The Risk Dimension of Asset Returns in Risk Parity Portfolios

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[1] The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.

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   - Which Risk Premium?
   - Which Risk Measure?
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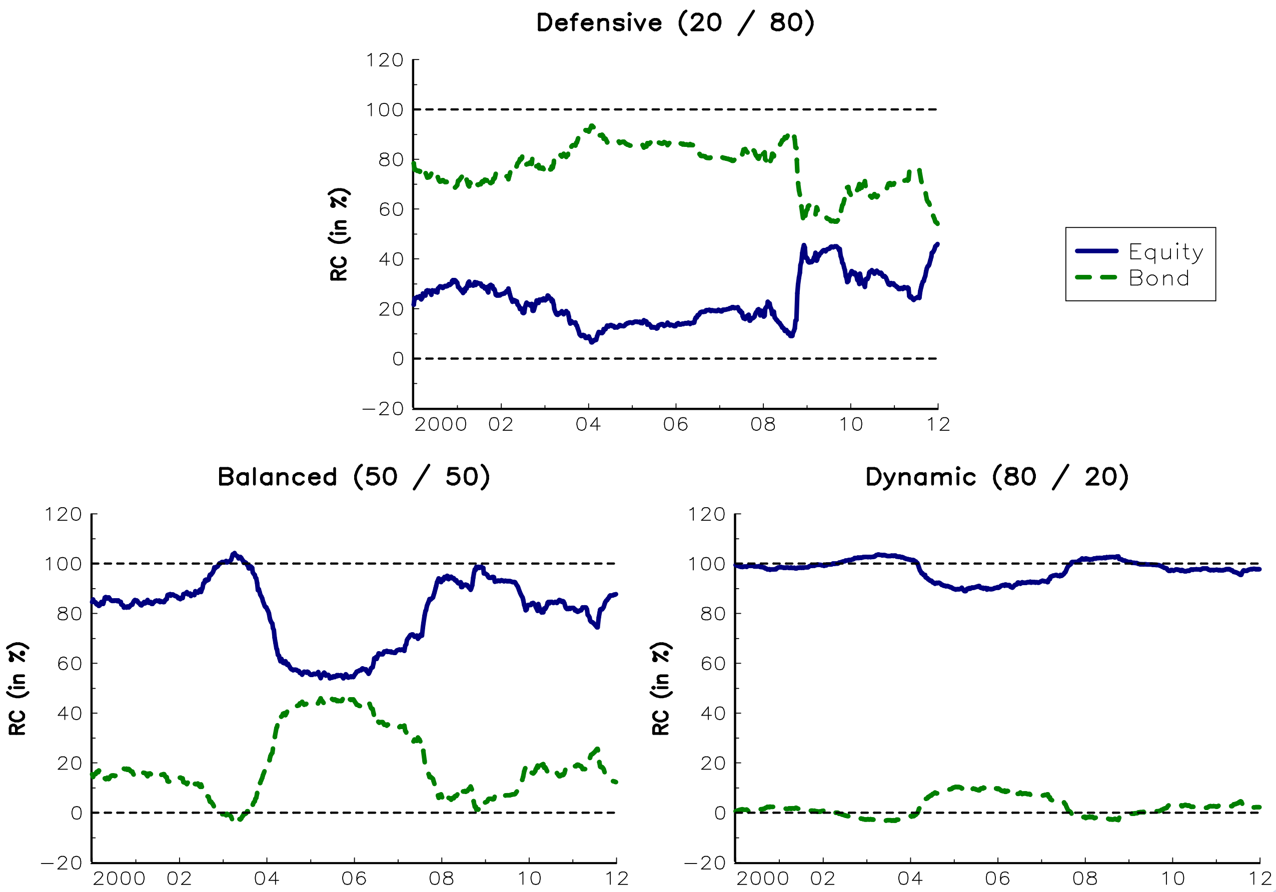
Motivations

- Which Diversification?
- Which Risk Factors?
- Which Risk Premium?
- Which Risk Measure?

Which diversification?

The case of diversified funds

Figure: Equity (MSCI World) and bond (WGBI) risk contributions



- Contrarian constant-mix strategy
- Deleverage of an equity exposure
- Low risk diversification
- No mapping between fund profiles and investor profiles
- Static weights
- Dynamic risk contributions

Diversified funds
=
Marketing idea?

Which risk factors?

How to be sensitive to Σ and not to Σ^{-1}?

MVO portfolios are of the following form: $x^\star \propto f\left(\Sigma^{-1}\right)$.

The important quantity is then the information matrix $\mathcal{I} = \Sigma^{-1}$ and the eigendecomposition of $\mathcal{I}$ is:

$$V_i\left(\mathcal{I}\right) = V_{n+1-i}\left(\Sigma\right) \quad \text{and} \quad \lambda_i\left(\mathcal{I}\right) = \frac{1}{\lambda_{n+1-i}\left(\Sigma\right)}$$

If we consider the following example: $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$, we obtain:

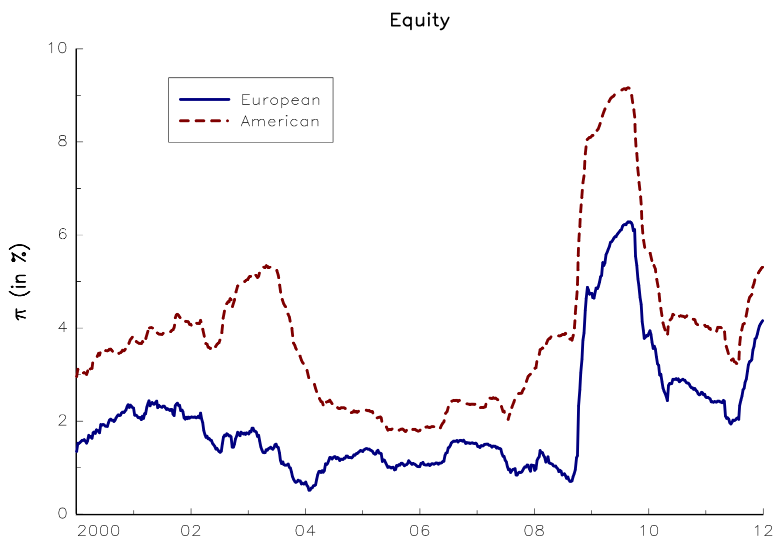
	Covariance matrix Σ			Information matrix $\mathcal{I}$		
Asset / Factor	1	2	3	1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

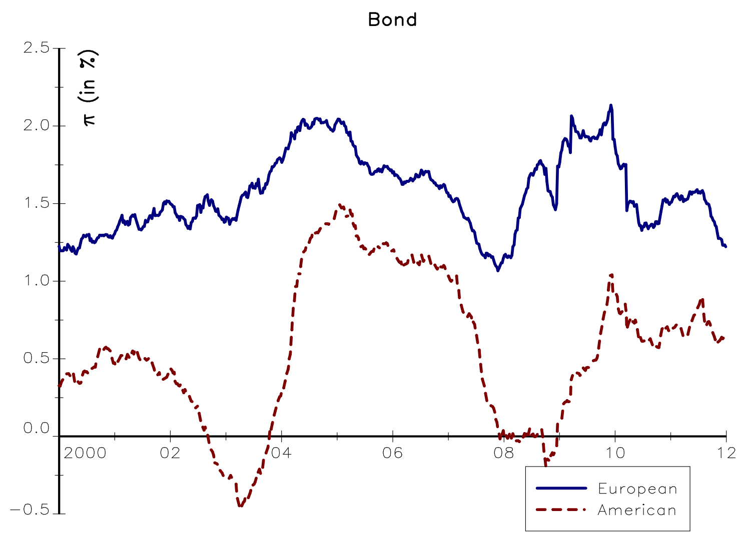
Which risk premium?
Allocation = bets on risk premium

CAPM

$$\pi^\star = \mathrm{SR}\,(x^\star \mid r) \cdot \frac{\partial\,\sigma\,(x^\star)}{\partial\,x}$$

Figure: Comparison of typical American and European institutional investors



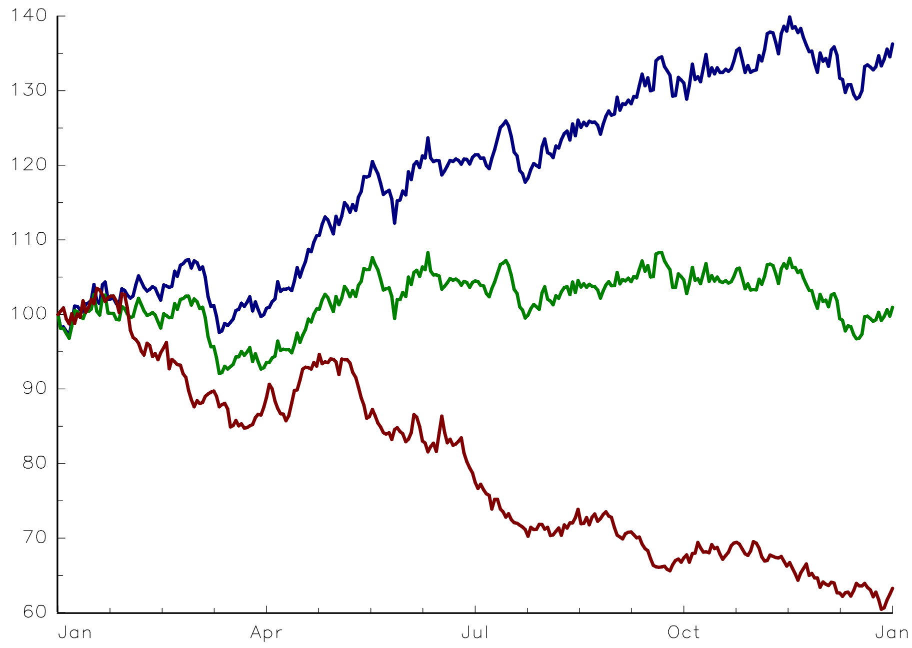


Are bonds a performance asset or a hedging asset?

Which risk measure?

- Equity smart beta
  - Stock volatility risk measure
  - Lyxor SmartIX ERC Equity Indices, etc.
- Fixed-income smart beta
  - Credit volatility risk measure
  - Lyxor RB EGBI, etc.
- Diversified funds
  - Asset volatility risk measure
  - Invesco IBRA Fund, etc.

Figure: 3 assets with a 20% volatility



Is the volatility the right risk measure for:
1. Strategic asset allocation?
2. Tactical asset allocation?

The risk parity (or risk budgeting) approach

- Definition
- Main Properties
- Using the Standard Deviation-Based Risk Measure
- Some Illustrations

Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathcal{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned} \mathcal{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial \mathcal{R}(x_1, \ldots, x_n)}{\partial x_i} \\ &= \sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n) \end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

2. Risk budgeting (RB)

$$\mathrm{RC}_i = b_i \cdot \mathcal{R}(x_1, \ldots, x_n)$$

Traditional risk parity with the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We assume that the risk measure $\mathcal{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$. We have:

$$\begin{aligned}
\frac{\partial\,\mathcal{R}(x)}{\partial\, x} &= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\
\mathrm{RC}_i(x_1, \ldots, x_n) &= x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
\sum_{i=1}^n \mathrm{RC}_i(x_1, \ldots, x_n) &= \sum_{i=1}^n x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sigma(x)
\end{aligned}$$

The risk budgeting portfolio is defined by this system of equations:

$$\left\{ \begin{array}{l}
x_i \cdot (\Sigma x)_i = b_i \cdot \left(x^\top \Sigma x\right) \\
x_i \geq 0 \\
\sum_{i=1}^n x_i = 1
\end{array} \right.$$

An example

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1$^{\text{st}}$ asset and the 2$^{\text{nd}}$ asset, 50% between the 1$^{\text{st}}$ asset and the 3$^{\text{rd}}$ asset and 30% between the 2$^{\text{nd}}$ asset and the 3$^{\text{rd}}$ asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

Existence and uniqueness

We consider the following risk budgeting problem:

$$\begin{cases} \mathrm{RC}_i(x) = b_i \mathcal{R}(x) \\ x_i \geq 0 \\ \sum_{i=1}^n b_i = 1 \\ \sum_{i=1}^n x_i = 1 \end{cases}$$

Theorem

- The RB portfolio exists and is unique if the risk budgets are strictly positive (and if $\mathcal{R}(x)$ is bounded below)
- The RB portfolio exists and may be not unique if some risk budgets are set to zero
- The RB portfolio may not exist if some risk budgets are negative

These results hold for convex risk measures: volatility, Gaussian VaR & ES, elliptical VaR, non-normal ES, Kernel historical VaR, Cornish-Fisher VaR, etc.

The RB portfolio is a long-only minimum risk (MR) portfolio subject to a constraint of weight diversification

Let us consider the following minimum risk optimization problem:

$$x^{\star}(c) = \arg\min \mathcal{R}(x)$$
$$\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} b_i \ln x_i \geq c \\ \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{cases}$$

- if $c = c^{-} = -\infty$, $x^{\star}(c^{-}) = x_{\mathrm{mr}}$ (no weight diversification)
- if $c = c^{+} = \sum_{i=1}^{n} b_i \ln b_i$, $x^{\star}(c^{+}) = x_{\mathrm{wb}}$ (no risk minimization)
- $\exists c^{0} : x^{\star}(c^{0}) = x_{\mathrm{rb}}$ (risk minimization and weight diversification)

$\Longrightarrow$ if $b_i = 1/n$, $x_{\mathrm{rb}} = x_{\mathrm{erc}}$ (variance minimization, weight diversification and perfect risk diversification[2])

[2] The Gini coefficient of the risk measure is then equal to 0.

The RB portfolio is located between the MR portfolio and the WB portfolio

- The RB portfolio is a combination of the MR and WB portfolios:

$$x_i / b_i = x_j / b_j \qquad \text{(wb)}$$
$$\partial_{x_i} \mathcal{R}(x) = \partial_{x_j} \mathcal{R}(x) \qquad \text{(mr)}$$

$$\mathrm{RC}_i / b_i = \mathrm{RC}_j / b_j \qquad \text{(rb)}$$

- The risk of the RB portfolio is between the risk of the MR portfolio and the risk of the WB portfolio:

$$\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{rb}}) \leq \mathcal{R}(x_{\mathrm{wb}})$$

  With risk budgeting, we always diminish the risk compared to the weight budgeting.

$\Rightarrow$ For the ERC portfolio, we retrieve the famous relationship:

$$\sigma(x_{\mathrm{mr}}) \leq \sigma(x_{\mathrm{erc}}) \leq \sigma(x_{\mathrm{ew}})$$

Introducing expected returns in RB portfolios

In the original paper of Maillard *et al.* (2010), the risk measure is the volatility:

$$\mathcal{R}(x) = \sigma(x) = \sqrt{x^\top \Sigma x}$$

Let us consider the standard deviation-based risk measure[3]:

$$\mathcal{R}(x) = -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} = -\mu(x) + c \cdot \sigma(x)$$

It encompasses three well-known risk measures:

- Gaussian value-at-risk with $c = \Phi^{-1}(\alpha)$
- Gaussian expected shortfall with $c = \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$
- Markowitz quadratic utility function with $c = \frac{\phi}{2}\sigma(x(\phi))$

We can easily compute the risk contribution of asset i:

$$\mathrm{RC}_i = x_i \left(\mu_i + c \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \right)$$

[3] The right specification is: $\mathcal{R}(x) = -(\mu(x) - r) + c \cdot \sigma(x)$.

Existence and uniqueness

Theorem

If $c > \mathrm{SR}^+ = \mathrm{SR}(x^\star \mid r)$ where $x^\star$ is the tangency portfolio, the RB portfolio exists and is unique[a].

[a] Because of the homogeneity property $\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x)$.

Remark

This contrasts with the result based on the volatility risk measure: in this case, the RB portfolio always exists and is unique.

Existence and uniqueness

Example

We consider four assets. Their volatilities are equal to 15%, 20%, 25% and 30% while the correlation matrix of asset returns is given by the following matrix:

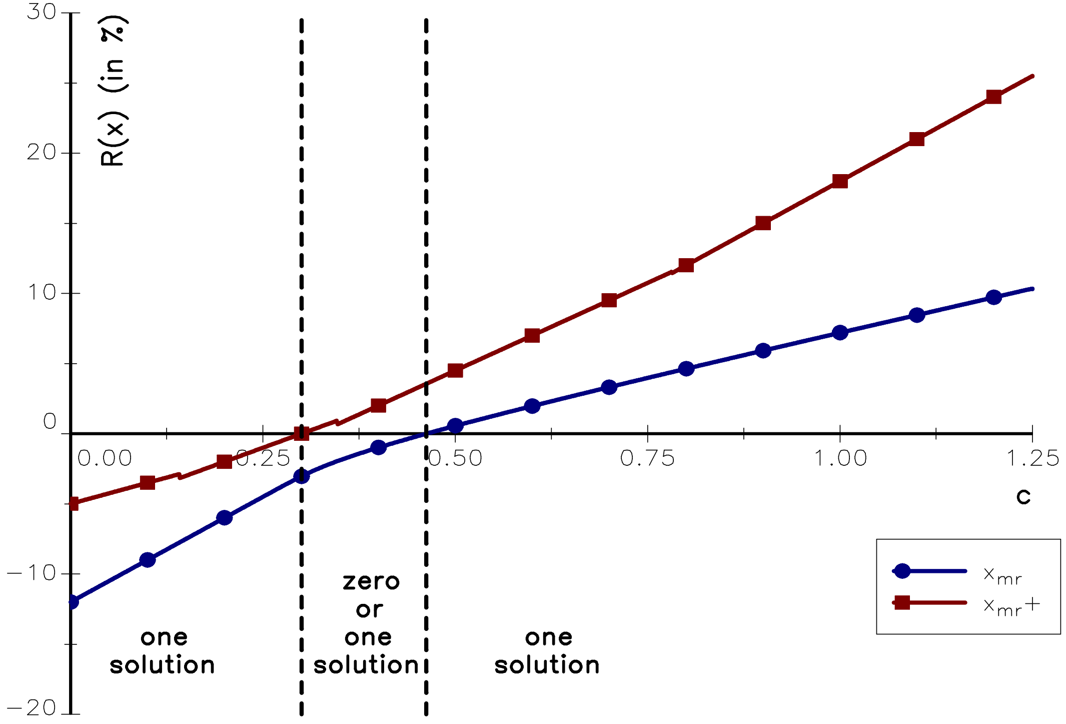
$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Here is the solution for the ERC portfolio:

	$\mu_i = 7\%$			$\mu_i = 25\%$		
c	0.40	$\Phi^{-1}(0.95)$	$\Phi^{-1}(0.99)$	0.40	$\Phi^{-1}(0.95)$	$\Phi^{-1}(0.99)$
1		43.54	42.06	19.78		56.82
2		28.18	28.11	21.89		29.75
3		15.05	15.82	27.63		7.34
4		13.23	14.01	30.70		6.08
SR^+	0.557			1.991		

Existence and uniqueness

If the expected returns are 5%, 6%, 8% and 12%, we obtain:



⇒ **We only consider RB portfolios with $c > \mathrm{SR}^+$.**

Numerical solution of the optimization problem

Cyclical coordinate descent method of Tseng (2001):

$$\arg\min f(x_1,\ldots,x_n) = f_0(x_1,\ldots,x_n) + \sum_{k=1}^{m} f_k(x_1,\ldots,x_n)$$

where f_0 is strictly convex and the functions f_k are non-differentiable. If we apply the CCD algorithm to the RB problem:

$$\mathcal{L}(x;\lambda) = \arg\min -\mu(x) + c\cdot\sigma(x) - \lambda\sum_{i=1}^{n} b_i \ln x_i$$

we obtain[4]:

$$x_i^\star = \frac{-c\gamma_i + \mu_i\sigma(x) + \sqrt{(c\gamma_i - \mu_i\sigma(x))^2 + 4cb_i\sigma_i^2\sigma(x)}}{2c\sigma_i^2}$$

$\Rightarrow$ It always converges[5] (Theorem 5.1, Tseng, 2001).

[4] with $\gamma_i = \sigma_i\sum_{j\neq i} x_j\rho_{i,j}\sigma_j$.

[5] With an Intel T8400 3 GHz Core 2 Duo processor, computational times are 0.13, 0.45 and 1.10 seconds for a universe of 500, 1000 and 1500 assets.

MVO portfolios vs RB portfolios

Relationships

Volatility risk measure

$$
\begin{aligned}
x^{\star}(\kappa) &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\
\text{u.c.} &\quad \begin{cases} \sum_{i=1}^{n} b_i \ln x_i \geq \kappa \\ \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{cases}
\end{aligned}
$$

The RB portfolio is a minimum variance portfolio subject to a constraint of weight diversification.

Generalized risk measure

$$
\begin{aligned}
x^{\star}(\kappa) &= \arg\min -x^{\top}\mu + c \cdot \sqrt{x^{\top} \Sigma x} \\
\text{u.c.} &\quad \begin{cases} \sum_{i=1}^{n} b_i \ln x_i \geq \kappa \\ \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{cases}
\end{aligned}
$$

The RB portfolio is a mean-variance portfolio subject to a constraint of weight diversification.

MVO portfolios vs RB portfolios

Differences

RB portfolios with expected returns

=

reformulation of MVO portfolios with regularization?

The answer is: NOT.

MVO

- 2D: Risk and Return (trade-off)
- $\mu(x)$ = return dimension or profile
- ER = arbitrage opportunity

$\Rightarrow$ Active/bets management

RB

- 1D: Risk (no trade-off)
- $\mu(x)$ = risk dimension or profile
- Expected returns = directional risks

$\Rightarrow$ Risk management

MVO portfolios vs RB portfolios

Stability (I)

Example

We consider a universe of three assets. The expected returns are respectively $\mu_1 = \mu_2 = 8\%$ and $\mu_3 = 5\%$. For the volatilities, we have $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$. Moreover, we assume that the cross-correlations are the same and we have $\rho_{i,j} = \rho = 80\%$.

Table: Optimal portfolio[6] with $\sigma^\star = 15\%$

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$	$\mathcal{VC}_i$	$\mathcal{VC}_i^\star$
1	38.3	30.3	11.6	50.0	7.3	49.0
2	20.2	30.3	6.1	26.4	3.9	25.8
3	41.5	13.2	5.5	23.6	3.8	25.2
Volatility					15.0	

[6] We consider the standard deviation-based risk measure with $c = 2$.

MVO portfolios vs RB portfolios

Stability (II)

1. MVO: the objective is to target a volatility of 15%.
2. RB: the objective is to target the budgets (50.0%, 26.4%, 23.6%).

What is the sensitivity of MVO/RB portfolios to the input parameters?

ρ			70%	90%		90%		
σ_2					18%	18%		
μ_1							20%	-20%
MVO	x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%	0.0%
	x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%	51.7%
	x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%	48.3%
RB	x_1	38.3%	37.5%	39.2%	36.7%	37.5%	49.1%	28.8%
	x_2	20.2%	20.4%	20.0%	23.5%	23.3%	16.6%	23.3%
	x_3	41.5%	42.1%	40.8%	39.7%	39.1%	34.2%	47.9%

$\Rightarrow$ RB portfolios are less sensitive to specification errors and expected returns than optimized portfolios (Σ vs Σ^{-1}; arbitrage factors vs directional risk).

MVO portfolios vs RB portfolios

Stability (III)

MVO portfolios with targeted volatility are not sensitive to linear transformation of expected returns:

$$x^{\star}\left(\mu;\Sigma \mid \sigma^{\star}\right)=x^{\star}\left(\alpha\mu+\beta;\Sigma \mid \sigma^{\star}\right)$$

RB portfolios are sensitive to linear transformation of expected returns:

$$x^{\star}\left(\mu;\Sigma \mid b\right)\neq x^{\star}\left(\alpha\mu+\beta;\Sigma \mid b\right)$$

	μ		$\mu+10\%$		2μ		3μ - 10%	
	MVO	RB	MVO	RB	MVO	RB	MVO	RB
x_1	38.3%	38.3%	38.3%	26.1%	38.3%	36.0%	38.3%	41.4%
x_2	20.2%	20.2%	20.2%	13.5%	20.2%	18.9%	20.2%	21.9%
x_3	41.5%	41.5%	41.5%	60.4%	41.5%	45.1%	41.5%	36.7%

Impact of expected returns on the RB portfolio

We consider an investment universe of 3 assets. Their volatilities are equal to 15%, 20% and 25%, whereas the correlation matrix C is:

$$C = \begin{pmatrix} 1.00 & & \\ 0.30 & 1.00 & \\ 0.50 & 0.70 & 1.00 \end{pmatrix}$$

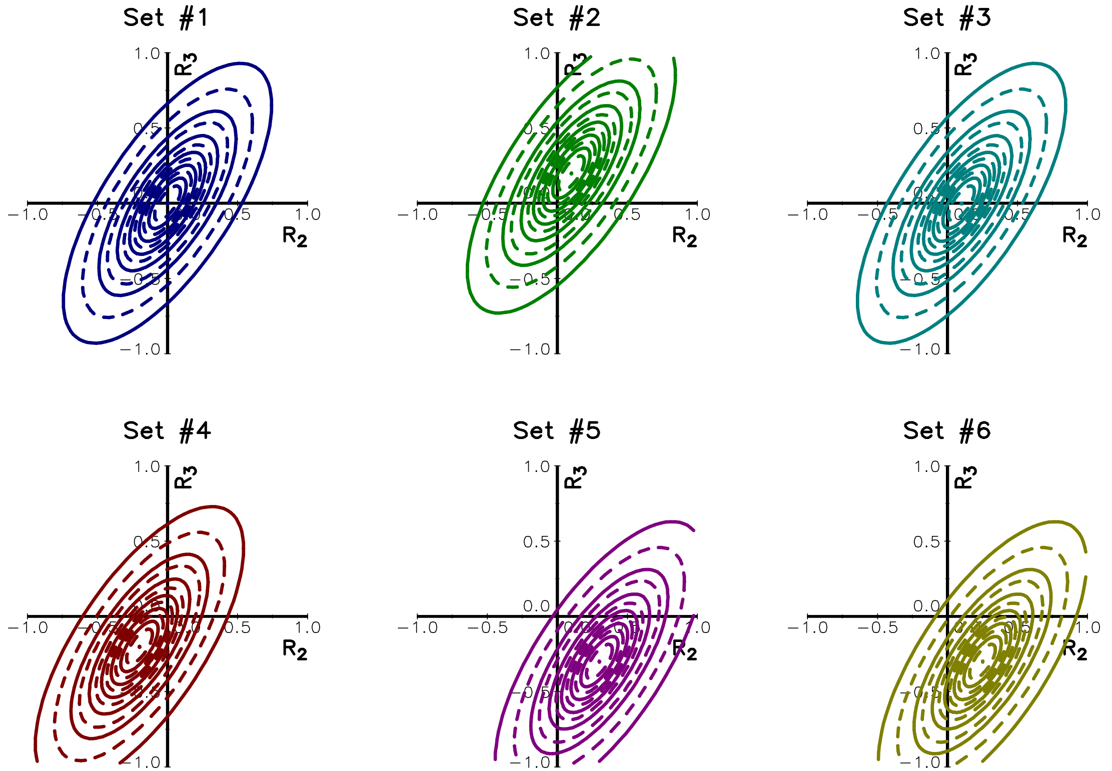
ERC portfolios[7] for 6 parameter sets of expected returns with $c = 2$:

Set	#1	#2	#3	#4	#5	#6
μ_1	0%	0%	20%	0%	0%	25%
μ_2	0%	10%	10%	−20%	30%	25%
μ_3	0%	20%	0%	−20%	−30%	−30%
x_1	45.25	37.03	64.58	53.30	29.65	66.50
x_2	31.65	33.11	24.43	26.01	63.11	31.91
x_3	23.10	29.86	10.98	20.69	7.24	1.59
$\mathcal{VC}_1^\star$	33.33	23.80	60.96	43.79	15.88	64.80
$\mathcal{VC}_2^\star$	33.33	34.00	23.85	26.32	75.03	33.10
$\mathcal{VC}_3^\star$	33.33	42.20	15.19	29.89	9.09	2.11

[7] $\mathcal{RC}_i^\star = 33.33\%$.

Impact of expected returns on the RB portfolio

Figure: Contour curves of the asset return distribution



⇒ Same volatility risk measure, but different directional risks

SAA and RP

- Long-term investment policy (10-30 years)
- Capturing the risk premia of asset classes (equities, bonds, real estate, natural resources, etc.)
- Top-down macro-economic approach (based on short-run disequilibrium and long-run steady-state)

ATP Danish Pension Fund

> *"Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk" (Henrik Gade Jepsen, CIO of ATP, IPE, June 2012).*

These risk budgets are then transformed into asset classes' weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities (Source: FTfm, June 10, 2012).

SAA in practice (March 2011)

Table: Expected returns, volatility and risk budgets[8] (in %)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
μ_i	4.2	3.8	5.3	9.2	8.6	11.0	8.8
σ_i	5.0	5.0	7.0	15.0	15.0	18.0	30.0
b_i	20.0	10.0	15.0	20.0	10.0	15.0	10.0

Table: Correlation matrix of asset returns (in %)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	100						
(2)	80	100					
(3)	60	40	100				
(4)	−10	−20	30	100			
(5)	−20	−10	20	90	100		
(6)	−20	−20	30	70	70	100	
(7)	0	0	10	20	20	30	100

[8] The investment universe is composed of seven asset classes: US Bonds 10Y (1), EURO Bonds 10Y (2), Investment Grade Bonds (3), US Equities (4), Euro Equities (5), EM Equities (6) and Commodities (7).

An example

Table: Long-term strategic portfolios

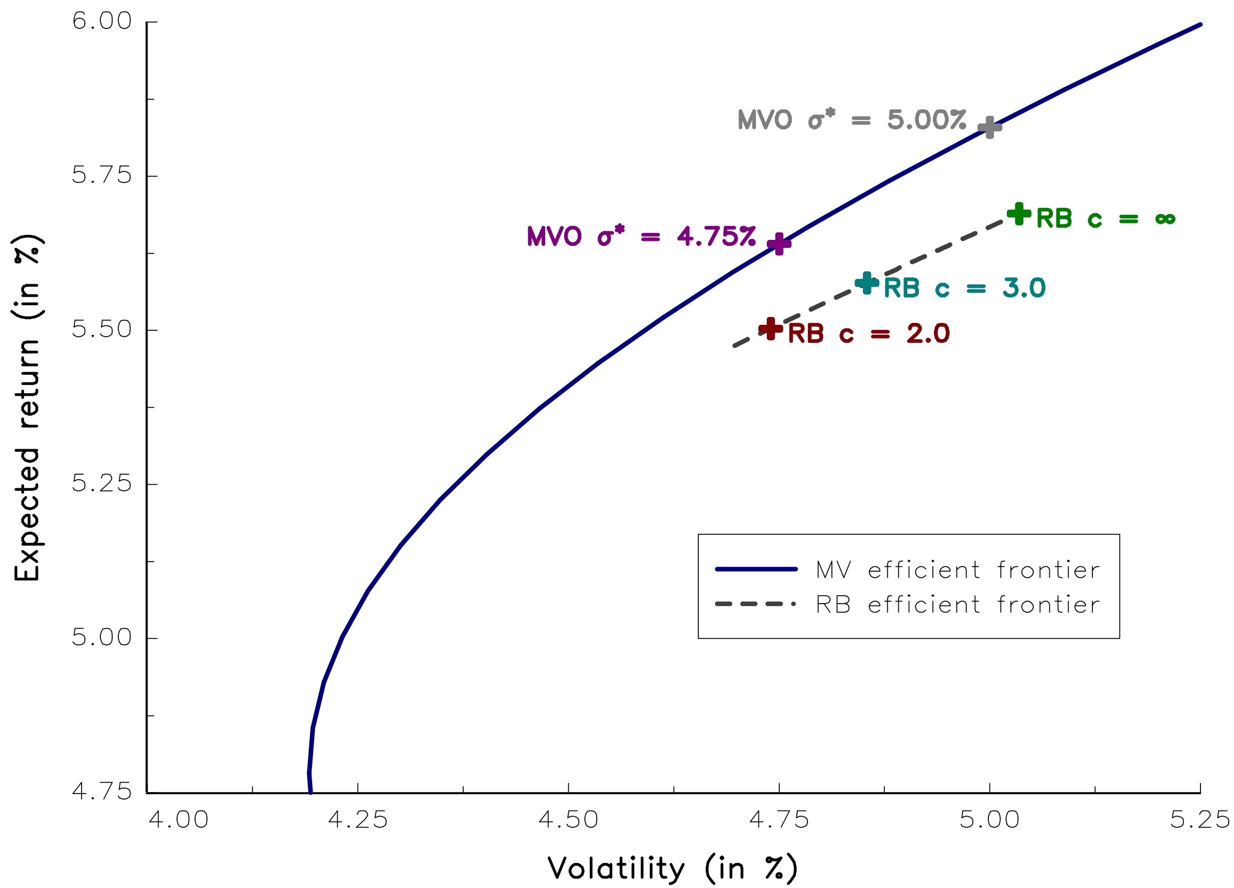
	RB						MVO			
	$c=\infty$		$c=3$		$c=2$		$\sigma^\star=4.75\%$		$\sigma^\star=5\%$	
	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$
(1)	36.8	20.0	38.5	23.4	39.8	26.0	60.5	38.1	64.3	34.6
(2)	21.8	10.0	23.4	12.3	24.7	14.1	14.0	7.4	7.6	3.2
(3)	14.7	15.0	13.1	14.0	11.7	12.8	0.0	0.0	0.0	0.0
(4)	10.2	20.0	9.5	18.3	8.9	17.1	5.2	10.0	5.5	10.8
(5)	5.5	10.0	5.2	9.2	4.9	8.6	5.2	9.2	5.5	9.8
(6)	7.0	15.0	6.9	14.5	7.0	14.4	14.2	33.7	16.0	39.5
(7)	3.9	10.0	3.4	8.2	3.0	6.9	1.0	1.7	1.1	2.1
$\mu(x)$	5.69		5.58		5.50		5.64		5.83	
$\sigma(x)$	5.03		4.85		4.74		4.75		5.00	
$\mathrm{SR}(x \mid r)$	1.13		1.15		1.16		1.19		1.17	

- RB portfolios have lower Sharpe ratios than MVO portfolios (by construction!), but the difference is small.
- RB portfolios are highly diversified, not MVO portfolios.
- Expected returns have some impact on the volatility contributions $\mathrm{VC}_i^\star$.

Efficient frontiers

- RB frontier is lower than MV frontier (because of the logarithmic barrier).
- $c = \infty$ corresponds to the RB portfolio with the highest volatility (and the highest expected return).
- $c \to \mathrm{SR}(x^\star | r)$ corresponds to the RB portfolio with the highest Sharpe ratio.

Figure: Efficient frontier of SAA portfolios



Risk parity and absolute return funds

The risk/return profile of risk parity funds is similar to that of diversified funds:

1. The drawdown is close to 20%;
2. The Sharpe ratio is lower than 0.5.

⇒ The (traditional) risk parity approach is not sufficient to build an absolute return fund.

How to transform it to an absolute return strategy?

1. By incorporating some views on economics and asset classes (global macro fund, e.g. the All Weather fund of Bridgewater)
2. By introducing trends and momentum patterns (long-only CTA)
3. By defining a more dynamic allocation (BL, time-varying risk budgets, etc.)

Calibrating the scaling factor

In a TAA model, the risk measure is no longer static:

$$\mathcal{R}_t(x_t) = -x_t^\top \mu_t + c_t \cdot \sqrt{x_t^\top \Sigma_t x_t}$$

c_t can not be constant because:

1. the solution may not exist[9].
2. this rule is time-inconsistent ($1Y \neq 1M$):

$$\begin{aligned} \mathcal{R}_t(x_t; c, h) &= -h \cdot x_t^\top \mu_t + c\sqrt{h} \cdot \sqrt{x_t^\top \Sigma_t x_t} \\ &= h \cdot \mathcal{R}_t(x_t; c', 1) \end{aligned}$$

with $c' = h^{-0.5} c$.

[9] There is no solution if $c = \Phi^{-1}(99\%)$ and the maximum Sharpe ratio is 3.

An illustration

- Investment universe: MSCI World TR Net index, Citigroup WGBI All Maturities index
- Empirical covariance matrix (260 days)
- Simple moving average based on the daily returns (260 days)
- Different rules:

$$c_t = \max\left(c_{\mathrm{ES}}(99.9\%), 2.00 \cdot \mathrm{SR}_t^+\right) \quad \text{(RP \#1)}$$

$$c_t = \max\left(c_{\mathrm{VaR}}(99\%), 1.10 \cdot \mathrm{SR}_t^+\right) \quad \text{(RP \#2)}$$

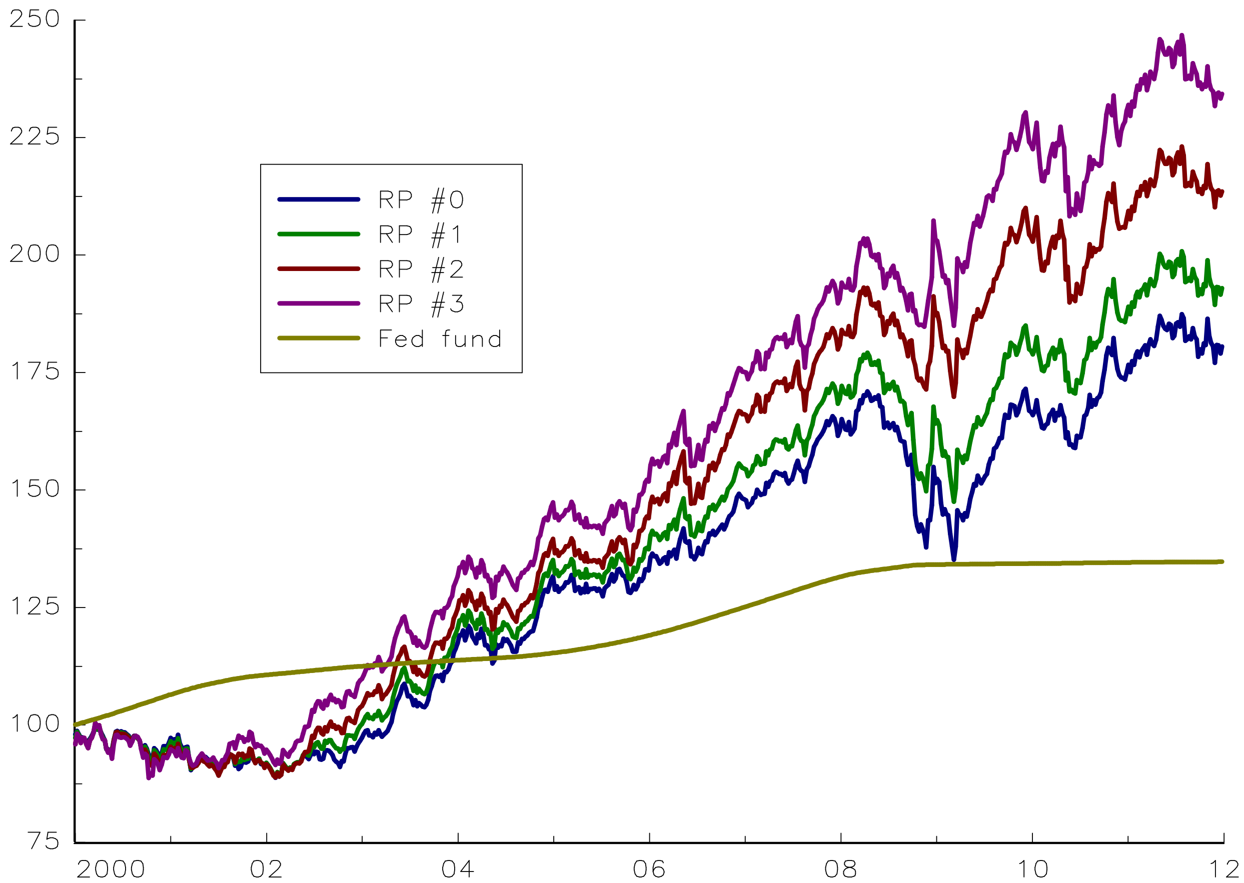
$$c_t = 1.10 \cdot \mathrm{SR}_t^+ \cdot \mathbf{1}\left\{\mathrm{SR}_t^+ > 0\right\} + \infty \cdot \mathbf{1}\left\{\mathrm{SR}_t^+ \leq 0\right\} \quad \text{(RP \#3)}$$

Table: Statistics of risk parity strategies

RP		$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2	τ
Static	#0	5.10	7.30	0.35	−21.39	0.07	2.68	0.30
Active	#1	5.68	7.25	0.44	−18.06	0.10	2.48	1.14
	#2	6.58	7.80	0.52	−12.78	0.05	2.80	2.98
	#3	7.41	8.00	0.61	−12.84	0.04	2.74	3.65

An illustration

Figure: Backtesting of RP strategies



The key issue is how to calibrate the scaling factor c_t in a out-of-sample framework ☹

Risk parity and time-varying risk premia

Optimality of the ERC portfolio

The ERC portfolio corresponds to the tangency portfolio if the Sharpe ratio is the same for all assets and the correlation is uniform.

The Sharpe ratio is constant if:

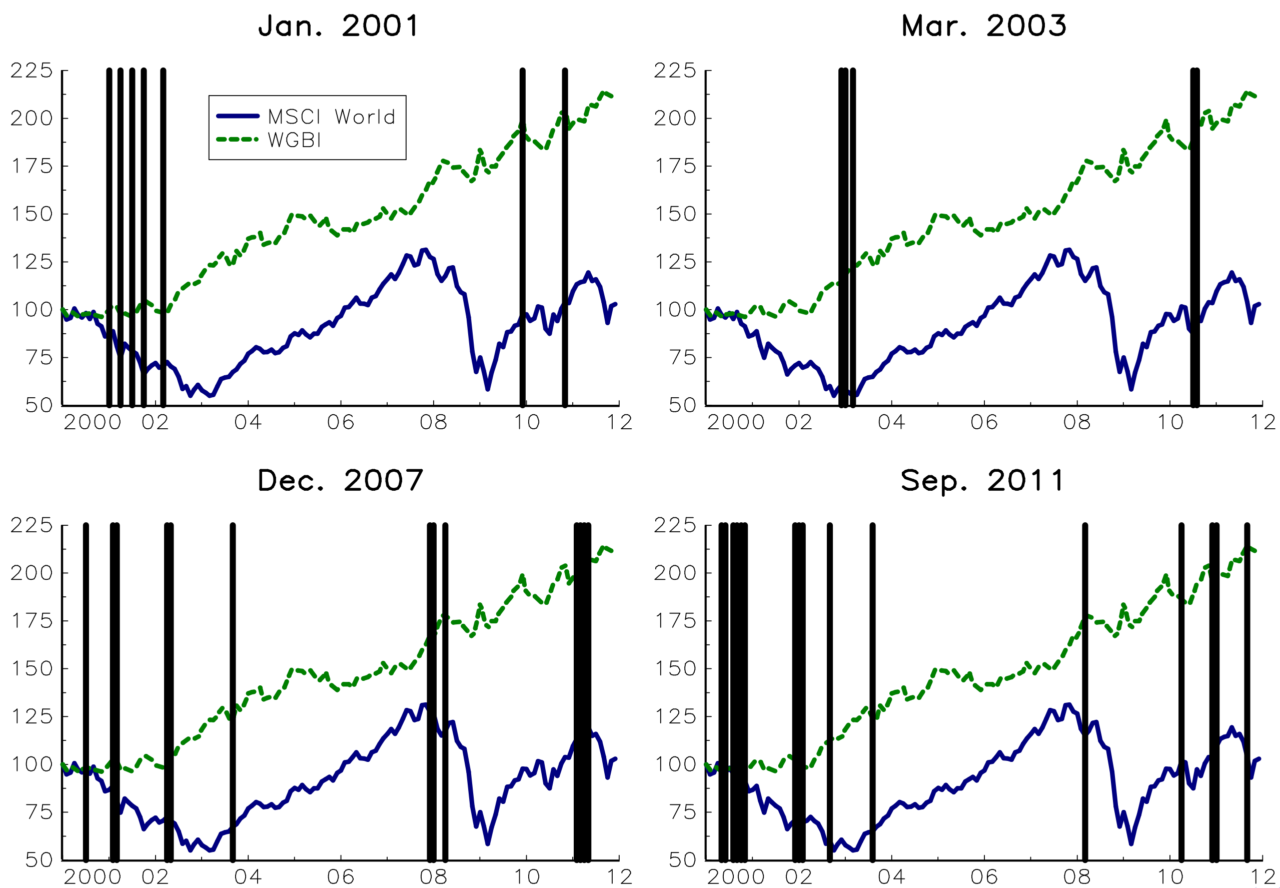
- the risk premia and the volatilities are constant;
- or the dynamic of the risk premia is the same as the dynamic of the volatilities.

$\Rightarrow$ Risk premia are time-varying:

- General framework: Lucas (1976), Engle, Lilien and Robins (1987), Cochrane (2005).
- Stocks: Campbell and Shiller (1988), Lettau and Ludvigson (2001).
- Bonds: Cochrane and Piazzesi (2002), Dai and Singleton (2002), Diebold (2006).

Same weight compositions, but different economic regimes

Figure: Equivalent ERC compositions (static risk parity)



- Dec. 2002 – Mar. 2003 ≡ Jul. – Aug. 2010 (26.5/73.5)
- Jul. 2000 ≡ Feb. – Mar. 2001 ≡ Apr. – May 2002 ≡ Sep. 2003 ≡ Dec. 2007 – Apr. 2008 ≡ Feb. – May 2011 (30/70)
- etc.

The rising interest rate challenge

30 years downward trend of interest rates

US 10-year sovereign interest rate:

Peak	30/09/1981	15.80%
Trough	25/07/2012	1.37%

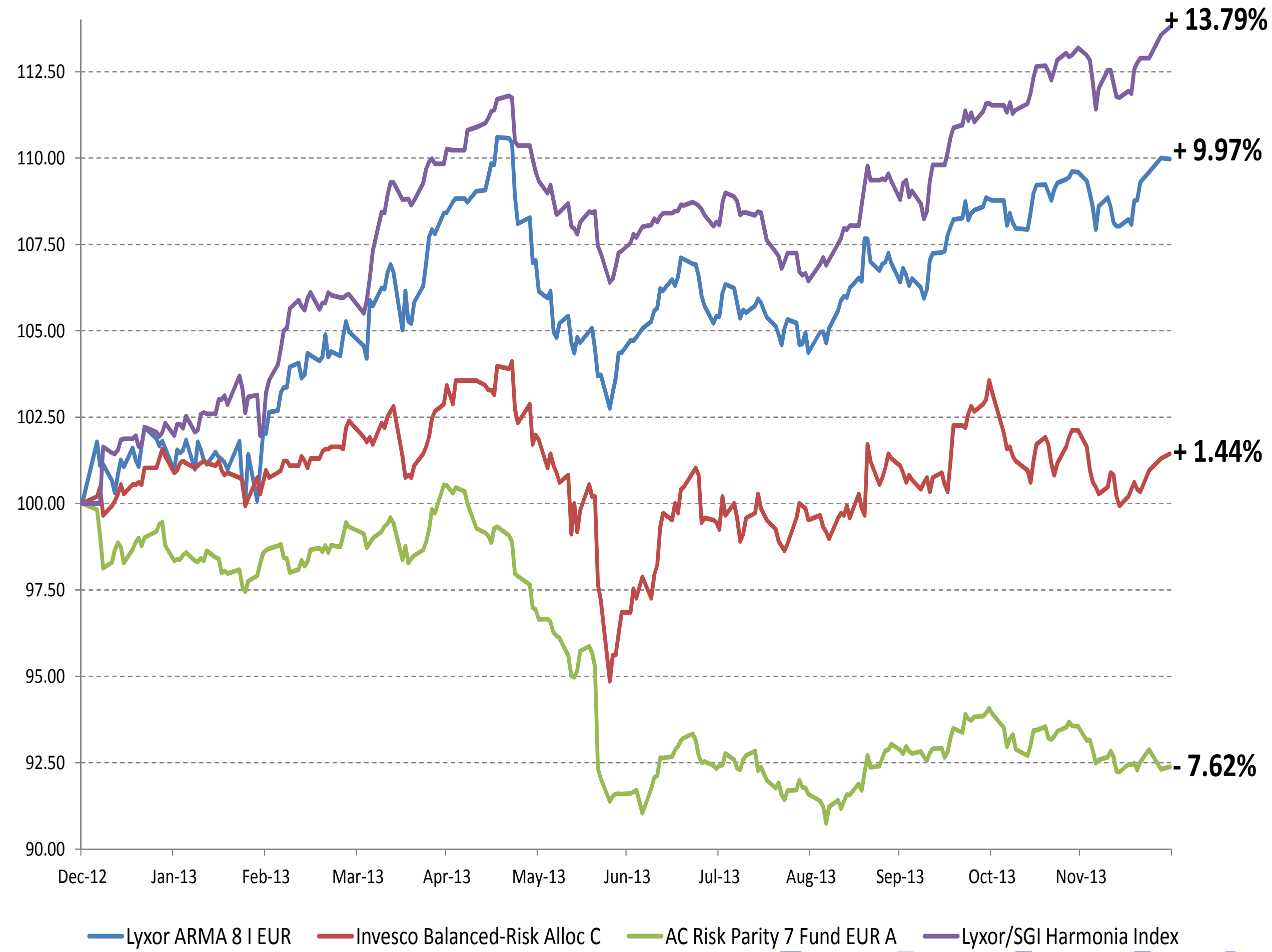
⇒ A significant component of the good performance of (static) risk parity funds.

- The right benchmark is certainly not the 60/40 asset mix policy.
- What will be the performance of risk parity funds if the interest rates rise?
  - Static risk parity vs active risk parity
  - 1994 scenario: fed fund = +300 bps / long rates = +250 bps
    ⇒ static: ☹, active: ☺
  - 1999 scenario: fed fund = +125 bps / long rates = +200 bps
    ⇒ static: ☺, active: ☺

One concept, several implementations, different performances!

- Choice of the investment universe
- Choice of the risk budgets
- Choice of the TAA model
- Choice of the leverage implementation
- Choice of the rebalancing frequency
- etc.

Figure: Performance of RP funds in 2013



Conclusion

- Risk parity based on the volatility risk measure = not the right answer to build absolute return fund.
- We propose a solution to incorporate discretionary views and trends into risk parity portfolios:
  - Expected returns = directional risks, and not performance opportunities.
  - It can be viewed as an active allocation strategy, but it remains a risk parity strategy.
- But it is not a magic allocation method:

  "It cannot free investors of their duty of making their own choices".

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