Asset Management & Sustainable Finance
Final Examination

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Oral examination: 27 March 2025

Remark 1 *The final exam consists of 2 exercises. Please write your answers completely*[1]*. Be specific about the different concepts and different statistics you are using. Define the optimization program associated with each portfolio. Also provide one Python program by exercise.*

- *Concerning risk decomposition*[2]*, present the results as follows:*

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

- *The report is a zipped file whose filename is yourname.zip if you are doing the project alone or yourname1-yourname2.zip if you are doing the project in groups of two.*
- *The zipped file contains three files:*
  1. *The PDF document containing the answers to the two exercises and a cover sheet with your names;*
  2. *The Python program of each exercise with an explicit filename, e.g. exercise1.py.*
- *The project seems very long. However, once you understand how to solve a mean-variance optimization problem with a QP solver, you can duplicate your code for many questions. For example, Question 2.(c) is a duplication of Question 2.(b), as are Questions 3.(a), 3.(b) and 3.(c) in Exercise 1. The same is true for Questions 3.(a), 3.(b), 3.(c) and 3.(d) in Exercise 2.*

[1] Read the questions carefully and answer all elements of the questions. For example, when I say "Find the portfolio x and compute its volatility $\sigma(x)$", you must give the numerical values of x and $\sigma(x)$. If you just give the numeric value of $\sigma(x)$, the answer is wrong because I don't know what the portfolio weights are.

[2] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^{\star}$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

1 Portfolio optimization and risk budgeting

We consider the CAPM model:

$$R_i - r = \beta_i \cdot (R_m - r) + \varepsilon_i$$

where R_i is the return of asset i, R_m is the return of the market portfolio, r is the risk-free asset, β_i is the beta of asset i with respect to the market portfolio and ε_i is the idiosyncratic risk. We assume that $R_m \sim \mathcal{N}\left(\mu_m, \sigma_m^2\right)$, $\varepsilon_i \sim \mathcal{N}\left(0, \tilde{\sigma}_i^2\right)$, $R_m \perp \varepsilon_i$ and $\varepsilon_i \perp \varepsilon_j$. We denote μ_m as the expected return of the market portfolio, σ_m as the volatility of the market portfolio and $\tilde{\sigma}_i$ as the idiosyncratic volatility.

We consider a universe of 6 stocks with the following parameter values:

Asset i	1	2	3	4	5	6
β_i	-0.30	0.30	0.70	1.00	1.50	2.50
$\tilde{\sigma}_i$	15%	16%	10%	11%	12%	14%

and $\sigma_m = 20\%$. The risk free return is set to 2% and we assume that the expected return of the market portfolio is equal to $\mu_m = 8\%$.

1. (a) Compute the vector μ of expected returns.
   (b) Compute the covariance matrix Σ of stock returns.
   (c) Deduce the vector σ of volatilities and the correlation matrix ρ of stock returns.
   (d) Compte the Sharpe ratio of each asset.

2. We consider **long/short MVO portfolios** such that $\sum_{i=1}^{n} x_i = 1$.

   (a) Give the QP formulation of the mean-variance optimization problem:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \left(\mu - r \mathbf{1}_6\right) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ -10 \leq x_i \leq 10 \end{array} \right. \end{aligned}$$

   (b) Using the γ-problem, find the optimal solution[3] $x^\star(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (c) Draw the efficient frontier by considering granular values[4] of γ.

   (d) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 10% and 15%. Give the corresponding value of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (e) Using the analytical formula, find the tangency portfolio. Compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (f) Using the efficient frontier with a fine grid of γ, find the tangency portfolio using the brute-force algorithm. Compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (g) Now consider the extended quadratic programming problem by including the risk-free asset in the investment universe. Formulate the extended QP problem, and solve it using a fine grid of $\gamma \in [0.5, 1.5]$. What do we observe when calculating the Sharpe ratio of the different optimized portfolios? How can this be explained? Which criterion can be used to implement the brute-force algorithm on the extended QP problem to find the tangency portfolio?

[3] You have to give the composition of each optimized portfolio.

[4] For instance, you can consider that $\gamma = -0.5, -0.4, \ldots, -0.1, 0, 0.05, 0.10, \ldots, 0.95, 1, 2, \ldots, 10$.

(h) Compare the three solutions (e), (f) and (g) in terms of risk tolerance γ, weights $x^\star(\gamma)$, expected return $\mu(x^\star(\gamma))$, volatility $\sigma(x^\star(\gamma))$ and Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

3. We consider **long-only MVO portfolios** such that $\sum_{i=1}^n x_i = 1$ and $0 \leq x_i \leq 1$.

   (a) Using the γ-problem, find the optimal solution[5] $x^\star(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (b) Compare the efficient frontier by considering granular values of γ with the long/short efficient frontier obtained in Question 2.(c). Comment on these results.

   (c) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 10% and 15%. Give the corresponding value of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

   (d) Find the long-only tangency portfolio $x^\star_{\mathrm{MSR}}$. Compute its expected return $\mu(x^\star_{\mathrm{MSR}})$, its volatility $\sigma(x^\star_{\mathrm{MSR}})$ and its Sharpe ratio $\mathrm{SR}(x^\star_{\mathrm{MSR}} \mid r)$. Compare these results with those obtained in the long/short case.

   (e) Compute the beta coefficient β_i of each asset with respect to the long-only tangency portfolio $x^\star_{\mathrm{MSR}}$. Deduce the implied expected return μ_i that is priced in by the market[6], and the corresponding alpha coefficient α_i of each asset.

4. We consider risk-budgeting portfolios.

   (a) Give the risk decomposition of the long-only tangency portfolio $x^\star_{\mathrm{MSR}}$ (MSR).

   (b) Give the risk decomposition of the equally-weighted portfolio (EW).

   (c) Give the risk decomposition of the long-only minimum variance portfolio (MV).

   (d) Give the risk decomposition of the long-only most diversified portfolio (MDP).

   (e) Compute the equal risk contribution portfolio using the CCD algorithm. Give its risk decomposition.

   (f) Compute the beta $\beta(x \mid b)$ of the portfolios MSR, EW, MV, MDP and ERC with respect to the benchmark b when b is the long-only tangency portfolio $x^\star_{\mathrm{MSR}}$. Same question when b is the EW portfolio. Comment on these results.

Remark 2 *To obtain a readable plot of the efficient frontier, it is important to focus on the most relevant section, i.e., where* $5\% \leq \sigma(x) \leq 30\%$ *and* $0 \leq \mu(x) \leq 12\%$.

[5] You have to give the composition of each optimized portfolio.

[6] We assume that the market portfolio is the long-only tangency portfolio $x^\star_{\mathrm{MSR}}$.

2 Equity portfolio optimization with ESG and climate risk objectives

We consider an investment universe of $n = 8$ stocks with two sectors ($\boldsymbol{\mathcal{S}}ector_1$ and $\boldsymbol{\mathcal{S}}ector_2$). The expected return μ_i and the volatility σ_i of each stock i are reported below:

Table 1: Financial and climate metrics of the investment universe

Stock i	Sector	b_i	μ_i	σ_i	S_i	$\mathcal{CI}_i$	$\mathcal{CM}_i$	$\mathcal{GI}_i$
1	$\boldsymbol{\mathcal{S}}ector_1$	9.50%	5.00%	20.0%	-2.0	80	-5.0%	5.0%
2	$\boldsymbol{\mathcal{S}}ector_2$	15.50%	5.50%	22.0%	$+2.5$	200	-7.5%	80.5%
3	$\boldsymbol{\mathcal{S}}ector_1$	5.50%	6.00%	25.0%	$+1.5$	390	-1.5%	15.0%
4	$\boldsymbol{\mathcal{S}}ector_1$	8.50%	4.00%	18.0%	$+2.0$	800	-2.0%	0.0%
5	$\boldsymbol{\mathcal{S}}ector_2$	10.00%	7.00%	45.0%	-1.0	60	$+8.0\%$	2.0%
6	$\boldsymbol{\mathcal{S}}ector_2$	25.00%	10.00%	80.0%	-0.5	120	-4.0%	0.0%
7	$\boldsymbol{\mathcal{S}}ector_2$	17.00%	8.75%	35.0%	-0.5	135	-7.0%	60.0%
8	$\boldsymbol{\mathcal{S}}ector_1$	9.00%	6.25%	40.5%	$+0.5$	580	$+2.0\%$	20.0%

The correlation matrix is equal to:

$$\mathbb{C} = (\rho_{i,j}) = \begin{pmatrix} 100\% & & & & & & & \\ 50\% & 100\% & & & & & & \\ 30\% & 30\% & 100\% & & & & & \\ 60\% & 60\% & 60\% & 100\% & & & & \\ 40\% & 30\% & 50\% & 30\% & 100\% & & & \\ 30\% & 20\% & 40\% & 70\% & 50\% & 100\% & & \\ 40\% & 60\% & 50\% & 60\% & 50\% & 60\% & 100\% & \\ 30\% & 30\% & 50\% & 30\% & 30\% & 30\% & 60\% & 100\% \end{pmatrix}$$

In Table 1, we report, for each stock, its weight b_i in the benchmark, the corresponding ESG score S_i, the Scope 1+2 carbon intensity $\mathcal{CI}_i$ in $tCO_2e/\$$ mn, the carbon momentum $\mathcal{CM}_i$, and the green intensity $\mathcal{GI}_i$ measured as the ratio of green capex to total capex over the past three years.

1. We consider the benchmark b.
   (a) Compute the covariance matrix Σ.
   (b) Compute the volatility $\sigma(b)$ of the benchmark.
   (c) Compute $\mathcal{CI}(b)$, $\mathcal{CM}(b)$, $\mathcal{GI}(b)$, and the ESG score $S(b)$.
   (d) Compute the carbon intensity, the carbon momentum, the green intensity and the ESG score for each sector[7].

2. The investor's decarbonization pathway follows the CTB trajectory, meaning that the carbon intensity of the investor's portfolio at time t must be less than a threshold $\mathcal{CI}^\star(t)$:

$$\mathcal{CI}(t, w) \leq \mathcal{CI}^\star(t) := (1 - 30\%)(1 - 7\%)^t \, \mathcal{CI}(b) \tag{1}$$

   The investor's objective is to minimize the volatility of the tracking error relative to the benchmark and to meet the decarbonization constraint based on Scope 1+2 emissions.

   (a) What is the optimization problem? Deduce the QP form.

[7] Note that the weights in a given sector must be renormalized to 100%.

(b) Compute the optimized portfolio $w^{\star}(t)$ for $t \in \{0, 1, 2, 5, 10\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the carbon momentum, the green intensity and the reduction rate:

$$\mathcal{R}(t, w) = 1 - \frac{\mathcal{CI}(t, w)}{\mathcal{CI}(b)}$$

Comment on these results.

(c) Compare the sector allocation of each optimized portfolio to the sector allocation of the benchmark. Comment on these figures.

(d) We consider the previous optimization problem and we impose the sector neutrality. Deduce the QP form. Compute the optimized portfolio $w^{\star}(t)$ for $t \in \{0, 1, 2, 5, 10\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the carbon momentum, the green intensity and the reduction rate. Verify that these portfolios are sector neutral.

3. The investor's objective is to minimize the volatility of the tracking error relative to the benchmark while incorporating ESG and climate risk constraints. At each step, the investor adds a new constraint, resulting in the accumulation of constraints.

   (a) The investor begins by adding a decarbonization constraint:

   $$\mathcal{CI}(w) \leq (1 - 30\%)\,\mathcal{CI}(b)$$

   Give the QP problem and find the optimal solution.

   (b) The investor adds a green intensity constraint:

   $$\mathcal{GI}(w) \geq (1 + 50\%)\,\mathcal{GI}(b)$$

   What does this constraint mean? Give the QP problem and find the optimal solution.

   (c) The investor adds a third constraint:

   $$\mathcal{CM}(w) \leq (1 + 50\%)\,\mathcal{CM}(b)$$

   What does this constraint mean? Give the QP problem and find the optimal solution.

   (d) The investor adds a fourth constraint:

   $$S(w) \geq S(b) + 0.5$$

   What does this constraint mean? Give the QP problem and find the optimal solution.

   (e) Compute the tracking error volatility, the carbon intensity, the carbon momentum, the green intensity, the ESG score, and the sector allocation for the optimized portfolios found in 3.(a), 3.(b), 3.(c), and 3.(d).

   (f) We look at the four constraints: $\mathcal{CI}(w) \leq (1 - \mathcal{R})\,\mathcal{CI}(b)$, $\mathcal{GI}(w) \geq (1 + 50\%)\,\mathcal{GI}(b)$, $\mathcal{CM}(w) \leq (1 + 50\%)\,\mathcal{CM}(b)$, and $S(w) \geq S(b) + 0.5$. What is the maximum level of reduction $\mathcal{R}$ to ensure that there is a solution to the optimization problem with the four constraints?

4. The investor's objective is to minimize the volatility of the tracking error relative to the benchmark while controlling the sector allocation.

   (a) Compute the carbon intensity of sectors $\mathcal{S}ector_1$ and $\mathcal{S}ector_2$.

   (b) What is the optimization problem if we impose to reduce the carbon intensity of $\mathcal{S}ector_1$ by 30% and the carbon intensity of $\mathcal{S}ector_2$ by 50%? Find the optimized portfolio.

(c) Formulate the QP problem if we also add the sector neutrality constraint. Find the optimized portfolio.

(d) Compute the tracking error volatility, the carbon intensity, the carbon momentum, the green intensity, the ESG score, and the sector allocation for the optimized portfolios found in 4.(b), and 4.(c).

(e) Comment on these results.