Risk Parity Final Examination

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Contents

Remark 1 ***The final examination on risk parity is composed of 4 exercises. Please write entirely your answers. Precise the different concepts and the different statistics you use. Provide also one excel file by question. Concerning risk[1] and performance[2] decompositions, present the results as follows:***

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

Asset	x_i	μ_i	$\mathcal{PC}_i$	$\mathcal{PC}_i^\star$
1				
2				
$\vdots$				
n				
$\mu(x)$			✓	

[1] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^\star$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

[2] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, μ_i is the expected return, $\mathcal{PC}_i$ is the nominal (ex-ante) performance contribution, $\mathcal{PC}_i^\star$ is the relative performance contribution and $\mu(x)$ is the expected return of the portfolio.

1 Risk-based portfolios

We consider an investment universe with 5 assets. We assume that their expected returns are 3%, 6%, 8%, 9% and 8%, and that their volatilities are 5%, 10%, 15%, 15% and 20%. The correlation matrix is given by:

$$\rho = \begin{pmatrix} 100\% & & & & \\ 50\% & 100\% & & & \\ 20\% & 20\% & 100\% & & \\ 50\% & 50\% & 70\% & 100\% & \\ 0\% & -20\% & -30\% & -30\% & 100\% \end{pmatrix}$$

1. We consider long-short portfolios x with $\sum_{i=1}^{5} x_i = 1$.
   (a) Compute the minimum variance portfolio.
   (b) Calculate the MVO portfolio if we target an ex-ante volatility of 5%.
   (c) Calculate the MVO portfolio if we target an ex-ante volatility of 10%.
   (d) Compare the three optimized portfolios in terms of expected return and volatility.
2. We restrict the analysis to long-only portfolios x meaning that $\sum_{i=1}^{5} x_i = 1$ and $x_i \geq 0$.
   (a) Compute the minimum variance portfolio.
   (b) Calculate the MVO portfolio if we target an ex-ante volatility of 5%.
   (c) Calculate the MVO portfolio if we target an ex-ante volatility of 10%.
   (d) Compare the three optimized portfolios in terms of expected return and volatility.
   (e) What is the impact of the long-only constraint?
3. We consider long-only portfolios. For each portfolio, compute the volatility decomposition.
   (a) Determine the equally weighted portfolio.
   (b) Compute the minimum variance portfolio.
   (c) Calculate the most diversified portfolio.
   (d) Find the ERC portfolio.
4. Draw the 4 previous risk-based portfolios on the efficient frontier. Comment on these results.

2 Implied risk premium

We suppose that the assets returns follow a linear factor model:

$$R_{i,t} = \beta_i R_{m,t} + \varepsilon_{i,t}$$

with $R_{m,t}$ the common factor, $\varepsilon_{i,t}$ the idiosyncratic factor, $R_{m,t} \perp \varepsilon_t$, $\operatorname{var}(F_t) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon_t) = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$. We consider a universe of 6 assets with the following parameter values:

i	1	2	3	4	5	6
β_i	0.30	0.50	0.70	1.00	1.20	1.50
$\tilde{\sigma}_i$	0.05	0.06	0.07	0.08	0.09	

and $\sigma_m = 8\%$.

1. What is the interpretation of the common factor?
2. (a) We assume that the volatility of the 6$^{\text{th}}$ asset is equal to 20%. Compute the value of the parameter $\tilde{\sigma}_6$.
   (b) Compute the volatilities of the assets and the corresponding correlation matrix.
3. We assume that the risk-free rate is equal to 2% and the expected returns μ_i are given by the following table:

i	1	2	3	4	5	6
μ_i	3.5%	4.5%	5.5%	7.0%	8.0%	9.5%

   (a) Find the tangency portfolio $x^\star$. Deduce its expected return $\mu(x^\star)$ and its volatility $\sigma(x^\star)$.
   (b) Compute the beta $\beta_i(x^\star)$ of each asset with respect to the tangency portfolio $x^\star$.
   (c) Compute the implied risk premium of each asset by considering the CAPM theory. Comment on these results.
4. We include the risk-free asset in the investment universe.
   (a) Give the vector μ of expected returns and the covariance matrix Σ with the seven-asset investment universe.
   (b) Draw the efficient frontier by considering the long-only constraint. Explain the obtained results.
   (c) An institutional investor targets an expected return of 9%. However, the regulation imposes that he cannot use short positions and leverage the portfolio.
      i. Explain why he cannot consider the tangency portfolio $x^\star$ obtained in Question 3(a).
      ii. Find his optimal portfolio $\tilde{x}$.
      iii. Compute the beta $\beta_i(\tilde{x})$ of each asset with respect to the optimal portfolio $\tilde{x}$.
      iv. Compute the implied risk premium of each asset for this investor.
      v. Draw the relationship between the beta $\beta_i(\tilde{x})$ and the alpha α_i of asset i. Explain why we don't retrieve the result of Frazzini and Pedersen (2010).

3 Risk allocation of bond portfolios

We assume that the zero-coupon rates from one to five years are equal to:

- $R_t(1Y) = 1.0\%$
- $R_t(2Y) = 1.5\%$
- $R_t(3Y) = 2.0\%$
- $R_t(4Y) = 2.5\%$
- $R_t(5Y) = 3.0\%$

We also assume that the risk factors are the variations of the zero-coupon rates:

$$\Delta_h R_t(T) = R_t(T) - R_{t-h}(T)$$

We note:

$$\Delta_h R_t = \begin{pmatrix} \Delta_h R_t(1Y) \\ \Delta_h R_t(2Y) \\ \Delta_h R_t(3Y) \\ \Delta_h R_t(4Y) \\ \Delta_h R_t(5Y) \end{pmatrix}$$

For this exercise, the holding period h is set to 10 days. Using historical data, we estimate the covariance matrix Σ of the five risk factors $\Delta_h R_t$. The volatilities (expressed in bps) are equal to:

$$\begin{pmatrix} \sigma_{1Y} \\ \sigma_{2Y} \\ \sigma_{3Y} \\ \sigma_{4Y} \\ \sigma_{5Y} \end{pmatrix} = \begin{pmatrix} 5.49 \\ 8.69 \\ 12.22 \\ 14.52 \\ 15.85 \end{pmatrix}$$

whereas the correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100.00\% & & & & \\ 79.32\% & 100.00\% & & & \\ 60.02\% & 95.24\% & 100.00\% & & \\ 49.54\% & 88.97\% & 98.36\% & 100.00\% & \\ 41.85\% & 82.81\% & 94.93\% & 98.98\% & 100.00\% \end{pmatrix}$$

In what follows, we assume that $\Delta_h R_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$. The risk measure corresponds to the Gaussian value-at-risk with a 99% confidence level and a 10-days holding period. It is expressed in euros.

1. We note $C^{(i)}(T)$ the value of the coupon with maturity T for the bond i. We also note ϖ_i the number of bonds i held in the portfolio.

   (a) Show that the P&L Π for the holding period h can be expressed as follows:

   $$\Pi = \sum_{i=1}^{n} \sum_{T=1Y}^{5Y} \varpi_i C^{(i)}(T) \left(B_{t+h}(T) - B_t(T)\right)$$

   where n is the number of different bonds and $B_t(T)$ is the zero-coupon price at time t with maturity T.

   (b) Compute the duration $D_t(T)$ associated to the zero-coupon bond $B_t(T)$. Deduce that:

   $$\Pi \simeq -\sum_{i=1}^{n} \sum_{T=1Y}^{5Y} \varpi_i C^{(i)}(T) D_t(T) B_t(T) \Delta_h R_t(T)$$

(c) Let $\delta(T)$ be the exposure of the P&L to the factor $\Delta_h R_t(T)$. Give the expression of $\delta(T)$ and show that the Gaussian value-at-risk is:

$$\mathrm{VaR}_\alpha(L) = \Phi^{-1}(\alpha)\sqrt{\delta^\top \Sigma \delta}$$

with:

$$\delta = \begin{pmatrix} \delta(1Y) \\ \delta(2Y) \\ \delta(3Y) \\ \delta(4Y) \\ \delta(5Y) \end{pmatrix}$$

(d) Let Π_i be the P&L of the bond i:

$$\Pi_i = -\sum_{T=1Y}^{5Y} C^{(i)}(T)\, D_t(T)\, B_t(T)\, \Delta_h R_t(T)$$

Compute the covariance matrix of $(\Pi_1, \ldots, \Pi_n)$. Deduce another expression of the Gaussian value-at-risk.

2. We consider that the bond portfolio is composed of 5 zero-coupon bonds with maturities $1Y, \ldots, 5Y$. This means that:

i	1	2	3	4	5
$C^{(i)}(1Y)$	1	0	0	0	0
$C^{(i)}(2Y)$	0	1	0	0	0
$C^{(i)}(3Y)$	0	0	1	0	0
$C^{(i)}(4Y)$	0	0	0	1	0
$C^{(i)}(5Y)$	0	0	0	0	1

(a) We assume that $\varpi_i = 1$. Compute the value-at-risk and the risk decomposition. Deduce the ratio between the risk measure and the value of the bond portfolio.

(b) We assume that the composition of the bond portfolio is the following:

$$\begin{pmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \\ \varpi_4 \\ \varpi_5 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 2 \end{pmatrix}$$

Compute the value-at-risk and the risk decomposition. Deduce the ratio between the risk measure and the value of the bond portfolio.

(c) We note N_i the notional[3] invested in the bond i. We have:

$$N_i = \varpi_i P_t^{(i)}$$

where $P_t^{(i)}$ is the price of the bond i:

$$P_t^{(i)} = \sum_{T=1Y}^{5Y} C^{(i)}(T)\, B_t(T)$$

Compute the price $P_t^{(i)}$ for the 5 different bonds. We assume that:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} = \begin{pmatrix} 5000 \\ 2000 \\ 1000 \\ 1000 \\ 1000 \end{pmatrix}$$

[3] N_i is expressed in euros.

Find the composition $(\varpi_1, \ldots, \varpi_5)$ of the bond portfolio. Compute then the value-at-risk and the risk decomposition. Deduce also the ratio between the risk measure and the value of the bond portfolio.

(d) Compute the composition $(\varpi_1, \ldots, \varpi_5)$ of the ERC portfolio if we assume that the portfolio value is equal to 1 million euros.

3. We consider that the bond portfolio is composed of 2 bonds with maturities $3Y$ and $5Y$:

i	1	2
$C^{(i)}(1Y)$	6	2
$C^{(i)}(2Y)$	6	2
$C^{(i)}(3Y)$	106	2
$C^{(i)}(4Y)$	0	2
$C^{(i)}(5Y)$	0	102

(a) We assume that $\varpi_1 = \varpi_2 = 1$. Compute the value-at-risk. Deduce then the ratio between the risk measure and the value of the bond portfolio.

(b) Compute the risk decomposition with respect to the fixing dates.

(c) Compute the risk decomposition with respect to the bonds.

(d) Repeat Questions 3(a), 3(b) and 3(c) if $\varpi_1 = 100$ and $\varpi_2 = 50$.

(e) Compute the composition (ϖ_1, ϖ_2) of the ERC portfolio if we assume that the portfolio value is equal to 1 million euros.

4 Strategic asset allocation

We consider a sovereign fund which is invested in the following asset classes:

- EQ-DM: equity developed markets
- EQ-EM: equity emerging markets
- BD-SB: sovereign bonds
- BD-IG: corporate IG bonds
- BD-HY: high yield bonds

Its current portfolio is:

$$x_0 = \begin{pmatrix} 25\% \\ 5\% \\ 45\% \\ 15\% \\ 10\% \end{pmatrix}$$

The department of the strategic asset allocation has to propose a new long-run allocation for the next 5 years. Its expectations in terms of performance, risk and correlation are the following:

Asset Class	μ_i	σ_i	$\rho_{i,j}$ EQ-DM	EQ-EM	BD-SB	BD-IG	BD-HY
EQ-DM	8%	15%	100%				
EQ-EM	10%	20%	70%	100%			
BD-SB	3%	5%	−20%	0%	100%		
BD-IG	5%	7%	30%	30%	50%	100%	
BD-HY	9%	10%	50%	50%	−10%	60%	100%

The risk-free rate is set to 2%. The objective of the sovereign fund is to define a **long-only** portfolio that generates a performance of 6% per year.

1. (a) What is the goal of strategic asset allocation?
   (b) Comment on the figures of performance, risk and correlation.
   (c) Compute the expected return $\mu(x_0)$ of the current allocation and the corresponding (ex-ante) performance decomposition.
2. We consider the mean-variance allocation approach with $x_i \geq 0\%$.
   (a) Compute the optimal portfolio if the sovereign fund targets a return of 6% per year.
   (b) Deduce the ex-ante performance decomposition.
   (c) Calculate the tracking error $\sigma\left(x^\star \mid x^0\right)$ and the turnover $\tau\left(x^\star \mid x^0\right)$. Comment on these results.
3. In order to limit the turnover, we impose the following constraints:

   $$|x_i - x_{0,i}| \leq 10\%$$

   (a) What mean these constraints?
   (b) Compute the optimal portfolio if the sovereign fund targets a return of 6% per year.
   (c) Deduce the ex-ante performance decomposition.
   (d) Calculate the tracking error $\sigma\left(x^\star \mid x^0\right)$ and the turnover $\tau\left(x^\star \mid x^0\right)$. Comment on these results.
4. We consider the tracking-error optimization model with the following constraint:

   $$\tau\left(x \mid x^0\right) \leq 1\%$$

(a) Compute the optimal portfolio.

(b) Deduce the ex-ante performance decomposition. Verify that the objective of the sovereign fund is reached.

(c) Calculate the tracking error $\sigma\left(x^{\star} \mid x^{0}\right)$ and the turnover $\tau\left(x^{\star} \mid x^{0}\right)$. Comment on these results.

(d) What do you think if we replace the tracking-error optimization model by the Black-Litterman allocation model?

5. We consider the risk budgeting approach.

   (a) Compute the implied expected return $\tilde{\mu}$ of the current allocation x_0 and the three optimized portfolios $x^{\star}$ obtained in Questions 2(a), 3(b) and 4(a) if we assume that these portfolios are optimal.

   (b) Compute the risk decomposition of the current allocation x_0. Which risk budgets should we increase or decrease in order to match the objective of the pension fund? What is the implicit assumption behind this approach?

   (c) The SAA department considers the following risk budgets:

$$b = \begin{pmatrix} 25\% \\ 5\% \\ 45\% \\ 15\% \\ 10\% \end{pmatrix}$$

      i. What is the rationale of this risk allocation?

      ii. Compute the risk budgeting portfolio and the corresponding risk decomposition. Verify that the objective of the sovereign fund is reached.

      iii. Compute the implied expected return $\tilde{\mu}$ of the RB portfolio. Deduce the corresponding (ex-ante) risk premium decomposition.

6. What portfolio do you prefer to define the long-run allocation of the sovereign fund? Justify your choice.

Remark 2 *The bonus question is:*

- *Find the Jagannathan-Ma implied covariance matrix of the optimized portfolios defined in Questions 2(a) and 3(b). Comment on these results.*