Beyond Risk Parity: Using Non-Gaussian Risk Measures and Risk Factors[1]

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Outline

The framework

Risk allocation

- How to allocate risk in a fair and effective way ?
- Litterman (1996), Denault (2001).

- It requires coherent and convex risk measures $\mathscr{R}(x)$ (Artzner *et al.*, 1999; Föllmer and Schied, 2002).

Subadditivity Homogeneity	Convexity
Monotonicity Translation invariance	

- It must satisfy some properties (Kalkbrener, 2005; Tasche, 2008).

  - Full allocation
  - RAPM compatible
  - Diversification compatible

Risk allocation with respect to P&L

Let $\Pi = \sum_{i=1}^{n} \Pi_i$ be the P&L of the portfolio. The risk-adjusted performance measure (RAPM) is defined by:

$$\mathrm{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathscr{R}(\Pi)} \text{ and } \mathrm{RAPM}(\Pi_i \mid \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathscr{R}(\Pi_i \mid \Pi)}$$

From an economic point of view, $\mathscr{R}(\Pi_i \mid \Pi)$ must satisfy two properties:

1. Risk contributions $\mathscr{R}(\Pi_i \mid \Pi)$ satisfy the full allocation property if:

$$\sum_{i=1}^{n} \mathscr{R}(\Pi_i \mid \Pi) = \mathscr{R}(\Pi)$$

2. They are RAPM compatible if there are some $\varepsilon_i > 0$ such that:

$$\mathrm{RAPM}(\Pi_i \mid \Pi) > \mathrm{RAPM}(\Pi) \Rightarrow \mathrm{RAPM}(\Pi + h\Pi_i) > \mathrm{RAPM}(\Pi)$$

   for all $0 < h < \varepsilon_i$.

In this case, Tasche (2008) shows that:

$$\mathscr{R}(\Pi_i \mid \Pi) = \left. \frac{\mathrm{d}}{\mathrm{d}h} \mathscr{R}(\Pi + h\Pi_i) \right|_{h=0}$$

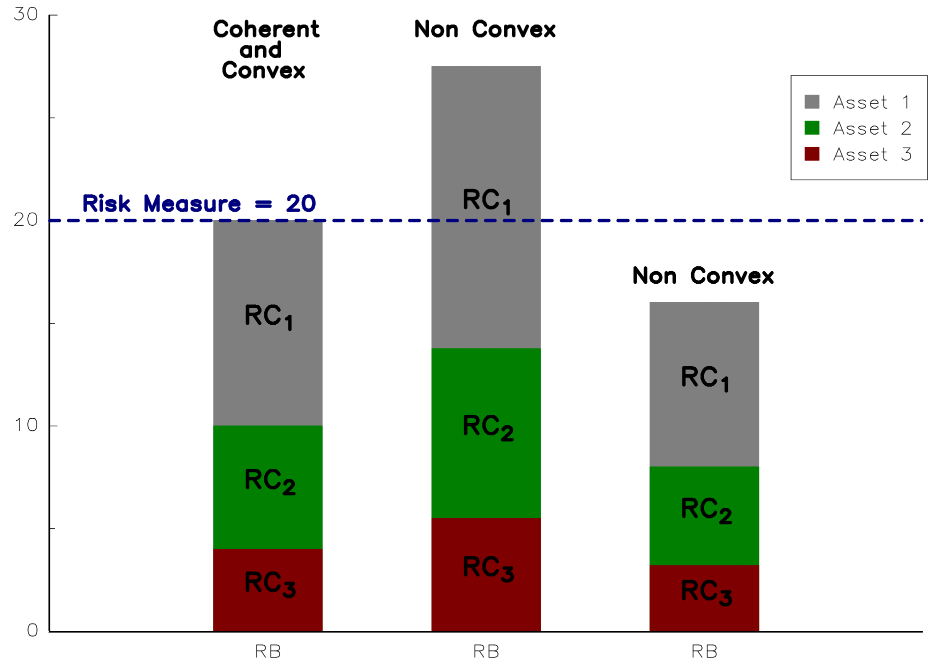
Risk allocation with respect to portfolio weights

With the previous framework, we obtain:

$$\mathrm{RC}_i = x_i \frac{\partial \mathscr{R}(x)}{\partial x_i}$$

and the risk measure satisfies the Euler decomposition:

$$\mathscr{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \mathscr{R}(x)}{\partial x_i} = \sum_{i=1}^{n} \mathrm{RC}_i$$



Some examples

Let $L(x)$ be the loss of the portfolio x.

- The volatility of the loss:

$$\sigma(L(x)) = \sigma(x)$$

- The standard deviation based risk measure:

$$\mathrm{SD}_c(x) = -\mu(x) + c \cdot \sigma(x)$$

- The value-at-risk:

$$\mathrm{VaR}_\alpha(x) = \inf\{\ell : \Pr\{L \leq \ell\} \geq \alpha\} = \mathbf{F}^{-1}(\alpha)$$

- The expected shortfall:

$$\begin{aligned}\mathrm{ES}_\alpha(x) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(x)\,\mathrm{d}u \\ &= \mathbb{E}\left[L(x) \mid L(x) \geq \mathrm{VaR}_\alpha(x)\right]\end{aligned}$$

Gaussian case

Volatility, value-at-risk and expected shortfall are equivalent.

Non-normal risk measures

For the value-at-risk, Gourieroux *et al.* (2000) shows that:

$$\mathrm{RC}_i = \mathbb{E}\left[L_i \mid L = \mathrm{VaR}_\alpha (L) \right]$$

whereas we have for the expected shortfall (Tasche, 2002):

$$\mathrm{RC}_i = \mathbb{E}\left[L_i \mid L \geq \mathrm{VaR}_\alpha (L) \right]$$

Example

EW portfolio with 2 assets (Clayton copula + student's t margins)[a]

	Vol	VAR	ES
$\mathscr{R}(x)$	24.51	18.32	35.99
$\mathrm{RC}_1(x)$	36.5%	34.2%	35.2%
$\mathrm{RC}_2(x)$	63.5%	65.8%	64.8%

[a] see Roncalli (2012).

Non-normal risk contributions

1. Value-at-risk with elliptical distributions (Carroll et al., 2001):

$$\mathrm{RC}_i = \mathbb{E}[L_i] + \frac{\mathrm{cov}(L, L_i)}{\sigma^2(L)} \left(\mathrm{VaR}_\alpha(L) - \mathbb{E}[L]\right)$$

2. Historical value-at-risk with non-elliptical distributions:

$$\mathrm{RC}_i = \mathrm{VaR}_\alpha(L) \frac{\sum_{j=1}^m \mathscr{K}\left(L^{(j)} - \mathrm{VaR}_\alpha(L)\right) L_i^{(j)}}{\sum_{j=1}^m \mathscr{K}\left(L^{(j)} - \mathrm{VaR}_\alpha(L)\right) L^{(j)}}$$

where $\mathscr{K}(u)$ is a kernel function (Epperlein and Smillie, 2006).

3. Value-at-risk with Cornish-Fisher expansion (Zangari, 1996):

$$\mathrm{VaR}_\alpha(L) = -x^\top \mu + z \cdot \sqrt{x^\top \Sigma x}$$

where:

$$z = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2 - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)\gamma_1^2$$

with $z_\alpha = \Phi^{-1}(\alpha)$, γ_1 is the skewness and γ_2 is the excess kurtosis[2].

[2] See Roncalli (2012) for the detailed formula of the risk contribution.

Properties of RB portfolios

Let us consider the **long-only** RB portfolio defined by:

$$\mathrm{RC}_i = b_i \mathscr{R}(x)$$

where b_i is the risk budget assigned to the i^{th} asset.

Bruder and Roncalli (2012) shows that:

- The RB portfolio exists if $b_i \geq 0$;
- The RB portfolio is unique if $b_i > 0$;
- The risk measure of the RB portfolio is located between those of the minimum risk portfolio and the weight budgeting portfolio:

$$\mathscr{R}(x_{\mathrm{mr}}) \leq \mathscr{R}(x_{\mathrm{rb}}) \leq \mathscr{R}(x_{\mathrm{wb}})$$

- If the RB portfolio is optimal[3], the performance contributions are equal to the risk contributions.

[3] In the sense of mean-risk quadratic utility function.

An example of RB portfolio

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

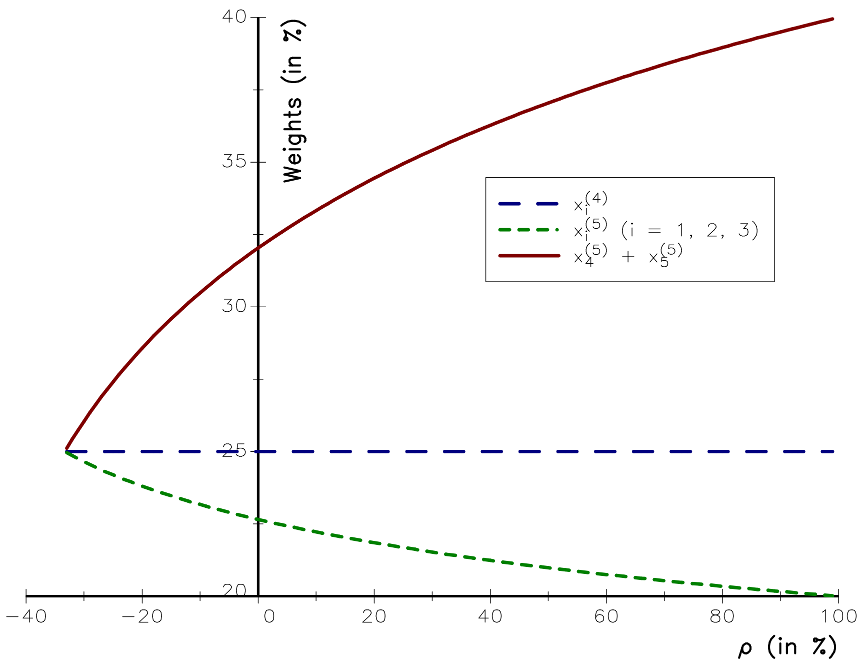
Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

On the importance of the asset universe

Example with 4 assets

- We assume equal volatilities and a uniform correlation ρ.
- The ERC portfolio is the EW portfolio: $x_1^{(4)} = x_2^{(4)} = x_3^{(4)} = x_4^{(4)} = 25\%$.
- We add a fifth asset which is perfectly correlated to the fourth asset.
- If $\rho = 0$, the ERC portfolio becomes $x_1^{(5)} = x_2^{(5)} = x_3^{(5)} = 22.65\%$ and $x_4^{(5)} = x_5^{(5)} = 16.02\%$.
- We would like that the allocation is $x_1^{(5)} = x_2^{(5)} = x_3^{(5)} = 25\%$ and $x_4^{(5)} = x_5^{(5)} = 12.5\%$.

Figure: 4 assets versus 5 assets



Which risk would you like to diversify?

- m primary assets $(\mathscr{A}'_1, \ldots, \mathscr{A}'_m)$ with a covariance matrix Ω.
- n synthetic assets $(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ which are composed of the primary assets.
- $W = (w_{i,j})$ is the weight matrix such that $w_{i,j}$ is the weight of the primary asset $\mathscr{A}'_j$ in the synthetic asset $\mathscr{A}_i$.

Example

- 6 primary assets and 3 synthetic assets.
- The volatilities of these assets are respectively 20%, 30%, 25%, 15%, 10% and 30%. We assume that the assets are not correlated.
- We consider three equally-weighted synthetic assets with:

$$W = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & & \\ & & 1/4 & 1/4 & 1/4 & 1/4 \\ & & 1/2 & 1/2 & & \end{pmatrix}$$

Which risk would you like to diversify?

Risk decomposition of portfolio #1

Along synthetic assets $\mathscr{A}_1, \ldots, \mathscr{A}_n$

$\sigma(x) = 10.19\%$	x_i	$\mathrm{MR}(\mathscr{A}_i)$	$\mathrm{RC}(\mathscr{A}_i)$	$\mathrm{RC}^\star(\mathscr{A}_i)$
$\mathscr{A}_1$	36.00%	9.44%	3.40%	33.33%
$\mathscr{A}_2$	38.00%	8.90%	3.38%	33.17%
$\mathscr{A}_3$	26.00%	13.13%	3.41%	33.50%

Along primary assets $\mathscr{A}'_1, \ldots, \mathscr{A}'_m$

$\sigma(y) = 10.19\%$	y_i	$\mathrm{MR}(\mathscr{A}'_i)$	$\mathrm{RC}(\mathscr{A}'_i)$	$\mathrm{RC}^\star(\mathscr{A}'_i)$
$\mathscr{A}'_1$	9.00%	3.53%	0.32%	3.12%
$\mathscr{A}'_2$	9.00%	7.95%	0.72%	7.02%
$\mathscr{A}'_3$	31.50%	19.31%	6.08%	59.69%
$\mathscr{A}'_4$	31.50%	6.95%	2.19%	21.49%
$\mathscr{A}'_5$	9.50%	0.93%	0.09%	0.87%
$\mathscr{A}'_6$	9.50%	8.39%	0.80%	7.82%

$\Rightarrow$ The portfolio seems well diversified on synthetic assets, but 80% of the risk is on assets 3 and 4.

Which risk would you like to diversify?

Risk decomposition of portfolio #2

Along synthetic assets $\mathscr{A}_1, \ldots, \mathscr{A}_n$

$\sigma(x) = 9.47\%$	x_i	$\mathrm{MR}(\mathscr{A}_i)$	$\mathrm{RC}(\mathscr{A}_i)$	$\mathrm{RC}^\star(\mathscr{A}_i)$
$\mathscr{A}_1$	48.00%	9.84%	4.73%	49.91%
$\mathscr{A}_2$	50.00%	9.03%	4.51%	47.67%
$\mathscr{A}_3$	2.00%	11.45%	0.23%	2.42%

Along primary assets $\mathscr{A}'_1, \ldots, \mathscr{A}'_m$

$\sigma(y) = 9.47\%$	y_i	$\mathrm{MR}(\mathscr{A}'_i)$	$\mathrm{RC}(\mathscr{A}'_i)$	$\mathrm{RC}^\star(\mathscr{A}'_i)$
$\mathscr{A}'_1$	12.00%	5.07%	0.61%	6.43%
$\mathscr{A}'_2$	12.00%	11.41%	1.37%	14.46%
$\mathscr{A}'_3$	25.50%	16.84%	4.29%	45.35%
$\mathscr{A}'_4$	25.50%	6.06%	1.55%	16.33%
$\mathscr{A}'_5$	12.50%	1.32%	0.17%	1.74%
$\mathscr{A}'_6$	12.50%	11.88%	1.49%	15.69%

$\Rightarrow$ This portfolio is more diversified than the previous portfolio if we consider primary assets.

The factor model

- n assets $\{\mathscr{A}_1, \ldots, \mathscr{A}_n\}$ and m risk factors $\{\mathscr{F}_1, \ldots, \mathscr{F}_m\}$.
- R_t is the $(n \times 1)$ vector of asset returns at time t and Σ its associated covariance matrix.
- $\mathscr{F}_t$ is the $(m \times 1)$ vector of factor returns at t and Ω its associated covariance matrix.
- We assume the following linear factor model:

$$R_t = A\mathscr{F}_t + \varepsilon_t$$

with $\mathscr{F}_t$ and ε_t two uncorrelated random vectors. The covariance matrix of ε_t is noted D. We have:

$$\Sigma = A\Omega A^\top + D$$

- The P&L of the portfolio x is:

$$\Pi_t = x^\top R_t = x^\top A\mathscr{F}_t + x^\top \varepsilon_t = y^\top \mathscr{F}_t + \eta_t$$

with $y = A^\top x$ and $\eta_t = x^\top \varepsilon_t$.

First route to decompose the risk

Let $B = A^\top$ and B^+ the Moore-Penrose inverse of B. We have therefore:

$$x = B^+ y + e$$

where $e = (I_n - B^+ B)x$ is a $(n \times 1)$ vector in the kernel of B.

We consider a convex risk measure $\mathscr{R}(x)$. We have:

$$\frac{\partial\,\mathscr{R}(x)}{\partial\,x_i} = \left(\frac{\partial\,\mathscr{R}(y,e)}{\partial\,y} B\right)_i + \left(\frac{\partial\,\mathscr{R}(y,e)}{\partial\,e}(I_n - B^+ B)\right)_i$$

Decomposition of the risk by m common factors and n idiosyncratic factors $\Rightarrow$ Identification problem!

Second route to decompose the risk

Meucci (2007) considers the following decomposition:

$$x = \begin{pmatrix} B^{+} & \tilde{B}^{+} \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \bar{B}^{\top} \bar{y}$$

where $\tilde{B}^{+}$ is any $n \times (n-m)$ matrix that spans the left nullspace of B^{+}.

Decomposition of the risk by m common factors and $n-m$ residual factors
$\Rightarrow$ Better identified problem.

Euler decomposition of the risk measure

Theorem

The risk contributions of common and residual risk factors are:

$$\begin{aligned}
\mathrm{RC}(\mathscr{F}_j) &= \left(A^\top x\right)_j \cdot \left(A^+ \frac{\partial\,\mathscr{R}(x)}{\partial\,x}\right)_j \\
\mathrm{RC}\left(\tilde{\mathscr{F}}_j\right) &= \left(\tilde{B}x\right)_j \cdot \left(\tilde{B}\frac{\partial\,\mathscr{R}(x)}{\partial\,x}\right)_j
\end{aligned}$$

They satisfy the Euler allocation principle:

$$\sum_{j=1}^{m} \mathrm{RC}(\mathscr{F}_j) + \sum_{j=1}^{n-m} \mathrm{RC}\left(\tilde{\mathscr{F}}_j\right) = \mathscr{R}(x)$$

$\Rightarrow$ Risk contribution with respect to risk factors (resp. to assets) are related to marginal risk of assets (resp. of risk factors).
$\Rightarrow$ The main important quantity is marginal risk, not risk contribution!

An example

We consider 4 assets and 3 factors. The loadings matrix is:

$$A = \begin{pmatrix} 0.9 & 0 & 0.5 \\ 1.1 & 0.5 & 0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are equal to 20%, 10% and 10%. We consider a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Along assets $\mathscr{A}_1, \ldots, \mathscr{A}_n$

	x_i	$\mathrm{MR}(\mathscr{A}_i)$	$\mathrm{RC}(\mathscr{A}_i)$	$\mathrm{RC}^\star(\mathscr{A}_i)$
$\mathscr{A}_1$	25.00%	18.81%	4.70%	21.97%
$\mathscr{A}_2$	25.00%	23.72%	5.93%	27.71%
$\mathscr{A}_3$	25.00%	24.24%	6.06%	28.32%
$\mathscr{A}_4$	25.00%	18.83%	4.71%	22.00%
$\sigma(x)$			21.40%	

Along factors $\mathscr{F}_1, \ldots, \mathscr{F}_m$ and $\tilde{\mathscr{F}}_1, \ldots, \tilde{\mathscr{F}}_{n-m}$

	y_i	$\mathrm{MR}(\mathscr{F}_i)$	$\mathrm{RC}(\mathscr{F}_i)$	$\mathrm{RC}^\star(\mathscr{F}_i)$
$\mathscr{F}_1$	100.00%	17.22%	17.22%	80.49%
$\mathscr{F}_2$	22.50%	9.07%	2.04%	9.53%
$\mathscr{F}_3$	35.00%	6.06%	2.12%	9.91%
$\tilde{\mathscr{F}}_1$	2.75%	0.52%	0.01%	0.07%
$\sigma(y)$			21.40%	

Beta contribution versus risk contribution

The linear model is:

$$\begin{pmatrix} R_{1,t} \\ R_{2,t} \\ R_{3,t} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \\ 0.3 & 0.5 \\ 0.8 & -0.2 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{1,t} \\ \mathcal{F}_{2,t} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

The factor volatilities are equal to 10% and 30%, while the idiosyncratic volatilities are equal to 3%, 5% and 2%.

If we consider the volatility risk measure, we obtain:

Portfolio	$(1/3, 1/3, 1/3)$		$(7/10, 7/10, -4/10)$	
Factor	β	$\mathrm{RC}^\star$	β	$\mathrm{RC}^\star$
$\mathcal{F}_1$	0.67	31%	0.52	3%
$\mathcal{F}_2$	0.33	69%	0.92	97%

The first portfolio has a bigger beta in factor 1 than in factor 2, but about 70% of its risk is explained by the second factor. For the second portfolio, the risk w.r.t the first factor is very small even if its beta is significant.

Matching the risk budgets

We consider the risk budgeting problem: $\mathrm{RC}(\mathcal{F}_j) = b_j \mathcal{R}(x)$. It can be formulated as a quadratic problem as in Bruder and Roncalli (2012):

$$
\begin{aligned}
(y^\star, \hat{y}^\star) &= \arg\min \sum_{j=1}^{m} \left(\mathrm{RC}(\mathcal{F}_j) - b_j \mathcal{R}(y, \tilde{y})\right)^2 \\
\text{u.c.} & \quad \begin{cases} \mathbf{1}^\top x = 1 \\ \mathbf{0} \preceq x \preceq \mathbf{1} \end{cases}
\end{aligned}
$$

This problem is tricky because the first order conditions are PDE!

Some special cases

- Positive factor weights ($y \geq 0$) with $m = n \Rightarrow$ a unique solution.
- Positive factor weights ($y \geq 0$) with $m < n \Rightarrow$ at least one solution.
- Positive asset weights ($x \geq 0$ or long-only portfolio) $\Rightarrow$ zero, one or more solutions.

The separation principle

The problem is unconstrained with respect to the residual factors $\tilde{\mathscr{F}}_t \Rightarrow$ we can solve the problem in two steps:

1. The first problem is $\tilde{\mathscr{R}}(y) = \inf_{\tilde{y}} \mathscr{R}(y, \tilde{y})$ and we obtain $\tilde{y} = \varphi(y)$;
2. The second problem is $y^\star = \arg\min \tilde{\mathscr{R}}(y)$.

The solution is then given by:

$$x^\star = B^+ y^\star + \tilde{B}^+ \varphi(y^\star)$$

The separation principle

Application to the volatility risk measure

We have:

$$\bar{\Omega} = \operatorname{cov}\left(\mathscr{F}_t, \tilde{\mathscr{F}}_t\right) = \left(\begin{array}{cc} \Omega & \Gamma^\top \\ \Gamma & \tilde{\Omega} \end{array}\right)$$

The expression of the risk measure becomes:

$$\mathscr{R}(y, \tilde{y}) = \bar{y}^\top \bar{\Omega} \bar{y} = y^\top \Omega y + \tilde{y}^\top \tilde{\Omega} \tilde{y} + 2 \tilde{y}^\top \Gamma^\top y$$

We obtain $\tilde{y} = \varphi(y) = -\tilde{\Omega}^{-1} \Gamma^\top y$ and the problem is thus reduced to $y^\star = \arg\min y^\top S y$ with $S = \Omega - \Gamma \tilde{\Omega}^{-1} \Gamma^\top$ the Schur complement of $\tilde{\Omega}$. Because we have $\Gamma^\top = (B^+)^\top \Sigma \tilde{B}^+$, we obtain:

$$x^\star = B^+ y^\star + \tilde{B}^+ \varphi(y^\star) = \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} \left(B^+\right)^\top \Sigma \tilde{B}^+\right) y^\star$$

Remark

If $\mathscr{F}_t$ and $\tilde{\mathscr{F}}_t$ are uncorrelated (e.g. PCA factors), a solution of the form $(y^\star, 0)$ exists and the (un-normalized) solution is given by $x^\star = B^+ y^\star$.

The separation principle

Adding long-only constraints

If we want to consider long-only allocations x, we must also include the following constraint:

$$x = B^{+}y + \tilde{B}^{+}\tilde{y} \succeq \mathbf{0}$$

- The solution may not exist even if φ is convex.
- The existence of the solution implies that there exists $\lambda = (\lambda_x, \lambda_y) \succeq \mathbf{0}$ such that:

$$\left(A^{+} - \left(\tilde{B}^{+}\right)^{\top} \Sigma \tilde{\Omega}^{-1} \left(\tilde{B}^{+}\right)^{\top}\right) \lambda_x + \lambda_y = 0$$

  We may show that this condition is likely to be verified for some non trivial $\lambda \in \mathbb{R}_{+}^{n+m}$. In such case, there exists $\zeta > 0$ such that $0 \leq \min y_j \leq \zeta$.

$\Rightarrow$ interpretation of this result with the convexity factor of the yield curve.

Matching the risk budgets

An example (Slide 18)

If $b = (49\%, 25\%, 25\%)$, $x^\star = (15.1\%, 39.4\%, 0.9\%, 45.6\%)$. $\Rightarrow$ It is a long-only portfolio.

Matching the risk budgets $b = (19\%, 40\%, 40\%)$

Optimal solution $(y^\star, \tilde{y}^\star)$

	y_i	$\mathrm{RC}(\mathcal{F}_i)$	$\mathrm{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	92.90%	4.45%	19.00%
$\mathcal{F}_2$	28.55%	9.36%	40.00%
$\mathcal{F}_3$	45.21%	9.36%	40.00%
$\tilde{\mathcal{F}}_1$	−23.57%	0.23%	1.00%
$\sigma(y)$		23.41%	

Corresponding portfolio $x^\star$

	x_i	RC_i	$\mathrm{RC}_i^\star$
$\mathcal{A}_1$	−26.19%	−3.70%	−15.81%
$\mathcal{A}_2$	32.69%	6.94%	29.63%
$\mathcal{A}_3$	14.28%	2.91%	12.45%
$\mathcal{A}_4$	79.22%	17.26%	73.73%
$\sigma(x)$		23.41%	

Imposing the long-only constraint with $b = (19\%, 40\%, 40\%)$

Optimal solution $(y^\star, \tilde{y}^\star)$

	y_i	$\mathrm{RC}(\mathcal{F}_i)$	$\mathrm{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	89.85%	6.19%	28.37%
$\mathcal{F}_2$	23.13%	6.63%	30.40%
$\mathcal{F}_3$	47.02%	8.99%	41.20%
$\tilde{\mathcal{F}}_1$	2.53%	0.01%	0.03%
$\sigma(y)$		21.82%	

Corresponding portfolio $x^\star$

	x_i	RC_i	$\mathrm{RC}_i^\star$
$\mathcal{A}_1$	0.00%	0.00%	0.00%
$\mathcal{A}_2$	32.83%	7.23%	33.15%
$\mathcal{A}_3$	0.00%	0.00%	0.00%
$\mathcal{A}_4$	67.17%	14.59%	66.85%
$\sigma(x)$		21.82%	

Managing the risk concentration

Concentration measures

Concentration index

Let $p \in \mathbf{R}_+^n$ such that $\mathbf{1}^\top p = 1$. A concentration index is a mapping function $\mathscr{C}(p)$ such that $\mathscr{C}(p)$ increases with concentration and verifies $\mathscr{C}(p^-) \leq \mathscr{C}(p) \leq \mathscr{C}(p^+)$ with $p^+ = \left\{\exists i_0 : p_{i_0}^+ = 1, p_i^+ = 0 \text{ if } i \neq i_0\right\}$ and $p^- = \left\{\forall i : p_i^- = 1/n\right\}$.

- The Herfindahl index

$$\mathscr{H}(p) = \sum_{i=1}^n p_i^2$$

- The Gini index $\mathscr{G}(p)$ measures the distance between the Lorenz curve of p and the Lorenz curve of p^-.
- The Shannon entropy is defined as follows[4]:

$$\mathscr{I}(p) = -\sum_{i=1}^n p_i \ln p_i$$

[4] Note that the concentration index is the opposite of the Shannon entropy.

Managing the risk concentration

Risk parity optimization

We would like to build a portfolio such that

$$\mathrm{RC}(\mathscr{F}_j) \simeq \mathrm{RC}(\mathscr{F}_k)$$

for $(j,k) \in \mathscr{J}$.

The optimization problem becomes:

$$\begin{aligned} x^\star &= \arg\min \mathscr{C}(p) \\ &\text{u.c.} \quad \begin{cases} \mathbf{1}^\top x = 1 \\ x \succeq \mathbf{0} \end{cases} \end{aligned}$$

with $p = \{\mathrm{RC}(\mathscr{F}_j), j \in \mathscr{J}\}$.

Managing the risk concentration

An example (Slide 18)

The lowest risk concentrated portfolio

$(\mathscr{H} \equiv \mathscr{G} \equiv \mathscr{I})$

Optimal solution $(y^\star, \tilde{y}^\star)$

	y_i	$\mathrm{RC}(\mathscr{F}_i)$	$\mathrm{RC}^\star(\mathscr{F}_i)$
$\mathscr{F}_1$	91.97%	7.28%	33.26%
$\mathscr{F}_2$	25.78%	7.28%	33.26%
$\mathscr{F}_3$	42.22%	7.28%	33.26%
$\tilde{\mathscr{F}}_1$	6.74%	0.05%	0.21%
$\sigma(y)$		23.41%	

Corresponding portfolio $x^\star$

	x_i	RC_i	$\mathrm{RC}_i^\star$
$\mathscr{A}_1$	0.30%	0.05%	0.22%
$\mathscr{A}_2$	39.37%	9.11%	41.63%
$\mathscr{A}_3$	0.31%	0.07%	0.30%
$\mathscr{A}_4$	60.01%	12.66%	57.85%
$\sigma(x)$		21.88%	

With some constraints

$(\mathscr{H} \not\equiv \mathscr{G} \not\equiv \mathscr{I})$

Optimal portfolios with $x_i \geq 10\%$

Criterion	$\mathscr{H}$	$\mathscr{G}$	$\mathscr{I}$
x_1	10.00%	10.00%	10.00%
x_2	22.08%	18.24%	24.91%
x_3	10.00%	10.00%	10.00%
x_4	57.92%	61.76%	55.09%
$\mathscr{H}^\star$	0.0436	0.0490	0.0453
$\mathscr{G}$	0.1570	0.1476	0.1639
$\mathscr{I}^\star$	2.8636	2.8416	2.8643

Solving invariance problems of Choueifaty *et al.* (2011)

The duplication invariance property

- $\Sigma^{(n)}$ is the covariance matrix of the n assets.
- $x^{(n)}$ is the RB portfolio with risk budgets $b^{(n)}$.
- We suppose now that we duplicate the last asset:

$$\Sigma^{(n+1)} = \begin{pmatrix} \Sigma^{(n)} & \Sigma^{(n)}\mathbf{e}_n \\ \mathbf{e}_n^\top \Sigma^{(n)} & 1 \end{pmatrix}$$

- We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = \mathbf{0}$ and $A = \begin{pmatrix} I_n & \mathbf{e}_n \end{pmatrix}^\top$.
- We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$.
- We have $x_i^{(n+1)} = x_i^{(n)}$ if $i < n$ and $x_n^{(n+1)} + x_{n+1}^{(n+1)} = x_n^{(n)}$.

$\Rightarrow$ The ERC portfolio verifies the duplication invariance property if the risk budgets are expressed with respect to factors and not to assets.

Solving invariance problems of Choueifaty *et al.* (2011)

The polico invariance property

- We introduce an asset $n+1$ which is a linear (normalized) combination α of the first n assets:

$$\Sigma^{(n+1)} = \begin{pmatrix} \Sigma^{(n)} & \Sigma^{(n)}\alpha \\ \alpha^\top \Sigma^{(n)} & \alpha^\top \Sigma^{(n)}\alpha \end{pmatrix}$$

- We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = \mathbf{0}$ and $A = \begin{pmatrix} I_n & \alpha \end{pmatrix}^\top$.
- We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$.
- We have $x_i^{(n)} = x_i^{(n+1)} + \alpha_i x_{n+1}^{(n+1)}$ if $i \leq n$.

$\Rightarrow$ RB portfolios (and so ERC portfolios) verifies the polico invariance property if the risk budgets are expressed with respect to factors and not to assets.

The Fama-French model

Framework

Capital Asset Pricing Model

$$\mathbb{E}[R_i] = R_f + \beta_i (\mathbb{E}[R_{\mathrm{MKT}}] - R_f)$$

where R_{MKT} is the return of the market portfolio and:

$$\beta_i = \frac{\mathrm{cov}(R_i, R_{\mathrm{MKT}})}{\mathrm{var}(R_{\mathrm{MKT}})}$$

Fama-French-Carhart model

$$\mathbb{E}[R_i] = \beta_i^{\mathrm{MKT}} \mathbb{E}[R_{\mathrm{MKT}}] + \beta_i^{\mathrm{SMB}} \mathbb{E}[R_{\mathrm{SMB}}] + \beta_i^{\mathrm{HML}} \mathbb{E}[R_{\mathrm{HML}}] + \beta_i^{\mathrm{MOM}} \mathbb{E}[R_{\mathrm{MOM}}]$$

where R_{SMB} is the return of small stocks minus the return of large stocks, R_{HML} is the return of stocks with high book-to-market values minus the return of stocks with low book-to-market values and R_{MOM} is the Carhart momentum factor.

The Fama-French model

Regression analysis

Results(⋆) using weekly returns from 1995-2012

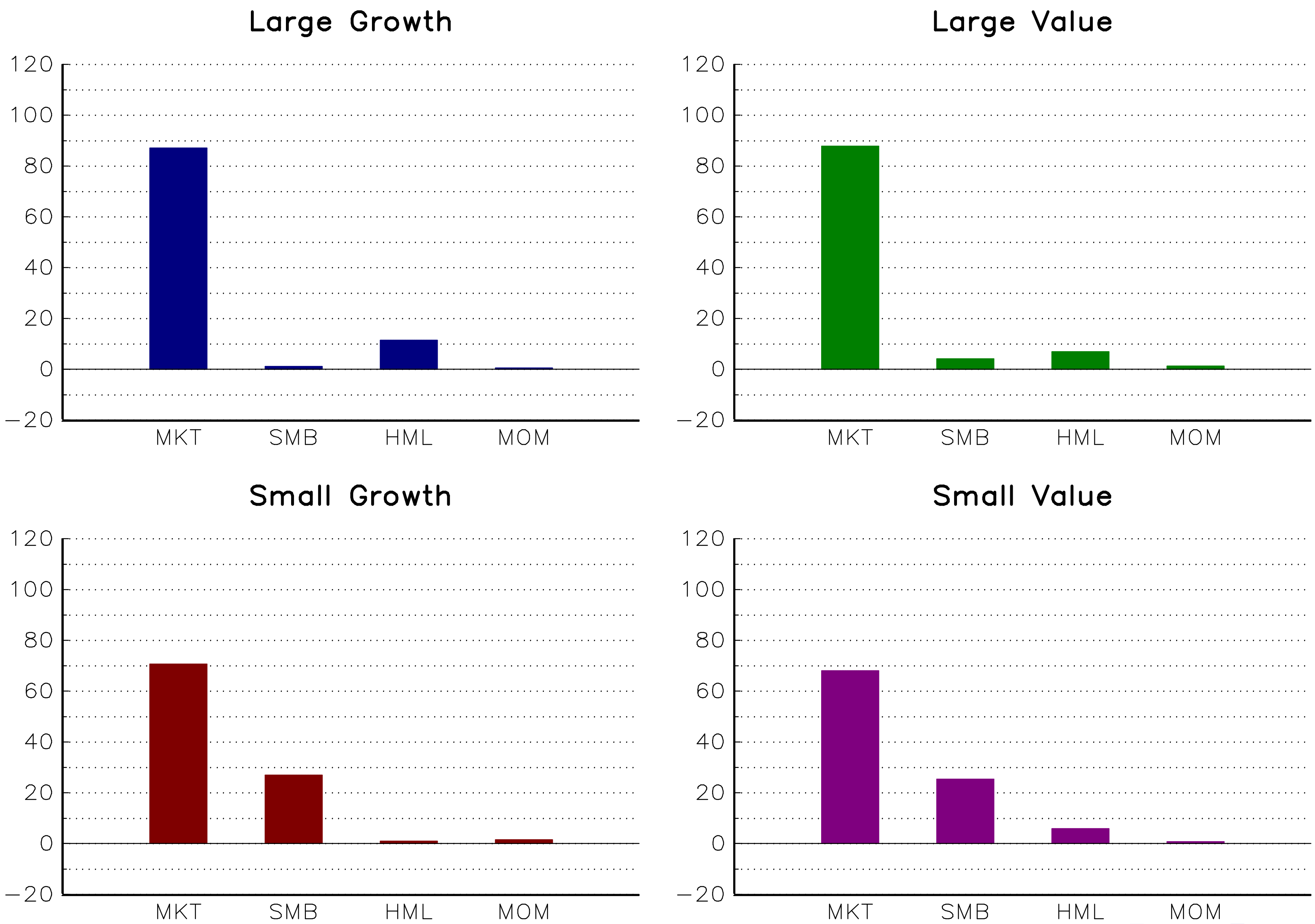
Index	β_i^{MKT}	β_i^{SMB}	β_i^{HML}	β_i^{MOM}
MSCI USA Large Growth	1.06	−0.12	−0.38	−0.07
MSCI USA Large Value	0.97	−0.21	0.27	−0.12
MSCI USA Small Growth	1.04	0.64	−0.12	0.15
MSCI USA Small Value	1.01	0.62	0.30	−0.10

(⋆)All the estimates are significant at the 95% confidence level.

Question: What is exactly the meaning of these figures?

The Fama-French model

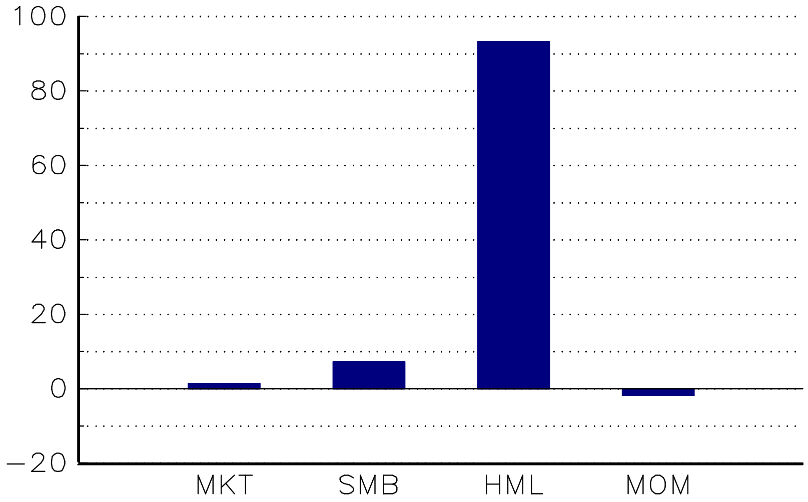
Risk contribution analysis



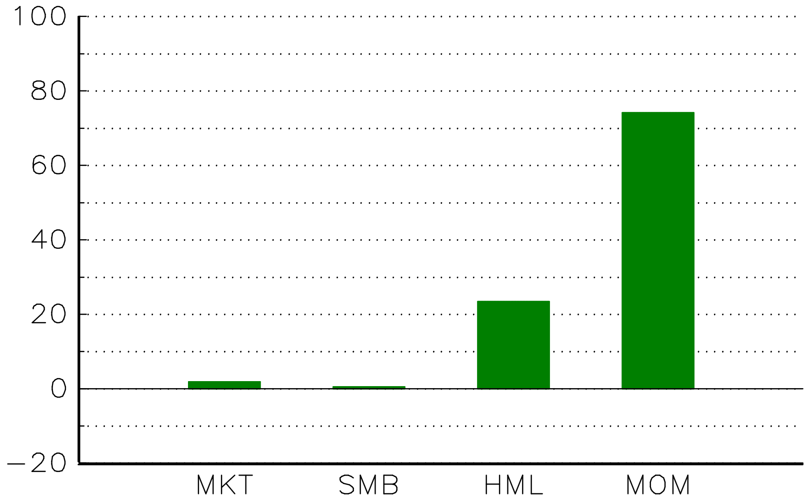
The Fama-French model

Risk analysis of long/short portfolios

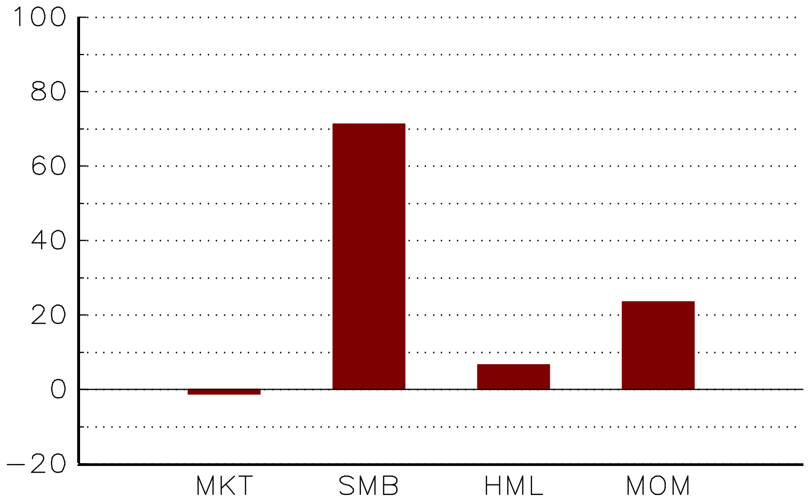
(100%, −100%, 0%, 0%)



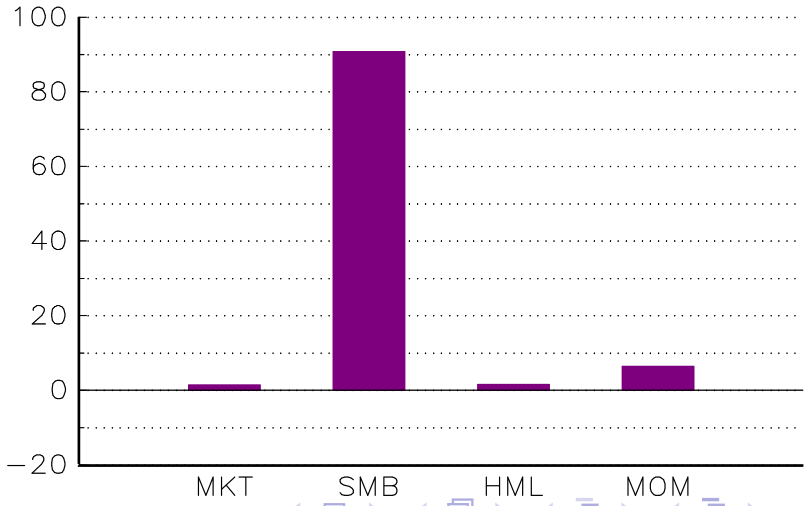
(0%, 0%, 100%, −100%)



(−100%, 0%, 100%, 0%)



(50%, 50%, −50%, −50%)



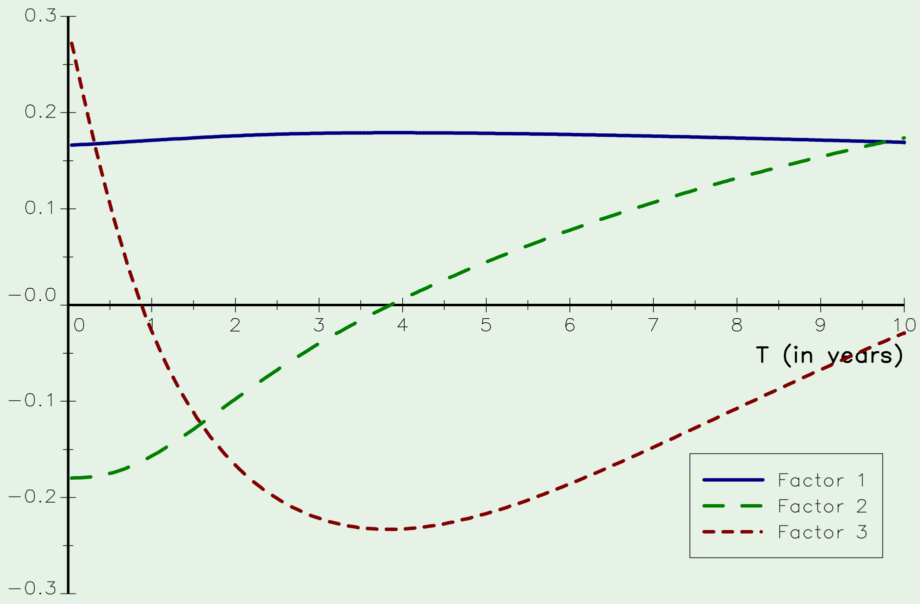
The risk factors of the yield curve

Principal component analysis

PCA factors

- Level
- Slope
- Convexity

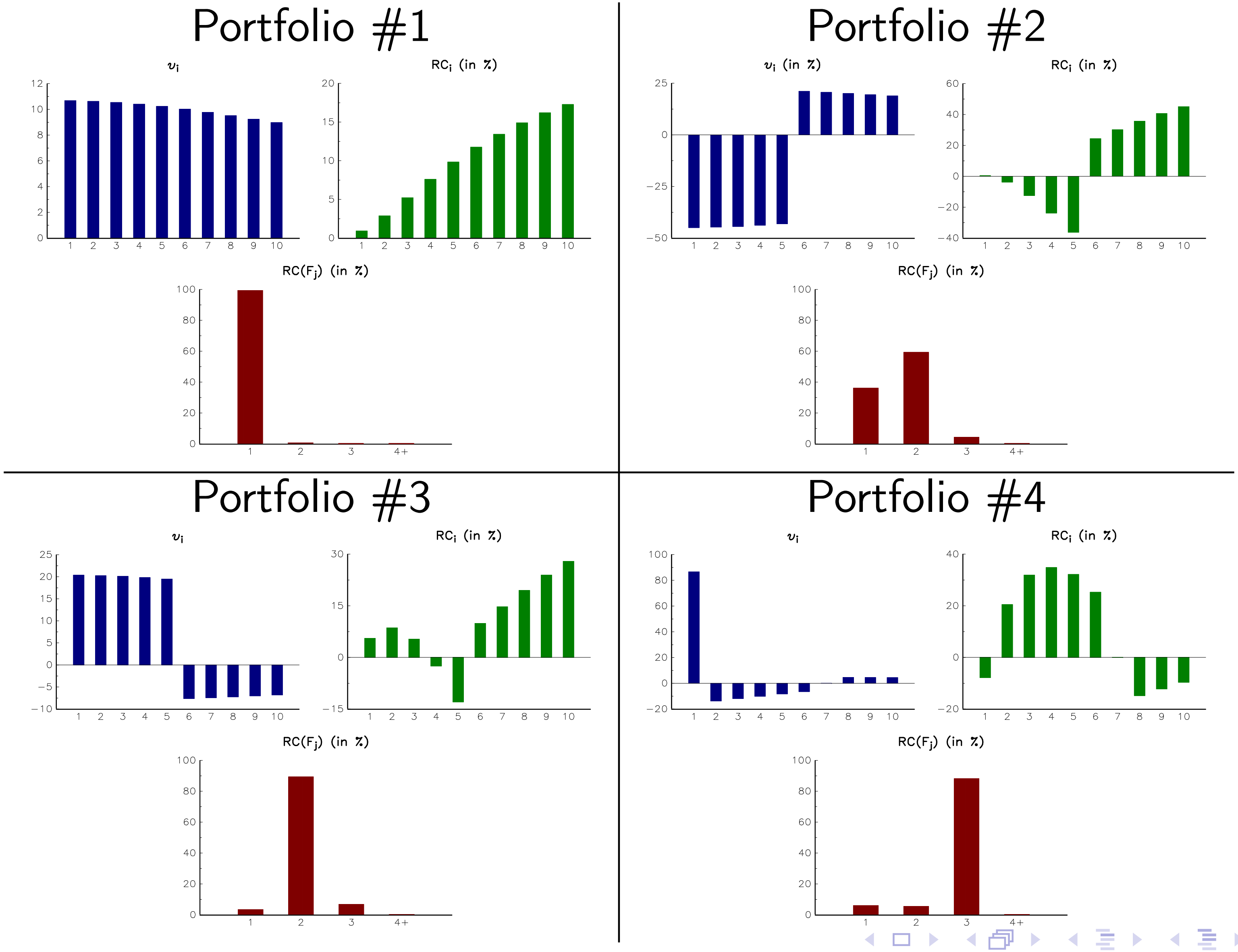
US yield curve (2003-2012)



Portfolio	Maturity (in years) 1	2	3	4	5	6	7	8	9	10
#1	1	1	1	1	1	1	1	1	1	1
#2	−2	−2	−2	−2	−2	1	1	1	1	1
#3	10	10	10	10	10	4	4	4	4	4
#4	53	−8	−7	−6	−5	−4	0	3	3	3

The risk factors of the yield curve

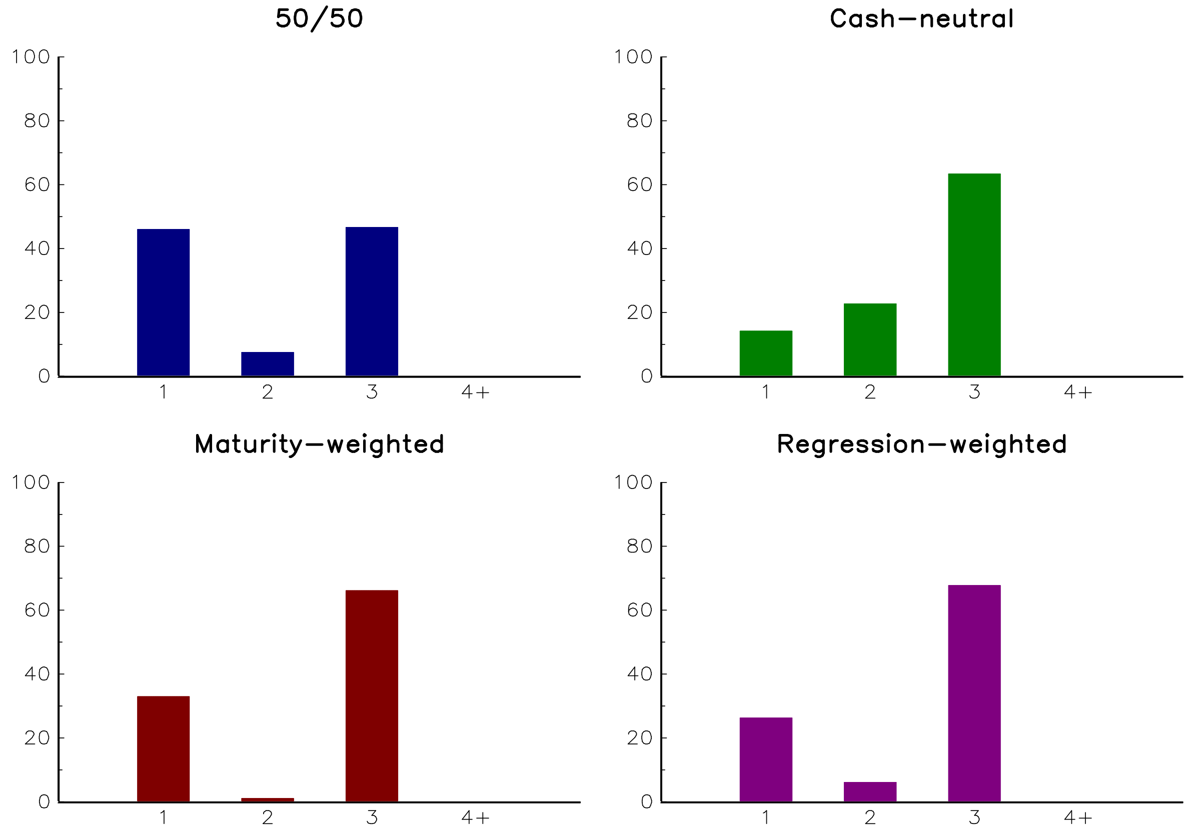
Risk decomposition of the four portfolios wrt zero-coupons and PCA risk factors



The PCA risk factors of the yield curve

Barbell portfolios (June 30, 2012 & US yield curve)

Maturity	50/50	Cash-neutral	Maturity-W.	Regression-W.
2Y	1.145	0.573	0.859	0.763
5Y	−1.000	−1.000	−1.000	−1.000
10Y	0.316	0.474	0.395	0.422



Diversifying a portfolio of hedge funds

The framework

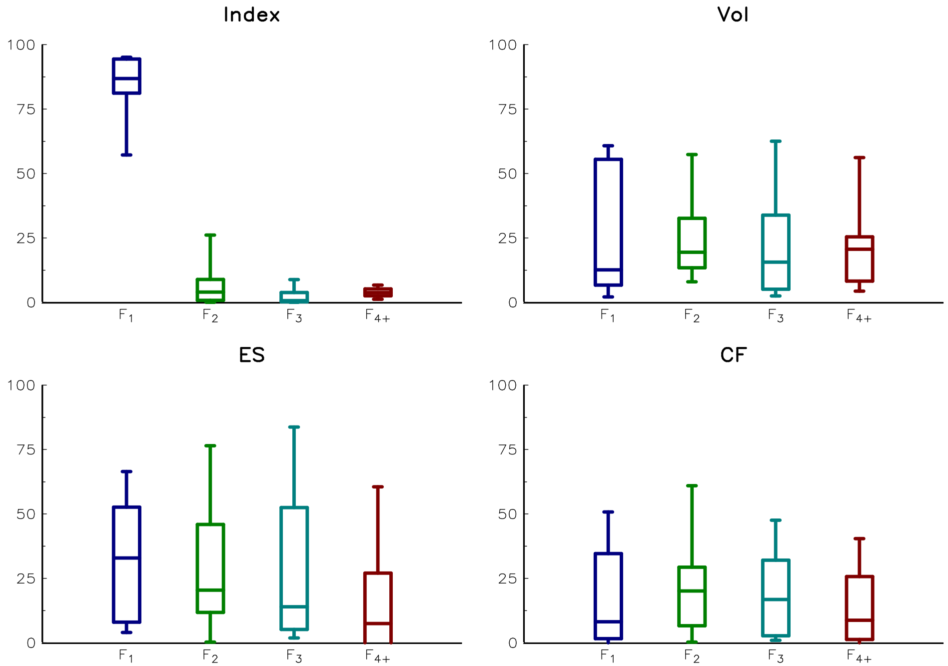
- We consider the Dow Jones Credit Suisse AllHedge index[5].
- We use three risk measures:
  1. Volatility;
  2. Expected shortfall with a 80% confidence level;
  3. Cornish-Fisher value-at-risk with a 99% confidence level.
- Factors are based on PCA (Fung and Hsieh, 1997).
- We consider two risk parity models.
  1. ERC portfolio.
  2. Risk factor parity (RFP) portfolio by minimizing the risk concentration between the first 4 PCA factors.

[5] This index is composed of 10 subindexes: (1) convertible arbitrage, (2) dedicated short bias, (3) emerging markets, (4) equity market neutral, (5) event driven, (6) fixed income arbitrage, (7) global macro, (8) long/short equity, (9) managed futures and (10) multi-strategy.

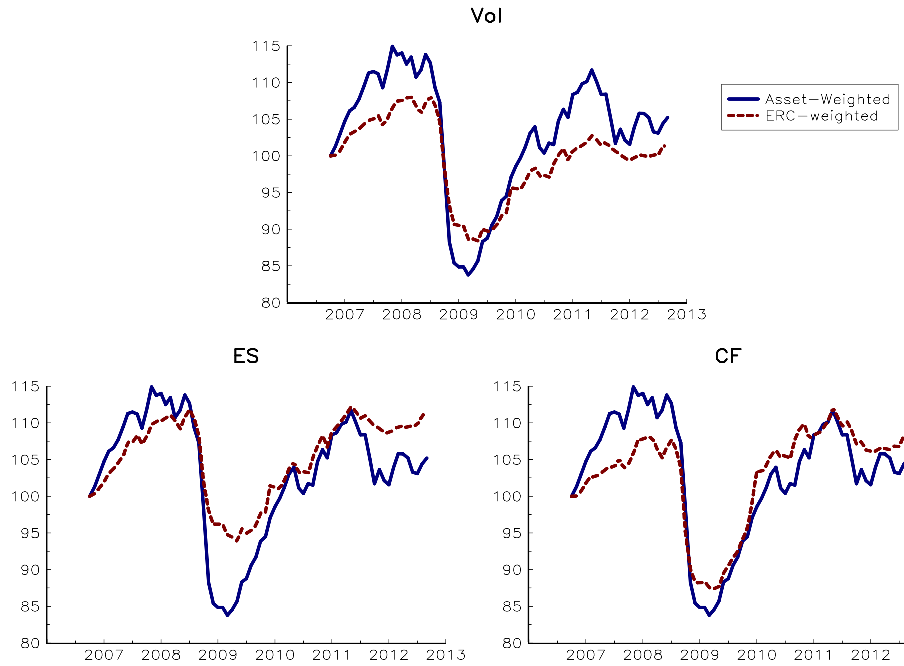
Diversifying a portfolio of hedge funds

The ERC approach

Risk decomposition in terms of factors



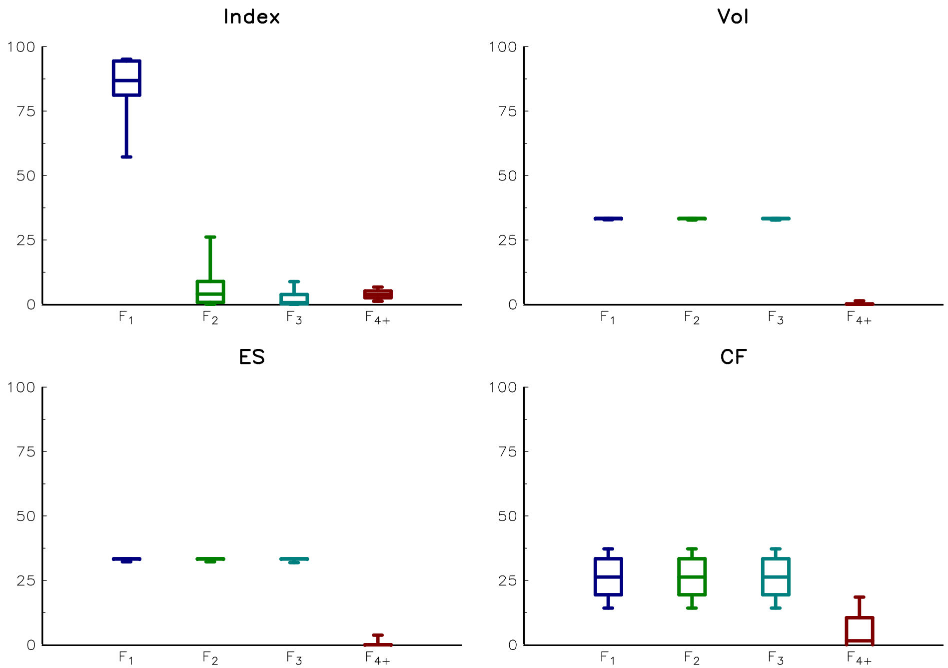
Simulated performance



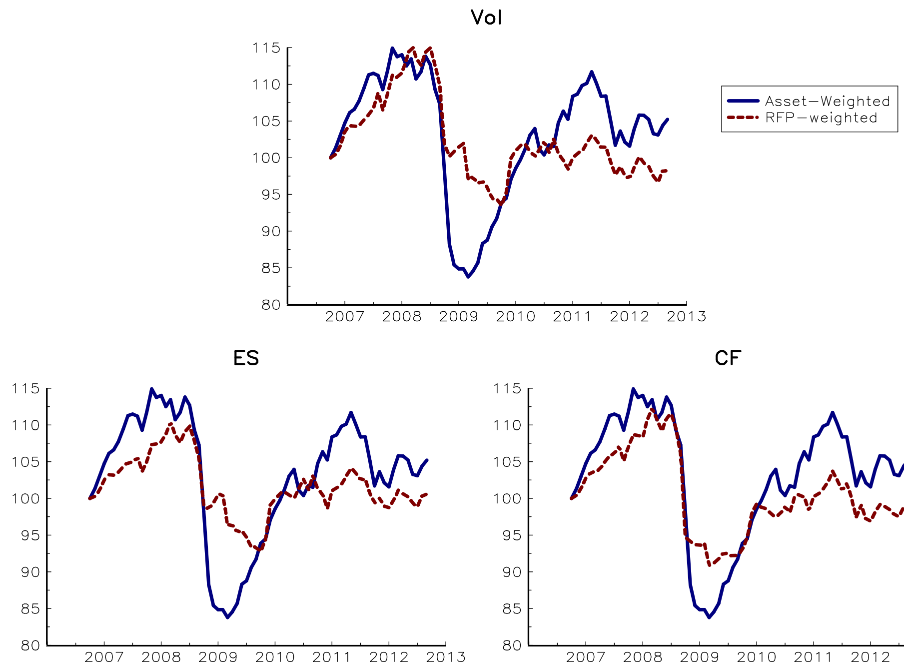
Diversifying a portfolio of hedge funds

The Risk Factor Parity (RFP) approach

Risk decomposition in terms of factors



Simulated performance



Strategic Asset Allocation

Back to the risk budgeting approach

Risk parity approach = a promising way for strategic asset allocation (see e.g. Bruder and Roncalli, 2012)

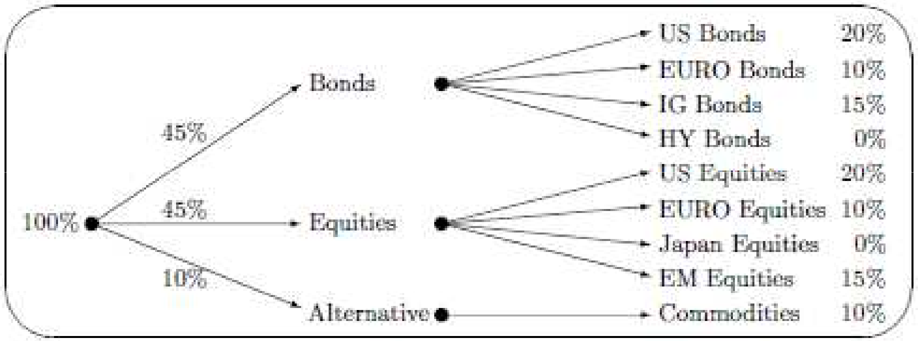
ATP Danish Pension Fund

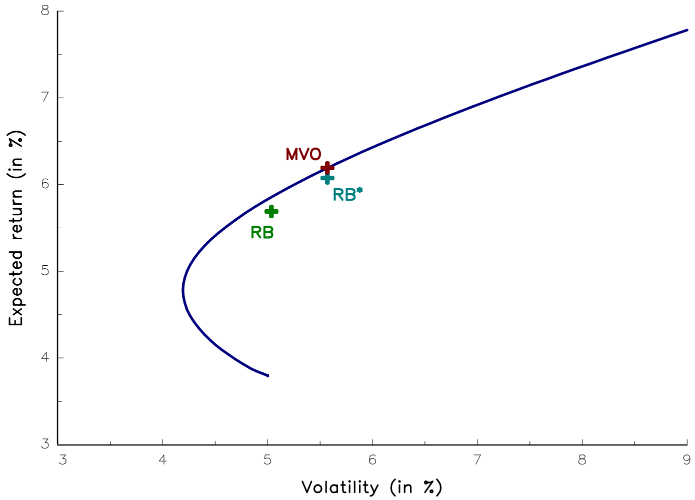
> *"Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk" (Henrik Gade Jepsen, CIO of ATP, IPE, June 2012).*

These risk budgets are then transformed into asset classes' weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities (Source: FTfm, June 10, 2012).

Strategic asset allocation

Risk budgeting policy of a pension fund





Asset class	RB		RB*		MVO	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
US Bonds	36.8%	20.0%	45.9%	18.1%	66.7%	25.5%
EURO Bonds	21.8%	10.0%	8.3%	2.4%	0.0%	0.0%
IG Bonds	14.7%	15.0%	13.5%	11.8%	0.0%	0.0%
US Equities	10.2%	20.0%	10.8%	21.4%	7.8%	15.1%
Euro Equities	5.5%	10.0%	6.2%	11.1%	4.4%	7.6%
EM Equities	7.0%	15.0%	11.0%	24.9%	19.7%	49.2%
Commodities	3.9%	10.0%	4.3%	10.3%	1.5%	2.7%

RB* = A BL portfolio with a tracking error of 1% wrt RB / MVO = Markowitz portfolio with the RB* volatility

Strategic Asset Allocation

The framework of risk factor budgeting

- Combining the risk budgeting approach to define the asset allocation and the economic approach to define the factors (Kaya *et al.*, 2011).
- Following Eychenne *et al.* (2011), we consider 7 economic factors grouped into four categories:
  1. activity: gdp & industrial production;
  2. inflation: consumer prices & commodity prices;
  3. interest rate: real interest rate & slope of the yield curve;
  4. currency: real effective exchange rate.
- Quarterly data from Datastream.
- ML estimation using YoY relative variations for the study period Q1 1999 – Q2 2012.
- Risk measure: volatility.

Strategic Asset Allocation

Allocation between asset classes

- 13 AC: equity (US, EU, UK, JP), sovereign bonds (US, EU, UK, JP), corporate bonds (US, EU), High yield (US, EU) and US TIPS.
- Three given portfolios:
  - Portfolio #1 is a balanced stock/bond asset mix.
  - Portfolio #2 is a defensive allocation with 20% invested in equities.
  - Portfolio #3 is an agressive allocation with 80% invested in equities.
- Portfolio #4 is optimized in order to take more inflation risk.

	Equity				Sovereign Bonds				Corp. Bonds		High Yield		TIPS
	US	EU	UK	JP	US	EU	UK	JP	US	EU	US	EU	US
#1	20%	20%	5%	5%	10%	5%	5%	5%	5%	5%	5%	5%	5%
#2	10%	10%			20%	15%	5%	5%	5%	5%	5%	5%	15%
#3	30%	30%	10%	10%	10%	10%							
#4	19.0%	21.7%	6.2%	2.3%		5.9%			24.1%	10.7%	2.6%	7.5%	

Factor	#1	#2	#3	#4
Activity	36.91%	19.18%	51.20%	34.00%
Inflation	12.26%	4.98%	9.31%	20.00%
Interest rate	42.80%	58.66%	32.92%	40.00%
Currency	7.26%	13.04%	5.10%	5.00%
Residual factors	0.77%	4.14%	1.47%	1.00%

Strategic Asset Allocation

Allocation within an asset class

Question: How to allocate between smart beta indices?

- Bond-like or equity-like?
- Sensitivity to economic risk factors?
- Behavior with respect to some economic scenarios?

Risk contributions of AW indices with respect to economic factors (Q4 1991 – Q3 2012)

Factors	S&P 100	EW	MV	MDP	ERC
Activity	72.13%	65.20%	25.29%	33.45%	52.29%
Inflation	18.10%	12.09%	8.38%	5.21%	4.59%
Interest rate	9.21%	22.08%	65.50%	59.65%	42.28%
Currency	0.57%	0.64%	0.83%	1.70%	0.85%

Risk contributions of AW indices with respect to economic factors (Q1 1999 – Q3 2012)

Factors	S&P 100	EW	MV	MDP	ERC
Activity	63.93%	64.87%	21.80%	34.44%	57.17%
Inflation	28.87%	22.76%	0.15%	12.38%	18.87%
Interest rate	5.96%	11.15%	73.34%	49.07%	22.19%
Currency	1.24%	1.21%	4.70%	4.11%	1.78%

Answer: Contact Lyxor ☺

Conclusion

- Risk factor contribution = a powerful tool.
- Risk budgeting with risk factors = be careful!
- PCA factors = some drawbacks (not always stable).

- Economic and risk factors = make more sense for long-term investment policy.
- Could be adapted to directional risk measure (e.g. expected shortfall).
- How to use this technology to hedge or be exposed to some economic risks?

- Our preliminary results open a door toward rethinking the long-term investment policy of pension funds.

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