Risk Factor, Risk Premium and Black-Litterman Model*

Natasha Abou Rjaily
Amundi Investment Institute
natasha.abourjaily@amundi.com

Thierry Roncalli
Amundi Investment Institute
thierry.roncalli@amundi.com

Jiali Xu
Amundi Investment Institute
jiali.xu@amundi.com

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Abstract

Risk factor models are now widely used by fund managers to construct portfolios and assess both return and risk based on the behavior of common risk factors to which the portfolios are exposed. However, fund managers often have subjective views on these risk factors that they may wish to incorporate into their asset allocation strategies. This study introduces an extension of the Black-Litterman model that allows views to be applied to risk factors rather than individual assets, greatly simplifying the process since the number of factors is typically much smaller than the number of assets in a portfolio. The concept of risk premia is central to portfolio allocation, but is typically assessed at the asset level. In our framework, risk premia are formulated and analyzed at the factor level. This theoretical advance allows the manager to calculate factor risk premia, formulate views based on these premia, and incorporate them into the portfolio optimization process to create an adjusted portfolio that is consistent with the manager's expectations.

This new framework has many applications. It allows fund managers to analyze the market's implied risk premia and identify the key drivers of market returns. In addition, the model facilitates comparisons between an actively managed portfolio and its benchmark by calculating how both are priced and identifying the factors that differentiate them. The approach can also be extended to incorporate economic factors, such as economic indicators or narratives, and can be applied to macroeconomic factor-mimicking portfolios. This article examines examples of each of these applications and analyzes the results obtained. Finally, given that the model involves several parameters that can be difficult to define, we provide practical guidance and demonstrate how varying these parameters can affect the final portfolio allocation.

Keywords: Factor model, risk premium, Black-Litterman model, minimum-variance portfolio, active management, tactical asset allocation.

JEL Classification: C02, G11, G12.

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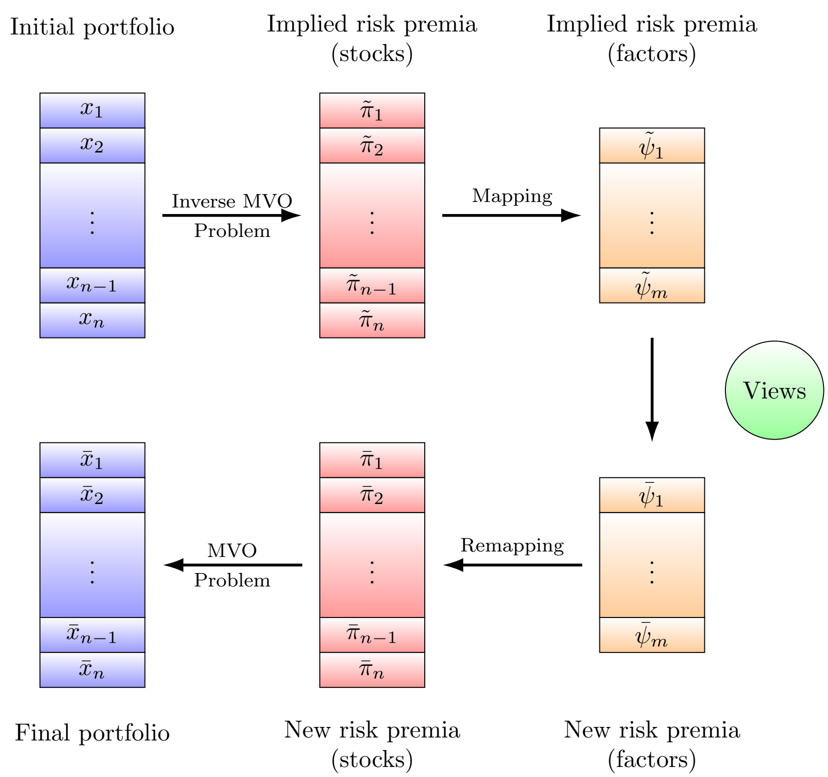
1 Introduction

The Black-Litterman model was developed in the early 1990s to address the problems faced by institutional investors in applying Modern Portfolio Theory in practice (Black and Litterman, 1991, 1992). This model uses a Bayesian framework and combines the assumption of market equilibrium with the views of the portfolio manager to produce a relevant allocation that reflects his expectations. Starting from an initial asset allocation and using an equilibrium argument, the Black-Litterman model first computes the implied risk premia of the assets by solving a reverse mean-variance optimization process, then modifies the allocation by taking into account the portfolio manager's views on the expected return of the assets, and finally derives the new optimized portfolio consistent with the portfolio manager's bets. It can also be seen as a synthesis of different approaches: portfolio optimization developed by Harry Markowitz, the CAPM theory of William Sharpe, the introduction of constraints in asset allocation, the Bayesian framework of active management, etc.

One challenge in applying the Black-Litterman model in practice arises when managing portfolios with a large number of assets, such as a mix of stocks and bonds. It becomes difficult for the portfolio manager to provide views on a large number of individual assets. As a result, the Black-Litterman model has traditionally been used by multi-asset portfolio managers who treat asset classes as the core securities in their portfolios. The underlying rationale is that it is more feasible for a manager to express views on a smaller, more manageable universe of securities, typically fewer than thirty. To address this limitation, it is natural to combine the Black-Litterman model with a multi-factor model, which is commonly used in finance as an asset pricing framework. In a multi-factor model, an asset's return is driven by its exposure to various underlying risk factors. These risk factors are defined as common patterns that help explain the variance in an asset's expected return. Examples include economic risk factors such as GDP, inflation, and interest rates; market risk factors such as value, momentum, and low beta; and statistical risk factors, such as the three-factor model of Litterman and Scheinkman (1991), which is based on principal component analysis and includes factors such as the level, steepness, and curvature of the yield curve. A multi-factor model can then be viewed as a dimensionality reduction technique, since a few factors (far fewer than the number of assets) can explain most of the variation in the covariance matrix of asset returns. Estimating such a model involves running multiple linear regressions to determine whether these factors can explain the returns of individual assets. This can be done using two main methods: time-series regression, as in the Fama-French factors (Fama and French, 1993), or cross-sectional regression, as in the Barra risk model (Fama and MacBeth, 1973). By incorporating a multi-factor model into the Black-Litterman model, portfolio managers can more easily and intuitively express their views on various risk factors rather than on a large number of individual assets.

In this paper, we introduce a new model called the Black-Litterman factor model to combine the Black-Litterman model with multi-factor models. Figure 1 shows the main steps of the Black-Litterman factor model. This model requires an initial portfolio allocation and a multi-factor asset pricing model. The first step is to compute the implied risk premia for the assets in the given portfolio, as we do in the classic Black-Litterman model. We then aggregate the implied risk premium from the asset level to the factor level. Next, using the Bayesian framework, we integrate the portfolio manager's views of the various factors to derive the posterior risk premia for the factors. The posterior risk premia of the assets are then computed, which can be viewed as the vector of expected returns in a traditional mean-variance optimization process. Constraints can also be imposed to find the appropriate asset allocation. The Black-Litterman factor model can provide a portfolio manager with factor-level information about his portfolio so that he can estimate how the portfolio is priced across risk factors and how the portfolio is performing relative to its benchmark. This information helps the manager develop views that are appropriate for the current market environment. The model then allows the portfolio manager to weight his views according to confidence levels and can combine current information about the portfolio with the portfolio manager's subjective views to assign new weights to the portfolio.

Figure 1: The Black-Litterman model applied to risk factors



This is not the first time that a Black-Litterman factor model has been proposed. For example, Kolm and Ritter (2021) developed a general framework for incorporating risk factors into the Black-Litterman model. The two approaches are very close, although we do not have the same prior distribution. In fact, our approach to defining the prior distribution is based on the mathematical model developed by Roncalli and Weisang (2016), while the prior of Kolm and Ritter (2021) is based on the Bayesian model developed by Kolm and Ritter (2017).

This paper is organized as follows. We begin with an overview of the multi-factor model, which we combine with the Black-Litterman model. In Section Two, we present the different steps of the Black-Litterman factor model, such as the calculation of the factor risk premia, the calculation of the posterior distribution of factor risk premia under the portfolio manager's views, and portfolio optimization. In Section Three, we show several applications of the Black-Litterman factor model in portfolio management: What is priced in by the market? How can a fund manager position his portfolio relative to a benchmark? What are the economic exposures of risk parity strategies? In this section, we also discuss parameter selection issues and other details of using the Black-Litterman factor model in practice. Finally, Section Four offers some concluding remarks.

2 The Black-Litterman factor model

2.1 Multi-factor model

We consider an investment universe of n assets. We assume that asset returns follow a multi-factor risk model:

$$R(t) - r = BF(t) + \varepsilon(t) \tag{1}$$

where $R(t) = \big(R_1(t), \ldots, R_n(t)\big)$ is the vector of asset returns, r is the return of the risk-free asset, $F(t) = \big(F_1(t), \ldots, F_m(t)\big)$ is the vector of risk factors, and $\varepsilon(t) = \big(\varepsilon_1(t), \ldots, \varepsilon_n(t)\big)$ is the vector of idiosyncratic risks. The loading matrix B collects all the beta values: $B = \big(\beta_{i,j}\big)$ where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. We assume that $\mathbb{E}\left[F(t)\right] = \psi$, $\operatorname{cov}\big(F(t)\big) = \Omega$, $\mathbb{E}\left[\varepsilon(t)\right] = \mathbf{0}_n$, $\operatorname{cov}\big(\varepsilon(t)\big) = D$ and $F(t) \perp \varepsilon(t)$. We deduce that the vector of risk premia is[1]:

$$\pi = B\psi \tag{2}$$

while the covariance matrix of asset returns is:

$$\Sigma = B\Omega B^\top + D \tag{3}$$

Given a portfolio x, its risk premium and volatility are:

$$\pi(x) = x^\top \pi = x^\top B\psi$$

and:

$$\sigma(x) = \sqrt{x^\top \Sigma x} = \sqrt{x^\top \left(B\Omega B^\top + D\right) x}$$

We can also calculate the vector of beta coefficients:

$$\beta(x) = B^\top x = \begin{pmatrix} \beta_1(x) \\ \vdots \\ \beta_m(x) \end{pmatrix}$$

where the j^{th} beta value is equal to $\beta_j(x) = \sum_{i=1}^n x_i \beta_{i,j}$.

To assess the importance of the common factors, we can calculate the proportion of the total variance accounted for by the common factors. This statistic is known as the R-squared coefficient, or the coefficient of determination if we are talking about linear regression. We have:

$$\mathfrak{R}_c^2 = \operatorname{diag}\left(B\Omega B^\top\right) \oslash \operatorname{diag}\left(B\Omega B^\top + D\right)$$

where $\oslash$ is the element by element division operator. For a portfolio x, we have:

$$\begin{aligned} R_x(t) &= x^\top R(t) \\ &= x^\top r + x^\top BF(t) + x^\top \varepsilon(t) \end{aligned}$$

We deduce that:

$$\mathfrak{R}_c^2(x) = \frac{x^\top \left(B\Omega B^\top\right) x}{x^\top \left(B\Omega B^\top + D\right) x}$$

[1] The vector of expected returns is given by:

$$\mu = r + \pi = r + B\psi$$

2.2 Computing the implied risk premia

2.2.1 Asset risk premia

Let μ and Σ be any vector of expected returns and covariance matrix[2]. We consider the following optimization problem:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top}\Sigma x - \gamma x^{\top}(\mu - r\mathbf{1}_n) \\ &\text{s.t.} \quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

where γ is the coefficient of risk tolerance. The optimal solution is:

$$x^{\star} = \gamma \Sigma^{-1}(\mu - r\mathbf{1}_n)$$

where $\gamma = \left(\mathbf{1}_n^{\top}\Sigma^{-1}(\mu - r\mathbf{1}_n)\right)^{-1}$. Given an initial allocation x, we deduce that this portfolio is optimal if the vector of implied risk premia is equal to:

$$\tilde{\pi} = \tilde{\mu} - r = \frac{1}{\gamma}\Sigma x$$

By assuming that we know the Sharpe ratio of the initial allocation, we deduce that:

$$\tilde{\pi} = \mathrm{SR}(x \mid r)\frac{\Sigma x}{\sqrt{x^{\top}\Sigma x}} \tag{4}$$

We retrieve one of the fundamental results of the capital asset pricing model. At the optimum, risk premia are proportional to marginal risk (Roncalli, 2013). Equation (4) is central to the Black-Litterman. It gives the risk premia required, or priced in, by the investor to hold portfolio x (Bourgeron *et al.*, 2019).

2.2.2 Factor risk premia

Following Meucci (2007) and Roncalli and Weisang (2016), we decompose the portfolio's asset exposures x by the portfolio's risk factor exposures y as follows:

$$x = B_y y + \breve{B}_y \breve{y}$$

where $B_y = \left(B^{\top}\right)^{+}$ is the Moore-Penrose inverse of $B^{\top}$ and $\breve{B}_y$ is any $n \times (n-m)$ matrix spanning the left null space of B_y. $\breve{y}$ corresponds to $n-m$ residual (or additional) factors that have no economic interpretation. It follows that:

$$\begin{cases} y = B_x x \\ \breve{y} = \breve{B}_x x \end{cases}$$

where $B_x = B^{\top}$ and $\breve{B}_x = \ker\left(B^{+}\right)^{\top}$. According to Roncalli and Weisang (2016), it is common to use this solution[3]: $\breve{B}_x = \left(\mathrm{null}\left(B^{+}\right)\right)^{+}\left(I_n - \left(B^{+}\right)^{\top}B^{\top}\right)$. In order to calculate the vector of factor risk premia, we use the relationship given in Equation (2). We deduce that:

$$\psi = B^{+}\pi$$

[2] The formulation is general, *i.e.* it is not necessarily derived from a multi-factor model.

[3] $\mathrm{null}(M)$ is the null space of the matrix M.

and:

$$\tilde{\psi} = B^{+}\tilde{\pi} = \mathrm{SR}\left(x \mid r\right) \frac{B^{+}\left(B\Omega B^{\top}+D\right)x}{\sqrt{x^{\top}\left(B\Omega B^{\top}+D\right)x}}$$

where B^{+} is the Moore-Penrose inverse of B. However, we can show that:

$$\tilde{\pi}\left(x\right) = x^{\top}\tilde{\pi} \neq y^{\top}\tilde{\psi}$$

Indeed, we have:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x} = B_y \frac{\partial\,\sigma\left(x\right)}{\partial\,y} + \breve{B}_y \frac{\partial\,\sigma\left(x\right)}{\partial\,\breve{y}} \tag{5}$$

We deduce that the marginal risk of the j^{th} risk factors is:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,y_j} = \left(B_{\sigma}\frac{\partial\,\sigma\left(x\right)}{\partial\,x}\right)_j$$

where $B_{\sigma} = B^{+}$. For the residual factors, we have:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,\breve{y}} = \left(\breve{B}_{\sigma}\frac{\partial\,\sigma\left(x\right)}{\partial\,x}\right)_j$$

where $\breve{B}_{\sigma} = \mathrm{null}\left(B^{+}\right)^{\top} = \breve{B}_x$. From Equation (5), we deduce that:

$$\tilde{\pi}\left(x\right) = y^{\top}\tilde{\psi} + \breve{y}^{\top}\breve{\upsilon} \tag{6}$$

where:

$$\left\{\begin{array}{l} y = B_x x \\ \breve{y} = \breve{B}_x x \\ \tilde{\psi} = B_{\sigma}\tilde{\pi} \\ \breve{\upsilon} = \breve{B}_{\sigma}\tilde{\pi} \end{array}\right. \tag{7}$$

Equations (6) and (7) are the core relationships for decomposing the risk premium of a portfolio with respect to the factor risk premia.

2.2.3 Some properties

Here are some interesting properties. The proofs are given in Appendix A.2.1 on page 45.

Property 1. *The factor exposures are equal to the beta sensitivities:*

$$y = \beta\left(x\right)$$

Property 2. *The covariance model is obtained by setting $B = I_n$ and $D = \mathbf{0}_{n,n}$. In this case, we have $R\left(t\right) = F\left(t\right)$ and $\Sigma = \Omega$. We have:*

$$\tilde{\psi} = \tilde{\pi}$$

Property 3. *The implied risk premium priced in by the tangent portfolio is equal to the vector of risk premia:*

$$\tilde{\pi} = \pi := \mu - r\mathbf{1}_n$$

Property 4. *The residual premium of the tangency portfolio is zero, which implies that the risk premium of the tangency portfolio is fully explained by the common risk factors:*

$$\begin{cases} \breve{\upsilon}\left(x^{\star}\right) = \breve{y}^{\top}\breve{\upsilon} = 0 \\ \tilde{\pi}\left(x^{\star}\right) = \tilde{\psi}\left(x^{\star}\right) = \dfrac{\pi^{\top}\Sigma^{-1}\pi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi} \end{cases}$$

Property 5. *If the portfolio is not optimal, the residual premium does not depend on the factor covariance Ω, but only on the idiosyncratic covariance D. The residual premium is then positive:*

$$\breve{\upsilon}\left(x\right) = x^{\top}\left(\breve{B}_x^{\top}\breve{B}_x D\right)x \geq 0$$

Property 6. *The factor risk premium of any portfolio is always positive:*

$$\tilde{\psi}\left(x\right) \geq 0$$

The last three properties are very interesting because we have the decomposition of the portfolio risk premium into the factor risk premium and the residual risk premium:

$$\tilde{\pi}\left(x\right) = \tilde{\psi}\left(x\right) + \breve{\upsilon}\left(x\right)$$

Using the traditional assumptions[4], the two terms are positive. If the portfolio is optimal and corresponds to the tangency portfolio, the portfolio risk premium has no residual premium. In all other cases, there is a positive residual premium. This is normal because the investor chooses a portfolio that is not optimal, which means that the investor believes that some idiosyncratic risk factors are being rewarded.

Property 7. *The asset risk premia admit a variance-covariance decomposition:*

$$\tilde{\pi} = \tilde{\pi}^{(\mathrm{var})} + \tilde{\pi}^{(\mathrm{cov})}$$

where $\tilde{\pi}^{(\mathrm{var})}$ depends on the asset variances and $\tilde{\pi}^{(\mathrm{cov})}$ depends on the asset covariances.

Property 8. *The asset risk premia admit a factor decomposition:*

$$\tilde{\pi} = \tilde{\pi}^{(\mathrm{factor})} + \tilde{\pi}^{(\mathrm{specific})}$$

where $\tilde{\pi}^{(\mathrm{factor})}$ depends on the common factors and $\tilde{\pi}^{(\mathrm{specific})}$ depends on the idiosyncratic factors.

Property 9. *The factor risk premia allow the following decompositions:*

$$\begin{cases} \tilde{\psi} = \tilde{\psi}^{(\mathrm{var})} + \tilde{\psi}^{(\mathrm{cov})} \\ \tilde{\psi} = \tilde{\psi}^{(\mathrm{factor})} + \tilde{\psi}^{(\mathrm{specific})} \end{cases}$$

where $\tilde{\psi}^{(\mathrm{var})}$, $\tilde{\psi}^{(\mathrm{cov})}$, $\tilde{\psi}^{(\mathrm{factor})}$ and $\tilde{\psi}^{(\mathrm{specific})}$ depend on the factor variances, factor covariances, common factors and idiosyncratic factors.

These three properties show that we can decompose the risk premium into two dimensions. The first approach considers a variance-covariance decomposition, while the second approach considers a factor decomposition between the contribution of common risk factors and the contribution of idiosyncratic risk factors.

[4] Both the Sharpe ratio $\mathrm{SR}\left(x \mid r\right)$ and the factor risk premia ψ are positive.

2.2.4 An example

Consider an investment universe with five assets and two factors. The loading matrix B is:

$$B = \begin{pmatrix} 0.5 & -0.3 \\ 1.2 & 0.6 \\ 1.4 & -0.3 \\ 0.7 & 0.3 \\ 0.8 & -0.9 \end{pmatrix}$$

The two factors are uncorrelated and their volatilities are 20% and 5%, respectively. We assume a diagonal matrix D with the following idiosyncratic volatilities: 12%, 10%, 8%, 10% and 12%. The expected returns of the factors are 4% and 1%, while the return of the risk-free asset is 2%. Below is the coefficient of determination for each asset and its decomposition between the two common risk factors.

Asset	$\mathfrak{R}_c^2$	$\mathfrak{R}_c^2(F_1)$	$\mathfrak{R}_c^2(F_2)$
#1	41.52%	40.61%	0.91%
#2	85.40%	84.09%	1.31%
#3	92.47%	92.21%	0.26%
#4	66.47%	65.72%	0.75%
#5	65.73%	60.92%	4.82%

For example, the common risk factors explain 41.52% of the risk of asset 1, with the first and second factors contributing 40.61% and 1.31% respectively. We note that the second factor has a low contribution and that the dynamics of the second and third assets are mainly driven by the common factors. In Table 1, we report several statistics[5] for five portfolios:

Portfolio	#1	#2	#3	#4	#5
x_1		20%	−50%	0%	50%
x_2		20%	100%	100%	−70%
x_3	$\propto \Sigma^{-1}(\mu - r\mathbf{1}_5)$	20%	50%	50%	−50%
x_4		20%	0%	0%	70%
x_5		20%	0%	−50%	100%

The first portfolio is the tangent portfolio:

$$x^\star = \frac{\Sigma^{-1}(\mu - r\mathbf{1}_5)}{\mathbf{1}_5^\top \Sigma^{-1}(\mu - r\mathbf{1}_5)}$$

The second portfolio corresponds to the equally-weighted portfolio, while the last three portfolios are long/short portfolios. For each portfolio, we compute the coefficient of determination $\mathfrak{R}_c^2(x)$, the asset risk premia $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$, the portfolio risk premium $\tilde{\pi}(x) = x^\top \tilde{\pi}$, the vector of sensitivities $\beta(x) = \big(\beta_1(x), \ldots, \beta_m(x)\big)$, the factor exposures $y = (y_1, \ldots, y_m)$, the factor risk premia $\tilde{\psi} = \left(\tilde{\psi}_1, \ldots, \tilde{\psi}_m\right)$, the factor risk premium of the portfolio $\tilde{\psi}(x) = y^\top \tilde{\psi}$ and the residual risk premium of the portfolio $\breve{\upsilon}(x) = \breve{y}^\top \breve{\upsilon}$. By definition, we have $\tilde{\psi}(x) + \breve{\upsilon}(x) = \tilde{\pi}(x)$.

In Table 1, we check that the estimated asset risk premia $\tilde{\pi}$ are exactly equal to the theoretical asset risk premia π in the case of the tangent portfolio. This portfolio has a beta of 1.25 and a factor exposure of 125% with respect to the first factor. In fact, we check

[5] All statistics except $\beta_j(x)$ are expressed in %.

Table 1: Risk premia of the two-factor model

Statistics	#1	#2	#3	#4	#5
$\tilde{\pi}_1$	1.70	2.11	1.46	1.89	-0.27
$\tilde{\pi}_2$	5.40	4.53	5.19	5.36	0.30
$\tilde{\pi}_3$	5.30	5.22	5.48	5.57	0.08
$\tilde{\pi}_4$	3.10	2.73	2.69	2.75	-0.21
$\tilde{\pi}_5$	2.30	3.21	2.96	2.50	-0.58
$\beta_1(x)$	1.25	0.92	1.65	1.50	0.00
$\beta_2(x)$	1.16	-0.12	0.60	0.90	-1.11
y_1	124.97	92.00	165.00	150.00	0.00
y_2	115.93	-12.00	60.00	90.00	-111.00
$\tilde{\psi}_1$	4.00	3.81	3.98	4.05	-0.02
$\tilde{\psi}_2$	1.00	-0.12	0.42	0.71	0.48
$\tilde{\psi}(x)$	6.16	3.52	6.82	6.71	-0.53
$\breve{\upsilon}(x)$	0.00	0.04	0.38	0.19	-0.58
$\tilde{\pi}(x)$	6.16	3.56	7.20	6.90	-1.11
$\mathfrak{R}_c^2(x)$	80.95	93.88	87.84	85.82	9.48
$\mathfrak{R}_c^2(x;F_1)$	76.82	93.78	87.12	83.94	0.00
$\mathfrak{R}_c^2(x;F_2)$	4.13	0.10	0.72	1.89	9.48

that the sensitivities and factor exposures are equal: $y = \beta(x) = B^\top x$. We find that the estimated factor risk premium $\tilde{\psi}_j$ is also equal to the theoretical factor risk premium ψ_i. This explains why the residual or specific premium is always zero and the risk premium of the portfolio is fully explained by the systematic risk factors. When the portfolio is not optimal, we have $\tilde{\pi} \neq \pi$, $\tilde{\psi} \neq \psi$ and $\breve{\upsilon}(x) > 0$. Focusing on the fifth portfolio, we notice that it has no exposure to the first factor. It is then normal for the implied risk premium of the risk factor to be close to zero.

Table 2: Decomposition of the asset risk premia (Portfolio #2)

Asset	$\tilde{\pi}_i$	$\tilde{\pi}_i^{(\text{var})}$	$\tilde{\pi}_i^{(\text{cov})}$	$\tilde{\pi}_i^{(\text{factor})}$	$\tilde{\pi}_i^{(\text{specific})}$
1	2.11	0.49	1.62	1.82	0.28
2	4.53	1.35	3.18	4.34	0.20
3	5.22	1.68	3.54	5.09	0.13
4	2.73	0.59	2.14	2.53	0.20
5	3.21	0.83	2.39	2.93	0.28

Table 3: Decomposition of the factor risk premia (Portfolio #2)

Factor	$\tilde{\psi}_j$	$\tilde{\psi}_j^{(\text{var})}$	$\tilde{\psi}_j^{(\text{cov})}$	$\tilde{\psi}_j^{(\text{factor})}$	$\tilde{\psi}_j^{(\text{specific})}$
1	3.81	3.63	0.19	3.63	0.19
2	-0.12	-0.03	-0.09	-0.03	-0.09

Above, we illustrate the decomposition of $\tilde{\pi}_i$ and $\tilde{\psi}_j$ in the case of the second portfolio. We check the identities:

$$\tilde{\pi}_i = \tilde{\pi}_i^{(\text{var})} + \tilde{\pi}_i^{(\text{cov})} = \tilde{\pi}_i^{(\text{factor})} + \tilde{\pi}_i^{(\text{specific})}$$

and:

$$\tilde{\psi}_j = \tilde{\psi}_j^{(\text{var})} + \tilde{\psi}_j^{(\text{cov})} = \tilde{\psi}_j^{(\text{factor})} + \tilde{\psi}_j^{(\text{specific})}$$

Since the covariance matrix Ω is diagonal, we get $\tilde{\psi}_j^{(\text{var})} = \tilde{\psi}_j^{(\text{factor})}$ and $\tilde{\psi}_j^{(\text{cov})} = \tilde{\psi}_j^{(\text{specific})}$.

2.3 Computing the posterior distribution of risk premia under the portfolio manager's view

2.3.1 Posterior distribution of asset risk premia

Let $\Pi(t) = R(t) - r\mathbf{1}_n$ be the vector of excess returns. Its conditional distribution is $\Pi(t) \mid \pi(t) \sim \mathcal{N}\left(\pi(t), \Sigma\right)$. Since an estimate of $\pi(t)$ is the vector $\tilde{\pi}$ of implied risk premia, we deduce that the prior distribution of $\pi(t)$ is:

$$\pi(t) \sim \mathcal{N}(\tilde{\pi}, \Gamma_\pi)$$

The unconditional distribution of $\Pi(t)$ becomes:

$$\begin{aligned}
\Pi(t) &\sim \mathcal{N}\left(\pi(t), \Sigma\right) \\
&\sim \pi(t) + \mathcal{N}(\mathbf{0}_n, \Sigma) \\
&\sim \mathcal{N}(\tilde{\pi}, \Gamma_\pi) + \mathcal{N}(\mathbf{0}_n, \Sigma) \\
&\sim \mathcal{N}(\tilde{\pi}, \Sigma + \Gamma_\pi)
\end{aligned}$$

We assume that the portfolio manager has some views $v_\pi(t)$ on $\pi(t)$:

$$P_\pi \pi(t) = v_\pi(t) + \epsilon_\pi(t)$$

where $\epsilon_\pi(t)$ is a noisy random vector, $\epsilon_\pi(t) \sim \mathcal{N}\left(\mathbf{0}_p, \Phi_\pi\right)$ and p is the number of views. We consider the random vector:

$$\begin{pmatrix} \pi(t) \\ v_\pi(t) \end{pmatrix} = \begin{pmatrix} I_n \\ P_\pi \end{pmatrix} \pi(t) - \begin{pmatrix} \mathbf{0}_{n,n} \\ I_p \end{pmatrix} \epsilon_\pi(t)$$

where $\pi(t) \perp \epsilon_\pi(t)$. It follows that the joint distribution of $\left(\pi(t), v_\pi(t)\right)$ is:

$$\begin{pmatrix} \pi(t) \\ v_\pi(t) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \tilde{\pi} \\ P_\pi \tilde{\pi} \end{pmatrix}, \begin{pmatrix} \Gamma_\pi & \Gamma_\pi P_\pi^\top \\ P_\pi \Gamma_\pi & P_\pi \Gamma_\pi P_\pi^\top + \Phi_\pi \end{pmatrix}\right)$$

We know that the conditional distribution of $\pi(t) \mid v_\pi(t) = v$ is Gaussian:

$$\pi(t) \mid v_\pi(t) = v_\pi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Gamma}_\pi\right)$$

where:

$$\begin{aligned}
\bar{\pi} &= \mathbb{E}\left[\pi(t) \mid v_\pi(t) = v_\pi\right] \\
&= \tilde{\pi} + \Gamma_\pi P_\pi^\top \left(P_\pi \Gamma_\pi P_\pi^\top + \Phi_\pi\right)^{-1} (v_\pi - P_\pi \tilde{\pi}) \\
&= \left(P_\pi^\top \Phi_\pi^{-1} P_\pi + \Gamma_\pi^{-1}\right)^{-1} \left(P_\pi^\top \Phi_\pi^{-1} v_\pi + \Gamma_\pi^{-1} \tilde{\pi}\right)
\end{aligned}$$

and:

$$\begin{aligned}
\bar{\Gamma}_\pi &= \Gamma_\pi - \Gamma_\pi P_\pi^\top \left(P_\pi \Gamma_\pi P_\pi^\top + \Phi_\pi\right)^{-1} P_\pi \Gamma_\pi \\
&= \left(P_\pi^\top \Phi_\pi^{-1} P_\pi + \Gamma_\pi^{-1}\right)^{-1}
\end{aligned}$$

The distribution of $\pi(t) \mid v_\pi(t) = v_\pi$ is called the posterior distribution of $\pi(t)$. Since $\psi(t) = B^+ \pi(t)$, we get:

$$\psi(t) \mid v_\pi(t) = v_\pi \sim \mathcal{N}\left(\bar{\psi}, \bar{\Gamma}_\psi\right)$$

where $\bar{\psi} = B^+ \bar{\pi}$ and $\bar{\Gamma}_\psi = B^+ \bar{\Gamma}_\pi {B^+}^\top$.

2.3.2 Posterior distribution of factor risk premia

We apply the previous framework to $\psi(t)$ instead of $\pi(t) = B\psi(t)$. The prior distribution of $\psi(t)$ is $\mathcal{N}\left(\tilde{\psi}, \Gamma_\psi\right)$, while the unconditional distribution of $\Pi(t)$ becomes:

$$\Pi(t) \sim \mathcal{N}\left(B\tilde{\psi}, \Sigma + B\Gamma_\psi B^\top\right)$$

We assume that the portfolio manager has some views $v_\psi(t)$ on $\psi(t)$:

$$P_\psi \psi(t) = v_\psi(t) + \epsilon_\psi(t)$$

where $\epsilon_\psi(t)$ is a noisy random vector, $\epsilon_\psi(t) \sim \mathcal{N}\left(\mathbf{0}_p, \Phi_\psi\right)$ and p is the number of views. It follows that the conditional distribution of $\psi(t)$ given the views $v_\psi(t)$ is Gaussian:

$$\psi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\psi}, \bar{\Gamma}_\psi\right)$$

where:

$$\begin{aligned}
\bar{\psi} &= \tilde{\psi} + \Gamma_\psi P_\psi^\top \left(P_\psi \Gamma_\psi P_\psi^\top + \Phi_\psi\right)^{-1} \left(v_\psi - P_\psi \tilde{\psi}\right) \\
&= \left(P_\psi^\top \Phi_\psi^{-1} P_\psi + \Gamma_\psi^{-1}\right)^{-1} \left(P_\psi^\top \Phi_\psi^{-1} v_\psi + \Gamma_\psi^{-1} \tilde{\psi}\right)
\end{aligned}$$

and:

$$\begin{aligned}
\bar{\Gamma}_\psi &= \Gamma_\psi - \Gamma_\psi P_\psi^\top \left(P_\psi \Gamma_\psi P_\psi^\top + \Phi_\psi\right)^{-1} P_\psi \Gamma_\psi \\
&= \left(P_\psi^\top \Phi_\psi^{-1} P_\psi + \Gamma_\psi^{-1}\right)^{-1}
\end{aligned}$$

We conclude that:

$$\pi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Gamma}_\pi\right)$$

where $\bar{\pi} = B\bar{\psi}$ and $\bar{\Gamma}_\pi = B\bar{\Gamma}_\psi B^\top$.

2.3.3 Some properties

Here are some interesting properties when we specify absolute views. The proofs are given in Appendix A.2.2 on page 49.

Property 10. *In the case of comprehensive absolute views, the conditional formulas for asset risk premia become:*

$$\left\{\begin{array}{l} \bar{\pi} = \tilde{\pi} + A_\pi \left(v_\pi - \tilde{\pi}\right) \\ \bar{\Gamma}_\pi = \left(I_n - A_\pi\right)\Gamma_\pi \end{array}\right.$$

where $A_\pi = \Gamma_\pi \left(\Gamma_\pi + \Phi_\pi\right)^{-1}$. *For factor risk premia, we have:*

$$\left\{\begin{array}{l} \bar{\psi} = \tilde{\psi} + A_\psi \left(v_\psi - \tilde{\psi}\right) \\ \bar{\Gamma}_\psi = \left(I_n - A_\psi\right)\Gamma_\psi \end{array}\right.$$

Property 11. *If the assets are independent of each other and we have only absolute views on the assets, these views do not affect the other assets. Moreover, the conditional covariance matrix* $\bar{\Gamma}_\pi$ *is diagonal and we have:*

$$\left(\bar{\Gamma}_\pi\right)_{i,i} = \frac{\left(\Gamma_\pi\right)_{i,i} \left(\Phi_\pi\right)_{i,i}}{\left(\Gamma_\pi\right)_{i,i} + \left(\Phi_\pi\right)_{i,i}}$$

This property also holds true when considering risk factors instead of assets.

Property 12. *When the certainty of the absolute views is very high, the conditional risk premia are equal to the views:*

$$\bar{\pi} = v_\pi$$

When implied risk premia and absolute views have the same uncertainty, conditional risk premia are the average of implied risk premia and views:

$$\bar{\pi} = \frac{v_\pi + \tilde{\pi}}{2}$$

This property also holds true when looking at risk factors rather than assets.

Property 13. *The conditional covariance matrix has an upper bound in the one-factor model, defined as:*

$$\bar{\Gamma}_\psi \leq \frac{1}{2}\left(\frac{\Gamma_\psi + \Phi_\psi}{2}\right)$$

Property 14. *In the two-factor model, when the absolute views are uncorrelated, the conditional risk premium on one factor is equal to a weighted average of the implied risk premium and the absolute view on that factor, and a correction term on the second view that depends on the correlation ρ between the two factors:*

$$\begin{cases} \bar{\psi}_1 = w_1\tilde{\psi}_1 + (1 - w_1)\, v_1 + \rho\lambda_1 \left(v_2 - \tilde{\psi}_2\right) \\ \bar{\psi}_2 = w_2\tilde{\psi}_2 + (1 - w_2)\, v_2 + \rho\lambda_2 \left(v_1 - \tilde{\psi}_1\right) \end{cases}$$

Property 15. *The conditional risk premia satisfy the following decomposition:*

$$\bar{\psi} = \underbrace{w \odot \tilde{\psi} + (\mathbf{1}_m - w) \odot v_\psi}_{\text{weighted average}} + \underbrace{\Lambda \left(v_\psi - \tilde{\psi}\right)}_{\text{correction term}}$$

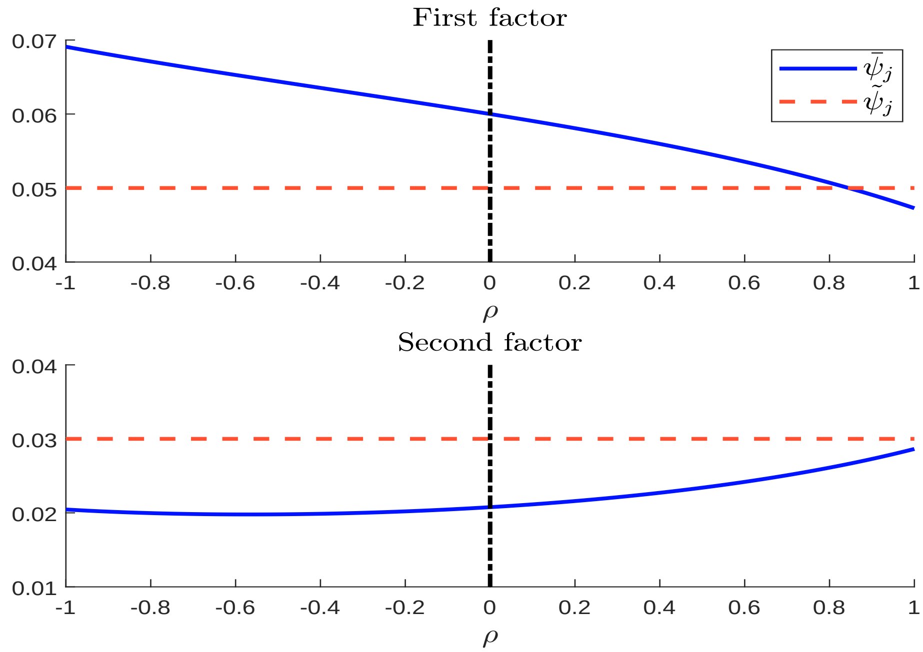
where $w \geq \mathbf{0}_m$.

Property 10 shows that the conditional formulas simplify a lot when we consider absolute views. The impact of the views is measured by the matrix A_ψ which depends on the covariance matrices Γ_ψ and Φ_ψ. Properties 14 and 15 demonstrate that the conditional risk premium is an average of the implied risk premium and the view plus a correction term that depends on the correlations of the implied risk premia and/or the views. Consider the following example: $\tilde{\psi} = (5\%, 3\%)$, $v_\psi = (7\%, 1.5\%)$, $\left(\Gamma_\psi\right)_{1,1} = 0.2^2$, $\left(\Gamma_\psi\right)_{2,2} = 0.1^2$, $\Phi_\psi = \operatorname{diag}\left(0.2^2, 0.15^2\right)$. Figure 2 shows the conditional risk premia $\bar{\psi}_1$ and $\bar{\psi}_2$ with respect to the correlation ρ. If there is no correlation, we verify that $\bar{\psi}_1$ and $\bar{\psi}_2$ are between the implied risk premia and the views. If we neglect the effect of ρ on the weights w_1 and w_2, it follows that $\bar{\psi}_1$ is a decreasing function of ρ and $\bar{\psi}_2$ is an increasing function of ρ since we have $v_1 - \bar{\psi}_1 = 2\% > 0$ and $v_2 - \bar{\psi}_2 = -1.5\% < 0$. These results are confirmed in Figure 2. It is very important to keep these cross effects in mind to understand the values of the conditional risk premia, since the coherence of the results with respect to the views depends on the cross correlation.

2.3.4 An example

We consider the two-factor model described on page 8. The current portfolio is the equally-weighted portfolio. For the specification of the covariance matrix Γ_π, we use the traditional rule $\Gamma_\pi = \tau\Sigma$ where $\tau = \frac{1}{2}$ (He and Litterman, 2002; Allaj, 2013). We assume that the

Figure 2: Conditional risk premia relative to the correlation ρ



portfolio manager has two views: $\pi_1 = 4\%$ and $\pi_4 = \pi_5$. The uncertainties of these views are 10% and 5%, respectively. This means that:

$$P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \; v_\pi = \begin{pmatrix} 4\% \\ 0\% \end{pmatrix} \text{ and } \Phi_\pi = \begin{pmatrix} 0.10^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}.$$

Using the prior distribution of $\pi(t)$ given in Table 4, we derive the posterior distribution of $\pi(t) \mid v_\pi(t) = v_\pi$ in Table 5 and $\psi(t) \mid v_\pi(t) = v_\pi$ in Table 6. As expected, the posterior risk premium of the first asset has increased: $\bar{\pi}_1 > \tilde{\pi}_1$ and the risk premium discrepancy between the fourth and fifth assets has decreased: $|\tilde{\pi}_4 - \tilde{\pi}_5| \ll |\bar{\pi}_4 - \bar{\pi}_5|$. In terms of risk factors, the two views greatly increase ψ_1 since we have $\tilde{\psi}_1 = 3.81\%$ and $\bar{\psi}_1 = 4.67\%$. The effect on the second factor is less important since we have $\tilde{\psi}_2 = -0.12\%$ and $\bar{\psi}_2 = -0.01\%$. Also, the posterior variance $\left(\bar{\Gamma}_\psi\right)_{2,2}$ is equal to 0.26%, which is about one-sixth of the posterior variance $\left(\bar{\Gamma}_\psi\right)_{1,1}$.

Table 4: Prior distribution $\pi(t) \sim \mathcal{N}(\tilde{\pi}, \Gamma_\pi)$

Asset	$\tilde{\pi}$	Γ_π				
#1	2.11	1.23	1.18	1.41	0.69	0.83
#2	4.53	1.18	3.43	3.34	1.70	1.85
#3	5.22	1.41	3.34	4.25	1.95	2.27
#4	2.73	0.69	1.70	1.95	1.49	1.09
#5	3.21	0.83	1.85	2.27	1.09	2.10

The previous analysis can be extended by considering the portfolio manager's views on risk factors. For the specification of the covariance matrix Γ_ψ, we use the rule $\Gamma_\psi = \sigma_\psi^2 I_m$ where $\sigma_\psi = 20\%$. We assume that we have absolute views on the two risk factors: $\psi_1 = 5\%$

Table 5: Posterior distribution $\pi(t) \mid v_\pi(t) = v_\pi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Gamma}_\pi\right)$

Asset	$\bar{\pi}$	$\bar{\Gamma}_\pi$				
#1	3.13	0.55	0.52	0.62	0.33	0.34
#2	5.51	0.52	2.80	2.58	1.36	1.37
#3	6.33	0.62	2.58	3.33	1.58	1.61
#4	3.48	0.33	1.36	1.58	1.16	1.09
#5	3.57	0.34	1.37	1.61	1.09	1.23

Table 6: Posterior distribution $\psi(t) \mid v_\pi(t) = v_\pi \sim \mathcal{N}\left(\bar{\psi}, \bar{\Gamma}_\psi\right)$

Factor	$\bar{\psi}$	$\bar{\Gamma}_\psi$	
#1	4.67	1.58	0.15
#2	−0.01	0.15	0.26

and $\psi_2 = 0\%$. The uncertainties of these views are 10% and 5%, respectively. This means that:

$$P_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; v_\psi = \begin{pmatrix} 5\% \\ 0\% \end{pmatrix} \text{ and } \Phi_\psi = \begin{pmatrix} 0.10^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}.$$

Using the prior distribution of $\psi(t)$ given in Table 7, we derive the posterior distribution of $\psi(t) \mid v_\psi(t) = v_\psi$ in Table 8 and $\pi(t) \mid v_\psi(t) = v_\psi$ in Table 9. As expected, the risk premium of the first factor has increased: $\bar{\psi}_1 > \tilde{\psi}_1$ and the posterior risk premium of the second factor is close to zero: $\bar{\psi}_2 \approx 0$. Regarding the asset risk premia, they have all increased because they are positively correlated with the first risk factor, whose risk premium has increased.

Table 7: Prior distribution $\psi(t) \sim \mathcal{N}\left(\tilde{\psi}, \Gamma_\psi\right)$

Factor	$\tilde{\psi}$	Γ_ψ	
#1	3.81	4.00	0.00
#2	−0.12	0.00	4.00

Table 8: Posterior distribution $\psi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\psi}, \bar{\Gamma}_\psi\right)$

Factor	$\bar{\psi}$	$\bar{\Gamma}_\psi$	
#1	4.76	0.80	0.00
#2	−0.01	0.00	0.24

Table 9: Posterior distribution $\pi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Gamma}_\pi\right)$

Asset	$\bar{\pi}$	$\bar{\Gamma}_\pi$				
#1	2.38	0.22	0.44	0.58	0.26	0.38
#2	5.71	0.44	1.24	1.30	0.71	0.64
#3	6.67	0.58	1.30	1.59	0.76	0.96
#4	3.33	0.26	0.71	0.76	0.41	0.38
#5	3.82	0.38	0.64	0.96	0.38	0.70

2.4 Portfolio optimization

2.4.1 Conditional distribution of excess returns

We recall that the vector of excess returns is defined as $\Pi(t) = R(t) - r\mathbf{1}_n \sim \mathcal{N}\left(\pi(t), \Sigma\right)$. We deduce that:

$$\Pi(t) = \pi(t) + \varepsilon(t)$$

where $\varepsilon(t) \sim \mathcal{N}(\mathbf{0}_n, \Sigma)$ is the noise process. We have seen that:

$$\pi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Gamma}_\pi\right)$$

where $\bar{\pi} = B\bar{\psi}$ and $\bar{\Gamma}_\pi = B\bar{\Gamma}_\psi B^\top$. Assuming that $\pi(t)$ and $\varepsilon(t)$ are independent, it follows that:

$$\begin{aligned} \mathbb{E}\left[\Pi(t) \mid v_\psi(t) = v_\psi\right] &= \mathbb{E}\left[\pi(t) \mid v_\psi(t) = v_\psi\right] + \mathbb{E}\left[\varepsilon(t) \mid v_\psi(t) = v_\psi\right] \\ &= \bar{\pi} \end{aligned}$$

and:

$$\begin{aligned} \operatorname{cov}\left(\Pi(t) \mid v_\psi(t) = v_\psi\right) &= \operatorname{cov}\left(\pi(t) \mid v_\psi(t) = v_\psi\right) + \operatorname{cov}\left(\varepsilon(t) \mid v_\psi(t) = v_\psi\right) \\ &= \bar{\Gamma}_\pi + \operatorname{cov}\left(\varepsilon(t)\right) \\ &= \bar{\Gamma}_\pi + \Sigma \end{aligned}$$

We conclude that:

$$\Pi(t) \mid v_\psi(t) = v_\psi \sim \mathcal{N}\left(\bar{\pi}, \bar{\Sigma}\right)$$

where:

$$\bar{\Sigma} = \Sigma + \bar{\Gamma}_\pi$$

2.4.2 Mean-variance optimization problem

As described in He and Litterman (2002), we can consider the following mean-variance optimization problem:

$$\begin{aligned} \bar{x}(\gamma) &= \arg\min \frac{1}{2} x^\top \bar{\Sigma} x - \gamma x^\top \bar{\pi} \\ & \text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned} \tag{8}$$

Alternatively, we can choose a benchmark b and use a minimum tracking-error problem:

$$\begin{aligned} \bar{x}(\gamma) &= \arg\min \frac{1}{2} (x-b)^\top \bar{\Sigma} (x-b) - \gamma (x-b)^\top \bar{\pi} \\ & \text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned} \tag{9}$$

In addition, we can add other constraints to Problems (8) and (9) to obtain more practical optimal portfolios, such as turnover control, weight bounds and asset class limits.

2.4.3 An example

We consider the two-factor model described on page 8. To understand the impact of views, we report the correlation matrix (in %) between asset returns below:

$$\rho = \begin{pmatrix} 100.00 & 57.34 & 61.68 & 50.83 & 51.84 \\ 57.34 & 100.00 & 87.46 & 75.33 & 69.05 \\ 61.68 & 87.46 & 100.00 & 77.40 & 76.08 \\ 50.83 & 75.33 & 77.40 & 100.00 & 61.36 \\ 51.84 & 69.05 & 76.08 & 61.36 & 100.00 \end{pmatrix}$$

Because assets are correlated, a particular view of one asset can affect other assets. The tangent portfolio is $x^\star = \left(-11.63\%, 95.86\%, 13.69\%, 50.76\%, -48.67\%\right)$, and the current portfolio is an equal-weighted portfolio. We will illustrate two different scenarios: views on assets and views on factors. To compute the Black-Litterman portfolio using the mean-variance optimization framework, we need to specify the risk tolerance coefficient γ. We assume that the Sharpe ratio of the equally-weighted portfolio is 0.30, and we set:

$$\gamma = \frac{\sigma\left(x_{\mathrm{ew}}\right)}{\mathrm{SR}\left(x_{\mathrm{ew}} \mid r\right)} = 0.6333$$

We also assume that $\Gamma_\pi = \tau\Sigma$ with $\tau = \dfrac{1}{2}$.

Implementing asset views

Case 1 We assume that the portfolio manager has two views: $\pi_1 = 4\%$ and $\pi_4 = \pi_5$. The uncertainties of these views are 10% and 5%, respectively. This means that:

$$P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \; v_\pi = \begin{pmatrix} 4\% \\ 0\% \end{pmatrix} \text{ and } \Phi_\pi = \begin{pmatrix} 0.10^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}$$

Results are given in Table 10 and 11. For each asset, we report the vector $\tilde{\pi}$ of implied (or prior) risk premia, the vector $\bar{\pi}$ of posterior risk premia, the long/short posterior solution $\bar{x}$ and the long-only posterior solution $\bar{x}_+$. We do the same thing with the risk factors. Since the fund manager has a more positive view on the first asset, the allocation to that asset increases. Since the fund manager believes that the last two assets have the same risk premium, the model increases the risk premium of the fourth asset from 4.37% to 4.78% and decreases the risk premium of the fifth asset from 5.15% to 4.91%. Thus, the allocation to the fourth asset increases while the allocation to the fifth asset decreases. Note also that the risk premia of Assets 2 and 3 increase because of their correlation with Asset 1.

Table 10: Impact of views on asset allocation (Case 1)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	3.69	20.00	41.86	41.86
#2	5.40	7.26	7.55	20.00	13.23	13.23
#3	5.30	8.35	8.63	20.00	8.48	8.48
#4	3.10	4.37	4.78	20.00	30.76	30.76
#5	2.30	5.15	4.91	20.00	5.66	5.66

Table 11: Impact of views on factor allocation (Case 1)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	6.34	92.00	74.75	74.75
#2	1.00	−0.20	0.10	−12.00	−3.03	−3.03

Case 2 This is a variation of Case 1. The assumptions are the same, except that we are more confident in the first view than in the second. We have:

$$P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix},\ v_\pi = \begin{pmatrix} 4\% \\ 0\% \end{pmatrix} \text{ and } \Phi_\pi = \begin{pmatrix} 0.05^2 & 0 \\ 0 & 0.10^2 \end{pmatrix}$$

Compared to the previous case (Table 10), the good confidence in the first view increases the allocation to Asset 1 (Table 12) from 41.86% to 47.32%. We also observe an increase in the allocation to Asset 3 due to the relatively high correlation between Assets 1 and 3.

Table 12: Impact of views on asset allocation (Case 2)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	3.89	20.00	47.32	47.32
#2	5.40	7.26	7.75	20.00	13.35	13.35
#3	5.30	8.35	8.88	20.00	14.00	14.00
#4	3.10	4.37	4.82	20.00	19.79	19.79
#5	2.30	5.15	5.17	20.00	5.54	5.54

Table 13: Impact of views on factor allocation (Case 2)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	6.52	92.00	77.56	77.56
#2	1.00	−0.20	−0.03	−12.00	−9.44	−9.44

Case 3 We assume that the portfolio manager has correct views on the expected returns of all assets: $\pi_1 = 1.7\%, \pi_2 = 5.4\%, \pi_3 = 5.3\%, \pi_4 = 3.1\%$ and $\pi_5 = 2.3\%$. Therefore, he formulates absolute views. The uncertainties of these absolute views are 5%. We have:

$$P_\pi = I_5,\ v_\pi = \pi \text{ and } \Phi_\pi = 0.05^2 I_5$$

Table 14: Impact of views on asset allocation (Case 3)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	1.86	20.00	26.50	26.43
#2	5.40	7.26	5.27	20.00	33.95	27.74
#3	5.30	8.35	5.39	20.00	−8.86	0.00
#4	3.10	4.37	3.08	20.00	47.23	45.82
#5	2.30	5.15	2.58	20.00	1.18	0.00

The optimal allocation $\bar{x}$ or $\bar{x}_+$ is very different from the current allocation x. The reason for this is that the fund manager has some strong bets, in particular Assets 3 and 5. For these two assets, the difference between what is priced in by the current portfolio and the manager's views is 305 and 285 basis points, respectively. Therefore, we observe a negative reallocation to these two assets in favor of the other three assets.

Table 15: Impact of views on factor allocation (Case 3)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	4.03	92.00	75.59	78.59
#2	1.00	−0.20	0.72	−12.00	28.19	22.46

Implementing factor views In this section, we illustrate the application of the Black-Litterman model with views on factors. For the specification of the covariance matrix Γ_ψ, we use the rule $\Gamma_\psi = \tau\Omega$ where $\tau = \frac{1}{2}$.

Case 1 We assume that the portfolio manager has a single view: $\psi_2 = 1\%$. The uncertainty of this view is 5%. This means that:

$$P_\psi = \begin{pmatrix} 0 & 1 \end{pmatrix}, \ v_\psi = 1\% \text{ and } \Phi_\psi = 0.05^2$$

Results are given in Tables 18 and 19. Since the fund manager is positive on the second factor, the exposure increases on the second factor and decreases on the first factor. Assets two and four have positive loadings on the second factor, while the other assets have negative loadings. But we must also consider the first factor. In fact, the second asset has a beta of 1.2, while the fourth asset has a beta of 0.7. Therefore, it makes sense to increase the allocation to the fourth asset. The allocation to the first asset also increases because it helps reduce the exposure to the first factor. Conversely, the allocation to the third asset decreases significantly because it has the highest beta and negative loadings on the second factor.

Table 16: Impact of views on factor allocation (Case 1)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	6.11	92.00	72.76	73.56
#2	1.00	−0.20	0.20	−12.00	2.37	0.70

Table 17: Impact of views on asset allocation (Case 1)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	2.99	20.00	29.75	30.09
#2	5.40	7.26	7.45	20.00	18.82	16.46
#3	5.30	8.35	8.49	20.00	−3.05	0.00
#4	3.10	4.37	4.34	20.00	40.10	39.96
#5	2.30	5.15	4.70	20.00	14.38	13.49

Case 2 We assume that the portfolio manager has two views: $\psi_2 = 1\%$ and $\psi_2 - \psi_1 = 2\%$. However, he is more confident in the first view, and the uncertainties of these views are 5% and 10%, respectively. This means that:

$$P_\psi = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \ v_\psi = \begin{pmatrix} 1\% \\ 2\% \end{pmatrix} \text{ and } \Phi_\psi = \begin{pmatrix} 0.10^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}$$

The addition of the second bet greatly reduces the allocation to the first factor. Therefore, it is normal for the allocations to the first and fourth assets to increase.

Table 18: Impact of views on factor allocation (Case 2)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	−0.81	92.00	20.57	57.74
#2	1.00	−0.20	0.32	−12.00	2.20	−6.77

Table 19: Impact of views on asset allocation (Case 2)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	−0.50	20.00	60.84	61.29
#2	5.40	7.26	−0.78	20.00	5.22	0.00
#3	5.30	8.35	−1.24	20.00	−60.44	0.00
#4	3.10	4.37	−0.47	20.00	70.10	38.71
#5	2.30	5.15	−0.94	20.00	24.27	0.00

Case 3 We assume that the portfolio manager has correct views on the expected return of the factors: $\psi_1 = 4\%$ and $\psi_2 = 1\%$. The uncertainties of these views are 5%. This means that:

$$P_\psi = I_2,\ v_\psi = \psi \text{ and } \Phi_\psi = 0.05^2 I_5$$

Table 20: Impact of views on factor allocation (Case 3)

Factor	ψ	$\tilde{\psi}$	$\bar{\psi}$	y	$\bar{y}$	$\bar{y}_+$
#1	4.00	6.11	4.23	92.00	74.56	74.92
#2	1.00	−0.20	0.20	−12.00	2.53	1.97

Table 21: Impact of views on asset allocation (Case 3)

Asset	π	$\tilde{\pi}$	$\bar{\pi}$	x	$\bar{x}$	$\bar{x}_+$
#1	1.70	3.37	2.06	20.00	28.67	28.73
#2	5.40	7.26	5.20	20.00	19.38	18.60
#3	5.30	8.35	5.87	20.00	−1.12	0.00
#4	3.10	4.37	3.02	20.00	39.12	39.03
#5	2.30	5.15	3.21	20.00	13.96	13.64

The posterior risk premia $\bar{\psi}$ change significantly to reflect the manager's views. This is also the case for the assets. In fact, the posterior risk premia $\bar{\pi}$ are now very close to the true values π. Again, the changes in the portfolio reflect the differences between the prior risk premia $\tilde{\pi}$ and the posterior risk premia $\bar{\pi}$.

Remark 1. *The previous examples show that the specification of a view on one asset or factor can have an impact on the other assets or factors due to the correlation between the assets or factors. It is very important to take the time to understand the values of $\bar{\pi}$ and $\bar{\psi}$ to be sure that the bets the manager has formulated are the bets he really wants to implement. For example, consider two bets on two factors that are highly correlated. The manager believes that the risk premium on the first factor will increase and the risk premium on the second factor will decrease. In this case, the manager formulates two conflicting bets. The outcome of the model is unpredictable and the solution cannot be satisfactory.*

3 Applications

3.1 What is priced in by the market?

Now let us see how the previous framework can be used to extract information from a given portfolio, strategy, or market. We consider the S&P 500 from December 1998 to May 2024. We dynamically estimate the four-factor Carhart model:

$$\begin{aligned} R_i(t) - r(t) &= \beta_i^{\mathrm{mkt}}(t)\left(R_{\mathrm{mkt}}(t) - r(t)\right) + \beta_i^{\mathrm{smb}}(t) R_{\mathrm{smb}}(t) + \beta_i^{\mathrm{hml}}(t) R_{\mathrm{hml}}(t) + \\ & \quad \beta_i^{\mathrm{wml}}(t) R_{\mathrm{wml}}(t) + \varepsilon_i(t) \end{aligned}$$

where $R_i(t)$ is the monthly return of stock i, $r(t)$ is the risk-free rate, $R_{\mathrm{mkt}}(t)$ is the US market risk factor, $R_{\mathrm{smb}}(t)$ is the US SMB risk factor, $R_{\mathrm{hml}}(t)$ is the US HML risk factor, $R_{\mathrm{wml}}(t)$ is the US WML risk factor, and $\varepsilon_i(t)$ is the idiosyncratic risk factor of stock i. The factor data are obtained from the Kenneth French data library[6]. At time t, for each stock in the S&P 500 index, we estimate the factor exposures $\beta_i^{\mathrm{mkt}}(t)$, $\beta_i^{\mathrm{smb}}(t)$, $\beta_i^{\mathrm{hml}}(t)$ and $\beta_i^{\mathrm{wml}}(t)$ and the residual volatility $\sigma\left(\varepsilon_i(t)\right)$ using a linear regression with a 36-month rolling window. Once the risk factor model is estimated at time t, we conduct a risk premium analysis of the S&P 500 index[7] and a factor decomposition.

3.1.1 Factor contribution to S&P 500 implied risk premium

Figure 3 shows the evolution of the implied risk premium $\tilde{\pi}(x)$ where x is the portfolio of the S&P 500 index at time t. We also decompose $\tilde{\pi}(x)$ into two components:

$$\tilde{\pi}(x) = \tilde{\psi}(x) + \breve{\upsilon}(x)$$

As expected, the implied risk premium $\tilde{\pi}(x)$ varies over time and increases during crisis periods as investors demand more premium to hold the portfolio and bear the higher risk. During normal periods, the portfolio risk premium is relatively low, less than 3.0% in 2006 and 2007 and ranging from 2.70% to 3.50% in the 2015-2019 period. However, this risk premium increases significantly during periods of stress, such as in 2011, when it peaked at 7.14%. We also find that factors and specific risk have a systematic positive contribution. In Figure 3, we see that idiosyncratic factors contribute only 2 to 13 bps. Thus, the idiosyncratic factors contribute almost nothing to the risk premium of the portfolio, and we can conclude that the common factors explain almost all of the implied risk premium of the portfolio.

Figure 4 shows the contribution of the four factors:

$$\tilde{\psi}(x) = \tilde{\psi}^{\mathrm{mkt}}(x) + \tilde{\psi}^{\mathrm{smb}}(x) + \tilde{\psi}^{\mathrm{hml}}(x) + \tilde{\psi}^{\mathrm{wml}}(x)$$

while we present statistics for specific dates in Table 22. Once again, we observe considerable variation in implied risk premia across the four factors. The market factor is the largest contributor. In contrast, the other three factors contribute little relative to the market, although their impact can sometimes be relatively significant or minimal, depending on the time period. The size factor (SMB) contributed positively to the portfolio's implied risk premium until early 2003, after which it started to have a negative impact, ranging from −26 bps in 2007 to zero bps in 2018. In general, the value factor (HML) shows a pattern of negative contribution during crisis periods. The highest positive contribution was +83 bps in February 2000, while the highest negative contribution was −48 bps in July 2001. The momentum factor (WML) shows two periods of significant contribution: the first from late

[6] The website is https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[7] We assume a constant Sharpe ratio $\mathrm{SR}\left(x \mid r\right) = 0.30$, which is a debatable hypothesis.

Figure 3: Implied risk premium of the S&P 500 index

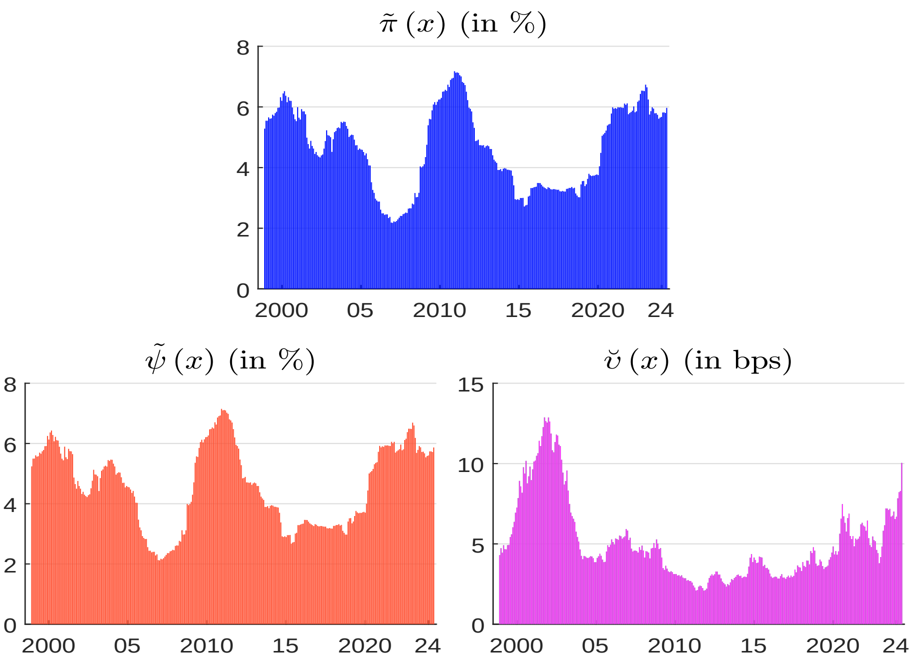
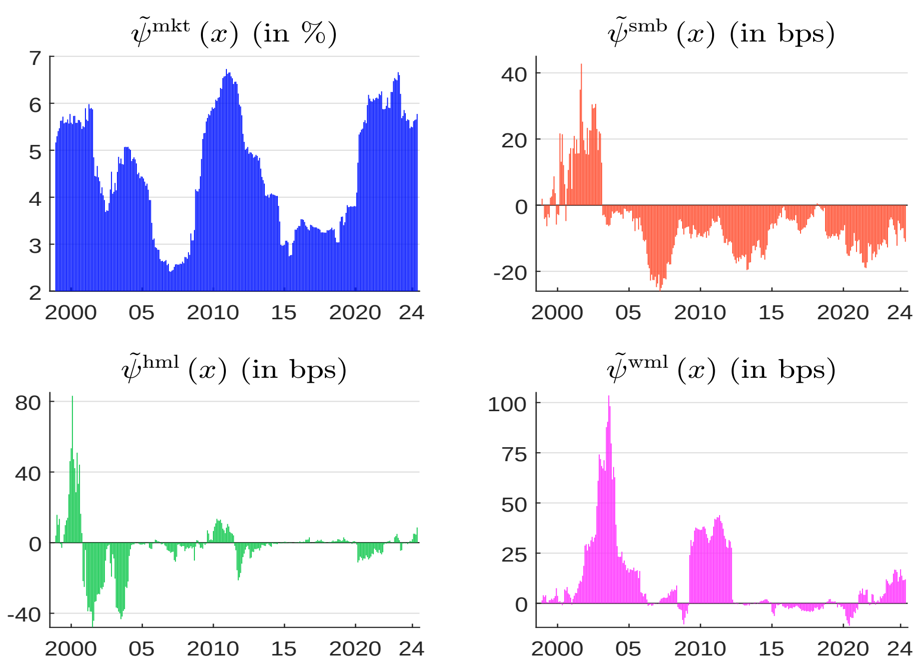


Figure 4: Contribution of MKT, SMB, HML and WML risk factors to the S&P 500 risk premium



2001 to 2005, with contributions ranging from 10 bps to 104 bps, and the second from 2009 to 2011, with a range of 24 bps to 43 bps. The highest contribution occurred in 2003, ranging from 71 bps to 104 bps. In late 2008 and mid-2020, momentum negatively impacted the portfolio's implied risk premium by −6 to −11 bps, while in other periods it made little or no contribution. Currently, the implied risk premium of the S&P 500 is 597 bps, of which 587 bps is attributable to common factors and 10 bps to idiosyncratic factors. Of the common factors, 577 bps are attributable to the market, while the remaining 10 bps come from the three long/short factors. The SMB factor contributes negatively with −11 bps, while the HML and Momentum factors contribute positively with +9 bps and +12 bps, respectively.

Table 22: Factor contribution to the S&P 500 implied risk premium (in bps)

t	$\tilde{\pi}(x)$	$\tilde{\psi}(x)$	$\breve{\upsilon}(x)$	$\tilde{\psi}^{\text{mkt}}(x)$	$\tilde{\psi}^{\text{smb}}(x)$	$\tilde{\psi}^{\text{hml}}(x)$	$\tilde{\psi}^{\text{wml}}(x)$
02/2000	645	637	8	556	−3	83	1
07/2001	576	565	11	587	16	−48	10
09/2001	478	465	12	445	43	−34	11
08/2003	533	526	7	471	−6	−42	104
03/2007	221	215	5	245	−26	−5	1
10/2008	404	399	5	417	−7	−2	−8
06/2010	653	650	3	613	−10	12	34
03/2017	329	326	3	336	−7	1	−3
04/2020	505	500	4	534	−15	−10	−8
05/2024	597	587	10	577	−11	9	12

3.1.2 Analysis of factor risk premia

In order to compare the implied risk premium across time and across factors, we need to normalize the risk factors. Indeed, it makes no sense to compare the implied risk premium of a given factor at two different times, or two given factors at the same time, if the volatility is completely different. Therefore, we normalize the risk factors such that we target the volatility σ':

$$F'_j(t) = \frac{\sigma'}{\sigma_j(t)} F_j(t)$$

In this case, we have $B' = B\Lambda$ where $\Lambda = \frac{1}{\sigma'} \operatorname{diag}\left(\sigma_1(t), \ldots, \sigma_m(t)\right)$. We deduce that[8]:

$$y' = \left(B'\right)^\top x = \Lambda^\top B^\top x = \Lambda y$$

and[9]:

$$\tilde{\psi}' = \left(B'\right)^+ \tilde{\pi} = \Lambda^{-1} B^+ \tilde{\pi} = \Lambda^{-1} \tilde{\psi}$$

Figures 5 and 6 show the evolution of the normalized factor risk premia and normalized exposures when σ' is set to 20%. Again, we see a lot of variation due to the dynamics of the stock market.

[8] We verify that the factor contribution does not change:

$$\tilde{\psi}'(x) = \left(y'\right)^\top \tilde{\psi}' = y^\top \tilde{\psi} = \tilde{\psi}(x)$$

[9] We use Property I on page 53.

Figure 5: Normalized factor risk premia (S&P 500 index, $\sigma' = 20\%$)

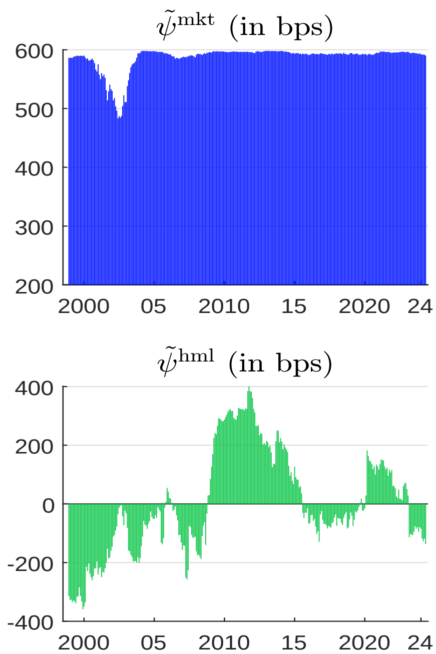
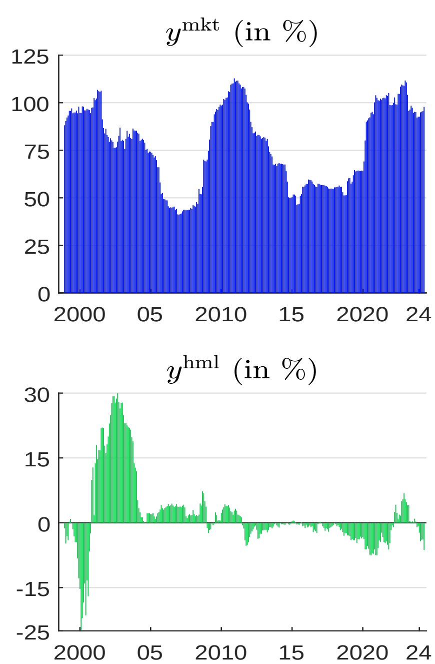


Figure 6: Normalized factor exposures (S&P 500 index, $\sigma' = 20\%$)



For the market risk factor, the normalized risk premium is nearly constant and follows this rule:

$$\tilde{\psi}^{\mathrm{mkt}} \approx \frac{\sigma^{\mathrm{mkt}}}{20\%} \times 6\%$$

However, this rule was not satisfied between 1999 and 2003. Specifically, $\tilde{\psi}^{\mathrm{mkt}}$ reached its lowest value in June 2002. The exposure of the S&P 500 index to the market risk factor is time-varying, but this reflects the normalization of the factor's volatility to 20%. Without this normalization, the exposure ranges from 80% to 113%, with a median value of 103%.

The S&P 500's exposure to the SMB factor is always negative because the portfolio is a cap-weighted index of the 500 largest US companies. As a result, the portfolio is heavily exposed to large-cap stocks and not small-cap stocks. The SMB premium priced into the S&P 500 Index was negative from 1999 through early 2003. The negative exposure and negative risk premium during this period explains the positive contribution of the SMB factor to the portfolio's implied risk premium. After 2003, the implied risk premium increased sharply, peaking at 3.83% in March 2007. Since the exposure is negative, the contribution of the SMB risk factor was negative between 2003 and 2024. In fact, investors are willing to bear this negative contribution in order to be exposed to large-cap companies.

Regarding the HML factor, the factor risk premium priced in by the S&P 500 is positive or negative depending on the period. In the period 1999-2008, the factor was negatively priced by the S&P 500, then it was positively priced between 2009 and 2015. It has been negatively priced again since 2023. In 1999 and 2000, the S&P 500 was negatively exposed to the HML factor, which explains the positive contribution during this period. However, the contribution became negative until 2007 due to the positive exposure of the HML factor. However, we note that the exposure of the HML factor has been relatively low since mid-2004. This calls into question the existence of the value factor over the last twenty years.

The momentum factor is almost always priced negatively, except for the periods from January 1999 to January 2001 and from May 2006 to May 2008. The WML exposure is very negative from 2001 to 2007 and from 2009 to mid-2012, which explains the positive contribution to the implied risk premium of the S&P 500 Index. However, the low exposure after 2012 also explains the very low contribution of this factor over the last twelve years.

The sign of $\tilde{\psi}^{\mathrm{hml}}$ and $\tilde{\psi}^{\mathrm{wml}}$ can sometimes be disturbing and their magnitude can be very different from the observed performance. To better understand these patterns, we perform the variance-covariance decomposition:

$$\tilde{\psi} = \tilde{\psi}^{(\mathrm{var})} + \tilde{\psi}^{(\mathrm{cov})}$$

Figure 7 shows the values of $\tilde{\psi}^{(\mathrm{var})}$ and $\tilde{\psi}^{(\mathrm{cov})}$ for the four risk factors. Unlike the vector $\tilde{\pi}^{(\mathrm{var})}$ which is always positive for a long-only portfolio, the sign of $\tilde{\psi}^{(\mathrm{var})}$ can be negative. In fact, we have:

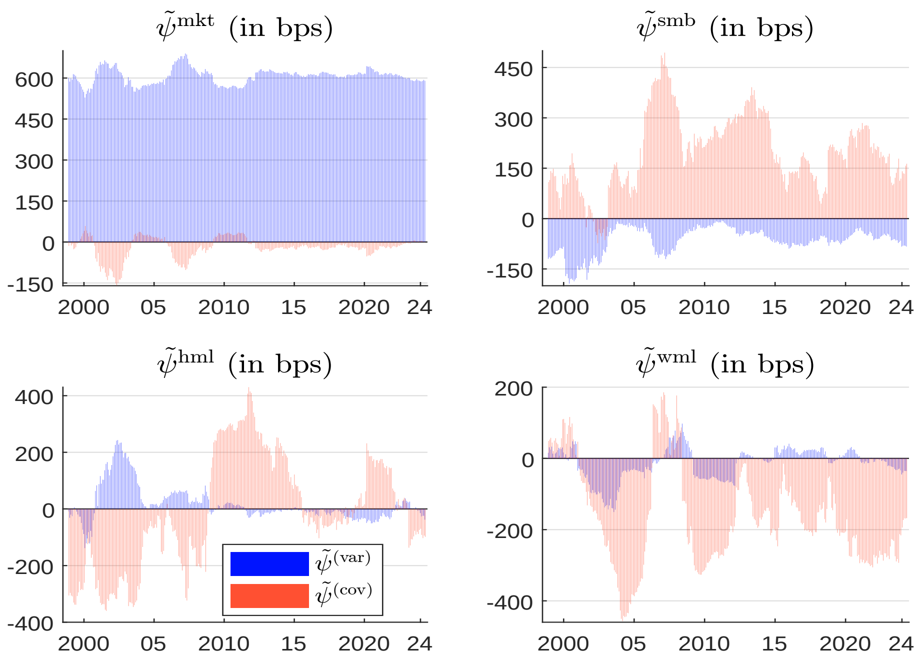
$$\tilde{\psi}^{(\mathrm{var})} = \varphi_x \left(B^{+} B \left(\omega \odot I_m\right) B^{\top} B_y \right) y$$

Since the matrix $\omega \odot I_m$ is positive semi-definite, we deduce that $B\left(\omega \odot I_m\right) B^{\top}$ and $B^{+} B \left(\omega \odot I_m\right) B^{\top} B_y$ are also two positive semi-definite matrices. Therefore, the sign of $\tilde{\psi}^{(\mathrm{var})}$ depends on the sign of y:

$$\operatorname{sgn} \tilde{\psi}^{(\mathrm{var})} = \operatorname{sgn} y$$

For example, the size factor always has a negative variance premium because it always has a negative exposure. It also has a positive covariance premium that dominates the variance premium most of the time, which explains why the SMB risk premium is generally positive. The variance component of the HML and WML risk factors is not significant compared to

Figure 7: Variance-covariance decomposition of factor risk premia (S&P 500 index, $\sigma' = 20\%$)



the covariance component. An interesting result is that the covariance component of these two risk factors is mainly explained by the correlation between market returns and factor returns. In other words, the S&P 500 Index does not have sufficient exposure to the HML and WML risk factors to generate a significant variance premium for these two factors. In addition, the market risk factor is ubiquitous. This implies that the HML and WML risk premia reflect the correlation of these factors with the market.

3.2 How to benchmark a portfolio and measure its active bets?

In this section, we illustrate how a fund manager can position his portfolio relative to a benchmark. To do this, we examine the MSCI USA Minimum Volatility portfolio relative to its benchmark, the MSCI USA Index, and identify the key active bets as of March 2024. We look at the factor model provided by Barra. This model corresponds to the Barra regional factor model USSLOW[10] and is based on 77 factors: the US country factor (which represents the market), 16 style factors[11] and 60 industry factors. To simplify the analysis of the results and for the purposes of this paper, we group[12] the 60 industry factors into 11 sector factors, which correspond to the GICS Level 1 sectors[13]. The factor model is then:

$$R_i(t) - r(t) = \beta_i^{\text{country}}(t) F^{\text{country}}(t) + \sum_{j=1}^{16} \beta_{i,j}^{\text{style}}(t) F_j^{\text{style}}(t) + \sum_{j=1}^{11} \beta_{i,j}^{\text{sector}}(t) F_j^{\text{sector}}(t) + \varepsilon_i(t)$$

[10] The exact name is the Barra US Total Market Equity Model for Long-Term Investors (USSLOW).

[11] These are Beta, Dividend Yield, Earnings Quality, Earnings Yield, Growth, Leverage, Liquidity, Long-Term Reversal, Management Quality, Mid-Capitalization, Momentum, Profitability, Prospect, Residual Volatility, Size, and Value. See Appendix B on page 56 for a description of these factors.

[12] The details of the grouping methodology can be found in Appendix A.3 on page 54.

[13] These are Communication Services, Consumer Discretionary, Consumer Staples, Energy, Financials, Health Care, Industrials, Information Technology, Materials, Real Estate, and Utilities.

3.2.1 Factor analysis

Table 23 shows that the MSCI USA MinVol portfolio has an implied risk premium of 394 bps, which is lower than the MSCI USA Index's implied risk premium of 530 bps. This is consistent with the lower risk of the MSCI USA MinVol portfolio relative to the risk level of the benchmark. Analyzing the factors contributing to the portfolios' implied risk premia, we find that the US country factor is the largest contributor in both portfolios. The market's contribution to both portfolios is almost the same (409 bps vs. 413 bps), even though the portfolios' risk premia are significantly different. We can conclude that the MSCI USA MinVol portfolio relies on long/short (L/S) factors to achieve its lower risk premium (and lower risk). This is evident from the fact that the L/S factors generate −20 bps for the minimum volatility portfolio, while they generate +113 bps for the cap-weighted portfolio. A decomposition between style and sector factors shows a gap of 30 bps for sectors and more than 100 bps for style factors. Thus, style factors play an important role in differentiating the MSCI USA MinVol portfolio from its benchmark. While style factors contribute positively to the implied risk premium of the MSCI USA Index (+28 bps), they have a significant negative impact of −74 bps in the MSCI USA MinVol portfolio. In fact, the MSCI USA MinVol portfolio sacrifices potential positive risk premia from style factors and even accepts a negative implied risk premium from these factors in order to meet its minimum volatility target. Idiosyncratic factors contribute only +4 bps to the implied risk premia in both portfolios, suggesting that common factors almost entirely explain the portfolios' implied risk premia.

Table 23: Factor contribution to the portfolio risk premium (in bps)

	MSCI USA Index	MSCI USA Minvol	Difference
Portfolio	530	394	−136
Factors	526	390	−137
Idiosyncratic	4	4	1
Market	413	409	−4
L/S factors	113	−20	−133
Idiosyncratic	4	4	1
Market	413	409	−4
Style factors	28	−74	−102
Sectors	85	55	−30
Idiosyncratic	4	4	1

To understand the factors driving the bets in the MSCI USA MinVol portfolio, we analyze the factor risk premia and the portfolio's exposure to these factors. Table 24 shows that among the style factors, the largest bet in both portfolios is on the Beta factor. Specifically, Beta[14] contributes +35 bps to the implied risk premium of the MSCI USA Index, while it has a significant negative contribution of −75 bps in the MSCI USA MinVol portfolio. A closer look at the exposures and risk premia reveals that the negative exposure in the MSCI USA MinVol portfolio (−75.6%), combined with the high implied risk premium (+99 bps), drives this negative contribution. In contrast, although the MSCI USA Index has a lower exposure to the Beta factor (+20.1%), the factor's contribution remains high because it is highly valued by the benchmark, with an implied risk premium of +175 bps. The remaining 15 factors contribute only −7 bps to the MSCI USA Index and +1 bps to the MSCI USA

[14]Beta explains common variations in stock returns due to different stock sensitivities to market or systematic risk that are not explained by the US country factor. It can be thought of as a high beta minus low beta risk factor.

MinVol portfolio. Thus, Beta is the primary contributor in both portfolios. It is also worth noting that the exposures to several factors such as Liquidity, Long-Term Reversal, Management Quality, Profitability, Prospect, Residual Volatility and Value are quite similar in both portfolios. However, the exposures to other factors such as Beta, Earnings Quality, Mid Capitalization and Size differ significantly.

Table 24: Factor risk premia, exposures and contributions

	Factor F_j	MSCI USA Index			MSCI USA Minvol		
		$\tilde{\psi}_j$ (in bps)	y_j (in %)	$y_j\tilde{\psi}_j$ (in bps)	$\tilde{\psi}_j$ (in bps)	y_j (in %)	$y_j\tilde{\psi}_j$ (in bps)
	US country (market)	414	99.9	413	409	100.0	409
Style factors	Beta	175	20.1	35	99	−75.6	−75
	Dividend Yield	3	−9.8	−0	7	1.3	0
	Earnings Quality	13	−2.9	−0	17	25.9	4
	Earnings Yield	−21	−6.4	1	−8	−2.9	0
	Growth	0	5.0	0	−1	−12.4	0
	Leverage	9	−14.3	−1	20	0.1	0
	Liquidity	13	−28.3	−4	8	−25.9	−2
	Long-Term Reversal	−14	−9.6	1	−16	−9.6	2
	Management Quality	−8	4.5	−0	−6	15.1	−1
	Mid Capitalization	6	−141.4	−9	19	−66.9	−12
	Momentum	−18	18.1	−3	−24	−3.5	1
	Profitability	−1	28.3	−0	2	17.3	0
	Prospect	7	−1.9	−0	4	−12.7	−1
	Residual Volatility	54	9.6	5	28	15.5	4
	Size	4	71.4	3	10	40.5	4
	Value	−3	−6.2	0	−2	−12.7	0
Sectors	Communication Services	69	8.9	6	−73	5.0	−4
	Consumer Discretionary	226	10.6	24	80	5.9	5
	Consumer Staples	34	5.5	2	56	10.9	6
	Energy	−158	4.0	−6	−50	2.4	−1
	Financials	74	10.5	8	67	13.6	9
	Health Care	−36	12.2	−4	14	16.7	2
	Industrials	44	8.8	4	40	10.2	4
	Information Technology	152	32.5	49	113	26.0	29
	Materials	52	2.4	1	54	2.5	1
	Real Estate	95	2.2	2	100	0.0	0
	Utilities	−15	2.2	−0	39	6.8	3

In terms of L/S sector factors, Information Technology and Consumer Discretionary are the largest contributors to the MSCI USA implied risk premium, adding +49 bps and +24 bps, respectively. In contrast, for the MSCI USA MinVol portfolio, Information Technology is the only major contributor at +29 bps. A closer look at exposures and risk premia reveals a heavy weighting to the Information Technology sector, which has the highest exposure in both portfolios (+32.5% in the MSCI USA Index and +26% in the MSCI USA MinVol portfolio) and a notably high implied risk premium of +152 bps in the MSCI USA Index and +113 bps in the MSCI USA MinVol portfolio. As for the Consumer Discretionary sector, despite having only +10.6% exposure to the benchmark, its high contribution to the MSCI USA Index is driven by its substantial implied risk premium of +226 bps. The remaining nine sectors provide similar contributions and have nearly identical exposures in both portfolios.

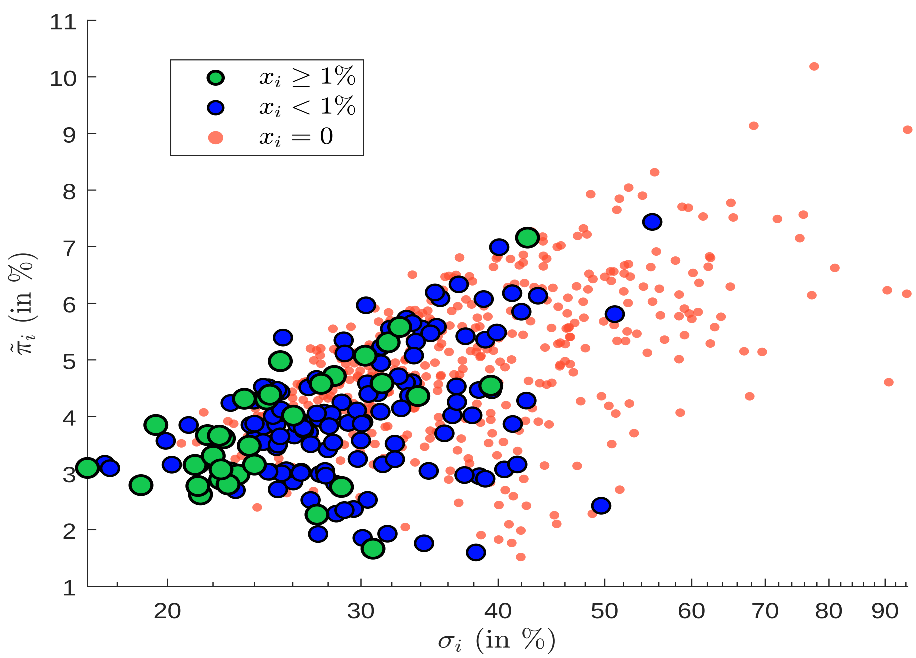
3.2.2 Stock analysis

The previous analysis can be extended to the individual constituents of the portfolio. In Figure 15 on page 58, we plot the implied risk premium $\tilde{\pi}_i^{(\text{minvol})}$ derived from the MinVol portfolio against the implied risk premium $\tilde{\pi}_i^{(\text{index})}$ derived from the cap-weighted index. We observe that they differ in magnitude, leading to the following linear regression model:

$$\tilde{\pi}_i^{(\text{minvol})} = 2\% + 0.54 \times \tilde{\pi}_i^{(\text{index})} + u_i$$

This model shows less dispersion in $\tilde{\pi}_i^{(\text{minvol})}$ compared to $\tilde{\pi}_i^{(\text{index})}$. This is confirmed by the empirical standard deviations: $\sigma(\tilde{\pi}_i^{(\text{index})}) = 2.34\%$ and $\sigma(\tilde{\pi}_i^{(\text{minvol})}) = 1.33\%$. Figures 16 and 17 on page 59 indicate no relationship between x_i and $\tilde{\pi}_i$, but a positive relationship between σ_i and $\tilde{\pi}_i$. Below, we present the relationship between x_i, σ_i, and $\tilde{\pi}_i^{(\text{minvol})}$. We note that a stock's weight in the MinVol portfolio is zero when its volatility is too high, while stocks with weights greater than 1% are low-volatility stocks.

Figure 8: Relationship between x_i, σ_i and $\tilde{\pi}$ (MSCI USA MinVol, March 2024)



Tables 25 and 26 show the risk premium of the 20 largest holdings in the MSCI USA Index and the MinVol portfolio. The risk premia in the two portfolios differ significantly because of the different allocations. For example, Microsoft has a risk premium of 6.02% in the MSCI USA Index and 4.98% in the MSCI MinVol portfolio. As explained earlier, when the implied risk premium derived from the cap-weighted index is low, it is higher for the MinVol portfolio. A typical example is Exxon Mobil (0.95% vs. 2.28%). Tables 27 and 28 show the top and bottom 10 implied risk premia. The MinVol portfolio does not hold any of the top 10 stocks in terms of implied risk premia priced in by the cap-weighted portfolio. This is due to the high risk of these stocks. We also note that a stock may have a high risk premium priced in by the MinVol portfolio even if it is not held by the portfolio.

Table 25: Risk premium in % of the 20 largest US companies (March 2024)

Stock	MSCI USA		MSCI MinVol		σ_i	$\mathfrak{R}_c^2$
	x_i	$\tilde{\pi}_i$	x_i	$\tilde{\pi}_i$		
MICROSOFT	6.45	6.02	1.51	4.98	25.33	78.93
APPLE	5.47	6.18	0.81	5.39	25.47	88.69
NVIDIA	4.85	9.63		6.30	47.13	57.98
AMAZON.COM	3.64	8.03		5.29	35.52	88.33
ALPHABET	3.64	6.80	0.11	4.66	32.08	89.18
META PLATFORMS	2.34	7.76		4.60	44.74	58.80
LILLY ELI	1.36	3.53	0.82	3.58	30.00	50.67
BROADCOM	1.28	9.45	2.15	7.16	42.53	65.51
JPMORGAN CHASE	1.26	3.14		3.20	21.55	77.39
BERKSHIRE HATHAWAY	1.20	2.93	1.58	3.14	21.19	71.48
TESLA	1.09	11.53		7.77	65.14	58.50
EXXON MOBIL	1.01	0.95	0.49	2.28	28.49	95.32
UNITEDHEALTH GROUP	0.99	1.80	1.26	3.01	22.81	56.02
VISA	0.96	3.86	0.88	3.72	26.89	46.52
MASTERCARD	0.88	4.43	0.71	4.25	28.81	49.87
PROCTER & GAMBLE	0.83	2.35	1.28	3.09	16.91	73.03
HOME DEPOT	0.83	7.06	0.84	5.36	38.93	56.31
JOHNSON & JOHNSON	0.83	1.78	1.33	2.79	18.92	68.26
MERCK	0.73	2.13	1.75	2.97	23.18	40.89
COSTCO WHOLESALE	0.71	3.56		3.64	23.65	53.96

Table 26: Risk premium in % of the 20 largest holdings of the MSCI USA MinVol portfolio (March 2024)

Stock	MSCI USA		MSCI MinVol		σ_i	$\mathfrak{R}_c^2$
	x_i	$\tilde{\pi}_i$	x_i	$\tilde{\pi}_i$		
BROADCOM	1.28	9.45	2.15	7.16	42.53	65.51
MERCK	0.73	2.13	1.75	2.97	23.18	40.89
WASTE MANAGEMENT	0.19	1.30	1.69	2.62	21.43	55.20
IBM	0.38	6.03	1.68	5.31	31.76	50.98
WASTENNECTIONS	0.10	2.85	1.67	3.66	22.29	47.29
REPUBLIC SERVICES	0.09	1.53	1.63	2.77	21.31	55.31
AMPHENOL	0.15	4.63	1.61	4.58	27.60	51.57
BERKSHIRE HATHAWAY	1.20	2.93	1.58	3.14	21.19	71.48
PROGRESSIVE	0.26	2.03	1.57	2.76	28.80	38.74
MICROSOFT	6.45	6.02	1.51	4.98	25.33	78.93
MOTOROLA SOLUTIONS	0.13	4.25	1.51	4.72	28.37	46.01
T-MOBILE US	0.18	−0.07	1.49	1.66	30.76	46.40
VERIZONMMUNICATIONS	0.38	2.37	1.48	2.89	22.36	88.03
ROPER TECHNOLOGIES	0.13	2.98	1.45	3.61	22.48	54.46
DUKE ENERGY	0.16	3.02	1.42	4.38	24.79	79.44
CISCO SYSTEMS	0.44	4.86	1.41	4.59	31.35	44.85
ACCENTURE PLC	0.47	6.02	1.38	5.07	30.26	56.62
WALMART	0.58	2.58	1.35	3.30	22.01	54.00
TEXAS INSTRUMENTS	0.34	6.26	1.35	5.58	32.54	57.02
JOHNSON & JOHNSON	0.83	1.78	1.33	2.79	18.92	68.26

Table 27: Top and bottom 10 implied risk premia in % priced in by the MSCI USA Index (March 2024)

Stock	MSCI USA		MSCI MinVol		σ_i	$\mathfrak{R}_c^2$
	x_i	$\tilde{\pi}_i$	x_i	$\tilde{\pi}_i$		
SUPER MICROMPUTER	0.11	13.30		9.07	94.34	42.29
ENPHASE ENERGY	0.04	13.09		10.18	77.56	51.17
CLOUDFLARE	0.06	11.85		9.14	68.33	42.87
TESLA	1.09	11.53		7.77	65.14	58.50
MARVELL TECHNOLOGY	0.13	11.07		8.31	55.51	57.51
ROKU	0.02	10.89		7.15	75.25	47.71
SAMSARA	0.01	10.63		7.65	51.29	75.16
PALANTIR TECHNOLOGIES	0.09	10.51		7.52	65.46	43.37
SIRIUS XM HOLDINGS	0.01	10.33		7.49	71.81	52.45
MONOLITHIC POWER SYSTEMS	0.07	10.24		7.85	51.56	55.72
⋮		⋮		⋮	⋮	
NRG ENERGY	0.03	0.13		1.83	40.03	34.51
NEUROCRINE BIOSCIENCES	0.03	0.07	0.41	1.76	34.24	27.84
T-MOBILE US	0.18	−0.07	1.49	1.66	30.76	46.40
OCCIDENTAL PETROLEUM	0.09	−0.15		2.10	45.21	50.48
PIONEER NATURAL RESOURCES	0.13	−0.20		1.90	38.58	58.17
PHILLIPS 66	0.16	−0.24	0.08	1.60	38.18	66.85
CENCORA	0.09	−0.24	0.73	1.92	27.42	58.85
DIAMONDBACK ENERGY	0.07	−0.34		1.77	41.14	58.26
CHENIERE ENERGY	0.08	−0.59	0.35	2.42	49.63	57.20
COTERRA ENERGY	0.05	−0.90		1.52	41.93	58.20

Table 28: Top and bottom 10 implied risk premia in % priced in by the MSCI USA MinVol portfolio (March 2024)

Stock	MSCI USA		MSCI MinVol		σ_i	$\mathfrak{R}_c^2$
	x_i	$\tilde{\pi}_i$	x_i	$\tilde{\pi}_i$		
ENPHASE ENERGY	0.04	13.09		10.18	77.56	51.17
CLOUDFLARE	0.06	11.85		9.14	68.33	42.87
SUPER MICROMPUTER	0.11	13.30		9.07	94.34	42.29
MARVELL TECHNOLOGY	0.13	11.07		8.31	55.51	57.51
ON SEMICONDUCTOR	0.07	9.88		8.04	52.57	53.12
BOSTON PROPERTIES	0.02	9.58		7.93	48.55	74.68
ATLASSIAN	0.07	9.88		7.90	54.12	51.02
MONOLITHIC POWER SYSTEMS	0.07	10.24		7.85	51.56	55.72
TESLA	1.09	11.53		7.77	65.14	58.50
LATTICE SEMICONDUCTOR	0.02	9.99		7.71	58.81	47.81
⋮		⋮		⋮	⋮	
UNITED THERAPEUTICS	0.02	0.54	0.09	1.93	31.71	29.00
CENCORA	0.09	−0.24	0.73	1.92	27.42	58.85
PIONEER NATURAL RESOURCES	0.13	−0.20		1.90	38.58	58.17
YTE	0.02	0.26	0.49	1.86	30.10	33.81
NRG ENERGY	0.03	0.13		1.83	40.03	34.51
DIAMONDBACK ENERGY	0.07	−0.34		1.77	41.14	58.26
NEUROCRINE BIOSCIENCES	0.03	0.07	0.41	1.76	34.24	27.84
T-MOBILE US	0.18	−0.07	1.49	1.66	30.76	46.40
PHILLIPS 66	0.16	−0.24	0.08	1.60	38.18	66.85
COTERRA ENERGY	0.05	−0.90		1.52	41.93	58.20

Table 29: List of economic narratives

Narrative	Themes
Back to the 70s	Inflation, taxation, employment, government policy, central bank intervention, commodity prices, exchange rate, macroeconomic risk
Environment	Natural disaster, environment law, green finance, green growth, health, green innovation, natural resource management, protest, resilience, adaption, mitigation
Geopolitical Risk	Aid groups, attacks, country at risk, crime, crisis, ideology, justice, politics, sanctions, tensions, trade barriers, war, weapons
Monetary	Easing, quantitative restriction, monetary policy, central banks, interest rates, financial markets
Roaring 20s	Innovation, productivity, growth, technology, savings, inequality
Secular Stagnation	Growth, inflation, productivity, demographics, macroeconomic risk, savings
Social	Discrimination, extreme parties, inequality, living together, supply chain, terror groups, unrest

Source: Blanqué *et al.* (2022).

3.3 Economic exposures of risk parity strategies

In this section, we will see how risk parity strategies are affected by economic factors. We analyze a global risk parity strategy whose portfolio consists of futures contracts on the stocks and bonds of different regions[15]. We consider portfolio returns from August 2020 through September 2024. The risk factor model uses seven factors based on economic narratives constructed around specific themes (Table 29):

1. Back to the 70s
2. Environment
3. Geopolitical Risk
4. Monetary
5. Roaring 20s
6. Secular Stagnation
7. Social

At each date, we estimate the following model using linear regression with a 52-week rolling window:

$$R_i(t) = \sum_{j=1}^{7} \beta_{i,j}(t)\, F_j^{\text{narratives}}(t) + \varepsilon_i(t)$$

where $R_i(t)$ is the weekly return of asset i, $F_j^{\text{narratives}}$ is the j^{th} economic risk factor, and $\varepsilon_i(t)$ is the idiosyncratic risk factor. Once the risk factor model is estimated at time t, we perform a risk premium analysis of the portfolio and a factor decomposition assuming a constant Sharpe ratio $\text{SR}\left(x \mid r\right) = 0.30$.

[15] The assets used are: EMU stocks, Hong Kong stocks, UK stocks, US stocks, Japanese stocks, emerging market stocks, 10-year Australian bonds, 10-year Canadian bonds, 10-year German bonds, 10-year UK bonds, and 10-year US bonds.

3.3.1 Analysis of the portfolio risk premium

Table 30 shows the implied risk premium $\tilde{\pi}(x)$ of the portfolio and its decomposition into the factor risk premium $\tilde{\psi}(x)$ and the idiosyncratic risk premium $\breve{\upsilon}(x)$. The portfolio's implied risk premium varies over time, ranging from 88 bps to 213 bps. The contribution of economic risk factors ranges from +65 bps to +211 bps. Idiosyncratic factors have a relatively high contribution to the portfolio's implied risk premium, ranging from 0 bps to +31 bps over the last 5 years. The contribution of these idiosyncratic factors can be as high as 29.34% of the portfolio's implied risk premium, as is the case on April 23, 2021. In addition, the economic risk factor model more or less explains the portfolio's return, as shown by the $\mathfrak{R}_c^2$ coefficient, which ranges from 22.59% to 57.71%.

Table 30: Risk premium contribution to the risk parity portfolio (in bps)

	2020 14/08	2021 23/04	2022 21/01	2022 01/04	2023 20/10	Range Min	 Max
$\tilde{\pi}(x)$	147	92	100	119	153	88	213
$\tilde{\psi}(x)$	137	65	87	107	138	65	211
$\breve{\upsilon}(x)$	9	27	13	13	15	0	31
$\tilde{\psi}^{\text{Back to the 70s}}(x)$	211	66	78	6	−233	−1 076	1 095
$\tilde{\psi}^{\text{Environment}}(x)$	−283	−2	−2	9	143	−585	438
$\tilde{\psi}^{\text{Geopolitical}}(x)$	−161	−33	−45	46	350	−391	553
$\tilde{\psi}^{\text{Monetary}}(x)$	233	104	−4	5	121	−335	802
$\tilde{\psi}^{\text{Roaring 20s}}(x)$	184	−4	43	9	−3	−539	397
$\tilde{\psi}^{\text{Secular Stagnation}}(x)$	−28	−46	−5	46	30	−714	1 039
$\tilde{\psi}^{\text{Social}}(x)$	−19	−20	22	−15	−270	−546	685
$\mathfrak{R}_c^2(x)$	32.81%	38.09%	44.38%	44.17%	27.41%	22.59%	57.71%

Looking at the factor contribution $\tilde{\psi}^j(x)$, all factors can make a significant contribution to the portfolio's implied risk premium depending on the period. In fact, the contributions range from −1076 bps to +1095 bps. Moreover, 71% of the time, at least 4 of the 7 factors contribute more than 20 bps (positive or negative). This implies that the economic factors have large negative and positive contributions in each period, which offset each other and result in the portfolio's implied risk premium. For example, looking at the 14 August 2020 results, going back to the 70s, environmental, geopolitical risk, monetary, and roaring 20s provide contributions of +211 bps, −283 bps, −161 bps, +233 bps, and +184 bps, respectively, while secular stagnation and social provide contributions of −28 bps and −19 bps, respectively. All these significant contributions, in both directions, result in a portfolio risk premium of only 147 bps.

A closer look at the contribution of each risk factor shows that it varies significantly in both directions. This is the case for geopolitical narratives (Figure 9). Prior to March 2021, this factor makes a large contribution with values between −161 bps and +180 bps. This contribution decreases in the following period until 2023, where it peaks positively in June with +490 bps and negatively in August with −320 bps.

3.3.2 Risk factor analysis

To understand the impact of economic narratives on the implied risk premia of the risk parity portfolio, we analyze the factor implied risk premia and the portfolio's exposure to the factors. To compare values over time and across factors, we normalize to a 10% volatility

Figure 9: Risk premium contribution of geopolitical narratives (smoothed curve)

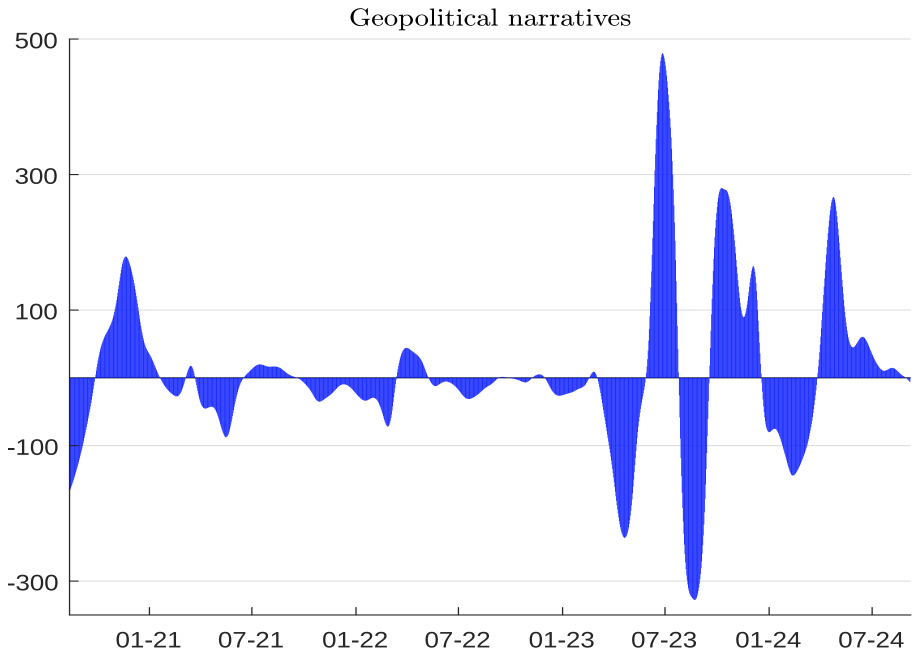


Table 31: Factor implied risk premia (in %)

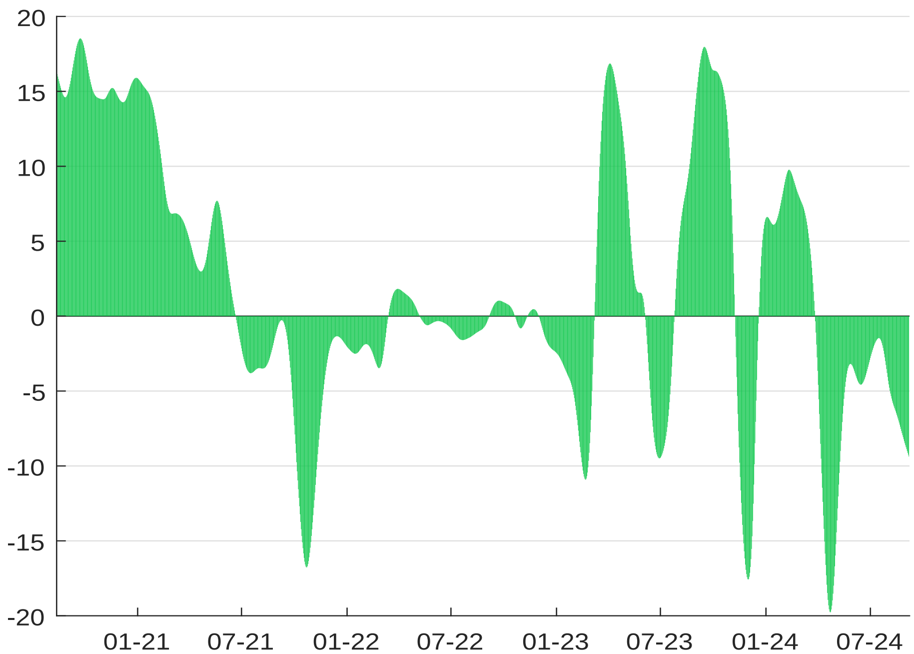
Factor	2020 14/08	2021 23/04	2022 21/01	2022 01/04	2023 20/10	Range Min	Range Max
Back to the 70s	16.3	5.3	−2.9	−0.2	9.1	−31.4	26.0
Environment	13.5	−0.8	−0.6	1.4	7.9	−31.2	23.7
Geopolitical Risk	16.8	3.4	−3.2	1.8	13.0	−36.2	21.3
Monetary	18.3	5.9	−0.2	1.3	8.1	−35.2	25.6
Roaring 20s	16.4	0.5	−2.8	−0.6	0.2	−30.6	22.8
Secular Stagnation	13.6	5.4	−0.5	1.9	13.3	−40.3	30.2
Social	19.8	3.9	−3.2	1.2	11.5	−47.5	29.9

Table 32: Factor exposure (in %)

Factor	2020 14/08	2021 23/04	2022 21/01	2022 01/04	2023 20/10	Range Min	Range Max
Back to the 70s	13.0	15.4	−27.3	−29.3	−25.5	−70.7	66.7
Environment	−20.9	−0.7	2.9	7.0	18.0	−24.0	30.5
Geopolitical Risk	−9.6	−12.6	13.9	25.3	27.0	−56.2	29.2
Monetary	12.7	17.9	17.5	3.9	15.0	−17.3	67.7
Roaring 20s	11.2	−3.0	−15.3	−14.9	−15.1	−37.0	19.1
Secular Stagnation	−2.0	−8.3	10.4	24.9	2.3	−40.0	56.4
Social	−1.0	−7.4	−6.8	−12.9	−23.5	−32.2	59.8

level. The results are shown in tables 31 and 32. Comparing the exposures in Table 32, we see a significant variation in different time periods. The risk parity portfolio is most of the time highly exposed to four or more factors simultaneously. If we look at the exposures in 20 October 2023, the portfolio had a low exposure to only one factor, secular stagnation. In addition, the risk parity portfolio has a wide range of exposure to all factors, with exposures ranging from -70.7% to $+67.7\%$.

Figure 10: Implied risk premium of geopolitical narratives (smoothed curve)



Looking at Table 31, the factor risk premia priced in by the risk parity portfolio vary significantly from period to period. In addition, the risk parity portfolio can price in all factors at once to a significant extent, as is the case in 8 August 2020. Indeed, the range of factor risk premia is from -47.5% to $+30.2\%$. Given these results, and given that $\tilde{\psi}^j(x) = y_j\tilde{\psi}_j$, we can understand why the risk parity portfolio has significant contributions from many economic factors in most of the periods in Table 30. In fact, the risk parity portfolio is heavily influenced by all economic narratives, given the high factor implied risk premia that it prices and its significant exposure to these factors. Moreover, given that factor risk premia and exposures can be negative or positive, the contributions are large and in both directions, resulting in a low portfolio implied risk premium. Focusing on the geopolitical factor (Figure 10), its implied risk premium varies from period to period. For example, in 2022, the factor had a relatively low implied risk premium ranging from -3.6% to $+2.4\%$, while after 2023, the risk premium is much higher in absolute value, ranging from -19.9% to $+19.1\%$.

Remark 2. *The previous illustration is sometimes difficult to interpret because the economic narrative factors are based on news. A more conventional exercise is to consider economic factor-mimicking portfolios (Jurczenko and Teiletche, 2023).*

3.4 Black-Litterman model in practice

3.4.1 The risk tolerance coefficient

Given an initial portfolio, the Black-Litterman model first calculates the implied risk premium as a prior estimate of the average return on assets. As explained in Section 2.2.1, we use the following formula to obtain the implied risk premia:

$$\tilde{\pi} = \frac{1}{\gamma}\Sigma x = \mathrm{SR}\left(x \mid r\right)\frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$$

where γ is the risk tolerance coefficient and $\mathrm{SR}\left(x \mid r\right)$ is the Sharpe ratio of the initial allocation. In practice, it is difficult to know the true value of γ or the Sharpe ratio, so we need to find a suitable method to select these parameters. Obviously, in the above formula, γ is the scaling factor, and its value will greatly affect the implied risk premia. The practical solution to this problem is to select a benchmark for the initial portfolio. Then, using the same value of γ, we separately calculate and compare their implied risk premia. In this case, the implied risk premium no longer has an absolute economic meaning, but rather a relative one, so the exact value of γ no longer matters. To illustrate this point, consider the MSCI USA MinVol portfolio and its benchmark, the MSCI USA Index, and select the Barra USSLOW model as the risk factor model. As shown in Table 33, different values of γ have the same degree of scaling for the implied risk premium of the portfolio and its benchmark. Thus, we can choose any value for γ and simply refer to the benchmark value when giving the view. When $\gamma = 0.40$, a view such as "*the expected return of the momentum factor is* -27 *bps*" is equivalent to a view such as "*the expected return of the momentum factor is* -13 *bps*" when $\gamma = 0.80$. By using this benchmark, portfolio managers can not only save themselves the trouble of choosing the value of γ, but can also more easily express their views on assets or factors based on the benchmark.

Table 33: Implied risk premia of style factors for different values of $\mathrm{SR}\left(x \mid r\right)$ and γ (in bps)

	USA	MinVol	USA	MinVol	USA	MinVol	USA	MinVol
$\mathrm{SR}\left(x \mid r\right)$	0.30		0.60		0.442	0.328	0.221	0.164
γ	0.589	0.438	0.295	0.219	0.40		0.80	
Beta	175	99	349	198	257	109	129	54
Dividend Yield	3	7	7	13	5	7	2	4
Earnings Quality	13	17	27	35	20	19	10	9
Earnings Yield	−21	−8	−42	−17	−31	−9	−15	−5
Growth	0	−1	0	−2	0	−1	0	−1
Leverage	9	20	17	41	13	22	6	11
Liquidity	13	8	26	16	19	9	9	4
Long-Term Reversal	−14	−16	−28	−33	−20	−18	−10	−9
Management Quality	−8	−6	−17	−11	−12	−6	−6	−3
Mid Capitalization	6	19	12	37	9	20	5	10
Momentum	−18	−24	−37	−49	−27	−27	−13	−13
Profitability	−1	2	−1	5	−1	3	−1	1
Prospect	7	4	14	9	10	5	5	2
Residual Volatility	54	28	108	56	80	31	40	15
Size	4	10	8	20	6	11	3	5
Value	−3	−2	−5	−5	−4	−3	−2	−1

Remark 3. *Choosing a constant value for the risk tolerance coefficient implies different values for the Sharpe ratio, and vice versa. Therefore, it is not neutral to assume a value for* γ *or* $\mathrm{SR}\left(x \mid r\right)$.

3.4.2 The impact of Γ_π

The basic assumption of the Black-Litterman model is that the prior distribution of $\pi(t)$ is:

$$\pi(t) \sim \mathcal{N}(\tilde{\pi}, \Gamma_\pi)$$

where Γ_π is the covariance matrix of the prior distribution, implying the degree of uncertainty in the implied risk premia. For the specification of Γ_π, we use the traditional rule $\Gamma_\pi = \tau\Sigma$, *i.e.* we accept the assumption that the structure of the covariance matrix Γ_π of the estimate is proportional to the covariance matrix Σ of the returns, in order to simplify the computation. However, the choice of τ is ambiguous in the literature. According to Black and Litterman (1992), τ should take a small value close to 0 because the uncertainty in the mean is smaller than the uncertainty in the returns. Therefore, as many academic papers suggest (Black and Litterman, 1992; Lee, 2000; He and Litterman, 2002; Idzorek, 2004), the value of τ is usually set between 1% and 50%. Furthermore, Blamont and Firoozy (2003) and Meucci (2010) interpret $\tau\Sigma$ as the standard error of the mean estimate of the implied risk premia $\tilde{\pi}$ and set τ to be inversely proportional to the total number of observations. In contrast to others, Satchell and Scowcroft (2000) argue that τ should be set to 100% under the assumption that the equilibrium excess returns conditional on the individual's forecasts are on average equal to the individual's forecast.

As shown in Figure 11, the covariance matrix of the posterior distribution is made up of the covariance matrix of the prior distribution Γ_π and the covariance matrix of the view distribution Φ. Therefore, their relative values are more important than their absolute values, and they should be determined together. Based on this idea, Idzorek (2004) proposed a method for determining these two parameters based on a confidence level from 0% to 100% specified by the portfolio manager. The underlying idea of Idzorek's method is that the final portfolio weight x_{final} caused by the manager's views should be between the initial portfolio x_{initial} and the portfolio that depends only on these views $x_{100\%\text{ sure}}$, where we are 100% sure about this view. In this last case, $\Phi = \mathbf{0}_{p,p}$, and the tilt follows a linear relationship:

$$\text{Confidence level} = \frac{x_{\text{final}} - x_{\text{initial}}}{x_{100\%\text{ sure}} - x_{\text{initial}}} \tag{10}$$

Given an arbitrary value of τ and a view with a confidence level of c%, for instance, the i-th line of the matrix P_π and the i^{th} element of the vector v_π, we will find the corresponding value $(\Phi_\pi)_{i,i}$ to satisfy Equation (10). We perform the same calculation for each view in the matrix P and get the value of $(\Phi_\pi)_{i,i}$ for each view and summarize them in a single matrix Φ_π. This approach generally simplifies the process of specifying the uncertainty in the prior information and the uncertainty in the views for the investor, who only needs to specify the confidence level for each view without having to determine the scalar values for τ and Φ_π. For more details on this method, see Idzorek (2004) and Walters (2014).

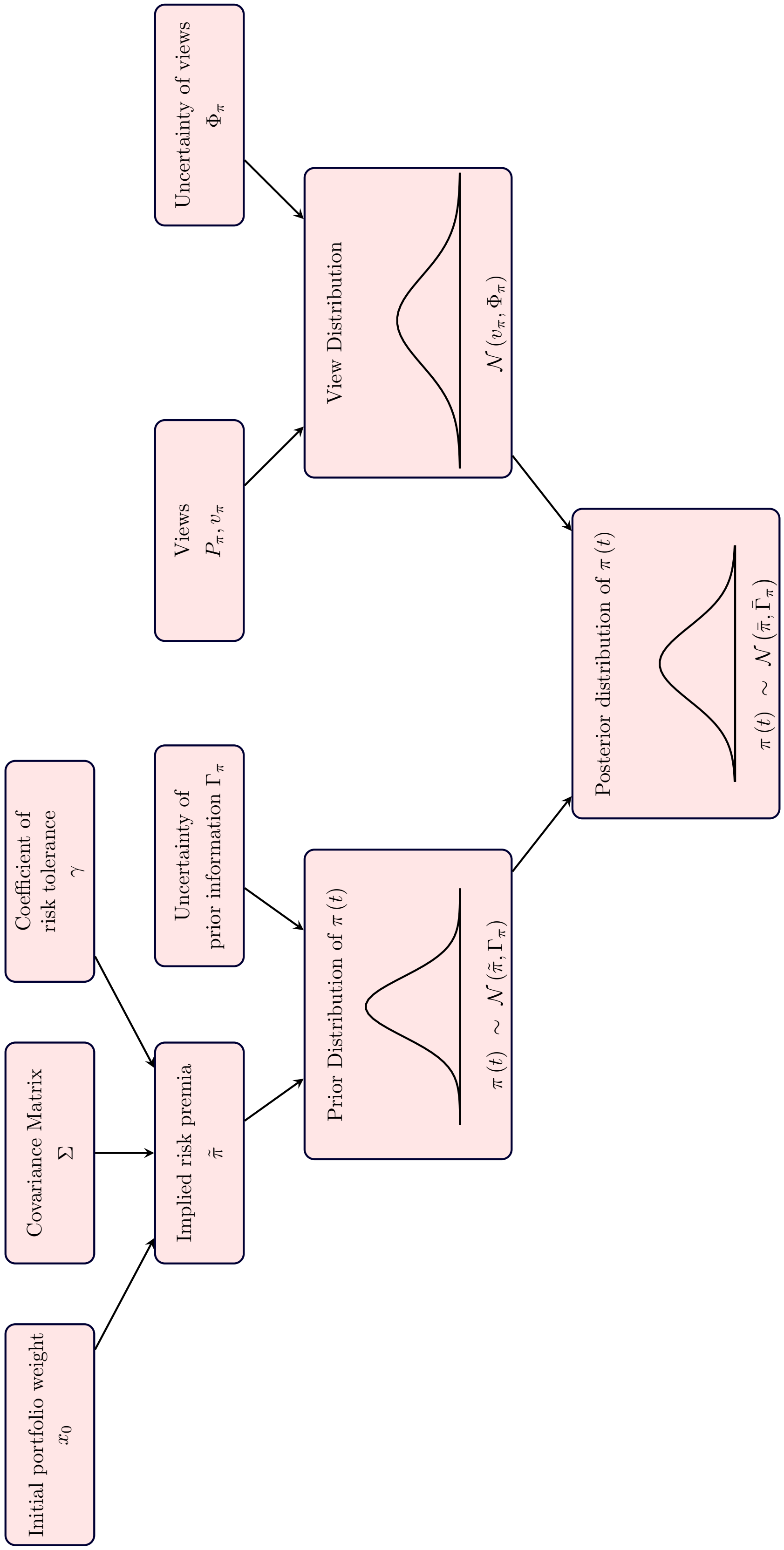
By analogy, for the Black-Litterman factor model presented in this paper, we can assume that the structure of Γ_ψ takes the same form as the covariance matrix Ω of the factor returns, *i.e.*, $\Gamma_\psi = \tau\Omega$ and we can use the same method to decide the value of Φ_ψ based on a confidence level from 0% to 100% for each view.

3.4.3 Optimization with Σ or $\bar{\Sigma}$?

Another thing that is not conclusively determined in the Black-Litterman model is whether to use the empirical covariance matrix Σ estimated from historical data or the covariance matrix $\bar{\Sigma}$ of the posterior distribution in the final optimization process. As explained in Section 2.4.1 on page 15, $\bar{\Sigma}$ is defined as:

$$\bar{\Sigma} = \Sigma + \bar{\Gamma}_\pi$$

Figure 11: Bayesian framework of the Black-Litterman model



Source: Idzorek (2004, Figure 1).

In the absence of views we have $\bar{\Gamma}_\pi = \Gamma_\pi$, and this reduces to:

$$\bar{\Sigma} = \Sigma + \Gamma_\pi = (1+\tau)\,\Sigma$$

Since τ is positive, the unconditional covariance matrix of asset returns $\bar{\Sigma}$ will be larger than the estimated covariance matrix of the historical data Σ when the Black-Litterman model is applied in the absence of views. This has been mentioned by He and Litterman (2002) and Kolm and Ritter (2017). This fact makes sense in theory, but it also means that the portfolio weight will change in the absence of views. In practice, we do not want this to happen. Therefore, Idzorek (2004) and others do not compute a new a posteriori covariance matrix $\bar{\Sigma}$, but instead use the known input covariance matrix Σ. In this case, the portfolio weight does not change in the absence of views. In addition, when we use Idzorek's method to calibrate the covariance matrix of the view distribution, using Σ makes the mathematics easier and allows us to use an analytical formula to obtain the result according to Walters (2014). For these two reasons, we recommend using the empirical covariance matrix in the final optimization process.

3.4.4 Impact of views

In this section, we will continue to use the MSCI USA Minimum Volatility Index as the portfolio to illustrate the application of the Black-Litterman factor model. We choose the Barra USSLOW model as the risk factor model and set the risk tolerance coefficient γ to 0.4. In Table 33 on page 35 we have reported the factor risk premia of this portfolio. The portfolio manager can express his view on these factors by referring to the benchmark.

Case 1 We assume that the portfolio manager has an absolute view on the liquidity factor: $v_{\text{Liquidity}} = 20$ bps. We use Idzorek's method and set the confidence level of this view to 50%. The optimized portfolio is derived from the following problem:

$$\begin{aligned} \bar{x}_+ &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \bar{\pi} \\ &\text{s.t.} \quad \begin{cases} \mathbf{1}_n^\top x = 1 \\ x \geq \mathbf{0}_n \end{cases} \end{aligned} \tag{11}$$

The results of this view are shown in Table 34. For each style factor, we report the vector of implied (or prior) risk premia $\tilde{\psi}$, the vector of posterior risk premia $\bar{\psi}$, the prior factor exposure y, the long/short posterior factor exposure $\bar{y}$, the long-only posterior factor exposure $\bar{y}_+$, and the vector of implied risk premia $\bar{\psi}_+$ inferred from the portfolio $\bar{x}_+$. Since the portfolio manager believes that the liquidity factor has a risk premium of 20 bps at the 50% confidence level[16], the model increases the risk premium of this factor from 9 bps to 13 bps and also increases the exposure from -25.9% to $+1.5\%$. In addition, the tracking error volatility of the portfolio $\bar{x}_+$ with respect to the current portfolio x is equal to 160 bps.

By changing the value of Φ_ψ we can analyse the effect of the confidence level on the volatility of the tracking error. We find that the minimum tracking error is 139 bps. This result is worrying as the lower bound of zero is never reached. Let us understand this problem. We have $\tilde{\pi} = \frac{1}{\gamma}\Sigma x$ and $\tilde{\psi} = B^+ \tilde{\pi}$. If we do not formulate any views, the posterior risk premia are equal to the prior risk premia:

$$\bar{\psi} = \tilde{\psi} = B^+ \tilde{\pi} = \frac{1}{\gamma} B^+ \Sigma x$$

Table 34: Impact of views on factor allocation (Case 1, confidence level = 50%)

Factor	$\tilde{\psi}$	$\bar{\psi}$ (in bps)	$\bar{\psi}_+$	y	$\bar{y}$ (in %)	$\bar{y}_+$
Beta	109	113	113	−75.6	−75.5	−74.5
Dividend Yield	7	8	8	1.3	4.6	4.1
Earnings Quality	19	19	19	25.9	27.3	25.7
Earnings Yield	−9	−10	−10	−2.9	−3.0	−2.6
Growth	−1	−1	−1	−12.4	−13.0	−12.4
Leverage	22	22	22	0.1	1.3	−0.1
Liquidity	9	14	13	−25.9	6.7	1.5
Long-Term Reversal	−18	−18	−18	−9.6	−10.0	−8.7
Management Quality	−6	−7	−7	15.1	15.1	12.8
Mid Capitalization	20	20	20	−66.9	−71.4	−70.1
Momentum	−27	−28	−28	−3.5	−4.1	−3.8
Profitability	3	1	1	17.3	14.4	13.0
Prospect	5	4	5	−12.7	−13.6	−11.5
Residual Volatility	31	35	35	15.5	17.6	17.0
Size	11	11	11	40.5	44.1	42.3
Value	−3	−3	−3	−12.7	−14.6	−12.8

We deduce that:

$$\bar{\pi} = B\bar{\psi} = \frac{1}{\gamma} BB^{+}\Sigma x \neq \tilde{\pi}$$

The solution of the MVO problem is then equal to:

$$\bar{x} = \gamma\Sigma^{-1}\bar{\pi} = \Sigma^{-1}BB^{+}\Sigma x$$

We do not obtain the equality $\bar{x} = x$ because $BB^{+} \neq I_n$. The tracking error variance of the optimized portfolio $\bar{x}$ with respect to the current portfolio is then:

$$\begin{aligned} \sigma^2\left(\bar{x} \mid x\right) &= \left(\bar{x} - x\right)^\top \Sigma \left(\bar{x} - x\right) \\ &= x^\top Q x \geq 0 \end{aligned}$$

where $Q = M^\top \Sigma^{-1} M$ and $M = \left(BB^{+} - I_n\right)\Sigma$. This means that when we formulate a Black-Litterman model with a risk factor model, there are two key effects:

- The effect of the risk factor model itself;
- The effect of active views on factor risk premia.

By considering the risk factor model $R(t) - r = BF(t) + \varepsilon(t)$, we are implicitly forming a view on assets, even if we do not make explicit bets on factor risk premia. Thus, the tracking-error volatility of a Black-Litterman portfolio $\bar{x}$ can be decomposed into two components:

$$\sigma\left(\bar{x} \mid x\right) = \underbrace{\sigma\left(\bar{x}^{(\text{no view})} \mid x\right)}_{\text{Effect of the factor model structure}} + \underbrace{\sigma\left(\bar{x} \mid \bar{x}^{(\text{no view})}\right)}_{\text{Effect of the active bets formulated by the fund manager}}$$

where $\bar{x}^{(\text{no view})}$ is the optimal Black-Litterman portfolio without active views. In the previous example, the total tracking error volatility was 160 pbs, of which 139 bps were due to the BARRA factor model, and 21 bps were attributed to the active bet on the liquidity risk factor.

Table 35: Impact of the confidence level of the view on posterior factor risk premia $\bar{\psi}$ in bps (Case 1)

Confidence level		1%	25%	50%	75%	99%
Beta	108.6	108.7	110.7	112.8	114.8	117.5
Dividend Yield	7.2	7.2	7.5	7.9	8.3	8.8
Earnings Quality	19.0	19.0	19.2	19.3	19.5	19.7
Earnings Yield	−9.1	−9.2	−9.6	−10.2	−10.6	−11.3
Growth	−1.2	−1.2	−1.1	−1.0	−0.9	−0.8
Leverage	22.3	22.3	22.4	22.4	22.5	22.5
Liquidity	8.9	9.0	11.4	14.1	16.5	19.9
Long-Term Reversal	−17.9	−17.9	−18.0	−18.1	−18.2	−18.3
Management Quality	−6.1	−6.1	−6.5	−6.9	−7.3	−7.9
Mid Capitalization	20.3	20.3	20.2	20.1	20.0	19.8
Momentum	−26.7	−26.7	−27.3	−27.8	−28.3	−29.0
Profitability	2.6	2.6	2.0	1.4	0.8	−0.0
Prospect	4.7	4.7	4.5	4.3	4.2	3.9
Residual Volatility	30.6	30.7	32.6	34.8	36.8	39.6
Size	10.9	10.9	11.0	11.2	11.3	11.5
Value	−2.6	−2.6	−2.7	−2.8	−2.9	−3.0
$\sigma\left(\bar{x} \mid x\right)$ (in bps)		138.7	144.4	159.8	178.7	208.9
$\sigma\left(\bar{x} \mid \bar{x}^{(\text{no view})}\right)$ (in bps)		0.1	5.6	21.0	40.0	70.1

Table 35 shows the effect of the confidence level of the view on the posterior factor risk premia $\bar{\psi}$. The more confident the portfolio manager is in his view of the liquidity factor, the closer the posterior risk premium will be to 20 bps. Since the covariance matrix Ω of the factors in the Barra model is not a diagonal matrix, a view on one factor can affect other factors. In our case, the factors beta and residual volatility are significantly affected when the confidence level of the view is high. In addition, the tracking error between the prior portfolio x and the posterior portfolio $\bar{x}_+$ increases with the confidence level.

Remark 4. *There are several practical ways to eliminate the effect of the risk factor model. The first approach is to impact only the allocation variation between the portfolio $\bar{x}^{(no\ view)}$ and $\bar{x}$. In this case, we have:*

$$\bar{x} \leftarrow x + \left(\bar{x} - \bar{x}^{(no\ view)}\right)$$

The second approach is to use the factor decomposition: $\tilde{\pi} = \tilde{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})}$. We then apply the views to the factor component only and calculate the posterior risk premia as follows:

$$\bar{\pi} = \bar{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})}$$

This means that we do not change the posterior risk premia due to idiosyncratic risks[17]*.*

[16] This is equivalent to set $\Phi_\psi = 5 \times 10^{-4}$.

[17] We have $\tilde{\pi}^{(\text{factor})} = \varphi_x B \Omega B^\top x$ and $\tilde{\pi}^{(\text{specific})} = \varphi_x D x$. It follows that $\tilde{\psi} = B^+ \tilde{\pi}^{(\text{factor})}$ and:

$$\bar{\psi} = \tilde{\psi} + \Gamma_\psi P_\psi^\top \left(P_\psi \Gamma_\psi P_\psi^\top + \Phi_\psi\right)^{-1} \left(v_\psi - P_\psi \tilde{\psi}\right)$$

We deduce that $\bar{\pi}^{(\text{factor})} = B\bar{\psi}$. Finally, the posterior risk premia are equal to $\bar{\pi} = \bar{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})}$. In the case of no views, we have $\bar{\psi} = \tilde{\psi}$ and $\bar{\pi} = B\bar{\psi} + \tilde{\pi}^{(\text{specific})} = B\tilde{\psi} + \tilde{\pi}^{(\text{specific})} = BB^+ \tilde{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})} = \varphi_x BB^+ B\Omega B^\top x + \tilde{\pi}^{(\text{specific})} = \varphi_x B\Omega B^\top x + \tilde{\pi}^{(\text{specific})} = \tilde{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})} = \tilde{\pi}$, implying that the portfolio weights do not change when we consider the mean-variance optimization problem without constraints.

Case 2 As mentioned above, a view on one factor can affect the risk premia of other factors. Suppose the portfolio manager wants to maintain the factor risk premia for beta and residual volatility. Therefore, we have three absolute views: $v_{\text{liquidity}} = 20$ bps, $v_{\text{res.vol}} = 31$ bps, and $v_{\text{beta}} = 109$ bps. The results are shown in Table 36. First, we set the confidence level for the second and third views to 50%. The model always increases the risk premium of the liquidity factor from 8.9 bps to 19.9 bps, but the risk premia of the residual volatility and beta factors reach 35.9 bps and 111.9 bps, respectively. Therefore, we perform a final portfolio optimization by setting the confidence level to 99% for all three views. In this case, we obtain a solution which satisfies the three views. However, this is not always the case. In this example, we were able do so because the risk factors are more or less uncorrelated.

Table 36: Impact of the confidence level of the views on $\bar{\psi}$ in bps (Case 2)

	View 1	1%	25%	50%	75%	99%	99%
Confidence level	View 2	50%	50%	50%	50%	50%	99%
	View 3	50%	50%	50%	50%	50%	99%
Beta	108.6	108.9	109.5	110.3	110.9	111.9	109.0
Dividend Yield	7.2	7.2	7.5	7.8	8.1	8.6	8.3
Earnings Quality	19.0	19.0	19.2	19.3	19.5	19.7	20.0
Earnings Yield	−9.1	−9.2	−9.5	−9.8	−10.1	−10.6	−9.9
Growth	−1.2	−1.2	−1.1	−1.0	−0.9	−0.7	−0.7
Leverage	22.3	22.3	22.4	22.6	22.7	22.9	23.5
Liquidity	8.9	9.0	11.3	13.9	16.4	19.9	19.9
Long-Term Reversal	−17.9	−17.9	−18.0	−18.0	−18.0	−18.1	−17.8
Management Quality	−6.1	−6.1	−6.4	−6.8	−7.2	−7.7	−7.5
Mid Capitalization	20.3	20.3	20.1	19.8	19.6	19.2	18.3
Momentum	−26.7	−26.8	−27.1	−27.5	−27.8	−28.3	−28.4
Profitability	2.6	2.6	2.1	1.6	1.1	0.5	1.0
Prospect	4.7	4.7	4.5	4.3	4.0	3.7	3.6
Residual Volatility	30.6	30.8	31.9	33.1	34.3	35.9	31.0
Size	10.9	10.8	11.2	11.7	12.1	12.7	14.1
Value	−2.6	−2.6	−2.6	−2.7	−2.7	−2.8	−2.3
$\sigma\left(\bar{x} \mid x\right)$ (in bps)		138.8	143.8	157.8	175.9	204.7	205.4
$\sigma\left(\bar{x} \mid \bar{x}^{(\text{no view})}\right)$ (in bps)		0.1	5.1	19.1	37.1	65.9	66.6

4 Conclusion

This article introduces an innovative approach that integrates the Black-Litterman model with risk factor models. This method allows fund managers to assess how risk factors are priced by the market/benchmark and, in turn, to understand how their own portfolio is priced relative to that benchmark. A key advantage of this approach is that it allows the manager to express views on the underlying risk factors rather than on the individual assets within the portfolio, which can be numerous and complex. By calculating the implicit risk premia of these factors, the manager can refine his views and make more informed adjustments to the portfolio in line with updated expectations. For instance, if the portfolio consists of 500 stocks and the risk model is based on 10 factors, this approach allows the manager to form views on the 10 factors instead of the 500 individual stocks. Of course, this approach requires the manager to use and be familiar with top-down factor construction, rather than traditional bottom-up stock selection.

The first part of this article develops the theoretical framework, linking asset risk premia to factor risk premia using a generic risk factor model. We also introduce the concept of factor exposures and how to calculate them. This allows us to decompose the implied risk premium of a portfolio into its factor and idiosyncratic components. Incorporating active views on risk factors is equivalent to computing the posterior distribution of factor risk premia. We can then optimize the portfolio according to the fund manager's views. The underlying framework closely mirrors the original Black-Litterman model. However, the structure of the factor model allows us to derive several key properties. For example, we prove that the tangent portfolio is the only portfolio for which the residual risk premium is zero. For all other portfolios, the residual risk premium remains positive, confirming that idiosyncratic risk is not rewarded in a well-diversified portfolio. In addition, we decompose both asset and factor risk premia into variance and covariance risk premia. This decomposition is used extensively by Portelli and Roncalli (2024) in multi-asset portfolios to explain the hedging asset status of bonds. It is particularly important for understanding why a risk factor may have a negative risk premium due to its covariance components.

Using the developed framework, the study of the US stock market by analyzing the S&P 500 index revealed that the market factor is the main driver of the portfolio's implied risk premium and that the remaining three factors (size, value and momentum) make relatively small contributions. We found that idiosyncratic risk contributes very little to the portfolio's risk premium, leading us to conclude that the Carhart risk factor model explains almost all of the implied risk premium of the US stock market. Our second study illustrates how an actively managed portfolio can be benchmarked. We consider the Barra factor model, the active portfolio corresponding to the MSCI USA MinVol Index, and the benchmark corresponding to the MSCI USA Index. In March 2024, the most important factors were the Beta style factor and the Information Technology and Consumer Discretionary sector factors. A bottom-up analysis of both portfolios shows that the MSCI MinVol portfolio can price a stock high but still not hold it due to its high risk. The methodology can also be applied to economic factors. The third illustration uses a global risk parity strategy and macroeconomic factors based on economic reports and daily news. The factor contributions vary significantly from period to period. This exercise can also be done with macroeconomic factor-mimicking portfolios.The article also discussed the impact of the parameters. In fact, the coefficient of risk tolerance γ, the covariance matrix of the prior distribution Γ_ψ, the covariance matrix of the views Φ_ψ, and the choice of using Σ or $\bar{\Sigma}$ all have a significant impact on the results of the optimization problem. Moreover, we show that not having a view on factor risk premia is not the same as having no view at all. Indeed, the Black-Litterman factor approach assumes that asset returns are driven by a risk factor model, and the use of such a model is inherently an active view. Therefore, when incorporating views, it is important to distinguish between the contribution of implicit views arising from the structure of the factor model and the contribution of explicit views on factor risk premia.

The methodology developed here is particularly valuable for fund managers seeking to assess portfolio risk within a risk factor framework. With the rise of factor investing (Cazalet *et al.*, 2014; Cazalet and Roncalli, 2014) and alternative risk premia strategies (Hamdan *et al.*, 2016; Roncalli, 2017), many fund managers now use risk factors, rather than individual assets or securities, as the fundamental building blocks of their portfolios. Similarly, multi-asset fund managers often view embedded macroeconomic risk factors as their true exposures and active bets. In these contexts, the Black-Litterman factor model provides a practical approach for incorporating such risk factors into portfolio management. Additionally, analyzing risk premia at the factor level is crucial for active management, as it provides insight into what is currently priced into the market or benchmark. For an active manager, success lies in positioning the portfolio relative to this market view.

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A Mathematical results

A.1 Conditional probability distribution in the Gaussian case

Let us consider a Gaussian random vector defined as follows:

$$\left(\begin{array}{c} X \\ Y \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_x \\ \mu_y \end{array}\right), \left(\begin{array}{cc} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{array}\right)\right)$$

The conditional probability distribution of Y given $X = x$ is a multivariate normal distribution. We have:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{y,x}\Sigma_{x,x}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{y,y|x} = \sigma^2\left[Y \mid X = x\right] = \Sigma_{y,y} - \Sigma_{y,x}\Sigma_{x,x}^{-1}\Sigma_{x,y}$$

A.2 Proofs of mathematical properties

A.2.1 Implied risk premium

We assume that $\psi_j \geq 0$. If this is not the case, we can always replace the j^{th} factor $F_j\left(t\right)$ by its opposite $-F_j\left(t\right)$.

Property 1. *The factor exposures are equal to the beta sensitivities:*

$$y = \beta\left(x\right)$$

Proof. We have $y = B_x x = B^\top x = \beta\left(x\right)$. ∎

Property 2. *The covariance model is obtained by setting $B = I_n$ and $D = \mathbf{0}_{n,n}$. In this case, we have $R\left(t\right) = F\left(t\right)$ and $\Sigma = \Omega$. We have:*

$$\tilde{\psi} = \tilde{\pi}$$

Proof. We have $B^+ = I_n$. ∎

Property 3. *The implied risk premium priced in by the tangent portfolio is equal to the vector of risk premia:*

$$\tilde{\pi} = \pi := \mu - r\mathbf{1}_n$$

Proof. The tangent portfolio is given by $x^\star = \gamma\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)$ where $\gamma = 1/\left(\mathbf{1}_n^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)\right)$. Since we have $\Sigma x^\star = \gamma\left(\mu - r\mathbf{1}_n\right)$, $\sigma^2\left(x\right) = {x^\star}^\top\Sigma x^\star = \gamma^2\left(\mu - r\mathbf{1}_n\right)^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)$ and $\mu\left(x\right) - r = {x^\star}^\top\left(\mu - r\mathbf{1}_n\right) = \gamma\left(\mu - r\mathbf{1}_n\right)^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)$, we conclude that:

$$\begin{aligned}
\tilde{\pi} &= \text{SR}\left(x^\star \mid r\right)\frac{\Sigma x^\star}{\sqrt{{x^\star}^\top\Sigma x^\star}} \\
&= \frac{\gamma\left(\mu - r\mathbf{1}_n\right)^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}{\sqrt{\gamma^2\left(\mu - r\mathbf{1}_n\right)^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}} \cdot \frac{\gamma\left(\mu - r\mathbf{1}_n\right)}{\sqrt{\gamma^2\left(\mu - r\mathbf{1}_n\right)^\top\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}} \\
&= \mu - r\mathbf{1}_n \\
&= \pi
\end{aligned}$$

∎

Property 4. *The residual premium of the tangency portfolio is zero, which implies that the risk premium of the tangency portfolio is fully explained by the common risk factors:*

$$\begin{cases} \breve{\upsilon}\left(x^{\star}\right) = \breve{y}^{\top}\breve{\upsilon} = 0 \\ \tilde{\pi}\left(x^{\star}\right) = \tilde{\psi}\left(x^{\star}\right) = \dfrac{\pi^{\top}\Sigma^{-1}\pi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi} \end{cases}$$

Proof. We have:

$$\tilde{\pi}\left(x^{\star}\right) = x^{\star\top}\tilde{\pi} = \frac{\pi^{\top}\Sigma^{-1}\pi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi} = \frac{\pi^{\top}\Sigma^{-1}B\psi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi}$$

and:

$$\tilde{\psi} = B^{+}\tilde{\pi} = B^{+}\pi$$

We deduce that:

$$\tilde{\psi}\left(x^{\star}\right) = y^{\top}\tilde{\psi} = x^{\star\top}BB^{+}\tilde{\pi} = x^{\star\top}BB^{+}B\psi = \frac{\pi^{\top}\Sigma^{-1}B\psi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi}$$

because $BB^{+}B = B$. We conclude that:

$$\tilde{\psi}\left(x^{\star}\right) = \frac{\pi^{\top}\Sigma^{-1}\pi}{\mathbf{1}_n^{\top}\Sigma^{-1}\pi} = \tilde{\pi}\left(x^{\star}\right)$$

and:

$$\breve{\upsilon}\left(x^{\star}\right) = \tilde{\pi}\left(x^{\star}\right) - \tilde{\psi}\left(x^{\star}\right) = 0$$

■

Property 5. *If the portfolio is not optimal, the residual premium does not depend on the factor covariance Ω, but only on the idiosyncratic covariance D. The residual premium is then positive:*

$$\breve{\upsilon}\left(x^{\star}\right) = x^{\top}\left(\breve{B}_x^{\top}\breve{B}_x D\right)x \geq 0$$

Proof. We have:

$$\begin{aligned} \breve{\upsilon}\left(x\right) &= \breve{y}^{\top}\breve{\upsilon} \\ &= \left(\breve{B}_x x\right)^{\top}\breve{B}_{\sigma}\tilde{\pi} \\ &= \mathrm{SR}\left(x \mid r\right) x^{\top}\breve{B}_x^{\top}\breve{B}_x\frac{\Sigma x}{\sqrt{x^{\top}\Sigma x}} \\ &= \varphi_x\, x^{\top}\left(\breve{B}_x^{\top}\breve{B}_x\Sigma\right)x \end{aligned}$$

where $\varphi_x = \dfrac{\mathrm{SR}\left(x \mid r\right)}{\sqrt{x^{\top}\Sigma x}} > 0$. We deduce that:

$$\begin{aligned} Q &= \breve{B}_x^{\top}\breve{B}_x\Sigma \\ &= \breve{B}_x^{\top}\breve{B}_x\left(B\Omega B^{\top} + D\right) \\ &= \breve{B}_x^{\top}\breve{B}_x B\Omega B^{\top} + \breve{B}_x^{\top}\breve{B}_x D \\ &= \breve{B}_x^{\top}\breve{B}_x D \end{aligned}$$

because $\ker\left(B^{+}\right) = \ker\left(B^{\top}\right)$ and $\breve{B}_x B = \left(B^{\top}\breve{B}_x^{\top}\right)^{\top} = \left(B^{\top}\ker\left(B^{\top}\right)\right)^{\top} = 0$. According to Ramuzat (2022), the product of a real symmetric positive semi-definite matrix and a real positive diagonal matrix is positive semi-definite. In our case, $\breve{B}_x^{\top}\breve{B}_x$ is the product of a real matrix and its transpose, so it's a real symmetric positive semi-definite matrix and D is a real positive diagonal matrix. Therefore, $Q = \breve{B}_x^{\top}\breve{B}_x D$ is a positive semi-definite matrix. We conclude that $\breve{\upsilon}(x)$ is the product of a positive scalar and a quadratic form $x^{\top}Qx$ and is therefore positive: $\forall x : \breve{\upsilon}(x) = \varphi_x\, x^{\top}Qx \geq 0$. ■

Property 6. *The factor risk premium of any portfolio is always positive:*

$$\tilde{\psi}(x) \geq 0$$

Proof. We have:

$$\begin{aligned}
\tilde{\psi}(x) &= y^{\top}\tilde{\psi} \\
&= (B_x x)^{\top} B_{\sigma}\tilde{\pi} \\
&= x^{\top}BB^{+}\tilde{\pi} \\
&= \varphi_x\, x^{\top}BB^{+}\Sigma x
\end{aligned}$$

We note $Q = BB^{+}\Sigma$. We have:

$$\begin{aligned}
Q &= BB^{+}\left(B\Omega B^{\top} + D\right) \\
&= B\Omega B^{\top} + BB^{+}D \\
&= Q_1 + Q_2
\end{aligned}$$

The first term $Q_1 = B\Omega B^{\top}$ defines a positive semi-definite matrix because Ω is positive semi-definite matrix. We deduce that $x^{\top}Q_1 x \geq 0$. For the second term $Q_2 = BB^{+}D$, we use the property that for any matrix $B \in \mathbb{R}_{n\times m}$, BB^{+} is an orthogonal projection matrix, which means that $BB^{+} = \left(BB^{+}\right)^{\top}$ and $BB^{+} = \left(BB^{+}\right)^{2}$. Furthermore, for all $y \in \mathbb{R}_n$, we have:

$$y^{\top}BB^{+}y = y^{\top}\left(BB^{+}\right)^{2}y = y^{\top}\left(BB^{+}\right)^{\top}\left(BB^{+}\right)y = \|BB^{+}y\|^{2} \geq 0$$

So BB^{+} is positive semi-definite. Then we apply the theorem that the product of a real symmetric positive semi-definite matrix and a real positive diagonal matrix is positive semi-definite (Ramuzat, 2022). We deduce that $x^{\top}Q_2 x \geq 0$. We conclude that:

$$\underbrace{x^{\top}B\Omega B^{\top}x}_{\geq 0} + \underbrace{x^{\top}BB^{+}Dx}_{\geq 0} \geq 0$$

and $\tilde{\psi}(x) \geq 0$. ■

Property 7. *The asset risk premia admit a variance-covariance decomposition:*

$$\tilde{\pi} = \tilde{\pi}^{(\mathrm{var})} + \tilde{\pi}^{(\mathrm{cov})}$$

where $\tilde{\pi}^{(\mathrm{var})}$ depends on the asset variances and $\tilde{\pi}^{(\mathrm{cov})}$ depends on the asset covariances.

Proof. We have:

$$\tilde{\pi}_i = \mathrm{SR}\left(x \mid r\right)\frac{(\Sigma x)_i}{\sqrt{x^{\top}\Sigma x}}$$

We deduce that $\tilde{\pi}_i^{(\mathrm{var})} = \varphi_x\, x_i\sigma_i^2$ and $\tilde{\pi}_i^{(\mathrm{cov})} = \varphi_x\, \sigma_i \sum_{j\neq i} x_j\rho_{i,j}\sigma_j$. ■

Property 8. *The asset risk premia admit a factor decomposition:*

$$\tilde{\pi} = \tilde{\pi}^{(\text{factor})} + \tilde{\pi}^{(\text{specific})}$$

where $\tilde{\pi}^{(\text{factor})}$ *depends on the common factors and* $\tilde{\pi}^{(\text{specific})}$ *depends on the idiosyncratic factors.*

Proof. We have:

$$\begin{aligned} \tilde{\pi} &= \operatorname{SR}\left(x \mid r\right) \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\ &= \varphi_x \, B \Omega B^\top x + \varphi_x \, D x \end{aligned}$$

We deduce that $\tilde{\pi}^{(\text{factor})} = \varphi_x \, B \Omega B^\top x$ and $\tilde{\pi}^{(\text{specific})} = \varphi_x \, D x$. ∎

Property 9. *The factor risk premia allow the following decompositions:*

$$\left\{ \begin{array}{l} \tilde{\psi} = \tilde{\psi}^{(\text{var})} + \tilde{\psi}^{(\text{cov})} \\ \tilde{\psi} = \tilde{\psi}^{(\text{factor})} + \tilde{\psi}^{(\text{specific})} \end{array} \right.$$

where $\tilde{\psi}^{(\text{var})}$, $\tilde{\psi}^{(\text{cov})}$, $\tilde{\psi}^{(\text{factor})}$ *and* $\tilde{\psi}^{(\text{specific})}$ *depend on the factor variances, factor covariances, common factors and idiosyncratic factors.*

Proof. We have:

$$\begin{aligned} \tilde{\psi} &= B_\sigma \tilde{\pi} \\ &= \varphi_x \, B_\sigma \Sigma x \\ &= \varphi_x \, B^+ \left(B \Omega B^\top + D \right) \left(B_y y + \breve{B}_y \breve{y} \right) \\ &= \varphi_x \left(B^+ B \Omega B^\top B_y y + B^+ B \Omega B^\top \breve{B}_y \breve{y} + B^+ D B_y y + B^+ D \breve{B}_y \breve{y} \right) \end{aligned}$$

The second term is equal to zero because $B^\top$ and B^+ have the same null space. We deduce that:

$$\tilde{\psi} = \varphi_x \left(B^+ B \Omega B^\top B_y y + B^+ D B_y y + B^+ D \breve{B}_y \breve{y} \right)$$

We get the following decomposition:

$$\left\{ \begin{array}{l} \tilde{\psi}^{(\text{factor})} = \varphi_x B^+ B \Omega B^\top B_y y \\ \tilde{\psi}^{(\text{specific})} = \varphi_x \left(B^+ D B_y y + B^+ D \breve{B}_y \breve{y} \right) \end{array} \right.$$

In this case, $\tilde{\psi}^{(\text{factor})}$ and $\tilde{\psi}^{(\text{specific})}$ depend on the covariance matrices Ω and D, respectively. This decomposition is equivalent to:

$$\left\{ \begin{array}{l} \tilde{\psi}^{(\text{factor})} = B_\sigma \tilde{\pi}^{(\text{factor})} \\ \tilde{\psi}^{(\text{specific})} = B_\sigma \tilde{\pi}^{(\text{specific})} \end{array} \right.$$

Another decomposition is:

$$\left\{ \begin{array}{l} \tilde{\psi}^{(\text{factor})} = \varphi_x \left(B^+ B \Omega B^\top B_y + B^+ D B_y \right) y \\ \tilde{\psi}^{(\text{specific})} = \varphi_x B^+ D \breve{B}_y \breve{y} \end{array} \right.$$

In this case, $\tilde{\psi}^{(\text{factor})}$ and $\tilde{\psi}^{(\text{specific})}$ depend on the factor exposures y and $\breve{y}$, respectively. Let $\omega = \operatorname{diag}(\Omega)$ be the variance of the factors. We have:

$$\Omega = \omega \odot I_m + (\Omega - \omega \odot I_m)$$

We deduce that:

$$\tilde{\psi} = \tilde{\psi}^{(\text{var})} + \tilde{\psi}^{(\text{cov})} = \varphi_x \left(B^+ B \Omega B^\top B_y y + B^+ D B_y y + B^+ D \breve{B}_y \breve{y} \right)$$

where:

$$\tilde{\psi}^{(\text{var})} = \varphi_x B^+ B (\omega \odot I_m) B^\top B_y y$$

and:

$$\tilde{\psi}^{(\text{cov})} = \varphi_x \left(B^+ B (\Omega - \omega \odot I_m) B^\top B_y y + B^+ D B_y y + B^+ D \breve{B}_y \breve{y} \right)$$

■

A.2.2 Conditional risk premium

Property 10. *In the case of comprehensive absolute views, the conditional formulas for asset risk premia become:*

$$\begin{cases} \bar{\pi} = \tilde{\pi} + A_\pi (v_\pi - \tilde{\pi}) \\ \bar{\Gamma}_\pi = (I_n - A_\pi) \Gamma_\pi \end{cases}$$

where $A_\pi = \Gamma_\pi (\Gamma_\pi + \Phi_\pi)^{-1}$. *For factor risk premia, we have:*

$$\begin{cases} \bar{\psi} = \tilde{\psi} + A_\psi \left(v_\psi - \tilde{\psi} \right) \\ \bar{\Gamma}_\psi = (I_n - A_\psi) \Gamma_\psi \end{cases}$$

where $A_\psi = \Gamma_\psi \left(\Gamma_\psi + \Phi_\psi \right)^{-1}$.

Proof. Since $P_\pi = I_n$, we have:

$$\begin{aligned} \bar{\pi} &= \tilde{\pi} + \Gamma_\pi P_\pi^\top \left(P_\pi \Gamma_\pi P_\pi^\top + \Phi_\pi \right)^{-1} (v_\pi - P_\pi \tilde{\pi}) \\ &= \tilde{\pi} + \Gamma_\pi (\Gamma_\pi + \Phi_\pi)^{-1} (v_\pi - \tilde{\pi}) \\ &= \tilde{\pi} + A_\pi (v_\pi - \tilde{\pi}) \end{aligned}$$

and:

$$\begin{aligned} \bar{\Gamma}_\pi &= \Gamma_\pi - \Gamma_\pi P_\pi^\top \left(P_\pi \Gamma_\pi P_\pi^\top + \Phi_\pi \right)^{-1} P_\pi \Gamma_\pi \\ &= \Gamma_\pi - \Gamma_\pi (\Gamma_\pi + \Phi_\pi)^{-1} \Gamma_\pi \\ &= (I_n - A_\pi) \Gamma_\pi \end{aligned}$$

where $A_\pi = \Gamma_\pi (\Gamma_\pi + \Phi_\pi)^{-1}$. If the absolute views are on the factors instead of the assets, we have $P_\psi = I_m$. We deduce that:

$$\begin{aligned} \bar{\psi} &= \tilde{\psi} + \Gamma_\psi P_\psi^\top \left(P_\psi \Gamma_\psi P_\psi^\top + \Phi_\psi \right)^{-1} \left(v_\psi - P_\psi \tilde{\psi} \right) \\ &= \tilde{\psi} + \Gamma_\psi \left(\Gamma_\psi + \Phi_\psi \right)^{-1} \left(v_\psi - \tilde{\psi} \right) \\ &= \tilde{\psi} + A_\psi \left(v_\psi - \tilde{\psi} \right) \end{aligned}$$

and:

$$\begin{aligned}\bar{\Gamma}_\psi &= \Gamma_\psi - \Gamma_\psi P_\psi^\top \left(P_\psi \Gamma_\psi P_\psi^\top + \Phi_\psi\right)^{-1} P_\psi \Gamma_\psi \\ &= \Gamma_\psi - \Gamma_\psi \left(\Gamma_\psi + \Phi_\psi\right)^{-1} \Gamma_\psi \\ &= \left(I_m - A_\psi\right) \Gamma_\psi\end{aligned}$$

where $A_\psi = \Gamma_\psi \left(\Gamma_\psi + \Phi_\psi\right)^{-1}$. ■

Property 11. *If the assets are independent of each other and we have only absolute views on the assets, these views do not affect the other assets. Moreover, the conditional covariance matrix $\bar{\Gamma}_\pi$ is diagonal and we have:*

$$\left(\bar{\Gamma}_\pi\right)_{i,i} = \frac{\left(\Gamma_\pi\right)_{i,i} \left(\Phi_\pi\right)_{i,i}}{\left(\Gamma_\pi\right)_{i,i} + \left(\Phi_\pi\right)_{i,i}}$$

This property also holds true when considering risk factors instead of assets.

Proof. If the assets are independent, Γ_π and Φ_π are diagonal matrices and $A_\pi = \Gamma_\pi \left(\Gamma_\pi + \Phi_\pi\right)^{-1}$ is also a diagonal matrix. Therefore, the j-th element of v_π only affects the j-th element of $\bar{\pi}$. This means that in the case where there are only absolute views and all assets are independent, the view on one asset does not affect the other assets. Indeed, we have:

$$\bar{\Gamma}_\pi = \left(P_\pi^\top \Phi_\pi^{-1} P_\pi + \Gamma_\psi^{-1}\right)^{-1} = \left(\Phi_\pi^{-1} + \Gamma_\pi^{-1}\right)^{-1} = \Gamma_\pi \left(\Gamma_\pi + \Phi_\pi\right)^{-1} \Phi_\pi$$

Since Γ_π and Φ_π are diagonal matrices, $\bar{\Gamma}_\pi$ is also a diagonal matrix with:

$$\left(\bar{\Gamma}_\pi\right)_{i,i} = \left(\Gamma_\pi\right)_{i,i} \left(\left(\Gamma_\pi\right)_{i,i} + \left(\Phi_\pi\right)_{i,i}\right)^{-1} \left(\Phi_\pi\right)_{i,i} = \frac{\left(\Gamma_\pi\right)_{i,i} \left(\Phi_\pi\right)_{i,i}}{\left(\Gamma_\pi\right)_{i,i} + \left(\Phi_\pi\right)_{i,i}}$$

This conclusion also holds for $\bar{\Gamma}_\pi$. ■

Property 12. *When the certainty of the absolute views is very high, the conditional risk premia are equal to the views:*

$$\bar{\pi} = v_\pi$$

When implied risk premia and absolute views have the same uncertainty, conditional risk premia are the average of implied risk premia and views:

$$\bar{\pi} = \frac{v_\pi + \tilde{\pi}}{2}$$

This property also holds true when looking at risk factors rather than assets.

Proof. When the certainty of the views is very high, we have $\Gamma_\pi + \Phi_\pi \approx \Gamma_\pi$ and $A_\pi = \Gamma_\pi \left(\Gamma_\pi + \Phi_\pi\right)^{-1} \approx \Gamma_\pi \Gamma_\pi^{-1} \approx I_n$. We conclude that the conditional risk premia are equal to the views:

$$\bar{\pi} \approx v_\pi$$

When the uncertainty in implied risk premia is equal to the uncertainty in the absolute views ($\Gamma_\pi = \Phi_\pi$), the conditional risk premia are the average of the implied risk premia and the views:

$$\bar{\pi} = \tilde{\pi} + \Gamma_\pi \left(2\Gamma_\pi\right)^{-1} \left(v_\pi - \tilde{\pi}\right) = \tilde{\pi} + \frac{1}{2} \left(v_\pi - \tilde{\pi}\right) = \frac{v_\pi + \tilde{\pi}}{2}$$

■

Property 13. *The conditional covariance matrix has an upper bound in the one-factor model, defined as:*

$$\bar{\Gamma}_\psi \leq \frac{1}{2}\left(\frac{\Gamma_\psi + \Phi_\psi}{2}\right)$$

Proof. In the one-factor model, we can only specify an absolute view: $P_\psi = 1$ and $v_\psi = v$. Since Γ_ψ and Φ_ψ are scalar values, and using Property 11, we get:

$$\bar{\Gamma}_\psi = \frac{\Gamma_\psi \Phi_\psi}{\Gamma_\psi + \Phi_\psi} \leq \frac{\Gamma_\psi + \Phi_\psi}{4}$$

because:

$$\begin{aligned}(a-b)^2 \geq 0 \quad &\Leftrightarrow \quad a^2 + b^2 - 2ab \geq 0 \\ &\Leftrightarrow \quad a^2 + b^2 + 2ab \geq 4ab \\ &\Leftrightarrow \quad (a+b)^2 \geq 4ab \\ &\Leftrightarrow \quad ab \leq \frac{1}{4}(a+b)^2\end{aligned}$$

In addition, we have three special cases:

- If $\Phi_\psi \ll \Gamma_\psi$, we have much more confidence in the given view than we do in the actual portfolio and we get:
$$\bar{\Gamma}_\psi = \frac{\Gamma_\psi \Phi_\psi}{\Gamma_\psi + \Phi_\psi} \approx \frac{\Gamma_\psi \Phi_\psi}{\Gamma_\psi} \approx \Phi_\psi$$
- If $\Phi_\psi \gg \Gamma_\psi$, we have much more confidence in the actual portfolio than we do in the given view and we get:
$$\bar{\Gamma}_\psi = \frac{\Gamma_\psi \Phi_\psi}{\Gamma_\psi + \Phi_\psi} \approx \frac{\Gamma_\psi \Phi_\psi}{\Phi_\psi} \approx \Gamma_\psi$$
- If $\Phi_\psi = \Gamma_\psi$, we have the same confidence in the actual portfolio as we do in the given view and we get:
$$\bar{\Gamma}_\psi = \frac{\Gamma_\psi \Phi_\psi}{\Gamma_\psi + \Phi_\psi} = \frac{\Gamma_\psi^2}{2\Gamma_\psi} = \frac{\Gamma_\psi}{2}$$

■

Property 14. *In the two-factor model, when the absolute views are uncorrelated, the conditional risk premium on one factor is equal to a weighted average of the implied risk premium and the absolute view on that factor, and a correction term on the second view that depends on the correlation ρ between the two factors:*

$$\begin{cases} \bar{\psi}_1 = w_1 \tilde{\psi}_1 + (1 - w_1) v_1 + \rho\lambda_1 \left(v_2 - \tilde{\psi}_2\right) \\ \bar{\psi}_2 = w_2 \tilde{\psi}_2 + (1 - w_2) v_2 + \rho\lambda_2 \left(v_1 - \tilde{\psi}_1\right) \end{cases}$$

Proof. We set $P_\psi = I_2$, $v_\psi = (v_1, v_2)$, $\Gamma_\psi = \begin{pmatrix} \gamma_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix}$ and $\Phi_\psi = \begin{pmatrix} \phi_1^2 & 0 \\ 0 & \phi_2^2 \end{pmatrix}$. We have:

$$\begin{aligned} A_\psi &= \Gamma_\psi \left(\Gamma_\psi + \Phi_\psi\right)^{-1} \\ &= \begin{pmatrix} \gamma_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix} \begin{pmatrix} \gamma_1^2 + \phi_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 + \phi_2^2 \end{pmatrix}^{-1} \end{aligned}$$

We use the inverse formula of the 2×2 matrix:

$$\begin{aligned} A_\psi &= \frac{1}{\xi}\begin{pmatrix} \gamma_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix}\begin{pmatrix} \gamma_2^2+\phi_2^2 & -\rho\gamma_1\gamma_2 \\ -\rho\gamma_1\gamma_2 & \gamma_1^2+\phi_1^2 \end{pmatrix} \\ &= \frac{1}{\xi}\begin{pmatrix} \gamma_1^2\gamma_2^2+\gamma_1^2\phi_2^2-\rho^2\gamma_1^2\gamma_2^2 & \rho\gamma_1\gamma_2\phi_1^2 \\ \rho\gamma_1\gamma_2\phi_2^2 & \gamma_1^2\gamma_2^2+\phi_1^2\gamma_2^2-\rho^2\gamma_1^2\gamma_2^2 \end{pmatrix} \end{aligned}$$

where:

$$\xi = \left(\gamma_1^2+\phi_1^2\right)\left(\gamma_2^2+\phi_2^2\right)-\rho^2\gamma_1^2\gamma_2^2 = \gamma_1^2\gamma_2^2+\gamma_1^2\phi_2^2+\phi_1^2\gamma_2^2+\phi_1^2\phi_2^2-\rho^2\gamma_1^2\gamma_2^2$$

Then, we have:

$$\bar{\psi}_1 = \tilde{\psi}_1 + \frac{\gamma_1^2\gamma_2^2+\gamma_1^2\phi_2^2-\rho^2\gamma_1^2\gamma_2^2}{\xi}\left(v_1-\tilde{\psi}_1\right) + \frac{\rho\gamma_1\gamma_2\phi_1^2}{\xi}\left(v_2-\tilde{\psi}_2\right)$$

and:

$$\bar{\psi}_2 = \tilde{\psi}_2 + \frac{\rho\gamma_1\gamma_2\phi_2^2}{\xi}\left(v_1-\tilde{\psi}_1\right) + \frac{\gamma_1^2\gamma_2^2+\phi_1^2\gamma_2^2-\rho^2\gamma_1^2\gamma_2^2}{\xi}\left(v_2-\tilde{\psi}_2\right)$$

Finally, we get[18]:

$$\bar{\psi}_1 = w_1\tilde{\psi}_1 + (1-w_1)\,v_1 + \rho\lambda_1\left(v_2-\tilde{\psi}_2\right)$$

where:

$$\begin{cases} w_1 = \dfrac{\phi_1^2\gamma_2^2+\phi_1^2\phi_2^2}{\xi} \\ \lambda_1 = \dfrac{\gamma_1\gamma_2\phi_1^2}{\xi} \end{cases}$$

and:

$$\bar{\psi}_2 = w_2\tilde{\psi}_2 + (1-w_2)\,v_2 + \rho\lambda_2\left(v_1-\tilde{\psi}_1\right)$$

where:

$$\begin{cases} w_2 = \dfrac{\gamma_1^2\phi_2^2+\phi_1^2\phi_2^2}{\xi} \\ \lambda_2 = \dfrac{\gamma_1\gamma_2\phi_2^2}{\xi} \end{cases}$$

We consider some special cases:

- Assuming that $\gamma_1 = \gamma_2 = \phi_1 = \phi_2$, we have:

$$\bar{\psi}_1 = \underbrace{\frac{2}{4-\rho^2}\tilde{\psi}_1 + \frac{2-\rho^2}{4-\rho^2}v_1}_{\text{weighted average}} + \underbrace{\frac{\rho}{4-\rho^2}\left(v_2-\tilde{\psi}_2\right)}_{\text{correction term}}$$

  We get the following values:

$$\bar{\psi}_1 = \begin{cases} \dfrac{v_1+\tilde{\psi}_1}{2} & \text{if} \quad \rho = 0 \\ \dfrac{2}{3}\tilde{\psi}_1 + \dfrac{1}{3}v_1 \pm \dfrac{1}{3}\left(v_2-\tilde{\psi}_2\right) & \text{if} \quad \rho = \pm 1 \end{cases}$$

[18] Note that w_1 and w_2 form a system of weights. In fact, we have:

$$w_1 = \frac{\phi_1^2\gamma_2^2+\phi_1^2\phi_2^2}{\phi_1^2\gamma_2^2+\phi_1^2\phi_2^2+\upsilon}$$

where $\upsilon = \left(1-\rho^2\right)\gamma_1^2\gamma_2^2+\gamma_1^2\phi_2^2$. Since all the terms are positive, $w_1 \geq 0$. Moreover, $w_1 \leq 0$ because $\upsilon \geq 0$. So we have $0 \leq w_1 \leq 1$, and this property holds for w_2 as well.

- Assuming that $\gamma_1 = \gamma_2 = \phi_1$ but $\phi_2 = 0$, we have:

$$\bar{\psi}_1 = \frac{1}{2-\rho^2}\tilde{\psi}_1 + \frac{1-\rho^2}{2-\rho^2}v_1 + \frac{\rho}{2-\rho^2}\left(v_2 - \tilde{\psi}_2\right)$$

and:

$$\bar{\psi}_2 = v_2$$

- Assuming that $\gamma_1 = \gamma_2 = \phi_1$ but $\phi_2 = \infty$, we have:

$$\bar{\psi}_1 = \frac{1}{2}\tilde{\psi}_1 + \frac{1}{2}v_1$$

and:

$$\bar{\psi}_2 = \tilde{\psi}_2 + \frac{\rho}{2}\left(v_1 - \tilde{\psi}_1\right)$$

■

Property 15. *The conditional risk premia satisfy the following decomposition:*

$$\bar{\psi} = \underbrace{w \odot \tilde{\psi} + (\mathbf{1}_m - w) \odot v_\psi}_{\textit{weighted average}} + \underbrace{\Lambda\left(v_\psi - \tilde{\psi}\right)}_{\textit{correction term}}$$

where $w \geq \mathbf{0}_m$.

Proof. We have:

$$\begin{aligned}
\bar{\psi} &= \tilde{\psi} + A_\psi\left(v_\psi - \tilde{\psi}\right) \\
&= I_m\tilde{\psi} + \operatorname{diag}\left(A_\psi\right)\left(v_\psi - \tilde{\psi}\right) + \left(A_\psi - \operatorname{diag}\left(A_\psi\right)\right)\left(v_\psi - \tilde{\psi}\right) \\
&= \left(I_m - \operatorname{diag}\left(A_\psi\right)\right)\tilde{\psi} + \operatorname{diag}\left(A_\psi\right)v_\psi + \left(A_\psi - \operatorname{diag}\left(A_\psi\right)\right)\left(v_\psi - \tilde{\psi}\right) \\
&= w \odot \tilde{\psi} + (\mathbf{1}_m - w) \odot v_\psi + \Lambda\left(v_\psi - \tilde{\psi}\right)
\end{aligned}$$

where $w = \operatorname{diag}\left(I_m - A_\psi\right) \geq \mathbf{0}_m$ and $\Lambda = A_\psi - \operatorname{diag}\left(A_\psi\right)$. We need to prove that $I_m - A_\psi = I_m - \Gamma_\psi\left(\Gamma_\psi + \Phi_\psi\right)^{-1}$ has positive diagonal elements. ■

A.2.3 Moore-Penrose inverse of the product of a full-rank matrix with a diagonal matrix

Property I. *Let* $A \in \mathbb{R}^{n\times m}$ *and* $\Lambda \in \mathbb{R}^{m\times m}$. *We assume that* Λ *is a diagonal matrix with* $\det\Lambda \neq 0$. *In the general case, it is difficult to prove that the Moore-Penrose inverse of the product* $A\Lambda$ *is the product of* Λ^{-1} *and* A^+. *However, if we assume that* A *is a full-rank matrix, we have:*

$$(A\Lambda)^+ = \Lambda^{-1}A^+$$

Proof. M^+ is a Moore-Penrose inverse of M if M^+ satisfies the four criteria: (1) $MM^+M = M$, (2) $M^+MM^+ = M^+$, (3) MM^+ is hermitian and (4) M^+M is hermitian. Let $B = \Lambda^{-1}A^+$ be the candidate Moore-Penrose matrix. In the general case, we can prove[19] (1),

[19] We have:

$$\begin{cases}
(A\Lambda)B(A\Lambda) = (A\Lambda)\Lambda^{-1}A^+(A\Lambda) = AA^+A\Lambda = A\Lambda \\
B(A\Lambda)B = \Lambda^{-1}A^+(A\Lambda)\Lambda^{-1}A^+ = \Lambda^{-1}A^+AA^+ = \Lambda^{-1}A^+ = B \\
(A\Lambda B)^* = \left(A\Lambda\Lambda^{-1}A^+\right)^* = \left(AA^+\right)^* = AA^+ = A\Lambda\Lambda^{-1}A^+ = A\Lambda B \\
(BA\Lambda)^* = \left(\Lambda^{-1}A^+A\Lambda\right)^* = \Lambda A^+A\Lambda^{-1} \neq \Lambda^{-1}A^+A\Lambda \neq BA\Lambda
\end{cases}$$

(2), and (3), but not (4). Now, assume that A is a full-rank matrix with $n > m$. In this case, we have $A^+ = \left(A^\top A\right)^{-1} A^\top$. Since we have already proved (1), (2), and (3), we only need to prove (4). We have

$$B = \Lambda^{-1} A^+ = \Lambda^{-1} \left(A^\top A\right)^{-1} A^\top$$

and:

$$BA\Lambda = \Lambda^{-1} \left(A^\top A\right)^{-1} A^\top A\Lambda = \Lambda^{-1}\Lambda = I_m$$

Since $I_m^* = I_m$, it follows that $(BA\Lambda)^* = BA\Lambda$. We conclude that $B = \Lambda^{-1}A^+$ is the Moore-Penrose inverse of $A\Lambda$. ■

A.3 Factor grouping methodology

Barra risk models typically use a mixture of different levels of the GICS hierarchy as industry factors. In practice, we prefer to focus on the GICS sector level (11 sectors) or industry group level (25 industry groups) and present the corresponding factor grouping method in this section. Mathematically, the cross-section multi-factor model takes the following form:

$$\begin{aligned} R_i(t) - r(t) &= \beta_i^{\text{country}} F^{\text{country}}(t) + \sum_{j=1}^{n_{\text{style}}} \beta_{i,j}^{\text{style}} F_j^{\text{style}}(t) + \\ & \sum_{j=1}^{n_{\text{industry}}} \beta_{i,j}^{\text{industry}} F_j^{\text{industry}}(t) + \varepsilon_i(t) \end{aligned} \tag{12}$$

where $R_i(t) - r(t)$ is the excess return of asset i at time t, $F^{\text{country}}(t)$ is the country risk factor, $F_j^{\text{industry}}(t)$ is a set of industry risk factors, $F_j^{\text{style}}(t)$ is a set of style risk factors and $\varepsilon_i(t)$ is the idiosyncratic risk factor. According to Barra (1998), the following optimization represents the cross-sectional weighted regression that the USSLOW model uses to estimate the model:

$$\hat{\beta} = \arg\min \sum_{i=1}^{n} \rho_i \varepsilon_i^2(t) \quad \text{s.t.} \quad \sum_{j=1}^{n_{\text{industry}}} \omega_j F_j^{\text{industry}}(t) = 0 \tag{13}$$

where $\rho_i \propto \sqrt{c_i}$ is the weight coefficient of stock i, c_i is the market capitalization of asset i, and $\omega_j = \sum_{i \in \text{industry}_j} c_i$ is the market capitalization of industry j. A simple approach for factor grouping would be to re-estimate the entire model, *i.e.*, instead of using 60 industry factors in the USSLOW model, we would simply use 11 industry factors. The advantage of this approach is that we use the same methodology as the Barra risk model and the interpretation of the new industry factors remains the same. However, this method requires a high degree of data integrity and cleanliness.

Instead of re-estimating the entire Barra risk model, we could explore an alternative approach where we re-estimate only a portion of the current model. For example, in the USSLOW model, we would like to replace the block of industry factors with a new one:

$$\begin{aligned} R_i(t) - r(t) &= \beta_i^{\text{country}} F^{\text{country}}(t) + \sum_{j=1}^{n_{\text{style}}} \beta_{i,j}^{\text{style}} F_j^{\text{style}}(t) + \\ & \sum_{j=1}^{n_{\text{sector}}} \beta_{i,j}^{\text{sector}} F_j^{\text{sector}}(t) + \varepsilon_i(t) \end{aligned} \tag{14}$$

With the other terms kept unchanged, we look for both $\beta_{i,j}^{\text{sector}}$ and $F_j^{\text{sector}}(t)$ to satisfy the equality condition:

$$\sum_{j=1}^{n_{\text{industry}}} \beta_{i,j}^{\text{industry}} F_j^{\text{industry}}(t) = \sum_{j=1}^{n_{\text{sector}}} \beta_{i,j}^{\text{sector}} F_j^{\text{sector}}(t)$$

The above system of equations can be solved sector by sector. For example, we want to merge 4 industries (*Banks*, *Diversified Financials*, *Insurance Brokers and Reinsurance* and *Life Health and Multi-line Insurance*) into a single sector called *Financials*:

$$\begin{pmatrix} \beta_1^{\text{banks}} \\ \beta_2^{\text{banks}} \\ \vdots \\ \beta_n^{\text{banks}} \end{pmatrix} F^{\text{banks}}(t) + \begin{pmatrix} \beta_1^{\text{div.fin}} \\ \beta_2^{\text{div.fin}} \\ \vdots \\ \beta_n^{\text{div.fin}} \end{pmatrix} F^{\text{div.fin}}(t) + \cdots = \begin{pmatrix} \beta_1^{\text{financials}} \\ \beta_2^{\text{financials}} \\ \vdots \\ \beta_n^{\text{financials}} \end{pmatrix} F^{\text{financials}}(t) \quad (15)$$

In this case, since we have $n+1$ unknown variables and n equations, we will get infinitely many solutions to this system of equations. Therefore, we must include an additional equation, such as:

$$\omega^{\text{banks}} F^{\text{banks}}(t) + \omega^{\text{div.fin}} F^{\text{div.fin}}(t) + \cdots = \omega^{\text{financials}} F^{\text{financials}}(t) \quad (16)$$

where ω^{banks}, $\omega^{\text{div.fin}}$, and $\omega^{\text{financials}} = \omega^{\text{banks}} + \omega^{\text{div.fin}} + \cdots$ are the market capitalization of industry factors *Banks*, *Diversified Financials* and sector *Financials*. Then the system of equations has a unique solution. By solving Equations (15) and (16), we get:

$$F^{\text{financials}}(t) = \frac{\omega^{\text{banks}} F^{\text{banks}}(t) + \omega^{\text{div.fin}} F^{\text{div.fin}}(t) + \cdots}{\omega^{\text{banks}} + \omega^{\text{div.fin}} + \cdots} \quad (17)$$

and:

$$\beta_i^{\text{financials}} = \frac{\beta_i^{\text{banks}} F^{\text{banks}}(t) + \beta_i^{\text{div.fin}} F^{\text{div.fin}}(t) + \cdots}{F^{\text{financials}}(t)} \quad (18)$$

The advantage of this approach is that we do not have to re-estimate the entire weighted regression problem. In this way, $F^{\text{country}}(t)$, $F_j^{\text{style}}(t)$ and $\varepsilon_i(t)$ remain unchanged. In addition, as shown in Equation (17), the new sector factor is a linear combination of the previous industry factors. Therefore, all style factors remain pure factors, *i.e.*, pure style factor portfolios have zero exposure to the new industry factors. The new sector factor returns can still be interpreted as pure industry factor portfolios, that are 100% long the respective sector and 100% short the country factor. However, the interpretation of the factor exposure $\beta_i^{\text{financials}}$ is less straightforward.

B Description of the Barra style factors

Table 37: List of style factors used in the USSLOW model

Style Factor	Description
Beta	Explains common variations in stock returns due to different stock sensitivities to market or systematic risk that cannot be explained by the US Country factor
Dividend Yield	Captures differences in stock returns attributable to stock's historical and predicted dividend-to-price ratios
Earnings Quality	Explains stock return differences due to uncertainty around company operating fundamentals (sales, earnings, cash flows) and the accrual components of their earnings
Earnings Yield	Describes stock return differences due to various ratios of the company's earnings relative to its price
Growth	Measures company growth prospects using historical sales growth and historical and predicted earnings growth
Leverage	Captures common variation in stock returns due to differences in the level of company leverage
Liquidity	Captures common variations in stock returns due to the amount of relative trading and differences in the impact of trading on stock returns
Long-Term Reversal	Explains common variation in returns related to a long-term (five years ex. recent thirteen months) stock price behavior
Management Quality	A combination of asset, investment, net issuance growth measures that captures common variation in stock returns of companies experiencing rapid growth or contraction of assets
Mid Capitalization	Explains the returns of mid-cap stocks relative to large and small-cap stocks by capturing deviations from linearity in the relationship between returns and the Size factor
Momentum	Explains common variation in stock returns related to recent (twelve months) stock price behavior
Profitability	A combination of profitability measures that characterizes efficiency of a firm's operations and total activities
Prospect	Explains common variation in stock returns that have exhibited a lottery-like behavior identified through a combination of stock return skewness over a long horizon and drawdown in returns over the recent period
Residual Volatility	Captures relative volatility in stock returns that is not explained by differences in stock sensitivities to market returns
Size	Captures differences in stock returns and risk due to differences in the market capitalization of companies
Value	Captures the extent to which a company is overpriced or underpriced using a combination of several relative valuation metrics and one structural valuation factor

Source: MSCI (2024) & Bayraktar *et al.* (2014).

C Additional results

Figure 12: Implied risk premium of style factors in bps (MSCI USA/MinVol Index, March 2024)

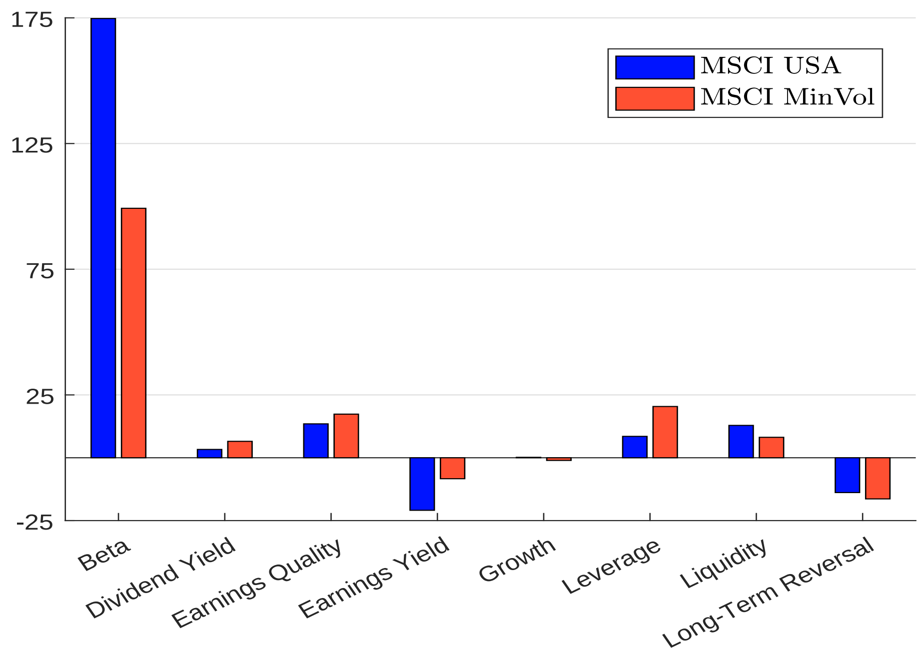

Figure 13: Implied risk premium of style factors in bps (MSCI USA/MinVol Index, March 2024)

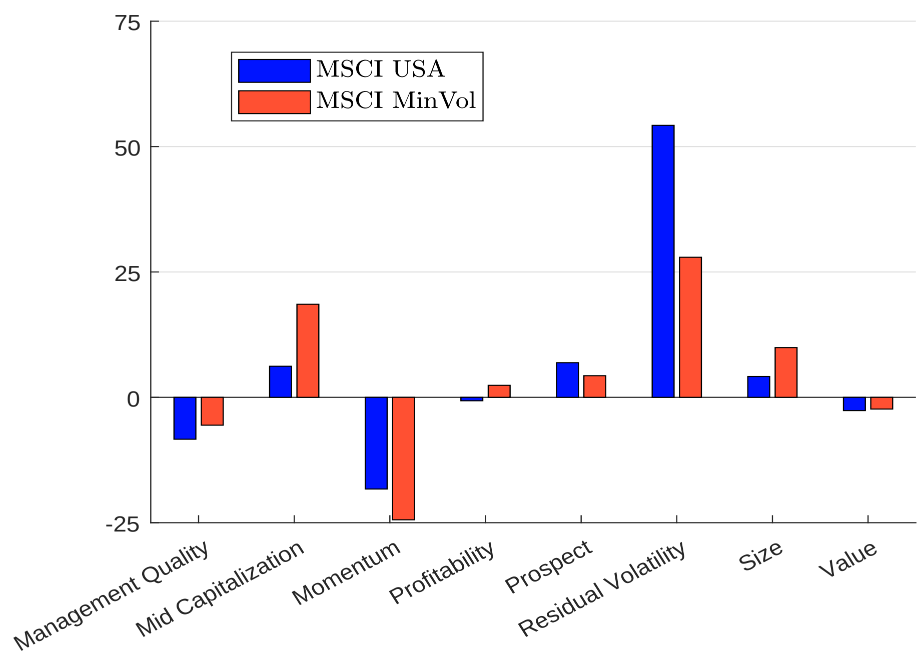

Figure 14: Implied risk premium of sectors in bps (MSCI USA/MinVol Index, March 2024)

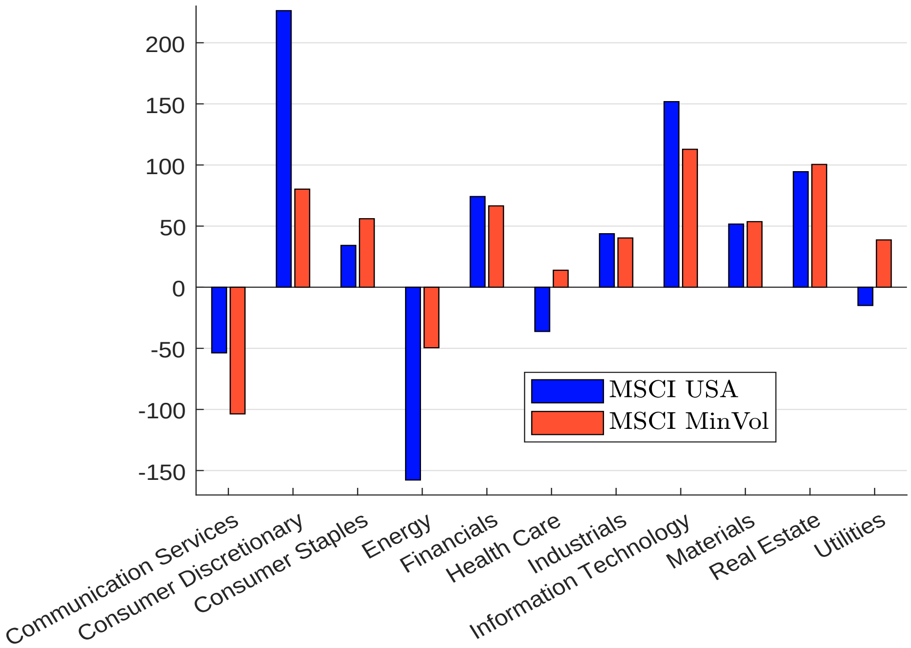

Figure 15: Asset risk premia (MSCI USA/MinVol Index, March 2024)

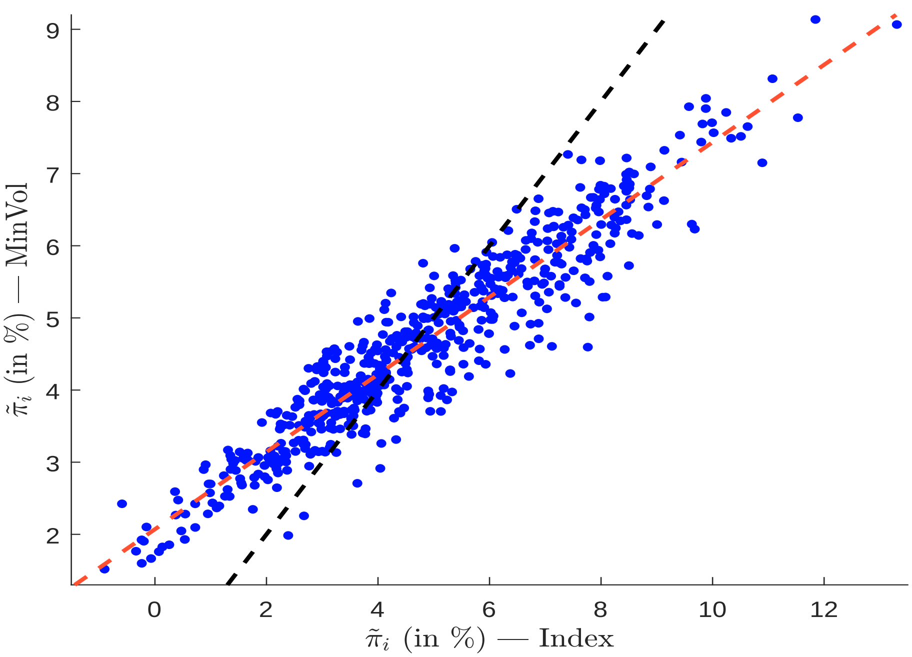

Figure 16: Relationship between x_i and $\tilde{\pi}$ (MSCI USA/MinVol Index, March 2024)

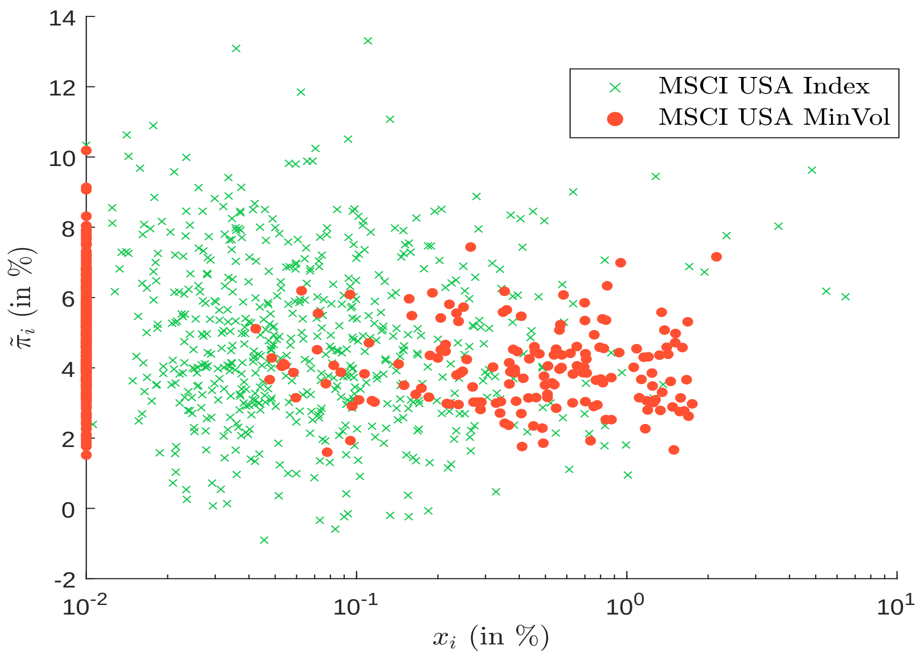

Figure 17: Relationship between σ_i and $\tilde{\pi}$ (MSCI USA/MinVol Index, March 2024)