Risk-Based Investing & Asset Management
Final Examination

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Remark 1 ***The final examination is composed of 3 exercises. Please write entirely your answers. Precise the different concepts and the different statistics you use. Define the optimization program associated to each portfolio. Provide also one excel file by question. Concerning risk[1] decompositions, present the results as follows:***

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

1 Portfolio optimization in a single-factor model

We assume a single-factor model:

$$R_{i,t} = \beta_i R_{m,t} + \varepsilon_{i,t}$$

where $R_{m,t}$ is the common risk factor, $\varepsilon_{i,t}$ the idiosyncratic factor, $R_{m,t} \perp \varepsilon_t$, $\operatorname{var}(F_t) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon_t) = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$. We consider a universe of 6 assets with the following parameter values:

i	1	2	3	4	5	6
β_i	-0.30	-0.10	0.50	0.90	1.40	1.80
$\tilde{\sigma}_i$	15%	16%	7%	8%	9%	10%

and $\sigma_m = 20\%$.

1. We assume that the common risk factor is the market portfolio. The return of the risk-free asset is equal to 2% whereas the market risk premium is equal to 5%.

   (a) Compute the covariance Σ of asset returns. Deduce the volatilities and the correlation matrix.

   (b) Using the CAPM, compute the expected return μ_i of the assets.

2. We consider long-short portfolios x with $\sum_{i=1}^n x_i = 1$.

   (a) Using different values of the parameter γ, draw the efficient frontier.

   (b) Compute the minimum variance portfolio.

   (c) Compute the MVO portfolio if we target an ex-ante volatility of 12%.

   (d) Compute the MVO portfolio if we target an ex-ante expected return of 6%.

[1] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^\star$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

(e) Compare the three optimized portfolios in terms of expected return, volatility and Sharpe ratio.

3. We now include the risk-free rate in the investment universe.

   (a) Deduce the new covariance matrix and the vector of expected returns.

   (b) Compute the MVO portfolio if we target an ex-ante volatility of 4%.

   (c) Compute the MVO portfolio if we target an ex-ante volatility of 10%.

   (d) Compute the MVO portfolio if we target an ex-ante expected return of 26.30%.

   (e) Compare the three optimized portfolios in terms of expected return, volatility and Sharpe ratio.

   (f) Deduce the tangency portfolio $x^\star$. Calculate its expected return, volatility and risk premium.

   (g) Deduce the beta $\beta_i(x^\star)$ and the risk premium $\pi_i(x^\star)$ of each asset i with respect to the tangency portfolio $x^\star$.

   (h) How do you explain that $\beta_i(x^\star) \neq \beta_i$, but $\pi_i(x^\star) = \mu_i - r$?

   (i) Show that the common risk factor of the single-factor model is a deleveraged version of the tangency portfolio. Give the value of the leverage ratio. Do you think that the common risk factor $R_{m,t}$ is the return of the market portfolio? Comment on these results.

2 Smart beta portfolios

We consider a capitalization-weighted equity index, which is composed of 8 stocks. The weights are equal to 23%, 19%, 17%, 13%, 9%, 8%, 6% and 5%. We assume that their volatilities are 21%, 20%, 40%, 18%, 35%, 23%, 13% and 29%. The correlation matrix is given by:

$$\rho = \begin{pmatrix}
100\% & & & & & & & \\
80\% & 100\% & & & & & & \\
70\% & 75\% & 100\% & & & & & \\
60\% & 65\% & 90\% & 100\% & & & & \\
70\% & 50\% & 70\% & 85\% & 100\% & & & \\
50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\
70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\
60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\%
\end{pmatrix}$$

We note Σ the covariance matrix. In what follows, we consider long-only portfolios.

1. For each portfolio, compute the volatility decomposition.

   (a) Give the volatility decomposition of the capitalization-weighted (CW) portfolio.

   (b) Determine the equally weighted (EW) portfolio.

   (c) Compute the minimum variance (MV) portfolio.

   (d) Compute the most diversified portfolio (MDP).

   (e) Find the ERC portfolio.

   (f) Compare the diversification ratio $\mathcal{DR}(x)$, the volatility $\sigma(x)$, the weight concentration[2] $\mathcal{H}^\star(x)$ and the risk concentration $\mathcal{H}^\star(\mathcal{RC})$ of the previous portfolios (CW, EW, MV, MDP and ERC).

2. We consider long-only minimum variance portfolios.

[2]We remind that the normalized Herfindahl index is defined as follows:

$$\mathcal{H}^\star(\pi) = \frac{n\sum_{i=1}^n \pi_i^2 - 1}{n-1}$$

where $\pi = (\pi_1, \ldots, \pi_n)$ is a vector satisfying $\pi_i \geq 0$ and $\sum_{i=1}^n \pi_i = 1$.

(a) Find the MV portfolio when we impose the upper bound $x_i \leq 50\%$.

(b) Find the MV portfolio when we impose the upper bound $x_i \leq 20\%$.

(c) We consider the statistic $n^{\star}(x)$ defined as follows:

$$n^{\star}(x) = \left(\sum_{i=1}^{n} x_i^2\right)^{-1}$$

Show that:

$$1 \leq n^{\star}(x) \leq n$$

Explain why the statistics $n^{\star}(x)$ is called the effective number of stocks. Compute $n^{\star}(x)$ for the following three portfolios:

Stock i	1	2	3	4	5	6	7	8
x_1	0.25	0.25	0.25	0.25	0.00	0.00	0.00	0.00
x_2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.65
x_3	0.20	0.10	0.20	0.10	0.05	0.05	0.15	0.15

Comments on these results.

(d) We consider the following constrained optimization program:

$$\begin{aligned} x^{\star}(\eta) &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\ \text{u.c.} & \quad \begin{cases} \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \\ n^{\star}(x) = \eta \end{cases} \end{aligned}$$

where η is the targeted effective number of stocks. Show that it is equivalent to solve this optimization program:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} (\Sigma + \lambda I_n) x \\ \text{u.c.} & \quad \begin{cases} \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{cases} \end{aligned} \tag{1}$$

where λ is a scalar to determine and I_n is the identity matrix.

(e) Solve the optimization program (1) when λ takes the following values: 0.01, 0.10, 0.50, 1.00, and 50. For each constrained minimum variance portfolio, calculate the corresponding effective number of stocks $n^{\star}(x)$.

(f) Using the bisection algorithm, find the optimal value of λ that satisfies:

$$n^{\star}(x) = 4$$

Give the optimal minimum variance portfolio.

3. We consider the CW portfolio as the benchmark.

(a) Describe the heuristic algorithm for performing index sampling.

(b) Using the heuristic algorithm, find the sampling portfolio with 7 stocks.

(c) Same question if we want to keep only 4 stocks.

(d) Give the elimination order of the stocks[3].

[3] From the first deleted stock to the eight deleted stock.

3 Alternative Risk Premia

1. The five equity risk factors
   (a) Define the equity risk factors: size, value, momentum, low beta and quality.
   (b) For each risk factor, give a metric or a ratio that allows to rank the stocks according to this factor.
2. The carry risk factor
   (a) How the carry risk premium is defined in the investment universe of equities?
   (b) Can the carry risk factor be extended to other asset classes: rates, currencies, credit and commodities?
3. The momentum risk factor
   (a) What is the difference between the cross-section momentum and the time-series momentum?
   (b) What is the main result obtained by Carhart (1977)?
4. The volatility risk premium
   (a) Define the VIX index.
   (b) What is the difference between the implied volatility and the realized volatility?
   (c) How do you explain the existence of the volatility risk premium in the universe of equities?
5. We provide the cumulative performance of 14 indices in the attached Excel file.
   (a) For each index, give the corresponding asset class (equities, rates, credit, currencies and commodities) and risk premium (carry, low beta, momentum, quality, value and volatility).
   (b) For each index, calculate the annualized performance and the realized volatility.
   (c) Using the Libor interest rates, deduce the Sharpe ratio for the 14 indices.
   (d) Calculate the empirical correlation matrix between these 14 indices.
   (e) Comment on these results.

References

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