The Risk Dimension of Asset Returns in Risk Parity Portfolios

Thierry Roncalli*

*Lyxor Asset Management¹, France & University of Évry, France

Workshop on Portfolio Management

University of Paris 6/Paris 7, April 3, 2014

¹The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.
Motivations
- Which Diversification?
- Which Risk Factors?
- Which Risk Premium?
- Which Risk Measure?

Risk Parity Approach
- Definition
- Main Properties
- Using the Standard Deviation-based Risk Measure

Applications
- Strategic Asset Allocation
- Tactical Asset Allocation
- Risk parity and time-varying risk premia
- One concept, several implementations, different performances!

Conclusion
Motivations

- Which Diversification?
- Which Risk Factors?
- Which Risk Premium?
- Which Risk Measure?
Which diversification?
The case of diversified funds

Figure: Equity (MSCI World) and bond (WGBI) risk contributions

- Contrarian constant-mix strategy
- Deleverage of an equity exposure
- Low risk diversification
- No mapping between fund profiles and investor profiles
- Static weights
- Dynamic risk contributions

Diversified funds = Marketing idea?
MVO portfolios are of the following form: $x^* \propto f(\Sigma^{-1})$.

The important quantity is then the information matrix $\mathcal{I} = \Sigma^{-1}$ and the eigendecomposition of $\mathcal{I}$ is:

$$V_i(\mathcal{I}) = V_{n+1-i}(\Sigma) \quad \text{and} \quad \lambda_i(\mathcal{I}) = \frac{1}{\lambda_{n+1-i}(\Sigma)}$$

If we consider the following example: $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$, we obtain:

<table>
<thead>
<tr>
<th>Asset / Factor</th>
<th>Covariance matrix $\Sigma$</th>
<th>Information matrix $\mathcal{I}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>65.35%</td>
<td>−72.29%</td>
</tr>
<tr>
<td>2</td>
<td>69.38%</td>
<td>69.06%</td>
</tr>
<tr>
<td>3</td>
<td>30.26%</td>
<td>−2.21%</td>
</tr>
<tr>
<td>Eigenv.</td>
<td>8.31%</td>
<td>0.84%</td>
</tr>
<tr>
<td>% cumulated</td>
<td>88.29%</td>
<td>97.20%</td>
</tr>
</tbody>
</table>

Thierry Roncalli
The Risk Dimension of Asset Returns in Risk Parity Portfolios 5 / 40
Which risk premium?  
Allocation = bets on risk premium

**CAPM**

\[
\pi^* = \text{SR} \left( x^* \mid r \right) \cdot \frac{\partial \sigma (x^*)}{\partial x}
\]

**Figure:** Comparison of typical American and European institutional investors

Are bonds a performance asset or a hedging asset?
Which risk measure?

- Equity smart beta
  - Stock volatility risk measure
  - Lyxor SmartIX ERC Equity Indices, etc.
- Fixed-income smart beta
  - Credit volatility risk measure
  - Lyxor RB EGBI, etc.
- Diversified funds
  - Asset volatility risk measure
  - Invesco IBRA Fund, etc.

Figure: 3 assets with a 20% volatility

Is the volatility the right risk measure for:
1. Strategic asset allocation?
2. Tactical asset allocation?
The risk parity (or risk budgeting) approach

- Definition
- Main Properties
- Using the Standard Deviation-Based Risk Measure
- Some Illustrations
Weight budgeting versus risk budgeting

Let \( x = (x_1, \ldots, x_n) \) be the weights of \( n \) assets in the portfolio. Let \( R(x_1, \ldots, x_n) \) be a coherent and convex risk measure. We have:

\[
R(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \cdot \frac{\partial R(x_1, \ldots, x_n)}{\partial x_i}
\]

\[
= \sum_{i=1}^{n} RC_i (x_1, \ldots, x_n)
\]

Let \( b = (b_1, \ldots, b_n) \) be a vector of budgets such that \( b_i \geq 0 \) and \( \sum_{i=1}^{n} b_i = 1 \). We consider two allocation schemes:

1. **Weight budgeting (WB)**
   \[ x_i = b_i \]

2. **Risk budgeting (RB)**
   \[ RC_i = b_i \cdot R(x_1, \ldots, x_n) \]
Traditional risk parity with the volatility risk measure

Let $\Sigma$ be the covariance matrix of the assets returns. We assume that the risk measure $\mathcal{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$. We have:

$$\frac{\partial \mathcal{R}(x)}{\partial x} = \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$$

$$\text{RC}_i(x_1, \ldots, x_n) = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

$$\sum_{i=1}^{n} \text{RC}_i(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sigma(x)$$

The risk budgeting portfolio is defined by this system of equations:

$$\begin{cases} x_i \cdot (\Sigma x)_i = b_i \cdot (x^\top \Sigma x) \\ x_i \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \end{cases}$$
Motivations
Risk Parity Approach
Applications
Conclusion

Definition
Main Properties
Using the Standard Deviation-based Risk Measure
Some Illustrations

An example

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.00%</td>
<td>29.40%</td>
<td>14.70%</td>
<td>70.43%</td>
</tr>
<tr>
<td>2</td>
<td>20.00%</td>
<td>16.63%</td>
<td>3.33%</td>
<td>15.93%</td>
</tr>
<tr>
<td>3</td>
<td>30.00%</td>
<td>9.49%</td>
<td>2.85%</td>
<td>13.64%</td>
</tr>
<tr>
<td>Volatility</td>
<td>20.87%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Weight budgeting (or traditional) approach

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.15%</td>
<td>28.08%</td>
<td>8.74%</td>
<td>50.00%</td>
</tr>
<tr>
<td>2</td>
<td>21.90%</td>
<td>15.97%</td>
<td>3.50%</td>
<td>20.00%</td>
</tr>
<tr>
<td>3</td>
<td>46.96%</td>
<td>11.17%</td>
<td>5.25%</td>
<td>30.00%</td>
</tr>
<tr>
<td>Volatility</td>
<td>17.49%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Risk budgeting approach

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weight</th>
<th>Marginal Risk</th>
<th>Risk Contribution Absolute</th>
<th>Risk Contribution Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.69%</td>
<td>27.31%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
<tr>
<td>2</td>
<td>32.44%</td>
<td>16.57%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
<tr>
<td>3</td>
<td>47.87%</td>
<td>11.23%</td>
<td>5.38%</td>
<td>33.33%</td>
</tr>
<tr>
<td>Volatility</td>
<td>16.13%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Existence and uniqueness

We consider the following risk budgeting problem:

\[
\begin{align*}
RC_i(x) &= b_i \mathcal{R}(x) \\
x_i &\geq 0 \\
\sum_{i=1}^{n} b_i &= 1 \\
\sum_{i=1}^{n} x_i &= 1
\end{align*}
\]

**Theorem**

- The RB portfolio exists and is unique if the risk budgets are strictly positive (and if \( \mathcal{R}(x) \) is bounded below)
- The RB portfolio exists and may be not unique if some risk budgets are set to zero
- The RB portfolio may not exist if some risk budgets are negative

These results hold for convex risk measures: volatility, Gaussian VaR & ES, elliptical VaR, non-normal ES, Kernel historical VaR, Cornish-Fisher VaR, etc.
The RB portfolio is a long-only minimum risk (MR) portfolio subject to a constraint of weight diversification.

Let us consider the following minimum risk optimization problem:

\[ x^*(c) = \arg\min_x R(x) \]

\[ \text{u.c. } \begin{cases} \sum_{i=1}^n b_i \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq 0 \end{cases} \]

- if \( c = c^- = -\infty \), \( x^*(c^-) = x_{\text{mr}} \) (no weight diversification)
- if \( c = c^+ = \sum_{i=1}^n b_i \ln b_i \), \( x^*(c^+) = x_{\text{wb}} \) (no risk minimization)
- \( \exists c^0 : x^*(c^0) = x_{\text{rb}} \) (risk minimization and weight diversification)

\[ \implies \text{if } b_i = 1/n, \ x_{\text{rb}} = x_{\text{erc}} \] (variance minimization, weight diversification and perfect risk diversification\(^2\))

\(^2\) The Gini coefficient of the risk measure is then equal to 0.
The RB portfolio is located between the MR portfolio and the WB portfolio

- The RB portfolio is a combination of the MR and WB portfolios:

\[
\frac{x_i}{b_i} = \frac{x_j}{b_j} \quad (\text{wb})
\]
\[
\partial x_i \mathcal{R}(x) = \partial x_j \mathcal{R}(x) \quad (\text{mr})
\]
\[
\frac{RC_i}{b_i} = \frac{RC_j}{b_j} \quad (\text{rb})
\]

- The risk of the RB portfolio is between the risk of the MR portfolio and the risk of the WB portfolio:

\[
\mathcal{R}(x_{mr}) \leq \mathcal{R}(x_{rb}) \leq \mathcal{R}(x_{wb})
\]

With risk budgeting, we always diminish the risk compared to the weight budgeting.

⇒ For the ERC portfolio, we retrieve the famous relationship:

\[
\sigma(x_{mr}) \leq \sigma(x_{erc}) \leq \sigma(x_{ew})
\]
Introducing expected returns in RB portfolios

In the original paper of Maillard et al. (2010), the risk measure is the volatility:

\[ \mathcal{R}(x) = \sigma(x) = \sqrt{x^\top \Sigma x} \]

Let us consider the standard deviation-based risk measure\(^3\):

\[ \mathcal{R}(x) = -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} = -\mu(x) + c \cdot \sigma(x) \]

It encompasses three well-known risk measures:

- Gaussian value-at-risk with \( c = \Phi^{-1}(\alpha) \)
- Gaussian expected shortfall with \( c = \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha} \)
- Markowitz quadratic utility function with \( c = \frac{\phi}{2} \sigma(x(\phi)) \)

We can easily compute the risk contribution of asset \( i \):

\[ \text{RC}_i = x_i \left( \mu_i + c \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \right) \]

\(^3\)The right specification is: \( \mathcal{R}(x) = -\left( \mu(x) - r \right) + c \cdot \sigma(x) \)
Existence and uniqueness

**Theorem**

If \( c > SR^+ = SR(x^* | r) \) where \( x^* \) is the tangency portfolio, the RB portfolio exists and is unique\(^a\).

\[^a\text{Because of the homogeneity property } \mathcal{R}(\lambda x) = \lambda \mathcal{R}(x).\]

**Remark**

*This contrasts with the result based on the volatility risk measure: in this case, the RB portfolio always exists and is unique.*
Existence and uniqueness

Example

We consider four assets. Their volatilities are equal to 15%, 20%, 25% and 30% while the correlation matrix of asset returns is given by the following matrix:

\[
C = \begin{pmatrix}
1.00 & 0.10 & 0.40 & 0.50 \\
0.10 & 1.00 & 0.70 & 0.40 \\
0.40 & 0.70 & 1.00 & 0.80 \\
0.50 & 0.40 & 0.80 & 1.00
\end{pmatrix}
\]

Here is the solution for the ERC portfolio:

<table>
<thead>
<tr>
<th>$c$</th>
<th>$\mu_i = 7%$</th>
<th>$\mu_i = 25%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Phi^{-1}(0.95)$</td>
<td>$\Phi^{-1}(0.99)$</td>
</tr>
<tr>
<td>1</td>
<td>43.54</td>
<td>42.06</td>
</tr>
<tr>
<td>2</td>
<td>28.18</td>
<td>28.11</td>
</tr>
<tr>
<td>3</td>
<td>15.05</td>
<td>15.82</td>
</tr>
<tr>
<td>4</td>
<td>13.23</td>
<td>14.01</td>
</tr>
<tr>
<td>SR$^+$</td>
<td>0.557</td>
<td></td>
</tr>
</tbody>
</table>
If the expected returns are 5%, 6%, 8% and 12%, we obtain:

⇒ We only consider RB portfolios with $c > SR^+$. 
Numerical solution of the optimization problem

Cyclical coordinate descent method of Tseng (2001):

$$\arg\min f (x_1, \ldots, x_n) = f_0 (x_1, \ldots, x_n) + \sum_{k=1}^{m} f_k (x_1, \ldots, x_n)$$

where $f_0$ is strictly convex and the functions $f_k$ are non-differentiable.

If we apply the CCD algorithm to the RB problem:

$$L(x; \lambda) = \arg\min -\mu(x) + c \cdot \sigma(x) - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

we obtain$^4$:

$$x_i^* = \frac{-c \gamma_i + \mu_i \sigma(x) + \sqrt{(c \gamma_i - \mu_i \sigma(x))^2 + 4cb_i \sigma_i^2 \sigma(x)}}{2c\sigma_i^2}$$

⇒ It always converges$^5$ (Theorem 5.1, Tseng, 2001).

---

$^4$with $\gamma_i = \sum_{j \neq i} x_j \rho_{i,j} \sigma_j$.

$^5$With an Intel T8400 3 GHz Core 2 Duo processor, computational times are 0.13, 0.45 and 1.10 seconds for a universe of 500, 1000 and 1500 assets.
Motivations
Risk Parity Approach
Applications
Conclusion
Definition
Main Properties
Using the Standard Deviation-based Risk Measure
Some Illustrations

MVO portfolios vs RB portfolios
Relationships

**Volatility risk measure**

\[ x^*(\kappa) = \arg \min_x \frac{1}{2} x^\top \Sigma x \]

\[ \text{u.c.} \quad \begin{cases} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ 1^\top x = 1 \\ x \geq 0 \end{cases} \]

The RB portfolio is a minimum variance portfolio subject to a constraint of weight diversification.

**Generalized risk measure**

\[ x^*(\kappa) = \arg \min_x -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} \]

\[ \text{u.c.} \quad \begin{cases} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ 1^\top x = 1 \\ x \geq 0 \end{cases} \]

The RB portfolio is a mean-variance portfolio subject to a constraint of weight diversification.
RB portfolios with expected returns

reformulation of MVO portfolios with regularization?

The answer is: NOT.

**MVO**
- 2D: Risk and Return (trade-off)
  - $\mu(x) =$ return dimension or profile
  - ER = arbitrage opportunity
  - $\Rightarrow$ Active/bets management

**RB**
- 1D: Risk (no trade-off)
  - $\mu(x) =$ risk dimension or profile
  - Expected returns = directional risks
  - $\Rightarrow$ Risk management
Example

We consider a universe of three assets. The expected returns are respectively $\mu_1 = \mu_2 = 8\%$ and $\mu_3 = 5\%$. For the volatilities, we have $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$. Moreover, we assume that the cross-correlations are the same and we have $\rho_{i,j} = \rho = 80\%$.

Table: Optimal portfolio\(^6\) with $\sigma^* = 15\%$

<table>
<thead>
<tr>
<th>Asset</th>
<th>$x_i$</th>
<th>$MR_i$</th>
<th>$RC_i$</th>
<th>$RC_i^*$</th>
<th>$VC_i$</th>
<th>$VC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>38.3</td>
<td>30.3</td>
<td>11.6</td>
<td>50.0</td>
<td>7.3</td>
<td>49.0</td>
</tr>
<tr>
<td>2</td>
<td>20.2</td>
<td>30.3</td>
<td>6.1</td>
<td>26.4</td>
<td>3.9</td>
<td>25.8</td>
</tr>
<tr>
<td>3</td>
<td>41.5</td>
<td>13.2</td>
<td>5.5</td>
<td>23.6</td>
<td>3.8</td>
<td>25.2</td>
</tr>
<tr>
<td>Volatility</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>15.0</td>
</tr>
</tbody>
</table>

\(^6\)We consider the standard deviation-based risk measure with $c = 2$. 
### MVO portfolios vs RB portfolios

**Stability (II)**

1. **MVO**: the objective is to target a volatility of 15%.
2. **RB**: the objective is to target the budgets (50.0%, 26.4%, 23.6%).

What is the sensitivity of MVO/RB portfolios to the input parameters?

<table>
<thead>
<tr>
<th></th>
<th>$\rho$</th>
<th>$\sigma_2$</th>
<th>$\mu_1$</th>
<th>70%</th>
<th>90%</th>
<th>18%</th>
<th>90%</th>
<th>20%</th>
<th>−20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>MVO</td>
<td>$x_1$</td>
<td>38.3%</td>
<td>38.3%</td>
<td>44.6%</td>
<td>13.7%</td>
<td>0.0%</td>
<td>56.4%</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>20.2%</td>
<td>25.9%</td>
<td>8.9%</td>
<td>56.1%</td>
<td>65.8%</td>
<td>0.0%</td>
<td>51.7%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>41.5%</td>
<td>35.8%</td>
<td>46.5%</td>
<td>30.2%</td>
<td>34.2%</td>
<td>43.6%</td>
<td>48.3%</td>
<td></td>
</tr>
<tr>
<td>RB</td>
<td>$x_1$</td>
<td>38.3%</td>
<td>37.5%</td>
<td>39.2%</td>
<td>36.7%</td>
<td>37.5%</td>
<td>49.1%</td>
<td>28.8%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>20.2%</td>
<td>20.4%</td>
<td>20.0%</td>
<td>23.5%</td>
<td>23.3%</td>
<td>16.6%</td>
<td>23.3%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>41.5%</td>
<td>42.1%</td>
<td>40.8%</td>
<td>39.7%</td>
<td>39.1%</td>
<td>34.2%</td>
<td>47.9%</td>
<td></td>
</tr>
</tbody>
</table>

⇒ RB portfolios are less sensitive to specification errors and expected returns than optimized portfolios ($\Sigma$ vs $\Sigma^{-1}$; arbitrage factors vs directional risk).

Thierry Roncalli

The Risk Dimension of Asset Returns in Risk Parity Portfolios
MVO portfolios vs RB portfolios
Stability (III)

MVO portfolios with targeted volatility are not sensitive to linear transformation of expected returns:

\[ x^* (\mu; \Sigma | \sigma^*) = x^* (\alpha \mu + \beta; \Sigma | \sigma^*) \]

RB portfolios are sensitive to linear transformation of expected returns:

\[ x^* (\mu; \Sigma | b) \neq x^* (\alpha \mu + \beta; \Sigma | b) \]

<table>
<thead>
<tr>
<th></th>
<th>( \mu )</th>
<th>( \mu + 10% )</th>
<th>2( \mu )</th>
<th>3( \mu - 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MVO</td>
<td>RB</td>
<td>MVO</td>
<td>RB</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>38.3%</td>
<td>38.3%</td>
<td>38.3%</td>
<td>26.1%</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>20.2%</td>
<td>20.2%</td>
<td>20.2%</td>
<td>13.5%</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>41.5%</td>
<td>41.5%</td>
<td>41.5%</td>
<td>60.4%</td>
</tr>
</tbody>
</table>
Impact of expected returns on the RB portfolio

We consider an investment universe of 3 assets. Their volatilities are equal to 15%, 20% and 25%, whereas the correlation matrix $C$ is:

$$
C = \begin{pmatrix}
1.00 \\
0.30 & 1.00 \\
0.50 & 0.70 & 1.00
\end{pmatrix}
$$

ERC portfolios\(^7\) for 6 parameter sets of expected returns with $c = 2$:

<table>
<thead>
<tr>
<th>Set</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0%</td>
<td>0%</td>
<td>20%</td>
<td>0%</td>
<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0%</td>
<td>10%</td>
<td>10%</td>
<td>-20%</td>
<td>30%</td>
<td>25%</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0%</td>
<td>20%</td>
<td>0%</td>
<td>-20%</td>
<td>-30%</td>
<td>-30%</td>
</tr>
<tr>
<td>$x_1$</td>
<td>45.25</td>
<td>37.03</td>
<td>64.58</td>
<td>53.30</td>
<td>29.65</td>
<td>66.50</td>
</tr>
<tr>
<td>$x_2$</td>
<td>31.65</td>
<td>33.11</td>
<td>24.43</td>
<td>26.01</td>
<td>63.11</td>
<td>31.91</td>
</tr>
<tr>
<td>$x_3$</td>
<td>23.10</td>
<td>29.86</td>
<td>10.98</td>
<td>20.69</td>
<td>7.24</td>
<td>1.59</td>
</tr>
<tr>
<td>$\mathcal{VC}_1^*$</td>
<td>33.33</td>
<td>23.80</td>
<td>60.96</td>
<td>43.79</td>
<td>15.88</td>
<td>64.80</td>
</tr>
<tr>
<td>$\mathcal{VC}_2^*$</td>
<td>33.33</td>
<td>34.00</td>
<td>23.85</td>
<td>26.32</td>
<td>75.03</td>
<td>33.10</td>
</tr>
<tr>
<td>$\mathcal{VC}_3^*$</td>
<td>33.33</td>
<td>42.20</td>
<td>15.19</td>
<td>29.89</td>
<td>9.09</td>
<td>2.11</td>
</tr>
</tbody>
</table>

\(^7\mathcal{RC}_i^* = 33.33\%.$
Impact of expected returns on the RB portfolio

**Figure:** Contour curves of the asset return distribution

⇒ Same volatility risk measure, but different directional risks
SAA and RP

- Long-term investment policy (10-30 years)
- Capturing the risk premia of asset classes (equities, bonds, real estate, natural resources, etc.)
- Top-down macro-economic approach (based on short-run disequilibrium and long-run steady-state)

ATP Danish Pension Fund

“Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk” (Henrik Gade Jepsen, CIO of ATP, IPE, June 2012).

These risk budgets are then transformed into asset classes’ weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities (Source: FTfm, June 10, 2012).
SAA in practice (March 2011)

Table: Expected returns, volatility and risk budgets\(^8\) (in %)

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_i)</td>
<td>4.2</td>
<td>3.8</td>
<td>5.3</td>
<td>9.2</td>
<td>8.6</td>
<td>11.0</td>
<td>8.8</td>
</tr>
<tr>
<td>(\sigma_i)</td>
<td>5.0</td>
<td>5.0</td>
<td>7.0</td>
<td>15.0</td>
<td>15.0</td>
<td>18.0</td>
<td>30.0</td>
</tr>
<tr>
<td>(b_i)</td>
<td>20.0</td>
<td>10.0</td>
<td>15.0</td>
<td>20.0</td>
<td>10.0</td>
<td>15.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Table: Correlation matrix of asset returns (in %)

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>80</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>60</td>
<td>40</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5)</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>90</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>20</td>
<td>20</td>
<td>30</td>
<td>70</td>
<td>70</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>(7)</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>30</td>
<td>100</td>
</tr>
</tbody>
</table>

---

\(^8\)The investment universe is composed of seven asset classes: US Bonds 10Y (1), EURO Bonds 10Y (2), Investment Grade Bonds (3), US Equities (4), Euro Equities (5), EM Equities (6) and Commodities (7).
**An example**

**Table:** Long-term strategic portfolios

<table>
<thead>
<tr>
<th></th>
<th>RB</th>
<th>MVO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c = \infty$</td>
<td>$c = 3$</td>
</tr>
<tr>
<td></td>
<td>$x_i$</td>
<td>$\text{VC}_i^*$</td>
</tr>
<tr>
<td>(1)</td>
<td>36.8</td>
<td>20.0</td>
</tr>
<tr>
<td>(2)</td>
<td>21.8</td>
<td>10.0</td>
</tr>
<tr>
<td>(3)</td>
<td>14.7</td>
<td>15.0</td>
</tr>
<tr>
<td>(4)</td>
<td>10.2</td>
<td>20.0</td>
</tr>
<tr>
<td>(5)</td>
<td>5.5</td>
<td>10.0</td>
</tr>
<tr>
<td>(6)</td>
<td>7.0</td>
<td>15.0</td>
</tr>
<tr>
<td>(7)</td>
<td>3.9</td>
<td>10.0</td>
</tr>
<tr>
<td>$\mu(x)$</td>
<td>5.69</td>
<td>5.58</td>
</tr>
<tr>
<td>$\sigma(x)$</td>
<td>5.03</td>
<td>4.85</td>
</tr>
<tr>
<td>$\text{SR}(x \mid r)$</td>
<td>1.13</td>
<td>1.15</td>
</tr>
</tbody>
</table>

- RB portfolios have lower Sharpe ratios than MVO portfolios (by construction!), but the difference is small.
- RB portfolios are highly diversified, not MVO portfolios.
- Expected returns have some impact on the volatility contributions $\text{VC}_i^*$.
- RB frontier is lower than MV frontier (because of the logarithmic barrier).

- \( c = \infty \) corresponds to the RB portfolio with the highest volatility (and the highest expected return).

- \( c \rightarrow \text{SR}(x^* | r) \) corresponds to the RB portfolio with the highest Sharpe ratio.

**Figure: Efficient frontier of SAA portfolios**

![Efficient frontier of SAA portfolios](image-url)
Risk parity and absolute return funds

The risk/return profile of risk parity funds is similar to that of diversified funds:

1. The drawdown is close to 20%;
2. The Sharpe ratio is lower than 0.5.

⇒ The (traditional) risk parity approach is not sufficient to build an absolute return fund.

How to transform it to an absolute return strategy?

1. By incorporating some views on economics and asset classes (global macro fund, e.g. the All Weather fund of Bridgewater)
2. By introducing trends and momentum patterns (long-only CTA)
3. By defining a more dynamic allocation (BL, time-varying risk budgets, etc.)
Calibrating the scaling factor

In a TAA model, the risk measure is no longer static:

\[ R_t(x_t) = -x_t^\top \mu_t + c_t \cdot \sqrt{x_t^\top \Sigma_t x_t} \]

\( c_t \) can not be constant because:

1. the solution may not exist\(^9\).
2. this rule is time-inconsistent (1Y \( \neq \) 1M):

\[
R_t(x_t; c, h) = -h \cdot x_t^\top \mu_t + c \sqrt{h} \cdot \sqrt{x_t^\top \Sigma_t x_t} \\
= h \cdot R_t(x_t; c', 1) \\
\]

with \( c' = h^{-0.5} c \).

\(^9\)There is no solution if \( c = \Phi^{-1}(99\%) \) and the maximum Sharpe ratio is 3.
An illustration

- Investment universe: MSCI World TR Net index, Citigroup WGBI All Maturities index
- Empirical covariance matrix (260 days)
- Simple moving average based on the daily returns (260 days)
- Different rules:
  \[
  c_t = \max \left( c_{ES} (99.9\%), 2.00 \cdot SR_t^+ \right) \quad \text{(RP #1)}
  \]
  \[
  c_t = \max \left( c_{Var} (99\%), 1.10 \cdot SR_t^+ \right) \quad \text{(RP #2)}
  \]
  \[
  c_t = 1.10 \cdot SR_t^+ \cdot 1 \{ SR_t^+ > 0 \} + \infty \cdot 1 \{ SR_t^+ \leq 0 \} \quad \text{(RP #3)}
  \]

Table: Statistics of risk parity strategies

<table>
<thead>
<tr>
<th>RP</th>
<th>( \mu_{1Y} )</th>
<th>( \sigma_{1Y} )</th>
<th>SR</th>
<th>MDD</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static #0</td>
<td>5.10</td>
<td>7.30</td>
<td>0.35</td>
<td>-21.39</td>
<td>0.07</td>
<td>2.68</td>
<td>0.30</td>
</tr>
<tr>
<td>#1</td>
<td>5.68</td>
<td>7.25</td>
<td>0.44</td>
<td>-18.06</td>
<td>0.10</td>
<td>2.48</td>
<td>1.14</td>
</tr>
<tr>
<td>Active #2</td>
<td>6.58</td>
<td>7.80</td>
<td>0.52</td>
<td>-12.78</td>
<td>0.05</td>
<td>2.80</td>
<td>2.98</td>
</tr>
<tr>
<td>#3</td>
<td>7.41</td>
<td>8.00</td>
<td>0.61</td>
<td>-12.84</td>
<td>0.04</td>
<td>2.74</td>
<td>3.65</td>
</tr>
</tbody>
</table>
The key issue is how to calibrate the scaling factor $c_t$ in an out-of-sample framework 😊
Optimality of the ERC portfolio

The ERC portfolio corresponds to the tangency portfolio if the Sharpe ratio is the same for all assets and the correlation is uniform.

The Sharpe ratio is constant if:

- the risk premia and the volatilities are constant;
- or the dynamic of the risk premia is the same as the dynamic of the volatilities.

⇒ Risk premia are time-varying:

Same weight compositions, but different economic regimes

Figure: Equivalent ERC compositions (static risk parity)

- etc.
The rising interest rate challenge

30 years downward trend of interest rates

<table>
<thead>
<tr>
<th></th>
<th>Peak</th>
<th>Trough</th>
</tr>
</thead>
<tbody>
<tr>
<td>US 10-year sovereign interest rate:</td>
<td>30/09/1981 15.80%</td>
<td>25/07/2012 1.37%</td>
</tr>
</tbody>
</table>

⇒ A significant component of the good performance of (static) risk parity funds.

- The right benchmark is certainly not the 60/40 asset mix policy.
- What will be the performance of risk parity funds if the interest rates rise?
  - Static risk parity vs active risk parity
    - 1994 scenario: fed fund = +300 bps / long rates = +250 bps
      ⇒ static: 😞, active: 😊
    - 1999 scenario: fed fund = +125 bps / long rates = +200 bps
      ⇒ static: 😊, active: 😊
One concept, several implementations, different performances!

- Choice of the investment universe
- Choice of the risk budgets
- Choice of the TAA model
- Choice of the leverage implementation
- Choice of the rebalancing frequency
- etc.

**Figure: Performance of RP funds in 2013**

Lyxor ARMA 8 I EUR
- 7.62%
Lyxor/SGI Harmonia Index
- 13.79%
Invesco Balanced-Risk Alloc C
+ 1.44%
AC Risk Parity 7 Fund EUR A
+ 9.97%

Thierry Roncalli

The Risk Dimension of Asset Returns in Risk Parity Portfolios
Risk parity based on the volatility risk measure = not the right answer to build absolute return fund.

We propose a solution to incorporate discretionary views and trends into risk parity portfolios:

- Expected returns = directional risks, and not performance opportunities.
- It can be viewed as an active allocation strategy, but it remains a risk parity strategy.

But it is not a magic allocation method:

“It cannot free investors of their duty of making their own choices”.

Thierry Roncalli
References

F. Barjou.
Active Risk Parity Strategies are Up to the Interest Rate Challenge.

S. Maillard, T. Roncalli and J. Teïletche.
The Properties of Equally Weighted Risk Contribution Portfolios.

L. Martellini, V. Milhau.

T. Roncalli.
Introduction to Risk Parity and Budgeting.

T. Roncalli.