcal{RC}_{0,i} + \rho\sigma_1(x)\,\mathcal{RC}_{1,i}}{\sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}}\end{aligned}$$

We then obtain:

$$\mathcal{RC}_i^{\star} = \frac{(1-\rho)\,\sigma_0^2(x)}{\sigma^2(x)}\mathcal{RC}_{0,i}^{\star} + \frac{\rho\sigma_1(x)}{\sigma^2(x)}\mathcal{RC}_{1,i}^{\star}$$

We verify that the risk contribution $\mathcal{RC}_i$ is a weighted average of $\mathcal{RC}_{0,i}^{\star}$ and $\mathcal{RC}_{1,i}^{\star}$.

Equally-weighted portfolio

Question 3

We suppose that the return of asset i satisfies the CAPM model:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the specific risk. We note $\beta = (\beta_1, \ldots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\operatorname{var}(R_m) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon) = D = \operatorname{diag}(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2)$.

Equally-weighted portfolio

Question 3.a

Calculate the volatility of the EW portfolio.

Equally-weighted portfolio

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

We deduce that:

$$\sigma(x) = \frac{1}{n}\sqrt{\sigma_m^2 \sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i\beta_j + \sum_{i=1}^{n}\tilde{\sigma}_i^2}$$

Equally-weighted portfolio

Question 3.b

Calculate the risk contribution $\mathcal{RC}_i$.

Equally-weighted portfolio

The risk contributions are equal to:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sigma(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}
\mathcal{RC}_i &= \frac{x_i \cdot \left(\sigma_m^2 \beta_i \sum_{j=1}^n x_j \beta_j + x_i \tilde{\sigma}_i^2\right)}{n^2 \sigma(x)} \\
&= \frac{\sigma_m^2 \beta_i \sum_{j=1}^n \beta_j + \tilde{\sigma}_i^2}{n^2 \sigma(x)} \\
&= \frac{n \sigma_m^2 \beta_i \bar{\beta} + \tilde{\sigma}_i^2}{n^2 \sigma(x)}
\end{aligned}$$

Equally-weighted portfolio

Question 3.c

Show that $\mathcal{RC}_i$ is approximately proportional to β_i if the number of assets is large. Illustrate this property using a numerical example.

Equally-weighted portfolio

When the number of assets is large and $\beta_i > 0$, we obtain:

$$\mathcal{RC}_i \simeq \frac{\sigma_m^2 \beta_i \bar{\beta}}{n \sigma(x)}$$

because $\bar{\beta} > 0$. We deduce that the risk contributions are approximately proportional to the beta coefficients:

$$\mathcal{RC}_i^{\star} \simeq \frac{\beta_i}{\sum_{j=1}^{n} \beta_j}$$

In Figure 38, we compare the exact and approximated values of $\mathcal{RC}_i^{\star}$. For that, we simulate β_i and $\tilde{\sigma}_i$ with $\beta_i \sim \mathcal{U}_{[0.5,1.5]}$ and $\tilde{\sigma}_i \sim \mathcal{U}_{[0,20\%]}$ whereas σ_m is set to 25%. We notice that the approximated value is very close to the exact value when n increases.

Equally-weighted portfolio

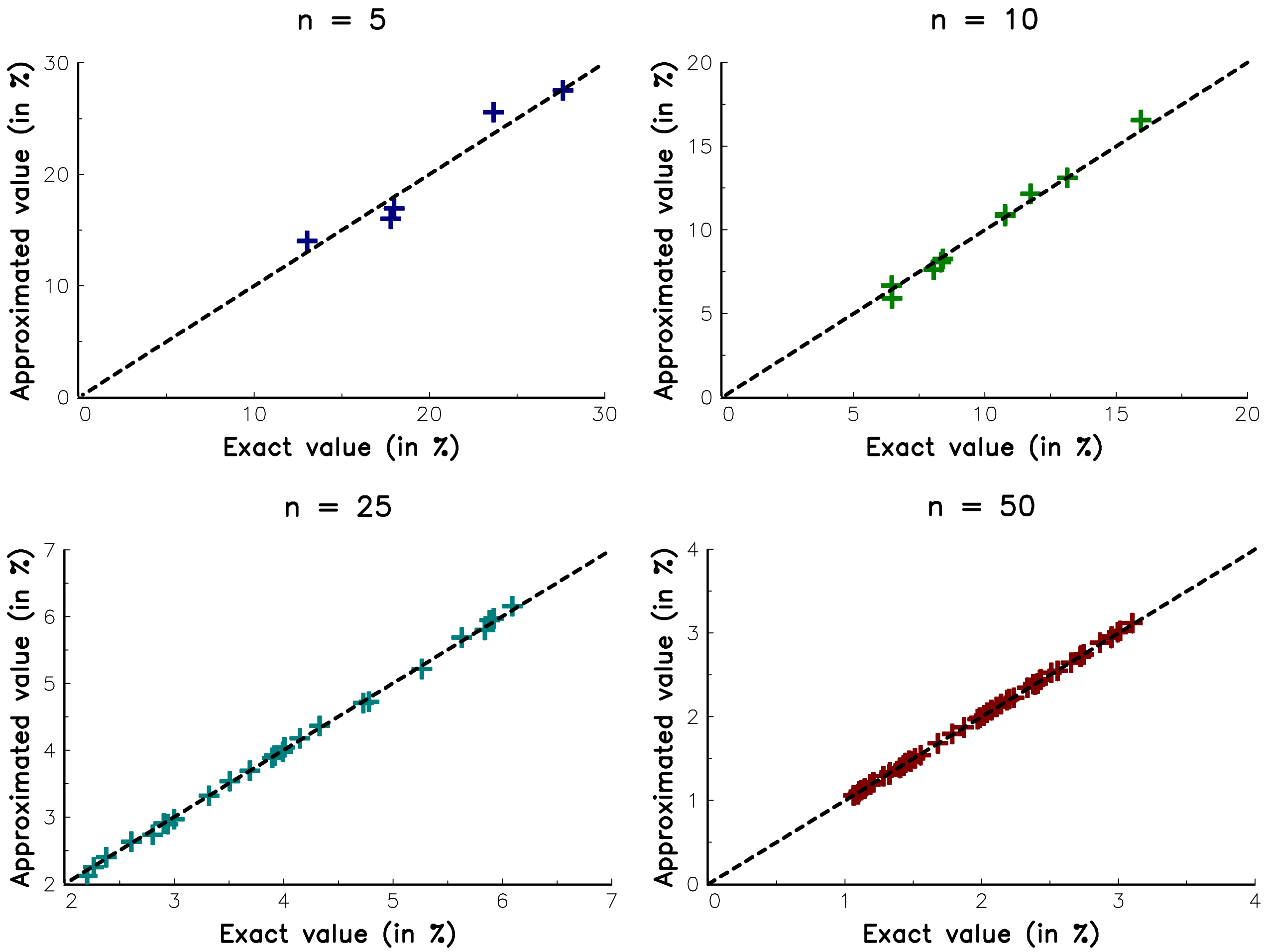


Figure 38: Comparing the exact and approximated values of $\mathcal{RC}_i^\star$

Most diversified portfolio

Exercise

We consider a universe of n assets. We note $\sigma = (\sigma_1, \ldots, \sigma_n)$ the vector of volatilities and Σ the covariance matrix.

Most diversified portfolio

Question 1

In what follows, we consider non-constrained optimized portfolios.

Most diversified portfolio

Question 1.a

Define the diversification ratioDiversification ratio $\mathcal{DR}(x)$ by considering a general risk measure $\mathcal{R}(x)$. How can one interpret this measure from a risk allocation perspective?

Most diversified portfolio

Let $\mathcal{R}(x)$ be the risk measure of the portfolio x. We note $\mathcal{R}_i = \mathcal{R}(\mathbf{e}_i)$ the risk associated to the i^{th} asset. The diversification ratio is the ratio between the weighted mean of the individual risks and the portfolio risk (TR-RPB, page 168):

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \mathcal{R}_i}{\mathcal{R}(x)}$$

If we assume that the risk measure satisfies the Euler allocation principle, we have:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \mathcal{R}_i}{\sum_{i=1}^{n} \mathcal{RC}_i}$$

Most diversified portfolio

Question 1.b

We assume that the weights of the portfolio are positive. Show that $\mathcal{DR}(x) \geq 1$ for all risk measures satisfying the Euler allocation principle. Find an upper bound of $\mathcal{DR}(x)$.

Most diversified portfolio

If $\mathcal{R}(x)$ satisfies the Euler allocation principle, we know that $\mathcal{R}_i \geq \mathcal{MR}_i$ (TR-RPB, page 78). We deduce that:

$$\begin{aligned} \mathcal{DR}(x) &\geq \frac{\sum_{i=1}^n x_i \mathcal{R}_i}{\sum_{i=1}^n x_i \mathcal{R}_i} \\ &\geq 1 \end{aligned}$$

Let x_{mr} be the portfolio that minimizes the risk measure. We have:

$$\mathcal{DR}(x) \leq \frac{\sup \mathcal{R}_i}{\mathcal{R}(x_{\mathrm{mr}})}$$

Most diversified portfolio

Question 1.c

We now consider the volatility risk measure. Calculate the upper bound of $\mathcal{DR}(x)$.

Most diversified portfolio

If we consider the volatility risk measure, the minimum risk portfolio is the minimum variance portfolio. We have (TR-RPB, page 164):

$$\sigma(x_{\mathrm{mv}}) = \frac{1}{\sqrt{\mathbf{1}_n^\top \Sigma \mathbf{1}_n}}$$

We deduce that:

$$\mathcal{DR}(x) \leq \sqrt{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \cdot \sup \sigma_i$$

Most diversified portfolio

Question 1.d

What is the most diversified portfolio (or MDP)? In which case does it correspond to the tangency portfolio? Deduce the analytical expression of the MDP and calculate its volatility.

Most diversified portfolio

The MDP is the portfolio which maximizes the diversification ratio when the risk measure is the volatility (TR-RPB, page 168). We have:

$$\begin{aligned} x^{\star} &= \arg\max \mathcal{DR}(x) \\ \text{u.c.} &\quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

Most diversified portfolio

If we consider that the risk premium $\pi_i = \mu_i - r$ of the asset i is proportional to its volatility σ_i, we obtain:

$$\begin{aligned}
\mathrm{SR}\left(x \mid r\right) &= \frac{\mu\left(x\right) - r}{\sigma\left(x\right)} \\
&= \frac{\sum_{i=1}^{n} x_i \left(\mu_i - r\right)}{\sigma\left(x\right)} \\
&= s \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sigma\left(x\right)} \\
&= s \cdot \mathcal{DR}\left(x\right)
\end{aligned}$$

where s is the coefficient of proportionality. Maximizing the diversification ratio is equivalent to maximizing the Sharpe ratio.

Most diversified portfolio

We recall that the expression of the tangency portfolio is:

$$x^{\star} = \frac{\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}{\mathbf{1}_n^{\top}\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}$$

We deduce that the weights of the MDP are:

$$x^{\star} = \frac{\Sigma^{-1}\sigma}{\mathbf{1}_n^{\top}\Sigma^{-1}\sigma}$$

The volatility of the MDP is then:

$$\begin{aligned}
\sigma\left(x^{\star}\right) &= \sqrt{\frac{\sigma^{\top}\Sigma^{-1}}{\mathbf{1}_n^{\top}\Sigma^{-1}\sigma}\Sigma\frac{\Sigma^{-1}\sigma}{\mathbf{1}_n^{\top}\Sigma^{-1}\sigma}} \\
&= \frac{\sqrt{\sigma^{\top}\Sigma^{-1}\sigma}}{\mathbf{1}_n^{\top}\Sigma^{-1}\sigma}
\end{aligned}$$

Most diversified portfolio

Question 1.e

Demonstrate then that the weights of the MDP are in some sense proportional to $\Sigma^{-1}\sigma$.

Most diversified portfolio

We recall that another expression of the unconstrained tangency portfolio is:

$$x^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\left(\mu\left(x^{\star}\right) - r\right)} \Sigma^{-1} \left(\mu - r \mathbf{1}_{n}\right)$$

We deduce that the MDP is also:

$$x^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)} \Sigma^{-1} \sigma$$

where $\bar{\sigma}\left(x^{\star}\right) = x^{\star\top}\sigma$. Nevertheless, this solution is endogenous.

Most diversified portfolio

Question 2

We suppose that the return of asset i satisfies the CAPM:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the specific risk. We note $\beta = (\beta_1, \ldots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\operatorname{var}(R_m) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon) = D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$.

Most diversified portfolio

Question 2.a

Compute the correlation $\rho_{i,m}$ between the asset return and the market return. Deduce the relationship between the specific risk $\tilde{\sigma}_i$ and the total risk σ_i of asset i.

Most diversified portfolio

We have:

$$\operatorname{cov}(R_i, R_m) = \beta_i \sigma_m^2$$

We deduce that:

$$\begin{aligned} \rho_{i,m} &= \frac{\operatorname{cov}(R_i, R_m)}{\sigma_i \sigma_m} \\ &= \beta_i \frac{\sigma_m}{\sigma_i} \end{aligned} \tag{1}$$

and:

$$\begin{aligned} \tilde{\sigma}_i &= \sqrt{\sigma_i^2 - \beta_i^2 \sigma_m^2} \\ &= \sigma_i \sqrt{1 - \rho_{i,m}^2} \end{aligned} \tag{2}$$

Most diversified portfolio

Question 2.b

Show that the solution of the MDP may be written as:

$$x_i^\star = \mathcal{DR}\left(x^\star\right) \frac{\sigma_i \sigma\left(x^\star\right)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star}\right) \tag{3}$$

with $\rho^\star$ a scalar to be determined.

Most diversified portfolio

We know that (TR-RPB, page 167):

$$\Sigma^{-1} = D^{-1} - \frac{1}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta} \tilde{\beta}^\top$$

where $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$.

Most diversified portfolio

We deduce that:

$$x^{\star} = \frac{\sigma^2 \left(x^{\star} \right)}{\bar{\sigma} \left(x^{\star} \right)} \left(D^{-1} \sigma - \frac{1}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta} \tilde{\beta}^\top \sigma \right)$$

and:

$$\begin{aligned}
x_i^{\star} &= \frac{\sigma^2 \left(x^{\star} \right)}{\bar{\sigma} \left(x^{\star} \right)} \left(\frac{\sigma_i}{\tilde{\sigma}_i^2} - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta}_i \right) \\
&= \frac{\sigma_i \sigma^2 \left(x^{\star} \right)}{\bar{\sigma} \left(x^{\star} \right) \tilde{\sigma}_i^2} \left(1 - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta} \frac{\sigma_m \tilde{\sigma}_i^2 \tilde{\beta}_i}{\sigma_i} \right) \\
&= \frac{\sigma_i \sigma^2 \left(x^{\star} \right)}{\bar{\sigma} \left(x^{\star} \right) \tilde{\sigma}_i^2} \left(1 - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta} \rho_{i,m} \right) \\
&= \mathcal{DR} \left(x^{\star} \right) \frac{\sigma_i \sigma \left(x^{\star} \right)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^{\star}} \right)
\end{aligned}$$

Most diversified portfolio

Using Equations (1) and (2), $\rho^{\star}$ is defined as follows:

$$
\begin{aligned}
\rho^{\star} &= \frac{\sigma_m^{-1} + \sigma_m \tilde{\beta}^{\top} \beta}{\tilde{\beta}^{\top} \sigma} \\
&= \left(1 + \sum_{j=1}^{n} \frac{\sigma_m^2 \beta_j^2}{\tilde{\sigma}_j^2} \right) \bigg/ \left(\sum_{j=1}^{n} \frac{\sigma_m \beta_j \sigma_j}{\tilde{\sigma}_j^2} \right) \\
&= \left(1 + \sum_{j=1}^{n} \frac{\rho_{j,m}^2}{1 - \rho_{j,m}^2} \right) \bigg/ \left(\sum_{j=1}^{n} \frac{\rho_{j,m}}{1 - \rho_{j,m}^2} \right)
\end{aligned}
$$

Most diversified portfolio

Question 2.c

In which case is the optimal weight $x_i^\star$ positive?

Most diversified portfolio

The optimal weight $x_i^\star$ is positive if:

$$1 - \frac{\rho_{i,m}}{\rho^\star} \geq 0$$

or equivalently:

$$\rho_{i,m} \leq \rho^\star$$

Most diversified portfolio

Question 2.d

Are the weights of the MDP a decreasing or an increasing function of the specific risk $\tilde{\sigma}_i$?

Most diversified portfolio

We recall that:

$$\begin{aligned}\rho_{i,m} &= \beta_i \frac{\sigma_m}{\sigma_i} \\ &= \frac{\beta_i \sigma_m}{\sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2}}\end{aligned}$$

If $\beta_i < 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ increases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We deduce that the weight is a decreasing function of the specific volatility $\tilde{\sigma}_i$. If $\beta_i > 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ decreases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We cannot conclude in this case.

Most diversified portfolio

Question 3

In this question, we illustrate that the MDP may be very different than the minimum variance portfolio.

Most diversified portfolio

Question 3.a

In which case does the MDP coincide with the minimum variance portfolio?

Most diversified portfolio

The MDP coincide with the MV portfolio when the volatility is the same for all the assets.

Most diversified portfolio

Question 3.b

We consider the following parameter values:

i	1	2	3	4
β_i	0.80	0.90	1.10	1.20
$\tilde{\sigma}_i$	0.02	0.05	0.15	0.15

with $\sigma_m = 20\%$. Calculate the unconstrained MDP with Formula (3). Compare it with the unconstrained MV portfolio. What is the result if we consider a long-only portfolio?

Most diversified portfolio

The formula cannot be used directly, because it depends on $\sigma(x^\star)$ and $\mathcal{DR}(x^\star)$. However, we notice that:

$$x_i^\star \propto \frac{\sigma_i}{\tilde{\sigma}_i^2}\left(1 - \frac{\rho_{i,m}}{\rho^\star}\right)$$

It suffices then to rescale these weights to obtain the solution. Using the numerical values of the parameters, $\rho^\star = 98.92\%$ and we obtain the following results:

	β_i	$\rho_{i,m}$	$x_i \in \mathbb{R}$ MDP	$x_i \in \mathbb{R}$ MV	$x_i \geq 0$ MDP	$x_i \geq 0$ MV
$x_1^\star$	0.80	99.23%	−27.94%	211.18%	0.00%	100.00%
$x_2^\star$	0.90	96.35%	43.69%	−51.98%	25.00%	0.00%
$x_3^\star$	1.10	82.62%	43.86%	−24.84%	39.24%	0.00%
$x_4^\star$	1.20	84.80%	40.39%	−34.37%	35.76%	0.00%
$\sigma(x^\star)$			24.54%	13.42%	23.16%	16.12%

Most diversified portfolio

Question 3.c

We assume that the volatility of the assets is 10%, 10%, 50% and 50% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.90 & 1.00 & & \\ 0.80 & 0.80 & 1.00 & \\ 0.00 & 0.00 & -0.25 & 1.00 \end{pmatrix}$$

Calculate the (unconstrained and long-only) MDP and MV portfolios.

Most diversified portfolio

The results are:

	$x_i \in \mathbb{R}$		$x_i \geq 0$	
	MDP	MV	MDP	MV
$x_1^{\star}$	−36.98%	60.76%	0.00%	48.17%
$x_2^{\star}$	−36.98%	60.76%	0.00%	48.17%
$x_3^{\star}$	91.72%	−18.54%	50.00%	0.00%
$x_4^{\star}$	82.25%	−2.98%	50.00%	3.66%
$\sigma(x^{\star})$	48.59%	6.43%	30.62%	9.57%

Most diversified portfolio

Question 3.d

Comment on these results.

Most diversified portfolio

These two examples show that the MDP may have a different behavior than the minimum variance portfolio. Contrary to the latter, the most diversified portfolio is not necessarily a low-beta or a low-volatility portfolio.

Computation of risk-based portfolios

Exercise

We consider a universe of five assets. Their expected returns are 6%, 10%, 6%, 8% and 12% whereas their volatilities are equal to 10%, 20%, 15%, 25% and 30%. The correlation matrix of asset returns is defined as follows:

$$\rho = \begin{pmatrix} 100\% & & & & \\ 60\% & 100\% & & & \\ 40\% & 50\% & 100\% & & \\ 30\% & 30\% & 20\% & 100\% & \\ 20\% & 10\% & 10\% & -50\% & 100\% \end{pmatrix}$$

We assume that the risk-free rate is equal to 2%.

Computation of risk-based portfolios

Question 1

We consider unconstrained portfolios. For each portfolio, compute the risk decomposition.

Computation of risk-based portfolios

Question 1.a

Find the tangency portfolio.

Computation of risk-based portfolios

To compute the unconstrained tangency portfolio, we use the analytical formula (TR-RPB, page 14):

$$x^{\star} = \frac{\Sigma^{-1}(\mu - r\mathbf{1}_n)}{\mathbf{1}_n^{\top}\Sigma^{-1}(\mu - r\mathbf{1}_n)}$$

We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	11.11%	6.56%	0.73%	5.96%
2	17.98%	13.12%	2.36%	19.27%
3	2.55%	6.56%	0.17%	1.37%
4	33.96%	9.84%	3.34%	27.31%
5	34.40%	16.40%	5.64%	46.09%

Computation of risk-based portfolios

Question 1.b

Determine the equally weighted portfolio.

Computation of risk-based portfolios

We obtain the following results for the equally weighted portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00%	7.47%	1.49%	13.43%
2	20.00%	15.83%	3.17%	28.48%
3	20.00%	9.98%	2.00%	17.96%
4	20.00%	9.89%	1.98%	17.80%
5	20.00%	12.41%	2.48%	22.33%

Computation of risk-based portfolios

Question 1.c

Compute the minimum variance portfolio.

Computation of risk-based portfolios

For the minimum variance portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	74.80%	9.08%	6.79%	74.80%
2	−15.04%	9.08%	−1.37%	−15.04%
3	21.63%	9.08%	1.96%	21.63%
4	10.24%	9.08%	0.93%	10.24%
5	8.36%	9.08%	0.76%	8.36%

Computation of risk-based portfolios

Question 1.d

Calculate the most diversified portfolio.

Computation of risk-based portfolios

For the most diversified portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	-14.47%	4.88%	-0.71%	-5.34%
2	4.83%	9.75%	0.47%	3.56%
3	18.94%	7.31%	1.38%	10.47%
4	49.07%	12.19%	5.98%	45.24%
5	41.63%	14.63%	6.09%	46.06%

Computation of risk-based portfolios

Question 1.e

Find the ERC portfolio.

Computation of risk-based portfolios

For the ERC portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	27.20%	7.78%	2.12%	20.00
2	13.95%	15.16%	2.12%	20.00
3	20.86%	10.14%	2.12%	20.00
4	19.83%	10.67%	2.12%	20.00
5	18.16%	11.65%	2.12%	20.00

Computation of risk-based portfolios

Question 1.f

Compare the expected return $\mu(x)$, the volatility $\sigma(x)$ and the Sharpe ratio $\mathrm{SR}(x \mid r)$ of the different portfolios. Calculate then the tracking error volatility $\sigma(x \mid b)$, the beta $\beta(x \mid b)$ and the correlation $\rho(x \mid b)$ if we assume that the benchmark b is the tangency portfolio.

Computation of risk-based portfolios

We recall the definition of the statistics:

$$\begin{aligned}
\mu(x) &= \mu^\top x \\
\sigma(x) &= \sqrt{x^\top \Sigma x} \\
\mathrm{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\
\sigma(x \mid b) &= \sqrt{(x-b)^\top \Sigma (x-b)} \\
\beta(x \mid b) &= \frac{x^\top \Sigma b}{b^\top \Sigma b} \\
\rho(x \mid b) &= \frac{x^\top \Sigma b}{\sqrt{x^\top \Sigma x}\sqrt{b^\top \Sigma b}}
\end{aligned}$$

Computation of risk-based portfolios

We obtain the following results:

Statistic	$x^\star$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.11%	9.67%	8.04%
$\sigma(x)$	12.24%	11.12%	9.08%	13.22%	10.58%
$\mathrm{SR}(x \mid r)$	60.96%	57.57%	45.21%	58.03%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	8.21%	4.06%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	55.01%	102.82%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	74.17%	95.19%	93.76%

We notice that all the portfolios present similar performance in terms of Sharpe Ratio. The minimum variance portfolio shows the smallest Sharpe ratio, but it also shows the lowest correlation with the tangency portfolio.

Computation of risk-based portfolios

Question 2

Same questions if we impose the long-only portfolio constraint.

Computation of risk-based portfolios

The tangency portfolio, the equally weighted portfolio and the ERC portfolio are already long-only. For the minimum variance portfolio, we obtain:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	65.85%	9.37%	6.17%	65.85%
2	0.00%	13.11%	0.00%	0.00%
3	16.72%	9.37%	1.57%	16.72%
4	9.12%	9.37%	0.85%	9.12%
5	8.32%	9.37%	0.78%	8.32%

Computation of risk-based portfolios

For the most diversified portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	0.00%	5.50%	0.00%	0.00%
2	1.58%	9.78%	0.15%	1.26%
3	16.81%	7.34%	1.23%	10.04%
4	44.13%	12.23%	5.40%	43.93%
5	37.48%	14.68%	5.50%	44.77%

Computation of risk-based portfolios

The results become:

Statistic	$x^{\star}$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.68%	9.19%	8.04%
$\sigma(x)$	12.24%	11.12%	9.37%	12.29%	10.58%
$\text{SR}(x \mid r)$	60.96%	57.57%	49.99%	58.56%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	7.04%	3.44%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	62.74%	96.41%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	82.00%	96.06%	93.76%

Building a carry trade exposure

Question 1

We would like to build a carry trade strategy using a *cash neutral* portfolio with equal weights and a notional amount of $100 mn. We use the data given in Table 17. The holding period is equal to three months.

Table 17: Three-month interest rates (March, 15$^{\text{th}}$ 2000)

Currency	AUD	CAD	CHF	EUR	GBP
Interest rate (in %)	5.74	5.37	2.55	3.79	6.21
Currency	JPY	NOK	NZD	SEK	USD
Interest rate (in %)	0.14	5.97	6.24	4.18	6.17

Building a carry trade exposure

Question 1.a

Build the carry trade exposure with two funding currencies and two asset currencies.

Building a carry trade exposure

We rank the currencies according to their interest rate from the lowest to the largest value:

1. JPY	2. CHF	3. EUR	4. SEK	5. CAD
6. AUD	7. NOK	8. USD	9. GBP	10. NZD

We deduce that the carry trade portfolio is:

1. long \$50 mn on NZD
2. long \$50 mn on GBP
3. short \$50 mn on JPY
4. short \$50 mn on CHF

Building a carry trade exposure

Question 1.b

Same question with five funding currencies and two asset currencies.

Building a carry trade exposure

The portfolio becomes:

1. long \$50 mn on NZD and GBP
2. short \$20 mn on JPY, CHF, EUR, SEK and CAD

Building a carry trade exposure

Question 1.c

What is the specificity of the portfolio if we use five funding currencies and five asset currencies.

Building a carry trade exposure

The portfolio is:

1. long \$20 mn on NZD, GBP, USD, NOK and AUD
2. short \$20 mn on JPY, CHF, EUR, SEK and CAD

The asset notional is not equal to the funding notional, because the funding notional is equal to \$100 mn and the asset notional is equal to \$80 mn. Indeed, we don't need to invest the \$20 mn USD exposure since the portfolio currency is the US dollar.

Building a carry trade exposure

Question 1.d

Calculate an approximation of the carry trade P&L if we assume that the spot foreign exchange rates remain constant during the next three months.

Building a carry trade exposure

If we consider the last portfolio, we have:

$$\begin{aligned}
\text{PnL} &\approx 20 \times \frac{1}{4}(6.24\% + 6.21\% + 6.17\% + 5.97\% + 5.74\%) - \\
&\quad 20 \times \frac{1}{4}(0.14\% + 2.55\% + 3.79\% + 4.18\% + 5.37\%) \\
&= \$0.71 \text{ mn}
\end{aligned}$$

If the spot foreign exchange rates remain constant during the next three months, the quarterly P&L is approximated equal to \$710 000.

Building a carry trade exposure

Question 2

We consider the data given in Tables 18 and 19.

Table 18: Three-month interest rates (March, 21$^{\text{th}}$ 2005)

Currency	BRL	CZK	HUF	KRW	MXN
Interest rate (in %)	18.23	2.45	8.95	3.48	8.98
Currency	PLN	SGD	THB	TRY	TWD
Interest rate (in %)	6.63	1.44	2.00	19.80	1.30

Table 19: Annualized volatility of foreign exchange rates (March, 21$^{\text{th}}$ 2005)

Currency	BRL	CZK	HUF	KRW	MXN
Volatility (in %)	11.19	12.57	12.65	6.48	6.80
Currency	PLN	SGD	THB	TRY	TWD
Volatility (in %)	11.27	4.97	4.26	11.61	4.12

Building a carry trade exposure

Question 2.a

Let Σ be the covariance matrix of the currency returns. Which expected returns are used by the carry investor? Write the mean-variance optimization problem if we assume a cash neutral portfolio.

Building a carry trade exposure

Let $\mathcal{C}_i$ and $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ be the carry of Currency i and the vector of carry values. The carry investor assumes that $\mu_i = \mathcal{C}_i$. We deduce that the mean-variance optimization problem is:

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mathcal{C} \\ & \text{u.c.} \quad \mathbf{1}_n^\top x = 0 \end{aligned}$$

The constraint $\mathbf{1}_n^\top x = 0$ indicates that the portfolio is cash neutral. If we target a portfolio volatility $\sigma^\star$, we use the bisection algorithm in order to find the optimal value of γ such that:

$$\sigma(x^\star(\gamma)) = \sigma^\star$$

Building a carry trade exposure

Question 2.b

By assuming a zero correlation between the currencies, calibrate the cash neutral portfolio when the objective function is to target a 3% portfolio volatility.

Building a carry trade exposure

We obtain the following solution:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	15.05%	−1.28%	4.11%	−1.57%	14.30%
Currency	PLN	SGD	THB	TRY	TWD
Weight	2.76%	−13.59%	−14.42%	15.52%	−20.87%

Building a carry trade exposure

Question 2.c

Same question if we use the following correlation matrix:

$$\rho = \begin{pmatrix}
1.00 & & & & & & & & & \\
0.30 & 1.00 & & & & & & & & \\
0.38 & 0.80 & 1.00 & & & & & & & \\
0.00 & 0.04 & 0.08 & 1.00 & & & & & & \\
0.50 & 0.30 & 0.34 & 0.12 & 1.00 & & & & & \\
0.35 & 0.70 & 0.78 & 0.06 & 0.30 & 1.00 & & & & \\
0.33 & 0.49 & 0.56 & 0.29 & 0.27 & 0.53 & 1.00 & & & \\
0.30 & 0.34 & 0.34 & 0.38 & 0.29 & 0.35 & 0.53 & 1.00 & & \\
0.43 & 0.39 & 0.48 & 0.10 & 0.38 & 0.41 & 0.35 & 0.43 & 1.00 & \\
0.03 & 0.07 & 0.06 & 0.63 & 0.09 & 0.07 & 0.30 & 0.40 & 0.20 & 1.00
\end{pmatrix}$$

Building a carry trade exposure

The solution becomes:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	13.69%	−9.45%	4.58%	17.31%	6.56%
Currency	PLN	SGD	THB	TRY	TWD
Weight	2.07%	−17.79%	−20.86%	17.98%	−14.10%

Building a carry trade exposure

Question 2.d

Calculate the carry of this optimized portfolio. For each currency, deduce the maximum value of the devaluation (or revaluation) rate that is compatible with a positive P&L.

Building a carry trade exposure

The carry of the portfolio is equal to:

$$\mathcal{C}(x) = \sum_{i=1}^{n} x_i \cdot \mathcal{C}_i$$

We find $\mathcal{C}(x) = 6.7062\%$ per year. We deduce that the maximum value of the devaluation or revaluation rate D_i that is compatible with a positive P&L is equal to:

$$D_i = \frac{6.7062\%}{4} = 1.6765\%$$

This figure is valid for an exposure of 100%.

Building a carry trade exposure

By considering the weights, we deduce that:

$$D_i = -\frac{\mathcal{C}(x)}{4x_i}$$

Finally, we obtain the following compatible devaluation (negative sign $-$) and revaluation (positive sign $+$) rates:

Currency	BRL	CZK	HUF	KRW	MXN
D_i	-12.25%	$+17.75\%$	-36.64%	-9.69%	-25.55%
Currency	PLN	SGD	THB	TRY	TWD
D_i	-81.08%	$+9.43\%$	$+8.04\%$	-9.32%	$+11.89\%$

Building a carry trade exposure

Question 2.e

Repeat Question 2.b assuming that the volatility target is equal 5%. Calculate the leverage ratio. Comment on these results.

Building a carry trade exposure

We obtain the following results:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	25.08%	−2.13%	6.84%	−2.62%	23.83%
Currency	PLN	SGD	THB	TRY	TWD
Weight	4.60%	−22.65%	−24.03%	25.86%	−34.78%

The leverage ratio of this portfolio is equal to $\sum_{i=1}^{n} |x_i| = 172.43\%$, whereas it is equal to 103.47% and 124.37% for the portfolios of Questions 2.b and 2.c. This is perfectly normal because the leverage is proportional to the volatility.

Building a carry trade exposure

Question 2.f

Find the analytical solution of the optimal portfolio $x^{\star}$ when we target a volatility $\sigma^{\star}$.

Building a carry trade exposure

The Lagrange function is equal to:

$$\mathcal{L}(x;\lambda_0) = \frac{1}{2}x^\top \Sigma x - \gamma x^\top \mathcal{C} + \lambda_0 \left(\mathbf{1}_n^\top x - 0\right)$$

The first-order condition is equal to:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0)}{\partial\,x} = \Sigma x - \gamma \mathcal{C} + \lambda_0 \mathbf{1}_n = \mathbf{0}_n$$

It follows that:

$$x = \Sigma^{-1}\left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n\right)$$

Building a carry trade exposure

The cash neutral constraint implies that:

$$\mathbf{1}_n^\top \Sigma^{-1} \left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n\right) = 0$$

We deduce that:

$$\lambda_0 = \gamma \frac{\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

Therefore, the optimal solution is equal to:

$$x^\star = \frac{\gamma \Sigma^{-1}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)$$

Building a carry trade exposure

The volatility of the optimal portfolio is equal:

$$\begin{aligned}
\sigma^2(x^\star) &= x^{\star\top}\Sigma x^\star \\
&= \left(\gamma\mathcal{C}^\top - \lambda_0\mathbf{1}_n^\top\right)\Sigma^{-1}\Sigma\Sigma^{-1}\left(\gamma\mathcal{C} - \lambda_0\mathbf{1}_n\right) \\
&= \left(\gamma\mathcal{C}^\top - \lambda_0\mathbf{1}_n^\top\right)\Sigma^{-1}\left(\gamma\mathcal{C} - \lambda_0\mathbf{1}_n\right) \\
&= \gamma^2\mathcal{C}^\top\Sigma^{-1}\mathcal{C} + \lambda_0^2\mathbf{1}_n^\top\Sigma^{-1}\mathbf{1}_n - 2\gamma\lambda_0\mathcal{C}^\top\Sigma^{-1}\mathbf{1}_n \\
&= \gamma^2\left(\mathcal{C}^\top\Sigma^{-1}\mathcal{C} - \frac{\left(\mathbf{1}_n^\top\Sigma^{-1}\mathcal{C}\right)^2}{\mathbf{1}_n^\top\Sigma^{-1}\mathbf{1}_n}\right) \\
&= \frac{\gamma^2}{\mathbf{1}_n^\top\Sigma^{-1}\mathbf{1}_n}\left(\left(\mathbf{1}_n^\top\Sigma^{-1}\mathbf{1}_n\right)\left(\mathcal{C}^\top\Sigma^{-1}\mathcal{C}\right) - \left(\mathbf{1}_n^\top\Sigma^{-1}\mathcal{C}\right)^2\right)
\end{aligned}$$

Building a carry trade exposure

We deduce that:

$$\gamma = \frac{\sqrt{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \sigma\left(x^\star\right)$$

Finally, we obtain:

$$\begin{aligned} x^\star &= \sigma\left(x^\star\right) \frac{\Sigma^{-1}\left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2 \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \\ &= \sigma^\star \frac{\Sigma^{-1}\left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2 \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \end{aligned}$$

Building a carry trade exposure

Question 2.g

We assume that the correlation matrix is the identity matrix I_n. Find the expression of the threshold value $\mathcal{C}^{\star}$ such that all currencies with a carry $\mathcal{C}_i$ larger than $\mathcal{C}^{\star}$ form the long leg of the portfolio.

Building a carry trade exposure

We recall that:

$$x^{\star} \propto \Sigma^{-1}\left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)$$

If $\rho = I_n$, we have:

$$\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n = \sum_{j=1}^{n} \frac{1}{\sigma_j^2}$$

and:

$$\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C} = \sum_{j=1}^{n} \frac{\mathcal{C}_j}{\sigma_j^2}$$

We deduce that:

$$x_i^{\star} \propto \frac{1}{\sigma_i^2}\left(\left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2}\right) \mathcal{C}_i - \left(\sum_{j=1}^{n} \frac{\mathcal{C}_j}{\sigma_j^2}\right)\right)$$

Building a carry trade exposure

The portfolio is long on the currency i if:

$$\mathcal{C}_i \geq \mathcal{C}^{\star}$$

where:

$$\mathcal{C}^{\star} = \left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2} \right)^{-1} \left(\sum_{j=1}^{n} \frac{\mathcal{C}_j}{\sigma_j^2} \right) = \sum_{j=1}^{n} \omega_j \mathcal{C}_j$$

and:

$$\omega_j = \frac{\sigma_j^{-2}}{\sum_{k=1}^{n} \sigma_k^{-2}}$$

$\mathcal{C}^{\star}$ is the weighted mean of the carry values and the weights are inversely proportional to the variance of the currency returns.

Main references

- Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting, Chapman and Hall/CRC Financial Mathematics Series, Chapter 3.
- Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting — Companion Book, Chapman and Hall/CRC Financial Mathematics Series, Chapter 3.
- Roncalli, T. (2017)
Alternative Risk Premia: What Do We Know?, in Jurczenko, E. (Ed.), *Factor Investing and Alternative Risk Premia*, ISTE Press – Elsevier.

References I

ACHARYA, V.V., and Pedersen, L.H. (2005)
Asset Pricing with Liquidity Risk, *Journal of Financial Economics*, 77(2), pp. 375-410.

BAKU, E., FORTES, R., HERVÉ, K., LEZMI, E., MALONGO, H., RONCALLI, T., and XU, J. (2019)
Factor Investing in Currency Markets: Does it Make Sense?, *Amundi Working Paper*, 89, www.research-center.amundi.com.

BAKU, E., FORTES, R., HERVÉ, K., LEZMI, E., MALONGO, H., RONCALLI, T., and XU, J. (2020)
Factor Investing in Currency Markets: Does it Make Sense?, *Journal of Portfolio Management*, 46(2), pp. 141-155.

References II

Barberis, N., and Thaler, R.H. (2003)
A Survey of Behavioral Finance, in Constantinides, G.M., Harris, M. and Stulz, R.M. (Eds), *Handbook of the Economics of Finance*, 1(B), Elsevier, pp. 1053-1128.

Bektic, D., Wenzler, J.S., Wegener, M., Schiereck, D., and Spielmann, T. (2019)
Extending Fama-French Factors to Corporate Bond Markets, *Journal of Portfolio Management*, 45(3), pp. 141-158.

Bektic, D., Neugebauer, U., Wegener, M. and Wenzler, J.S. (2017)
Common Equity Factors in Corporate Bond Markets, in Jurczenko, E. (Ed.), *Factor Investing and Alternative Risk Premia*, ISTE Press-Elsevier.

References III

Ben Slimane, M., De Jong, M., Dumas, J-M., Fredj, H., Sekine, T., and Srb, M. (2019)
Traditional and Alternative Factors in Investment Grade Corporate Bond Investing, *Amundi Working Paper*, 78, www.research-center.amundi.com.

Black, F. (1972)
Capital Market Equilibrium with Restricted Borrowing, *Journal of Business*, 45(3), pp. 444-455.

Bruder, B., Dao, T.L., Richard, J.C., and Roncalli, T. (2011)
Trend Filtering Methods for Momentum Strategies, *SSRN*, www.ssrn.com/abstract=2289097.

Carhart, M.M. (1997)
On Persistence in Mutual Fund Performance, *Journal of Finance*, 52(1), pp. 57-82.

References IV

Cazalet, Z., and Roncalli, T. (2014)
Facts and Fantasies About Factor Investing, *SSRN*,
www.ssrn.com/abstract=2524547.

Choueifaty, Y., and Coignard, Y. (2008)
Toward Maximum Diversification, *Journal of Portfolio Management*, 35(1), pp. 40-51.

Choueifaty, Y., Froidure, T., and Reynier, J. (2013)
Properties of the Most Diversified Portflio, *Journal of Investment Strategies*, 2(2), pp. 49-70.

Clarke, R.G., de Silva, H., and Thorley, S. (2011)
Minimum Variance Portfolio Composition, *Journal of Portfolio Management*, 37(2), pp. 31-45.

References V

- Clarke, R.G., de Silva, H., and Thorley, S. (2013)
  Risk Parity, Maximum Diversification, and Minimum Variance: An Analytic Perspective, *Journal of Portfolio Management*, 39(3), pp. 39-53.
- Cochrane, J.H. (2001)
  Asset Pricing, Princeton University Press.
- Cochrane, J.H. (2011)
  Presidential Address: Discount Rates, *Journal of Finance*, 66(4), pp. 1047-1108.
- DeMiguel, V., Garlappi, L., and Uppal, R. (2009)
  Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?, *Review of Financial Studies*, 22(5), pp. 1915-1953.

References VI

FAMA, E.F., and FRENCH, K.R. (1992)
The Cross-Section of Expected Stock Returns, *Journal of Finance*, 47(2), pp. 427-465.

FAMA, E.F., and FRENCH, K.R. (1993)
Common Risk Factors in the Returns on Stocks and Bonds, *Journal of Financial Economics*, 33(1), pp. 3-56.

FRAZZINI, A, and PEDERSEN, L.H. (2014)
Betting Against Beta, *Journal of Financial Economics*, 111(1), pp. 1-25.

GRINBLATT, M., TITMAN, S., and WERMERS, R. (1995)
Momentum Investment Strategies, Portfolio Performance, and Herding: A Study of Mutual Fund Behavior, *American Economic Review*, 85(5), pp. 1088-1105.

References VII

HAMDAN, R., PAVLOWSKY, F., RONCALLI, T., and ZHENG, B. (2016)
A Primer on Alternative Risk Premia, *SSRN*, www.ssrn.com/abstract=2766850.

HAUGEN, R.A. and BAKER, N.L. (1991)
The Efficient Market Inefficiency of Capitalization-weighted Stock Portfolios, *Journal of Portfolio Management*, 17(3), pp. 35-40.

HARVEY, C.R., LIU, Y., and ZHU, H. (2016)
...and the Cross-Section of Expected Returns, *Review of Financial Studies*, 29(1), pp. 5-68.

HOUWELING, P., and VAN ZUNDERT, J. (2017)
Factor Investing in the Corporate Bond Market, *Financial Analysts Journal*, 73(2), pp. 100-115.

References VIII

- ILMANEN, A. (2011)
  Expected Returns: An Investor's Guide to Harvesting Market Rewards, Wiley.

- ISRAEL, R., PALHARES, D., and RICHARDSON, S.A. (2018)
  Common Factors in Corporate Bond and Bond Fund Returns, *Journal of Investment Management*, 16(2), pp. 17-46

- JENSEN, M.C. (1968)
  The Performance of Mutual Funds in the Period 1945-1964, *Journal of Finance*, 23(2), pp. 389-416.

- JURCZENKO, E., MICHEL, T. and TEÏLETCHE, J. (2015)
  A Unified Framework for Risk-based Investing, *Journal of Investment Strategies*, 4(4), pp. 1-29.

References IX

Jusselin, P., Lezmi, E., Malongo, H., Masselin, C., Roncalli, T., and Dao, T-L. (2017)
Understanding the Momentum Risk Premium: An In-Depth Journey Through Trend-Following Strategies, *SSRN*, www.ssrn.com/abstract=3042173.

Koijen, R.S.J., Moskowitz, T.J., Pedersen, L.H., and Vrugt, E.B. (2018)
Carry, *Journal of Financial Economics*, 127(2), pp. 197-225.

Lucas, R.E. (1978)
Asset Prices in an Exchange Economy, *Econometrica*, 46(6), pp. 1429-1445.

MacDonald, R. (1995)
Long-Run Exchange Rate Modeling: a Survey of the Recent Evidence, *IMF Staff Papers*, 42(3), pp. 437-489.

References X

- Menkhoff, L., Sarno, L., Schmeling, M., and Schrimpf, A. (2016)
  Currency Value, *Review of Financial Studies*, 30(2), pp. 416-441.
- Roncalli, T. (2017)
  ESG & Factor Investing: A New Stage Has Been Reached, *Amundi ViewPoint*, www.research-center.amundi.com.
- Roncalli, T. (2020)
  Handbook of Financial Risk Management, Chapman and Hall/CRC Financial Mathematics Series.
- Scherer, B. (2011)
  A Note on the Returns from Minimum Variance Investing, *Journal of Empirical Finance*, 18(4), pp. 652-660.