Course 2023-2024 in Financial Risk Management
Lecture 3. Credit Risk

Thierry Roncalli*

*Amundi Asset Management[1]

*University of Paris-Saclay

September 2023

[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of risk management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   4
4. **Keywords**
   Finance, Risk Management, Applied Mathematics, Statistics
5. **Hours**
   Lectures: 36h, Training sessions: 15h, HomeWork: 30h
6. **Evaluation**
   There will be a final three-hour exam, which is made up of questions and exercises
7. **Course website**
   `http://www.thierry-roncalli.com/RiskManagement.html`

Objective of the course

The objective of the course is twofold:

1. knowing and understanding the financial regulation (banking and others) and the international standards (especially the Basel Accords)
2. being proficient in risk measurement, including the mathematical tools and risk models

Class schedule

Course sessions

- September 15 (6 hours, AM+PM)
- September 22 (6 hours, AM+PM)
- September 19 (6 hours, AM+PM)
- October 6 (6 hours, AM+PM)
- October 13 (6 hours, AM+PM)
- October 27 (6 hours, AM+PM)

Tutorial sessions

- October 20 (3 hours, AM)
- October 20 (3 hours, PM)
- November 10 (3 hours, AM)
- November 10 (3 hours, PM)
- November 17 (3 hours, PM)

Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry, Room 209 IDF

Agenda

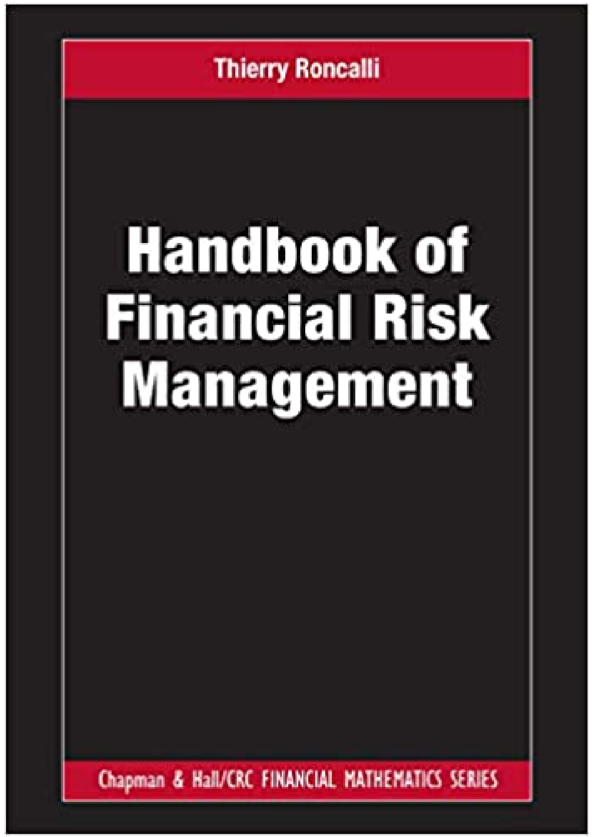
- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Textbook

- Roncalli, T. (2020), *Handbook of Financial Risk Management*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagement.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagementBook.html`

Agenda

- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- **Lecture 3: Credit Risk**
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models

The loan market

$\Rightarrow$ Banking intermediation (retail banks and corporate investment banks) $\neq$ financial market of debt securities (money market, bonds, notes, etc.)

Counterparties

- Sovereign
- Financial
- Corporate
- Retail

Products

- Mortgage and housing debt, consumer credit (auto loans, credit cards, revolving credit), student loans
- Revolving credit facilities (for corporates), corporate loans and other credit lines

$\Rightarrow$ Differences in terms of products and maturities (retail $\neq$ corporate)

Credit decision process

- Segmentation (retail banking)
- Pricing of the credit spread (commercial and investment banking)

The loan market

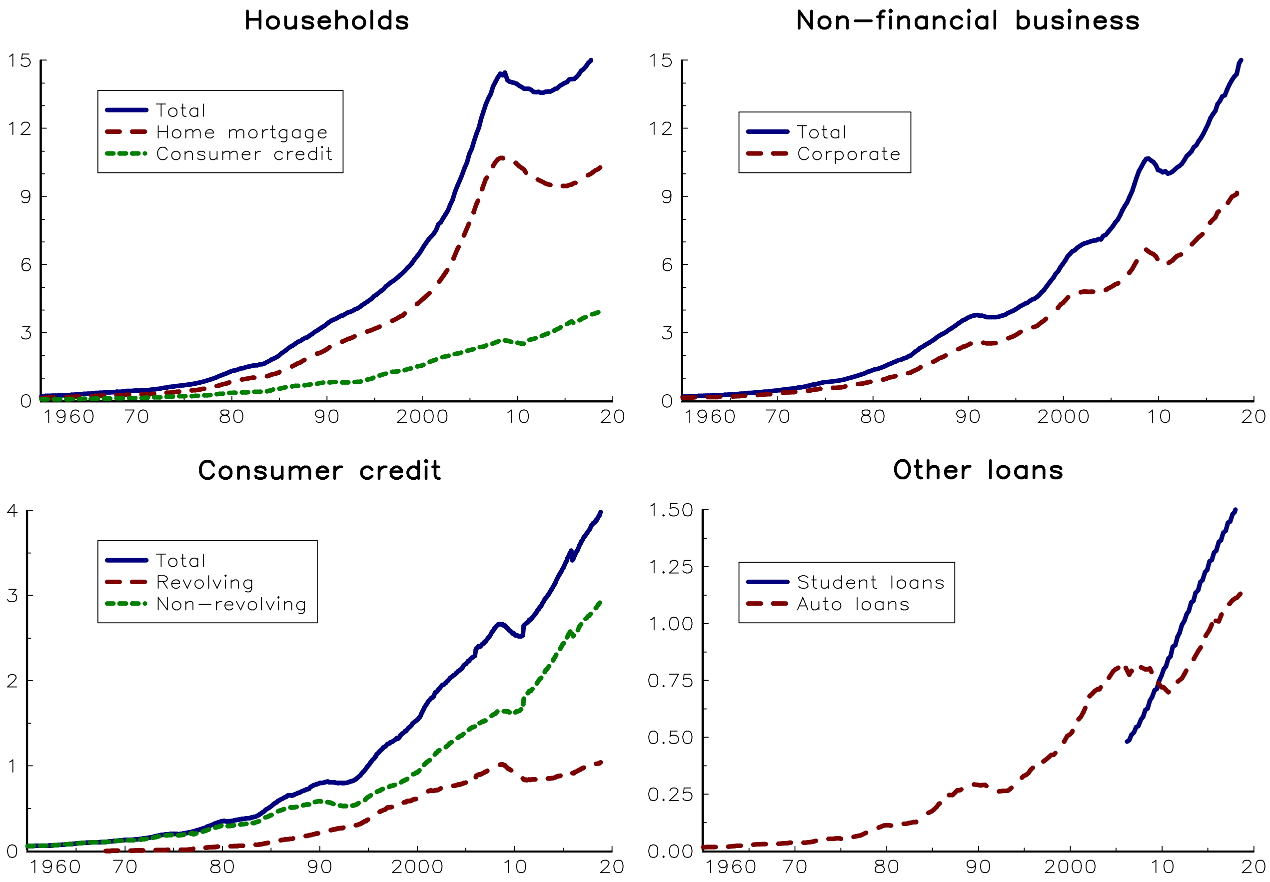


Figure: Credit debt outstanding in the United States (in $ tn)

The loan market

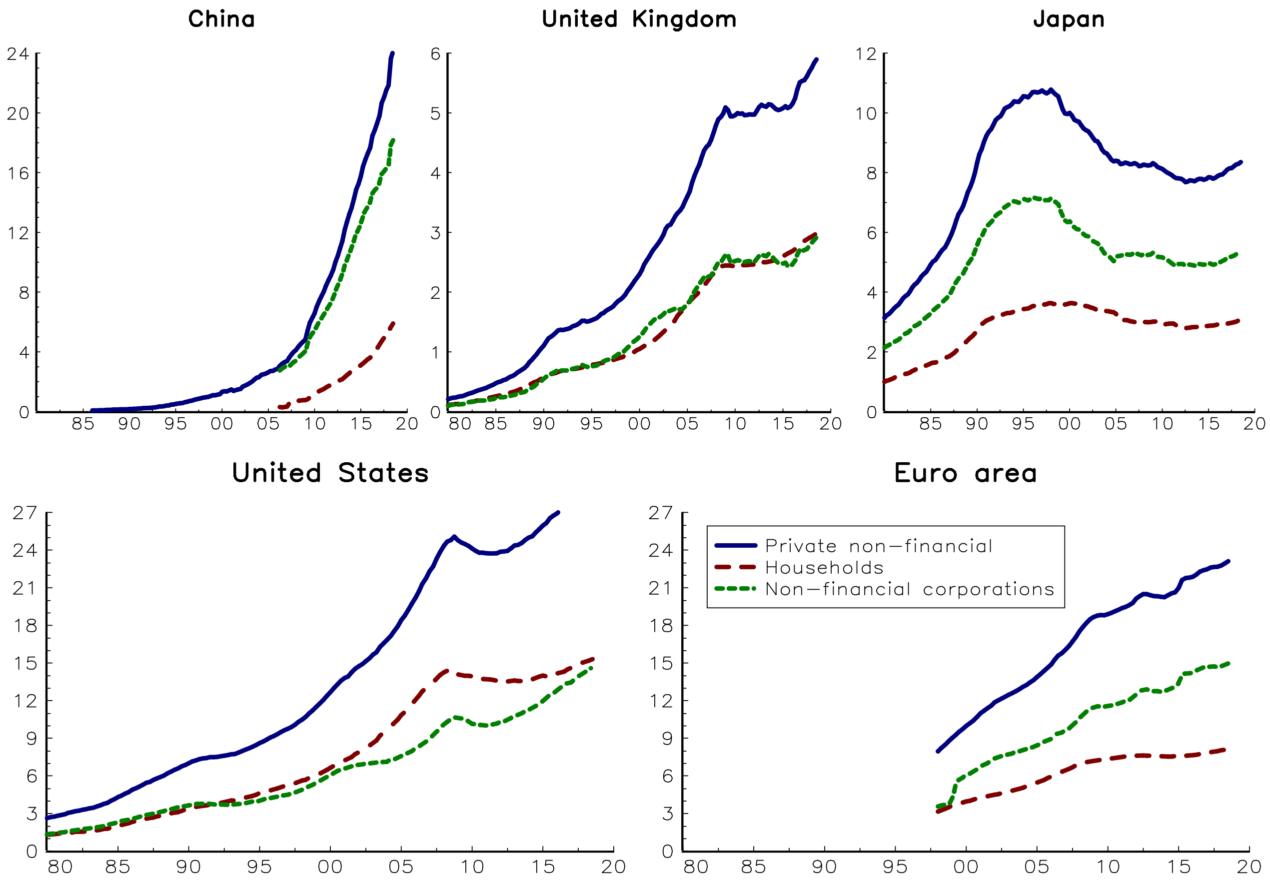


Figure: Credit to the private non-financial sector (in $ tn)

The bond market

Issuance $\neq$ outstanding:

- Primary market
- Secondary market

Three main sectors

- Central and local governments
- Financials
- Corporates

Statistics of the bond market

Table: Debt securities by residence of issuer (in $ bn)

		Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2017
Canada	Gov.	682	841	1 149	1 264
	Fin.	283	450	384	655
	Corp.	212	248	326	477
	Total	1 180	1 544	1 863	2 400
France	Gov.	1 236	1 514	1 838	2 258
	Fin.	968	1 619	1 817	1 618
	Corp.	373	382	483	722
	Total	2 576	3 515	4 138	4 597
Germany	Gov.	1 380	1 717	2 040	1 939
	Fin.	2 296	2 766	2 283	1 550
	Corp.	133	174	168	222
	Total	3 809	4 657	4 491	3 712
Italy	Gov.	1 637	1 928	2 069	2 292
	Fin.	772	1 156	1 403	834
	Corp.	68	95	121	174
	Total	2 477	3 178	3 593	3 299

Statistics of the bond market

Table: Debt securities by residence of issuer (in $ bn)

		Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2017
Japan	Gov.	6 336	6 315	10 173	9 477
	Fin.	2 548	2 775	3 451	2 475
	Corp.	1 012	762	980	742
	Total	9 896	9 852	14 604	12 694
Spain	Gov.	462	498	796	1 186
	Fin.	434	1 385	1 442	785
	Corp.	15	19	19	44
	Total	910	1 901	2 256	2 015
UK	Gov.	798	1 070	1 674	2 785
	Fin.	1 775	3 127	3 061	2 689
	Corp.	452	506	473	533
	Total	3 027	4 706	5 210	6 011
US	Gov.	6 459	7 487	12 072	17 592
	Fin.	12 706	17 604	15 666	15 557
	Corp.	3 004	3 348	3 951	6 137
	Total	22 371	28 695	31 960	39 504

Statistics of the bond market

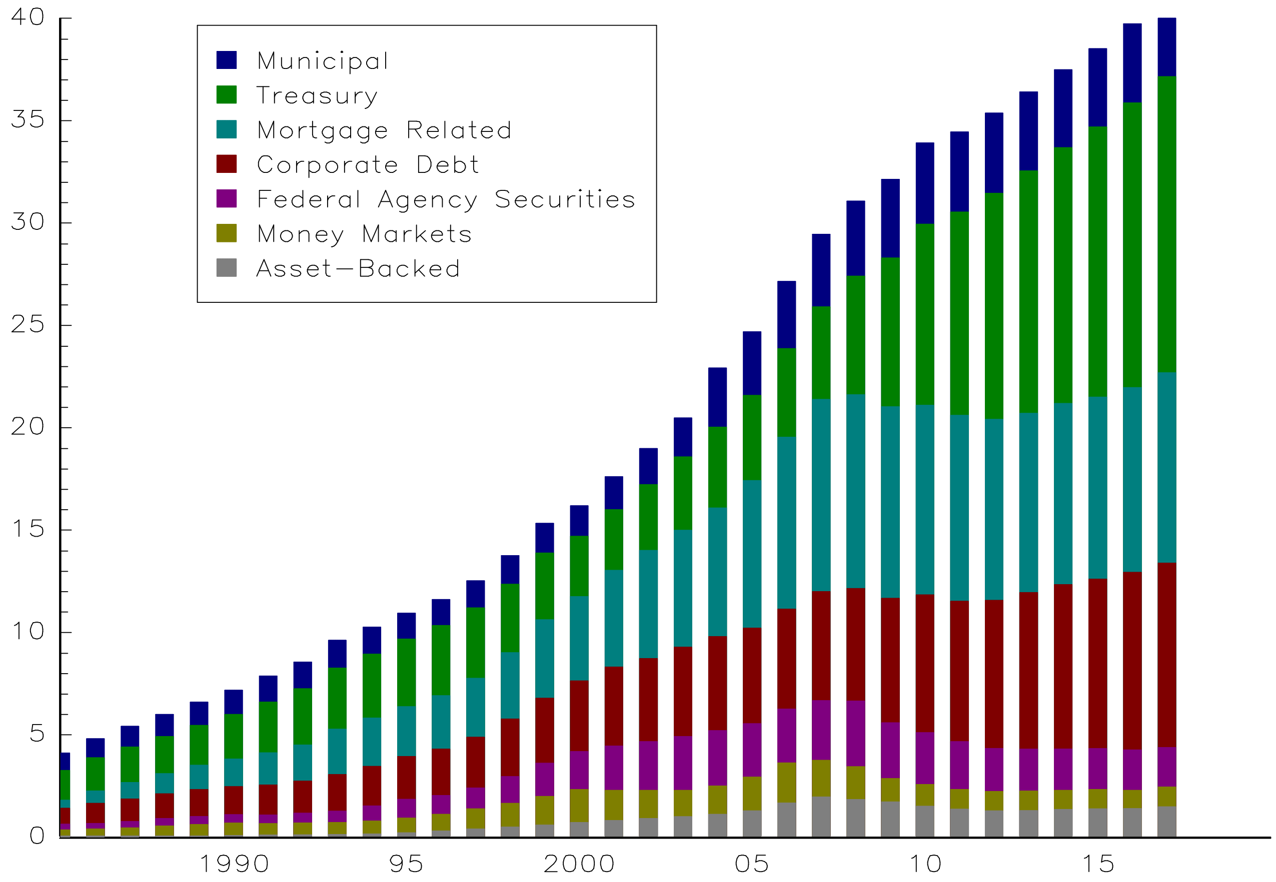


Figure: US bond market outstanding (in $ tn)

Statistics of the bond market

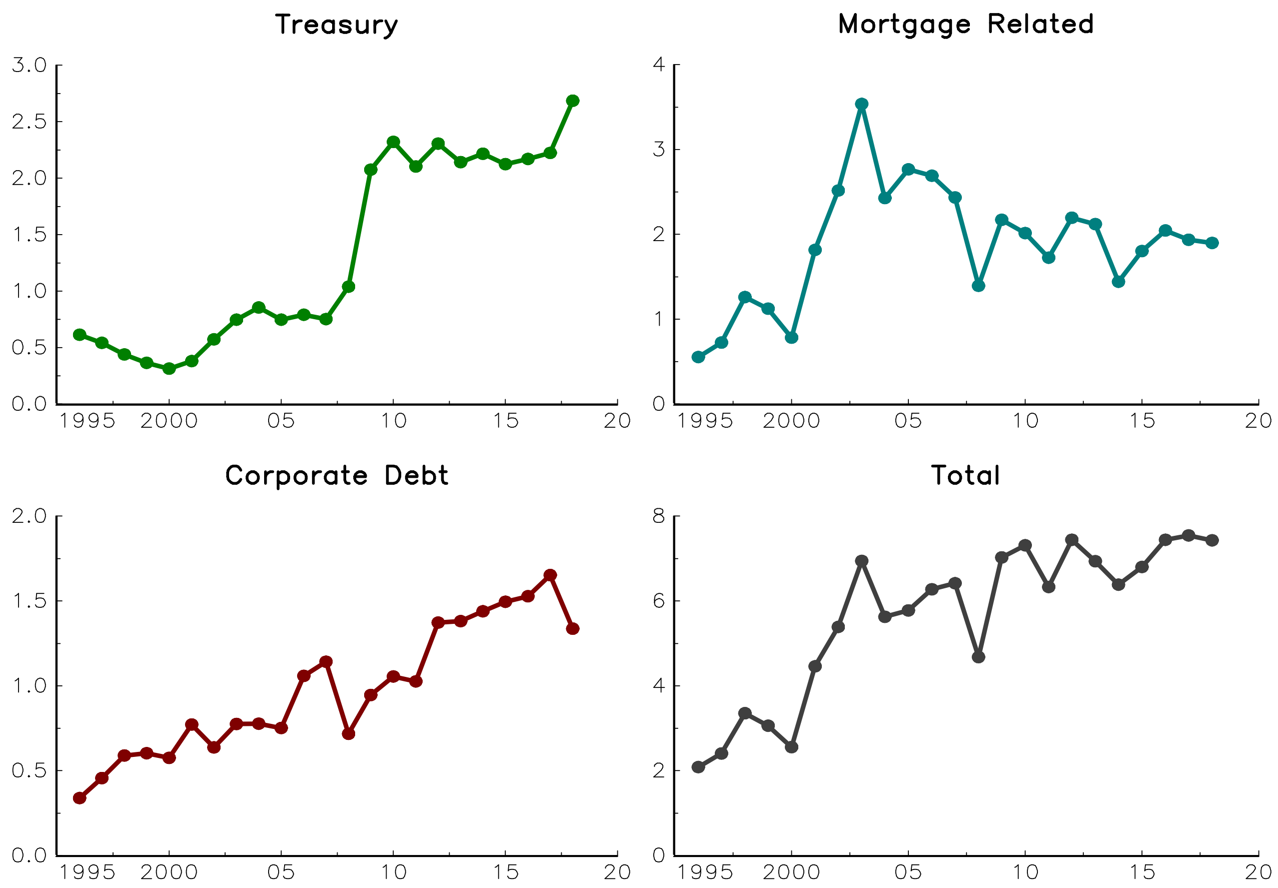


Figure: US bond market issuance (in $ tn)

Statistics of the bond market

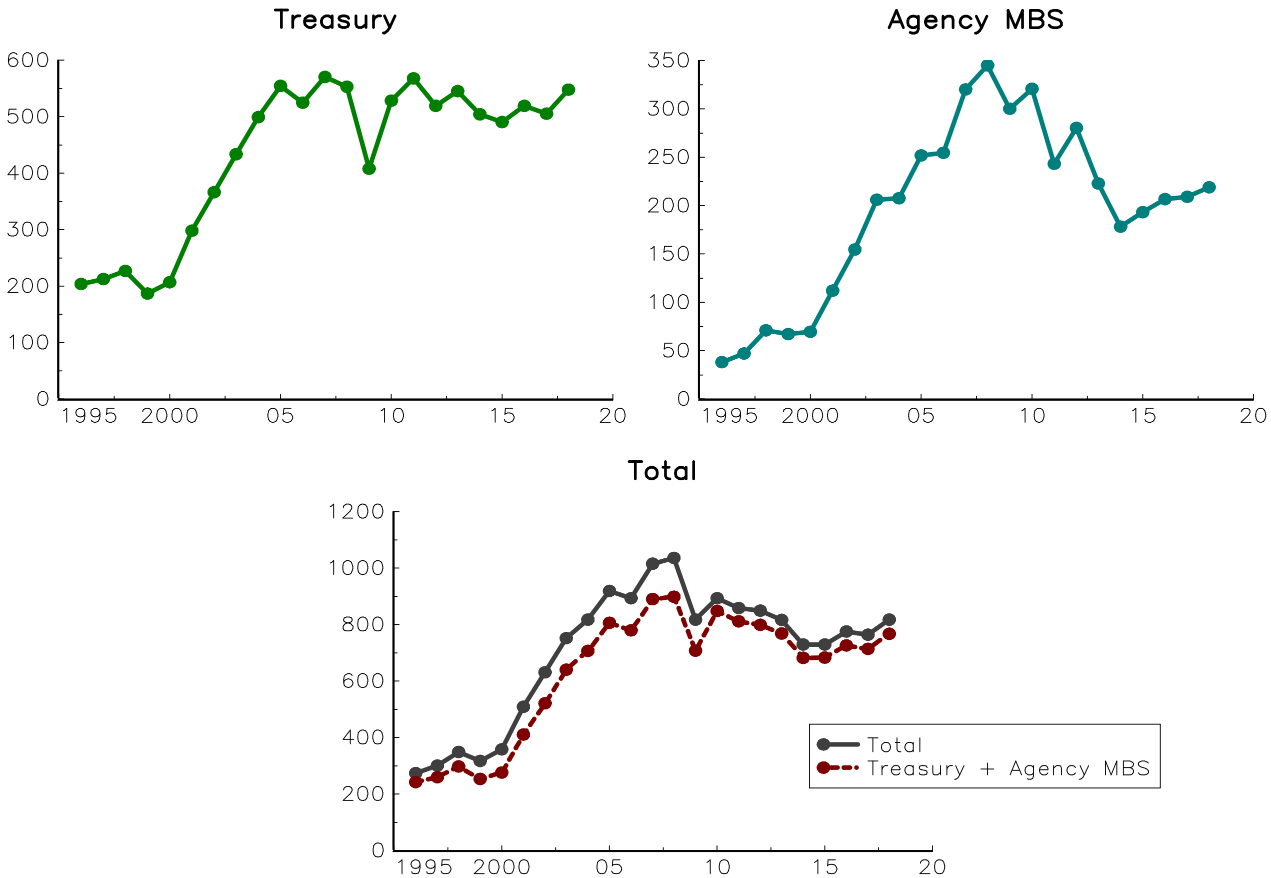


Figure: Average daily trading volume in US bond markets (in $ bn)

Bond pricing (without default risk)

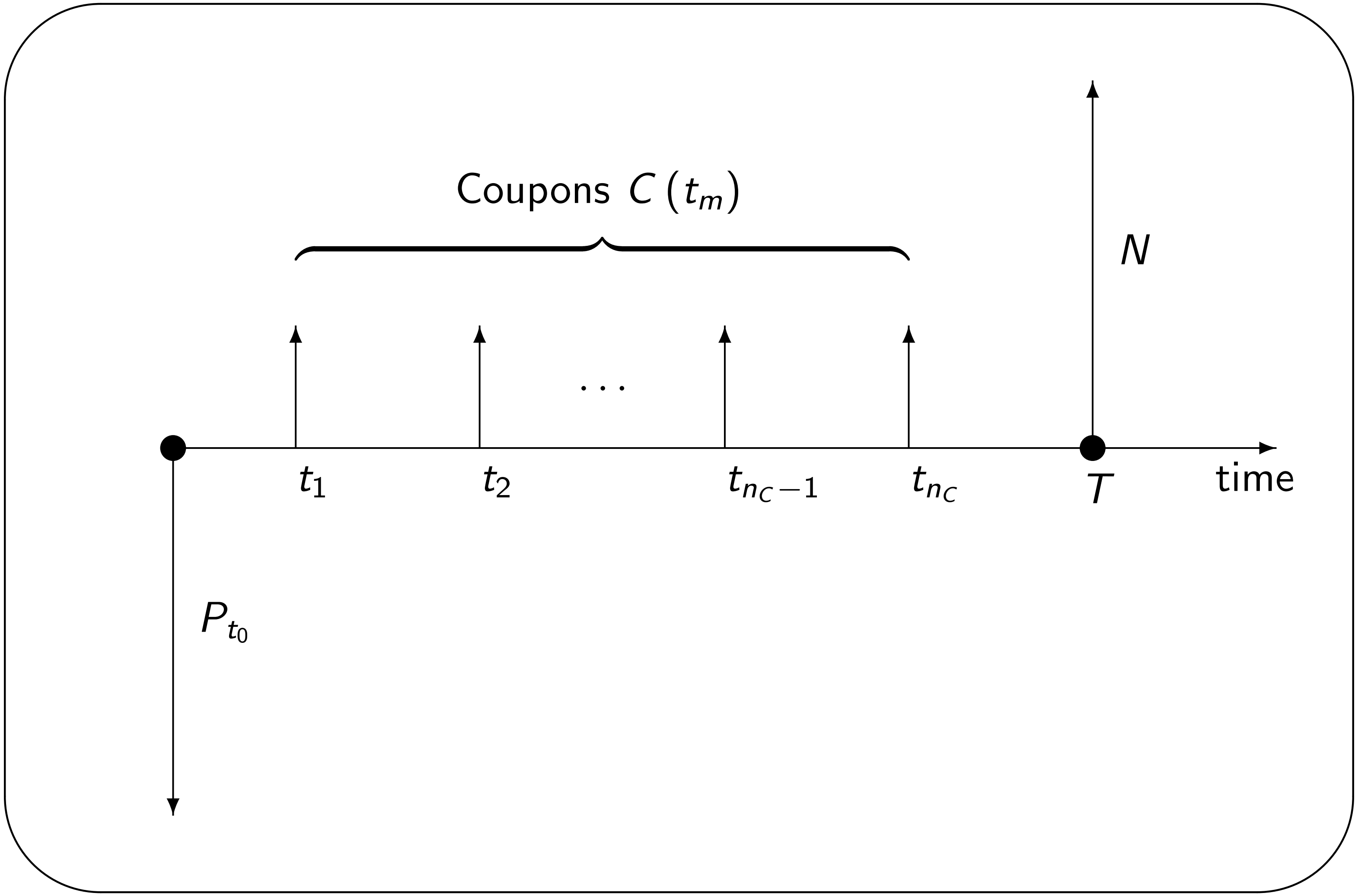


Figure: Cash flows of a bond with a fixed coupon rate

Bond pricing (without default risk)

The price of the bond at the inception date t_0 is the sum of the present values of all the expected coupon payments and the par value:

$$P_{t_0} = \sum_{m=1}^{n_C} C(t_m) \cdot B_{t_0}(t_m) + N \cdot B_{t_0}(T)$$

where $B_t(t_m)$ is the discount factor at time t for the maturity date t_m

Bond pricing (without default risk)

If we take into account the accrued interests, we have:

$$P_t + AC_t = \sum_{t_m \geq t} C(t_m) \cdot B_t(t_m) + N \cdot B_t(T)$$

Here, AC_t is the accrued coupon:

$$AC_t = C(t_c) \cdot \frac{t - t_c}{365}$$

and t_c is the last coupon payment date with $c = \{m : t_{m+1} > t, t_m \leq t\}$

- $P_t + AC_t$ is called the '*dirty price*'
- P_t is called the '*clean price*'

Impact of the term structure

3 main movements:

1. The movement of level corresponds to a parallel shift of interest rates.
2. A twist in the slope of the yield curve indicates how the spread between long and short interest rates moves.
3. A change in the curvature of the yield curve affects the convexity of the term structure.

Impact of the term structure

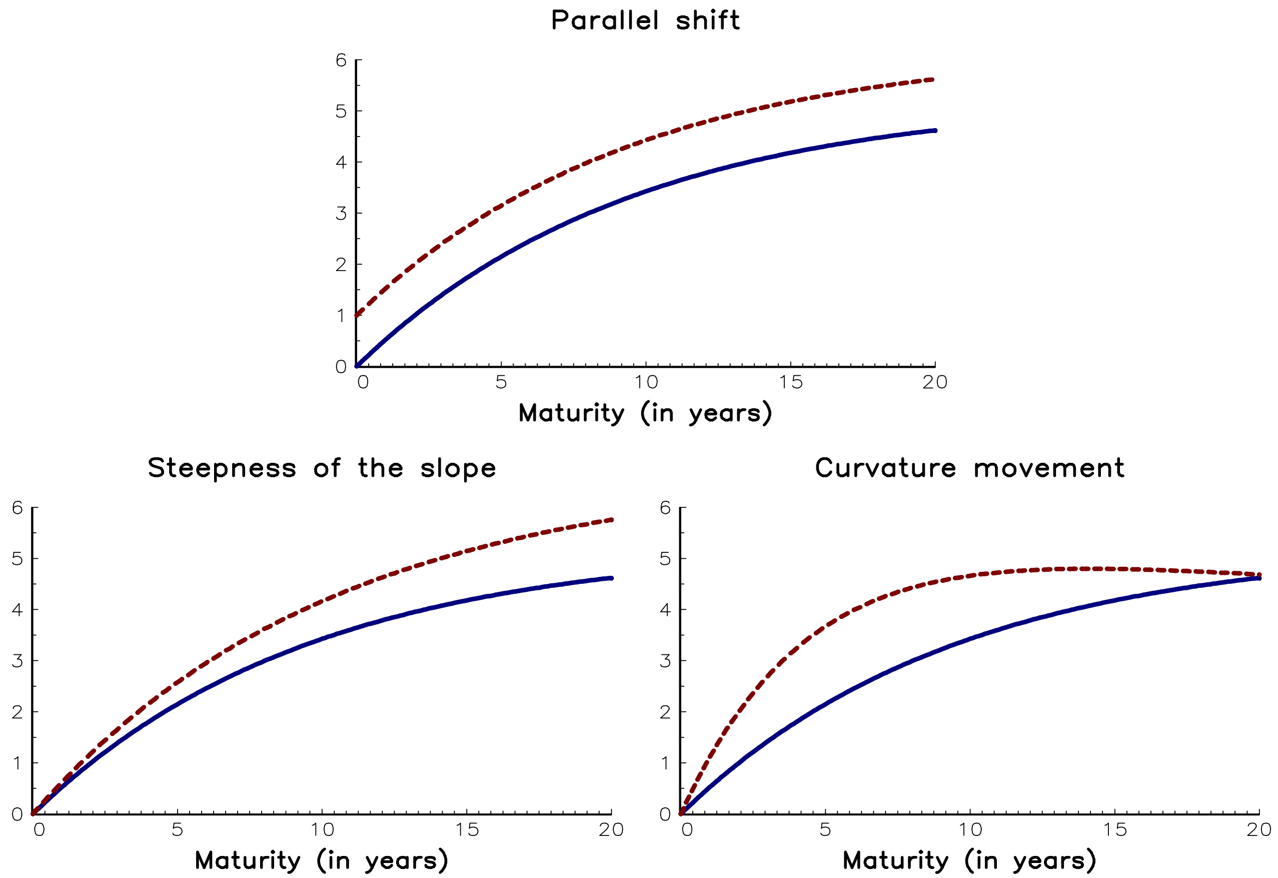


Figure: Movements of the yield curve

Yield to maturity

The yield to maturity y of a bond is the constant discount rate which returns its market price:

$$\sum_{t_m \geq t} C(t_m) e^{-(t_m - t)y} + N e^{-(T-t)y} = P_t + AC_t$$

The sensitivity S is the derivative of the clean price P_t with respect to the yield to maturity y:

$$S = \frac{\partial P_t}{\partial y} = -\sum_{t_m \geq t} (t_m - t) C(t_m) e^{-(t_m - t)y} - (T - t) N e^{-(T-t)y}$$

$\Rightarrow$ It indicates how the P&L of a long position on the bond moves when the yield to maturity changes:

$$\Pi \approx S \cdot \Delta y$$

Yield to maturity

Example

We assume that the zero-coupon rates are equal to 0.52% (1Y), 0.99% (2Y), 1.42% (3Y), 1.80% (4Y) and 2.15% (5Y). We consider a bond with a constant annual coupon of 5%. The nominal of the bond is \$100. We would like to price the bond when the maturity T ranges from 1 to 5 years.

The price of the four-year bond is equal to:

$$P_t = \frac{5}{(1+0.52\%)} + \frac{5}{(1+0.99\%)^2} + \frac{5}{(1+1.42\%)^3} + \frac{105}{(1+1.80\%)^4} = \$112.36$$

Yield to maturity

Table: Price, yield to maturity and sensitivity of bonds

T	$R_t(T)$	$B_t(T)$	P_t	y	S
1	0.52%	99.48	104.45	0.52%	−104.45
2	0.99%	98.03	107.91	0.98%	−210.86
3	1.42%	95.83	110.50	1.39%	−316.77
4	1.80%	93.04	112.36	1.76%	−420.32
5	2.15%	89.82	113.63	2.08%	−520.16

Yield to maturity

Table: Impact of a parallel shift of the yield curve on the bond with five-year maturity

ΔR (in bps)	$\breve{P}_t$	ΔP_t	$\hat{P}_t$	ΔP_t	$S \times \Delta y$
−50	116.26	2.63	116.26	2.63	2.60
−30	115.20	1.57	115.20	1.57	1.56
−10	114.15	0.52	114.15	0.52	0.52
0	113.63	0.00	113.63	0.00	0.00
10	113.11	−0.52	113.11	−0.52	−0.52
30	112.08	−1.55	112.08	−1.55	−1.56
50	111.06	−2.57	111.06	−2.57	−2.60

$$
\begin{aligned}
\breve{P}_t &= \sum_{t_m \geq t} C(t_m) e^{-(t_m - t)(R_t(t_m) + \Delta R)} + N e^{-(T-t)(R_t(T) + \Delta R)} \\
\hat{P}_t &= \sum_{t_m \geq t} C(t_m) e^{-(t_m - t)\left(y + \Delta R\right)} + N e^{-(T-t)\left(y + \Delta R\right)}
\end{aligned}
$$

Bond pricing (with default risk)

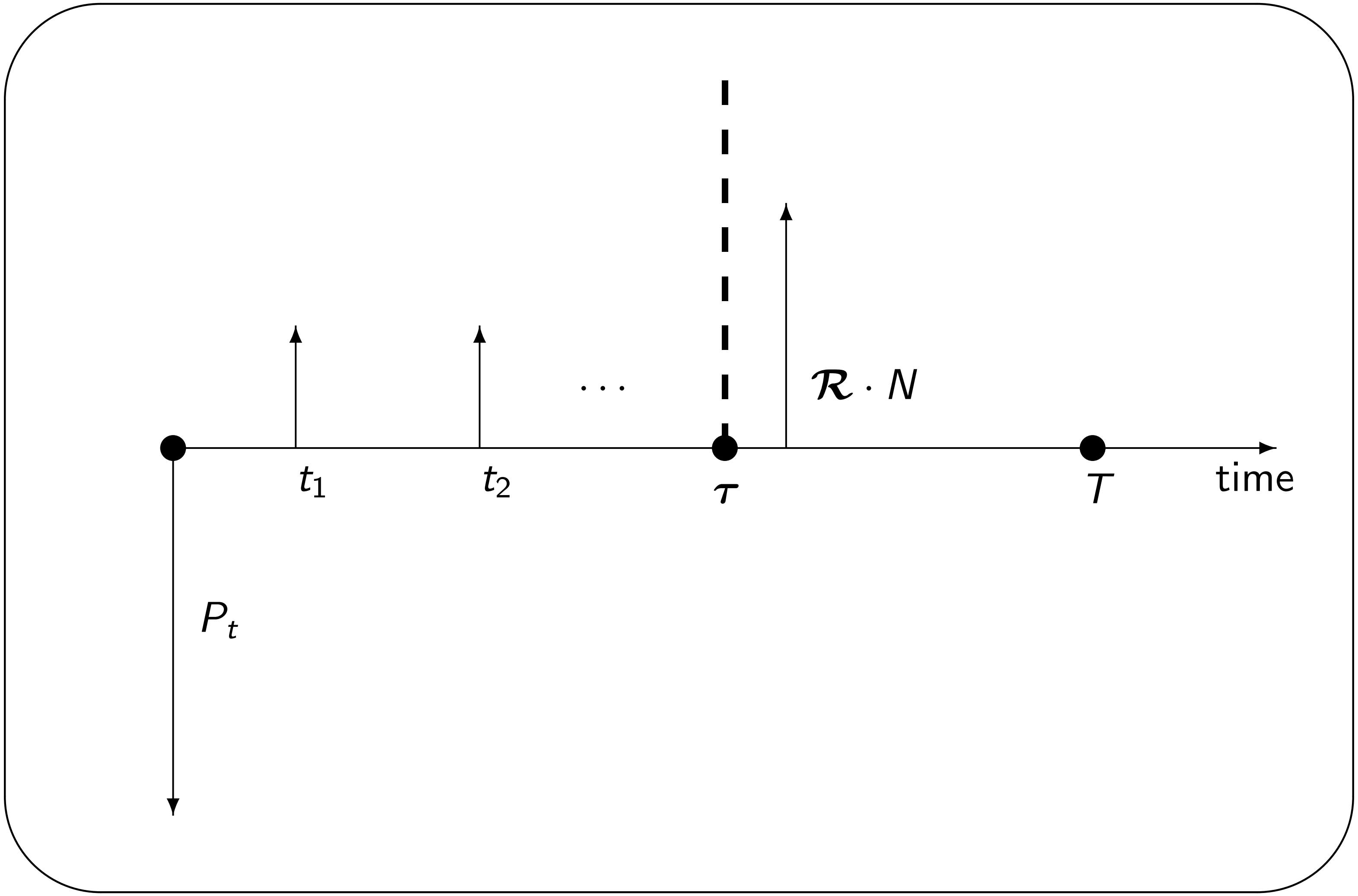


Figure: Cash flows of a bond with default risk

Bond pricing (with default risk)

- the coupons $C(t_m)$ if the bond issuer does not default before the coupon date t_m:

$$\sum_{t_m \geq t} C(t_m) \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\}$$

- the notional if the bond issuer does not default before the maturity date:

$$N \cdot \mathbb{1}\{\boldsymbol{\tau} > T\}$$

- the recovery part if the bond issuer defaults before the maturity date:

$$\boldsymbol{\mathcal{R}} \cdot N \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\}$$

  where $\boldsymbol{\mathcal{R}}$ is the corresponding recovery rate

$$\begin{aligned} SV_t &= \sum_{t_m \geq t} C(t_m) \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} + \\ & \quad N \cdot e^{-\int_t^T r_s \,\mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} > T\} + \boldsymbol{\mathcal{R}} \cdot N \cdot e^{-\int_t^{\boldsymbol{\tau}} r_s \,\mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\} \end{aligned}$$

Bond pricing (with default risk)

Closed-form formula

$$
\begin{aligned}
P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) \mathbf{S}_t(t_m) + N B_t(T) \mathbf{S}_t(T) + \\
&\quad \mathcal{R} N \int_t^T B_t(u) f_t(u) \, \mathrm{d}u
\end{aligned}
$$

where $\mathbf{S}_t(u)$ is the survival function at time u and $f_t(u)$ the associated density function

Bond pricing (with default risk)

If we consider an exponential default time with parameter λ – $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$, we have $\mathbf{S}_t(u) = e^{-\lambda(u-t)}$, $f_t(u) = \lambda e^{-\lambda(u-t)}$ and:

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) e^{-\lambda(t_m - t)} + N B_t(T) e^{-\lambda(T-t)} + \\ & \lambda \mathcal{R} N \int_t^T B_t(u) e^{-\lambda(u-t)} \,\mathrm{d}u \end{aligned}$$

If we assume a flat yield curve – $R_t(u) = r$, we obtain:

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) e^{-(r+\lambda)(t_m - t)} + N e^{-(r+\lambda)(T-t)} + \\ & \lambda \mathcal{R} N \left(\frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda} \right) \end{aligned}$$

If the recovery rate is equal to zero, $y = r + \lambda$

Credit spread

The credit spread is equal to the difference between the yield to maturity with default risk y and the yield to maturity without default risk $y^{\star}$:

$$s = y - y^{\star}$$

Remark

In the previous case (exponential default time + flat yield curve + zero recovery), we have:

$$s = \lambda$$

If λ is relatively small (less than 20%), the credit spread is approximately equal to the annual default probability PD:

$$\mathrm{PD} = \mathbf{S}_t \left(t + 1 \right) = 1 - e^{-\lambda} \approx \lambda$$

Credit spread

We consider the previous example with a coupon of 4.5% and a 10-year maturity

Table: Computation of the credit spread s

$\mathcal{R}$ (in %)	λ (in bps)	PD (in bps)	P_t (in \$)	y (in %)	s (in bps)
0	0	0.0	110.1	3.24	0.0
	10	10.0	109.2	3.34	9.9
	200	198.0	93.5	5.22	198.1
	1000	951.6	50.4	13.13	988.9
40	0	0.0	110.1	3.24	0.0
	10	10.0	109.6	3.30	6.0
	200	198.0	99.9	4.41	117.1
	1000	951.6	73.3	8.23	498.8
80	0	0.0	110.1	3.24	0.0
	10	10.0	109.9	3.26	2.2
	200	198.0	106.4	3.66	41.7
	1000	951.6	96.3	4.85	161.4

Credit risk versus market risk

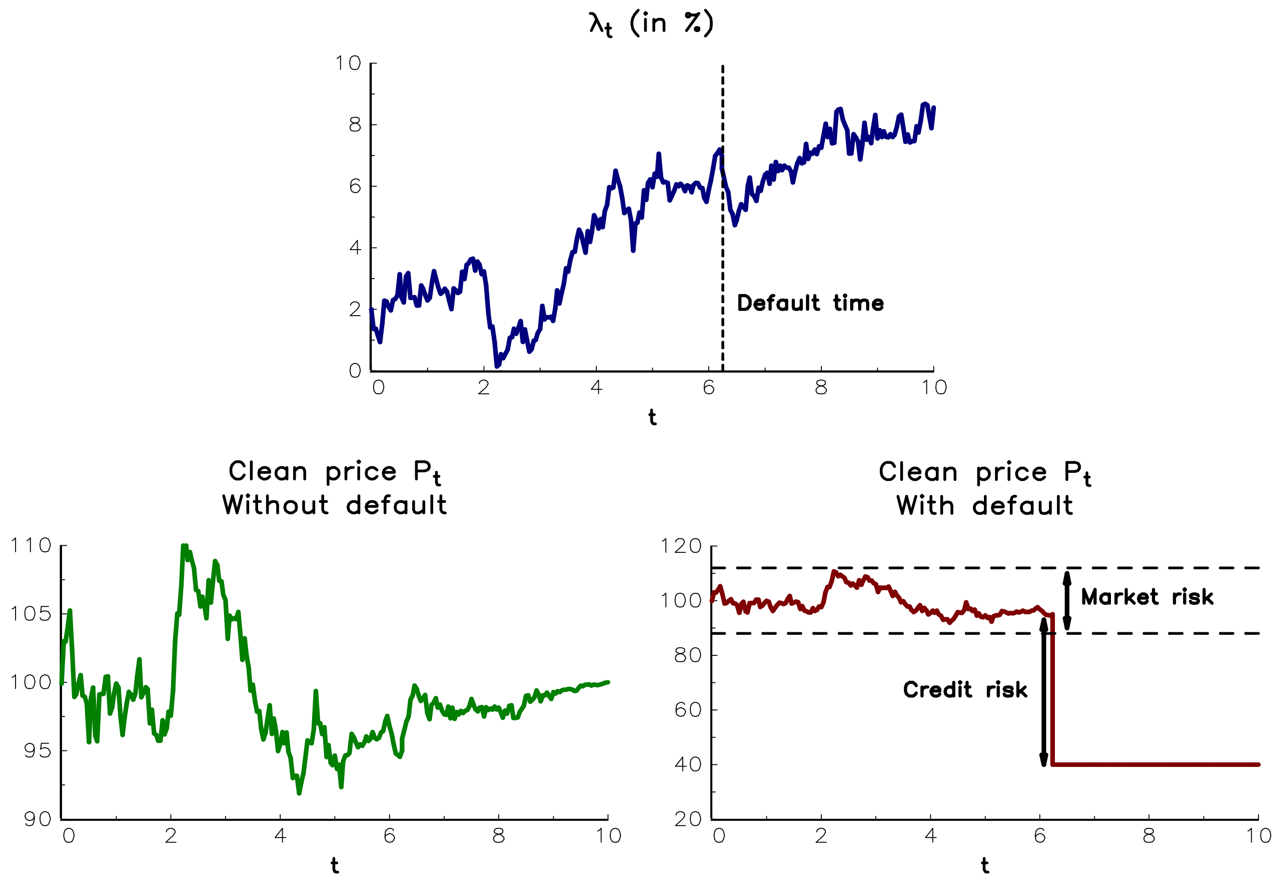


Figure: Difference between market and credit risks for a bond

Credit securitization

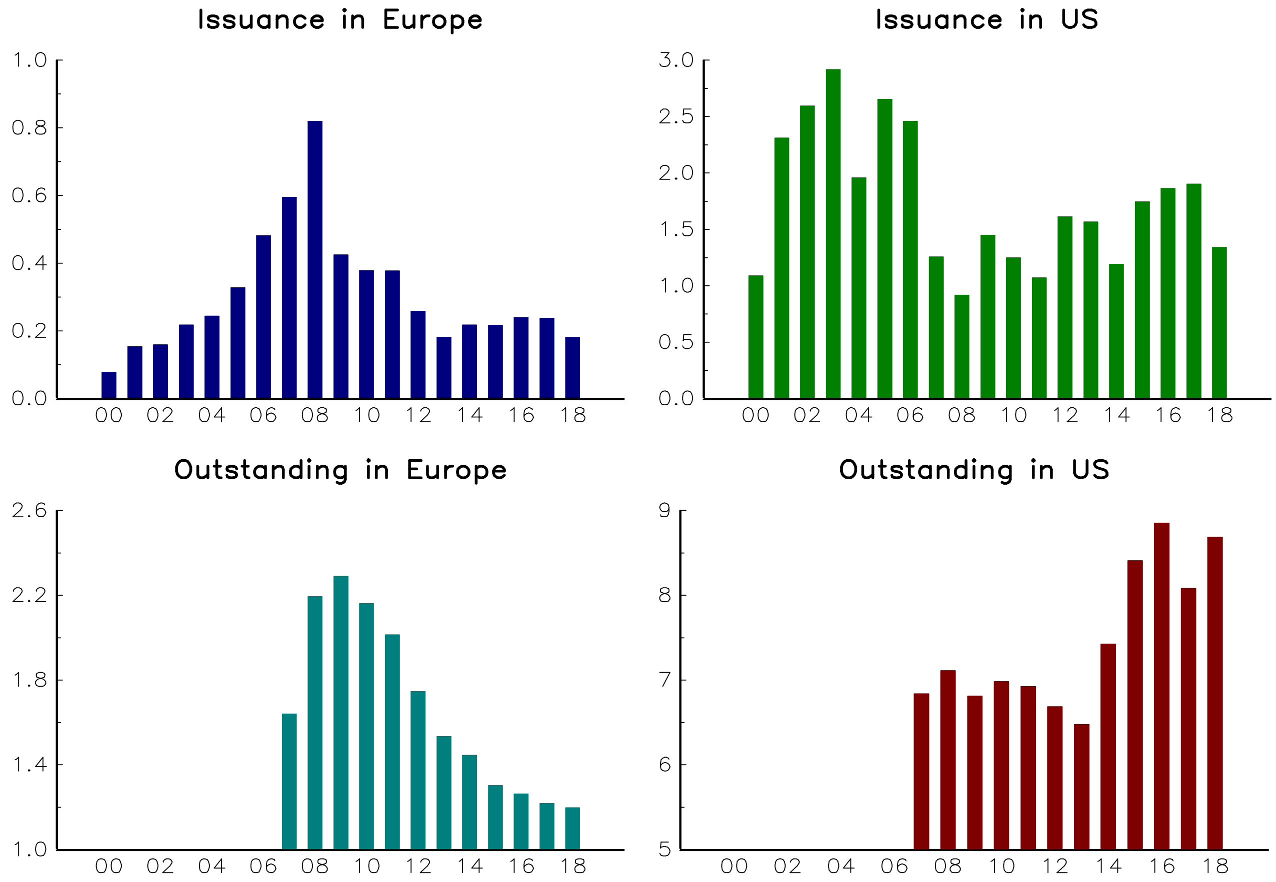


Figure: Securitization in Europe and US (in € tn)

Credit securitization

Collateral assets

- Mortgage-backed securities (MBS)
  - Residential mortgage-backed securities (RMBS)
  - Commercial mortgage-backed securities (CMBS)
- Collateralized debt obligations (CDO)
  - Collateralized loan obligations (CLO)
  - Collateralized bond obligations (CBO)
- Asset-backed securities (ABS)
  - Auto loans
  - Credit cards and revolving credit
  - Student loans

Credit securitization

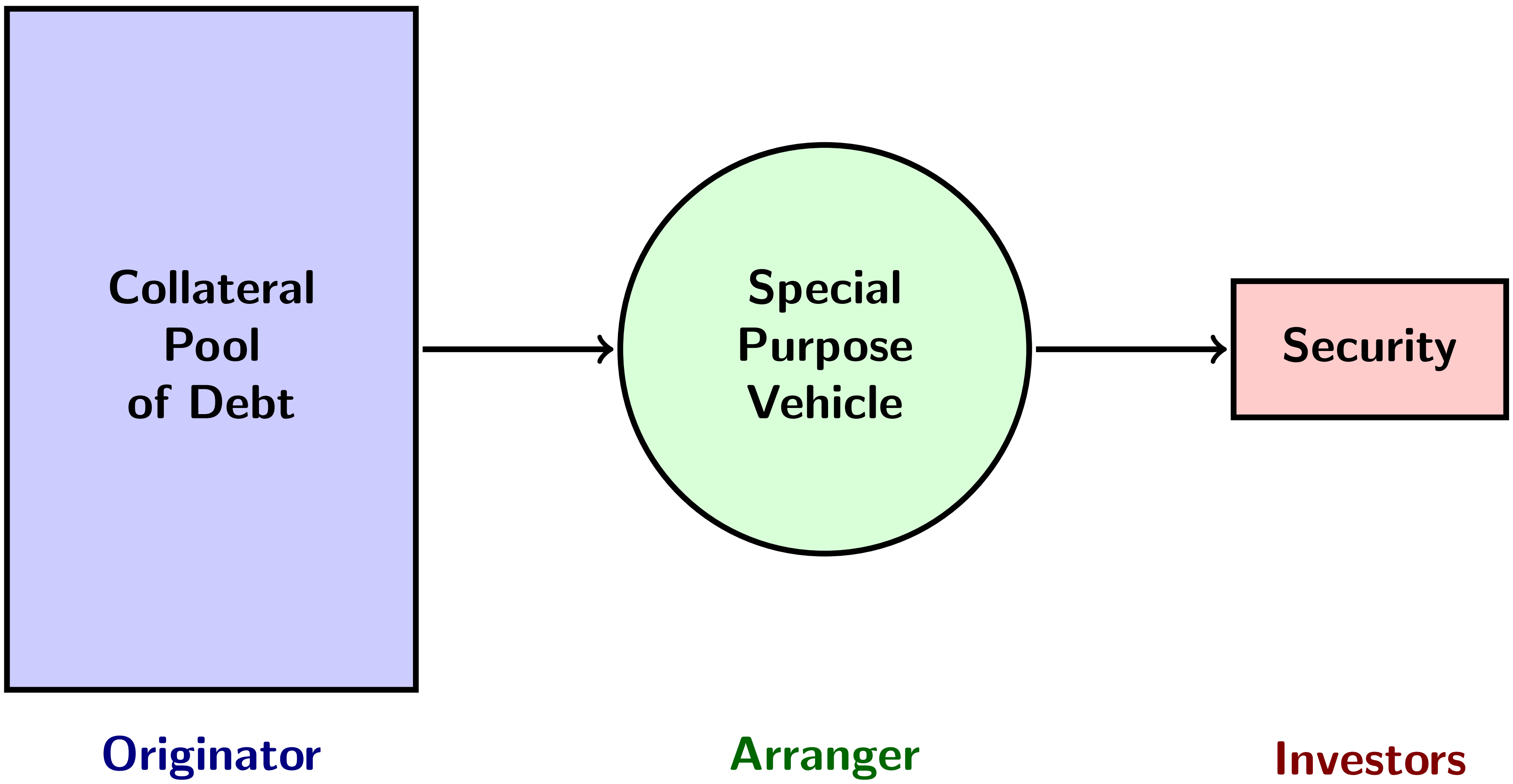


Figure: Structure of pass-through securities

Credit securitization

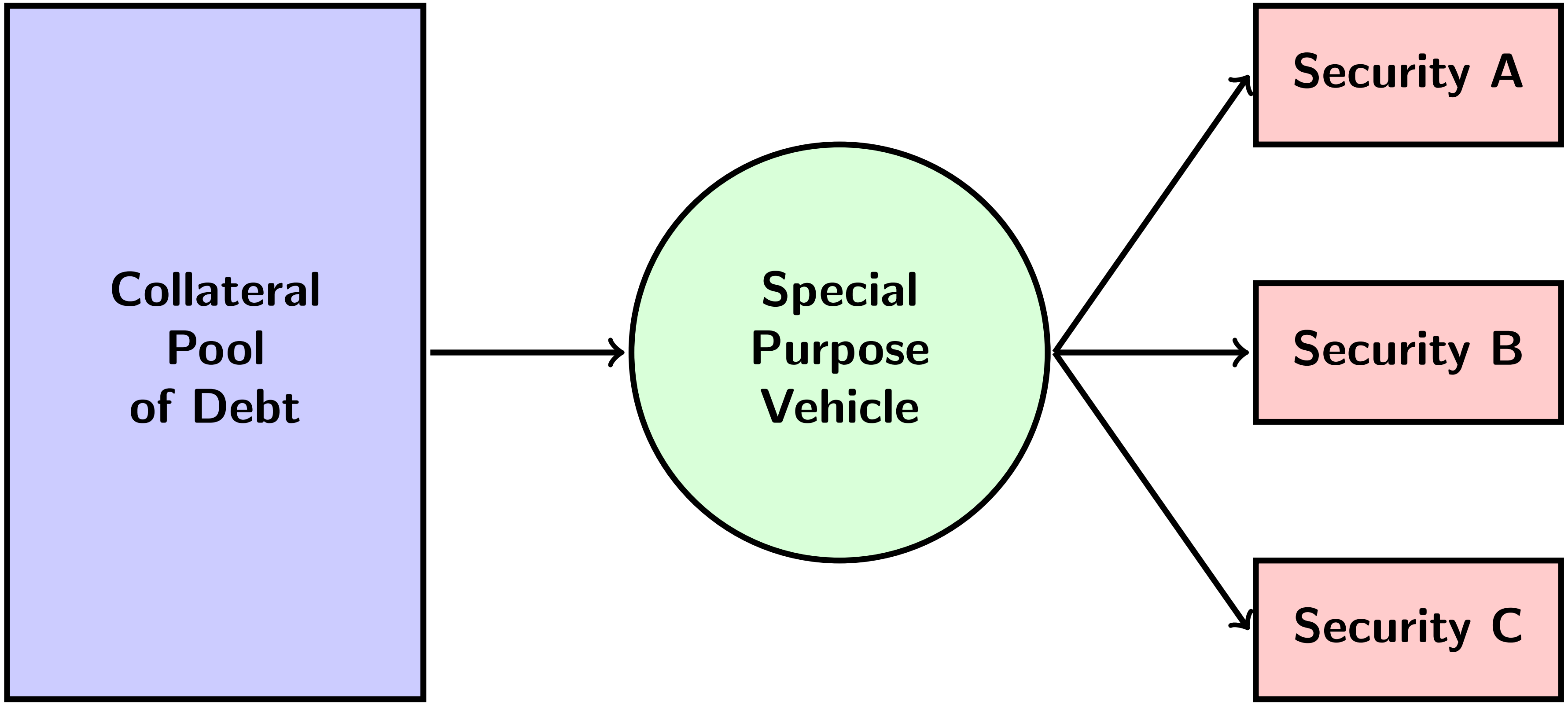


Figure: Structure of pay-through securities

Credit securitization

Table: US mortgage-backed securities

Year	Agency		Non-agency		Total
	MBS	CMO	CMBS	RMBS	(in $ bn)
			Issuance		
2002	57.5%	23.6%	2.2%	16.7%	2 515
2006	33.6%	11.0%	7.9%	47.5%	2 691
2008	84.2%	10.8%	1.2%	3.8%	1 394
2010	71.0%	24.5%	1.2%	3.3%	2 013
2012	80.1%	16.4%	2.2%	1.3%	2 195
2014	68.7%	19.2%	7.0%	5.1%	1 440
2016	76.3%	15.7%	3.8%	4.2%	2 044
2018	69.2%	16.6%	4.7%	9.5%	1 899
			Outstanding amount		
2002	59.7%	17.4%	5.6%	17.2%	5 289
2006	45.7%	14.9%	8.3%	31.0%	8 390
2008	52.4%	14.0%	8.8%	24.9%	9 467
2010	59.2%	14.6%	8.1%	18.1%	9 258
2012	64.0%	14.8%	7.2%	14.0%	8 838
2014	68.0%	13.7%	7.1%	11.2%	8 842
2016	72.4%	12.3%	5.9%	9.5%	9 023
2018	74.7%	11.3%	5.6%	8.4%	9 732

Credit securitization

Table: US asset-backed securities

Year	Auto Loans	CDO & CLO	Credit Cards	Equip-ement	Other	Student Loans	Total (in $ bn)
				Issuance			
2002	34.9%	21.0%	25.2%	2.6%	6.8%	9.5%	269
2006	13.5%	60.1%	9.3%	2.2%	4.6%	10.3%	658
2008	16.5%	37.8%	25.9%	1.3%	5.4%	13.1%	215
2010	46.9%	6.4%	5.2%	7.0%	22.3%	12.3%	126
2012	33.9%	23.1%	12.5%	7.1%	13.7%	9.8%	259
2014	25.2%	35.6%	13.1%	5.2%	17.0%	4.0%	393
2016	28.3%	36.8%	8.3%	4.6%	16.9%	5.1%	325
2018	20.8%	54.3%	6.1%	5.1%	10.1%	3.7%	517
				Outstanding amount			
2002	20.7%	28.6%	32.5%	4.1%	7.5%	6.6%	905
2006	11.8%	49.3%	17.6%	3.1%	6.0%	12.1%	1 657
2008	7.7%	53.5%	17.3%	2.4%	6.2%	13.0%	1 830
2010	7.6%	52.4%	14.4%	2.4%	7.1%	16.1%	1 508
2012	11.0%	48.7%	10.0%	3.3%	8.7%	18.4%	1 280
2014	13.2%	46.8%	10.1%	3.9%	9.8%	16.2%	1 349
2016	13.9%	48.0%	9.3%	3.7%	11.6%	13.5%	1 397
2018	13.3%	48.2%	7.4%	5.0%	16.0%	10.2%	1 677

Credit default swap

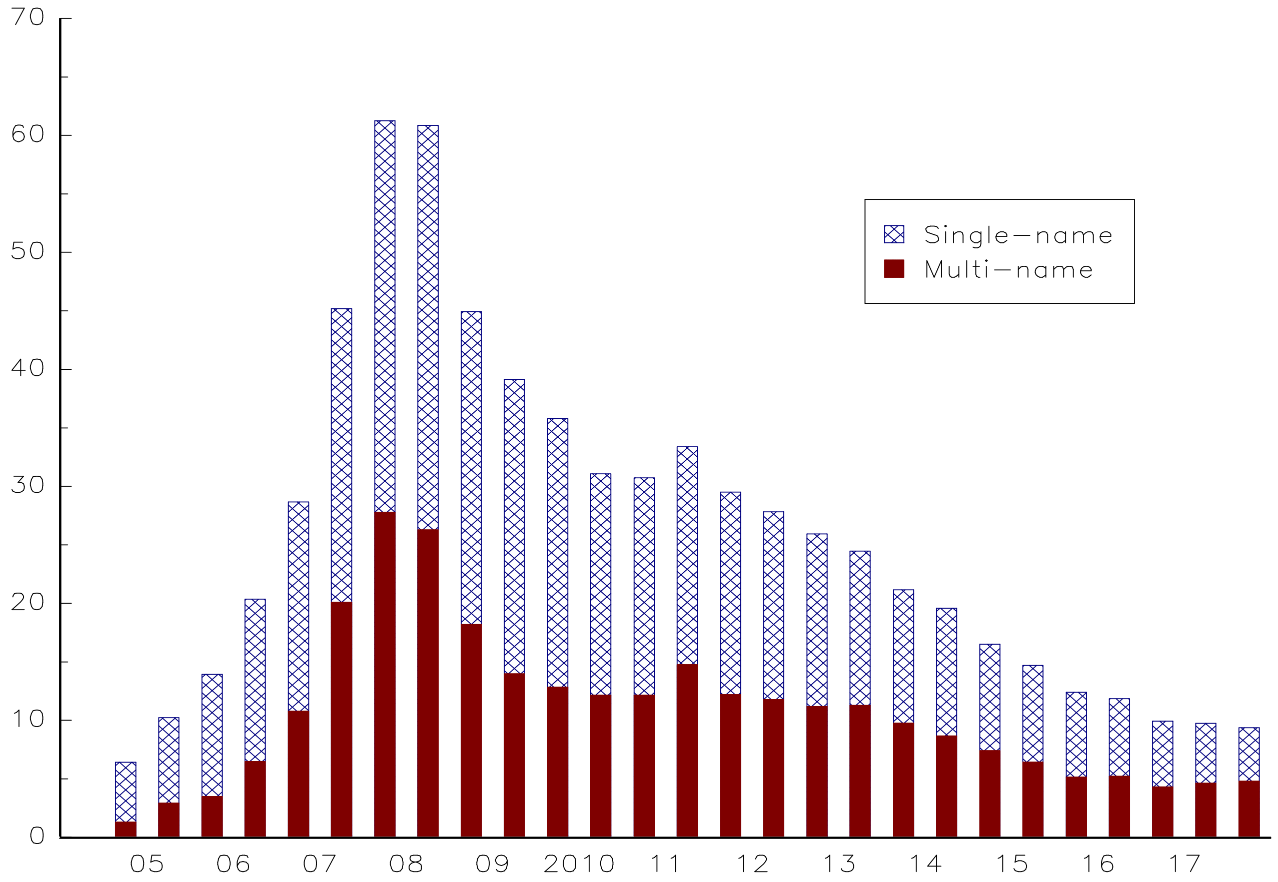


Figure: Outstanding amount of credit default swaps (in $ tn)

Credit default swap

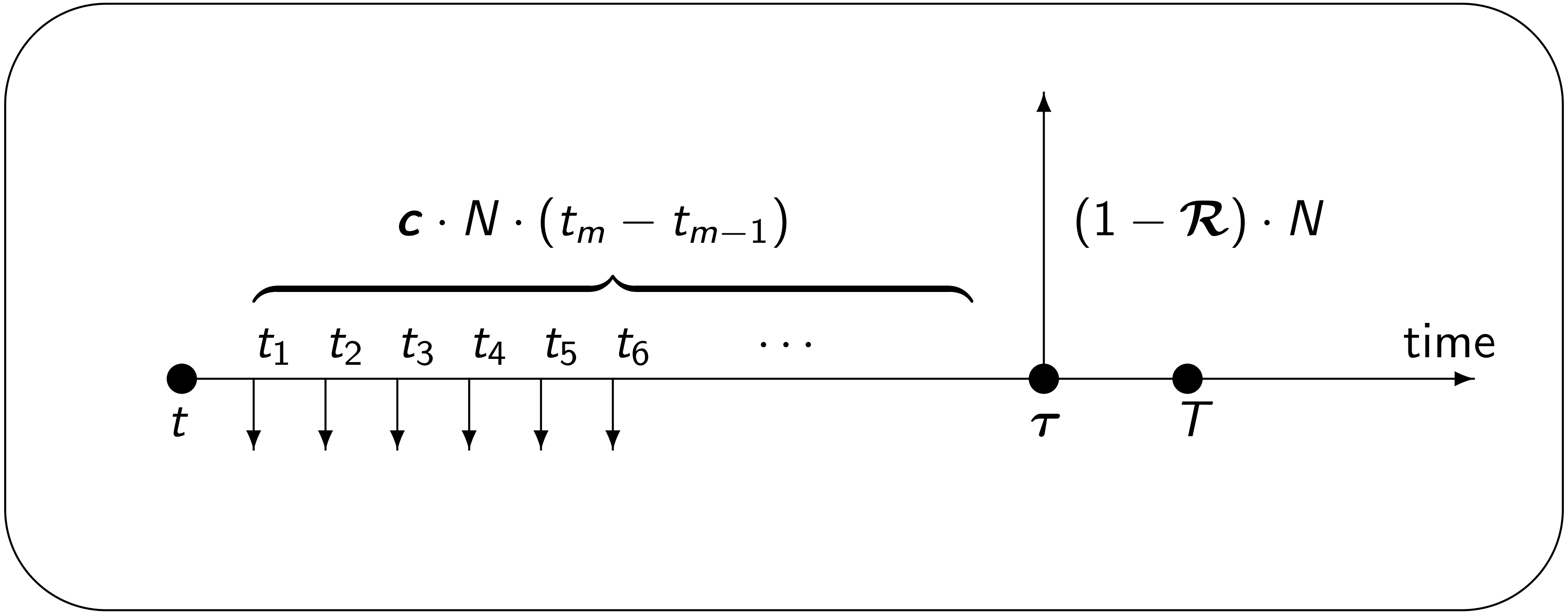


Figure: Cash flows of a single-name credit default swap

Credit default swap

Example

We consider a credit default swap, whose notional principal is \$10 mn, maturity is 5 years and payment frequency is quarterly. The credit event is the bankruptcy of the corporate entity A. We assume that the recovery rate is set to 40% and the coupon rate is equal to 2%

- 20 fixing dates: 3M, 6M, 9M, 1Y, . . . , 5Y
- Quarterly premium $= \$10\,\mathrm{mn} \times 2\% \times 0.25 = \$50\,000$
- No default $\Rightarrow$ the protection buyer will pay a total of $\$50\,000 \times 20 = \$1$ mn
- The corporate defaults two years and four months after the CDS inception date $\Rightarrow$ the protection buyer will pay $9 \times \$50\,000 = \$450\,000$ and the protection seller will pay the protection leg $(1 - 40\%) \times \$10\,\mathrm{mn} = \$6$ mn

Credit default swap

If we assume that the premium is not paid after the default time $\boldsymbol{\tau}$, the stochastic discounted value of the premium leg is:

$$SV_t(\mathcal{PL}) = \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot (t_m - t_{m-1}) \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s}$$

The present value of the premium leg is then:

$$\begin{aligned} PV_t(\mathcal{PL}) &= \mathbb{E}\left[\left.\sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot \Delta t_m \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s}\,\right|\, \mathcal{F}_t\right] \\ &= \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot \Delta t_m \cdot \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} > t_m\}\right] \cdot \mathbb{E}\left[e^{-\int_t^{t_m} r_s \,\mathrm{d}s}\right] \\ &= \boldsymbol{c} \cdot N \cdot \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m) \end{aligned}$$

where $\mathbf{S}_t(u)$ is the survival function at time u

Credit default swap

If we assume that the default leg is exactly paid at the default time $\boldsymbol{\tau}$, the stochastic discount value of the default (or protection) leg is:

$$SV_t\left(\mathcal{DL}\right) = \left(1 - \boldsymbol{\mathcal{R}}\right) \cdot N \cdot \mathbb{1}\left\{\boldsymbol{\tau} \leq T\right\} \cdot e^{-\int_t^{\boldsymbol{\tau}} r(s)\,\mathrm{d}s}$$

It follows that its present value is:

$$\begin{aligned}
PV_t\left(\mathcal{DL}\right) &= \mathbb{E}\left[\left(1 - \boldsymbol{\mathcal{R}}\right) \cdot N \cdot \mathbb{1}\left\{\boldsymbol{\tau} \leq T\right\} \cdot e^{-\int_t^{\boldsymbol{\tau}} r_s\,\mathrm{d}s} \,\middle|\, \mathcal{F}_t\right] \\
&= \left(1 - \boldsymbol{\mathcal{R}}\right) \cdot N \cdot \mathbb{E}\left[\mathbb{1}\left\{\boldsymbol{\tau} \leq T\right\} \cdot B_t\left(\boldsymbol{\tau}\right)\right] \\
&= \left(1 - \boldsymbol{\mathcal{R}}\right) \cdot N \cdot \int_t^T B_t\left(u\right) f_t\left(u\right)\,\mathrm{d}u
\end{aligned}$$

where $f_t\left(u\right)$ is the density function associated to the survival function $\mathbf{S}_t\left(u\right)$

Credit default swap

We deduce that the mark-to-market of the swap is:

$$\begin{aligned} P_t(T) &= PV_t(\mathcal{DL}) - PV_t(\mathcal{PL}) \\ &= (1-\mathcal{R}) N \int_t^T B_t(u) f_t(u) \,\mathrm{d}u - \boldsymbol{c} N \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m) \\ &= N \left((1-\mathcal{R}) \int_t^T B_t(u) f_t(u) \,\mathrm{d}u - \boldsymbol{c} \cdot \mathrm{RPV}_{01} \right) \end{aligned}$$

where $\mathrm{RPV}_{01} = \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)$ is called the risky PV01 and corresponds to the present value of 1 bp paid on the premium leg

CDS spread

The CDS spread $\mathcal{s}$ is the fair value coupon rate $\boldsymbol{c}$ in such a way that the initial value of the credit default swap is equal to zero $P_t = 0$:

$$\mathcal{s} = \frac{(1-\mathcal{R}) \int_t^T B_t(u) f_t(u) \,\mathrm{d}u}{\sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)}$$

Credit default swap

Three properties:

1. No default risk: $\mathbf{S}_t(u) = 1 \Rightarrow \mathcal{s} = 0$
2. Recovery rate is set to 100%: $\mathcal{R} = 1 \Rightarrow \mathcal{s} = 0$
3. $\mathcal{s}$ is a decreasing function of $\mathcal{R}$

If the premium leg is paid continuously, we obtain:

$$\mathcal{s} = \frac{(1 - \mathcal{R}) \int_t^T B_t(u) f_t(u) \, \mathrm{d}u}{\int_t^T B_t(u) \mathbf{S}_t(u) \, \mathrm{d}u}$$

Credit default swap

If the interest rates are equal to zero ($B_t(u) = 1$) and the default times is exponential with parameter λ – $\mathbf{S}_t(u) = e^{-\lambda(u-t)}$ and $f_t(u) = \lambda e^{-\lambda(u-t)}$, we get:

$$\mathcal{s} = \frac{(1-\mathcal{R}) \cdot \lambda \cdot \int_t^T e^{-\lambda(u-t)} \, \mathrm{d}u}{\int_t^T e^{-\lambda(u-t)} \, \mathrm{d}u} = (1-\mathcal{R}) \cdot \lambda$$

If λ is relatively small, the one-year default probability is equal to:

$$\mathrm{PD} = \Pr\{\boldsymbol{\tau} \leq t+1 \mid \boldsymbol{\tau} \leq t\} = 1 - \mathbf{S}_t(t+1) = 1 - e^{-\lambda} \simeq \lambda$$

Credit triangle relationship

$$\mathcal{s} \approx (1-\mathcal{R}) \cdot \mathrm{PD}$$

$\Rightarrow$ The spread is a decreasing function of the default probability

Credit default swap

- The first CDS was traded by J.P. Morgan in 1994
- Standardization: 2003 and 2014 ISDA
- Settlement: physical or cash

In the case of physical settlement, the protection buyer delivers a bond to the protection seller and receives the notional principal amount $\Rightarrow$ the price of the defaulted bond is equal to $\mathcal{R} \cdot N \Rightarrow$ the implied mark-to-market of the physical settlement is $N - \mathcal{R} \cdot N = (1 - \mathcal{R}) \cdot N$

Credit default swap

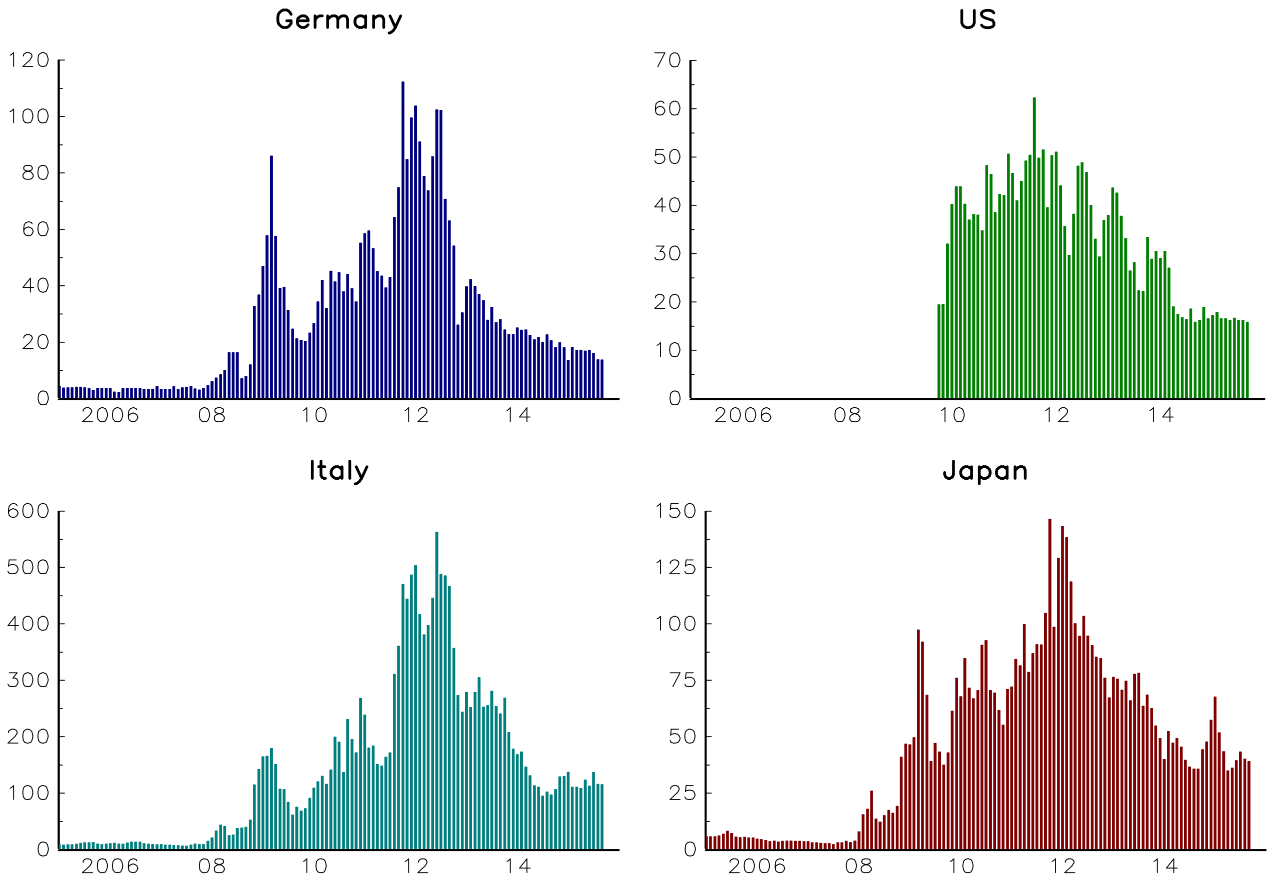


Figure: Evolution of some sovereign CDS spreads

Credit default swap

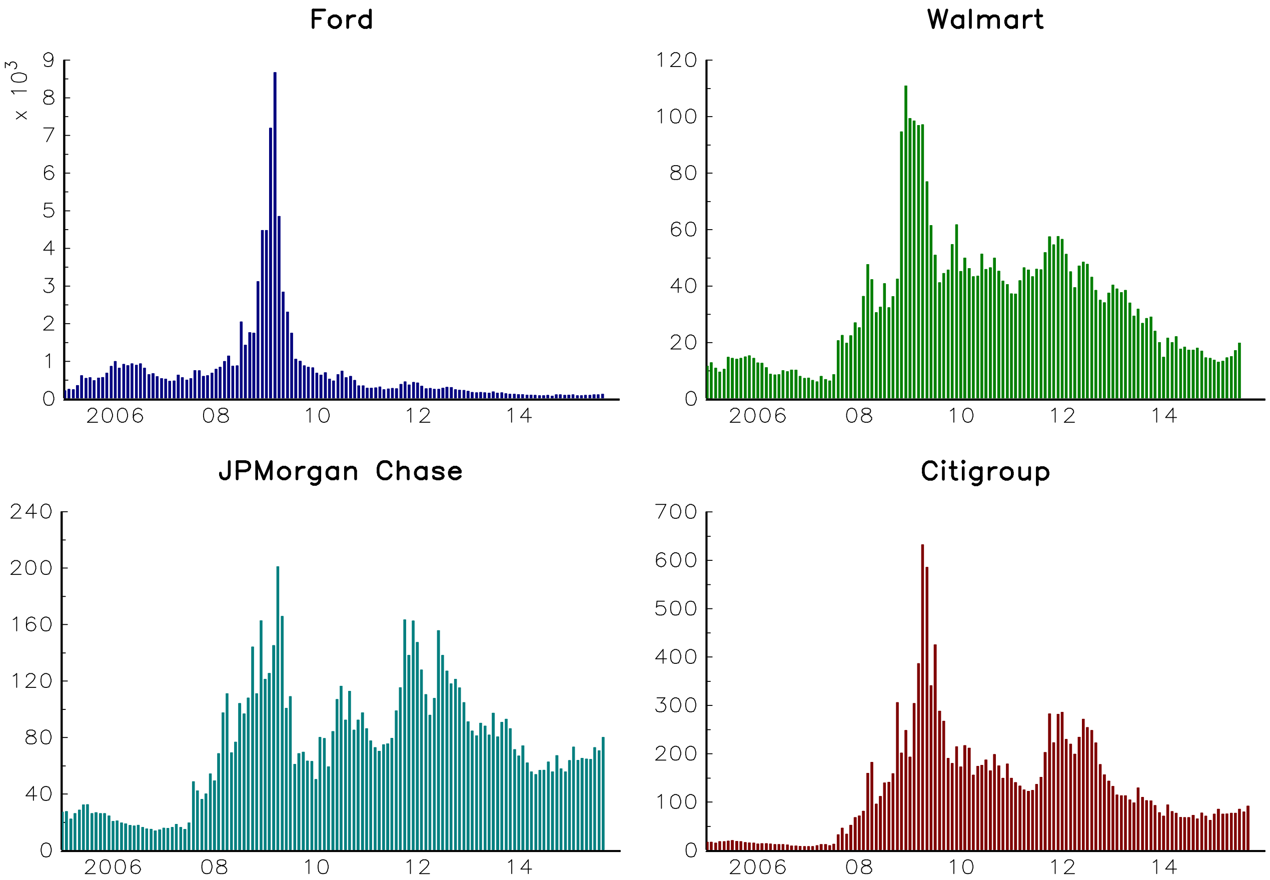


Figure: Evolution of some financial and corporate CDS spreads

Credit curve

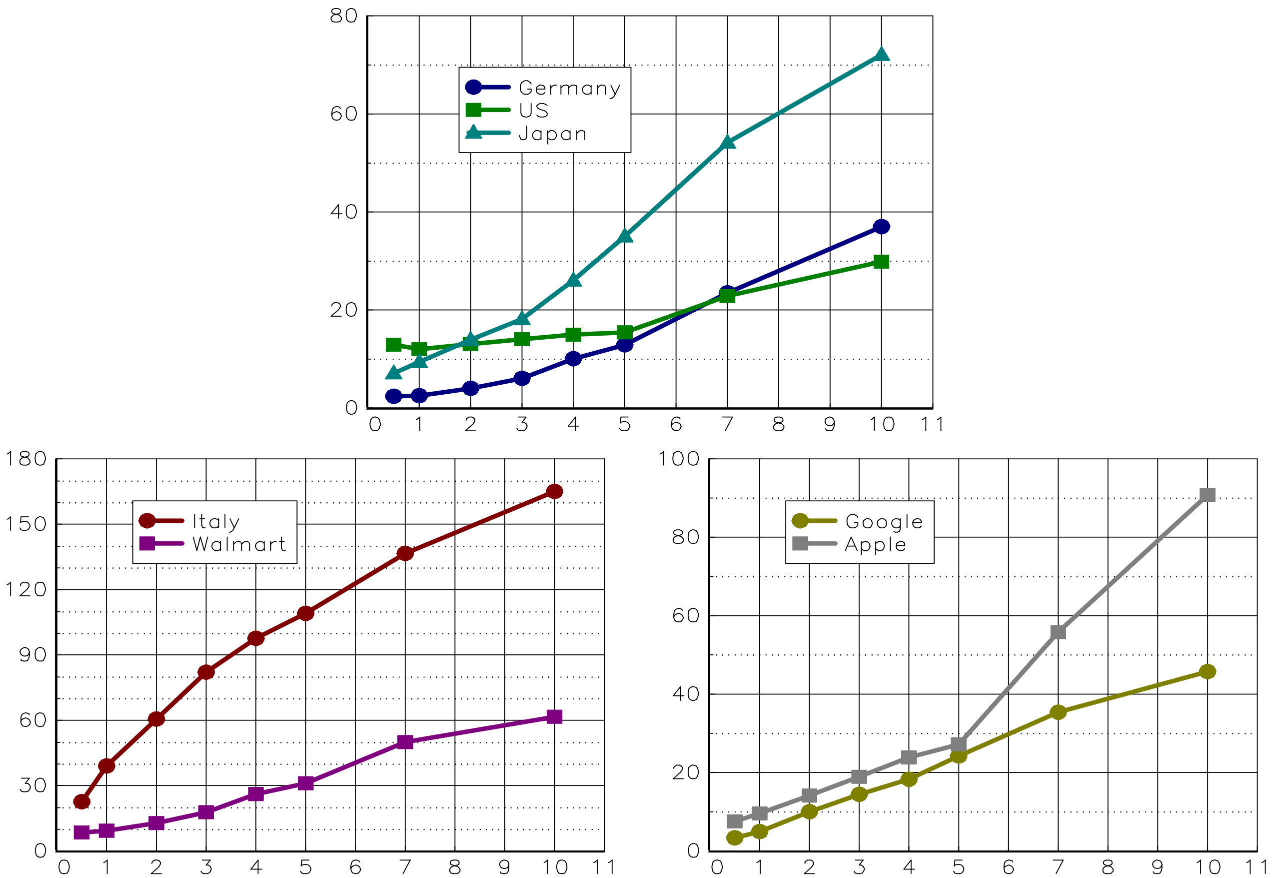


Figure: Example of CDS spread curves as of 17 September 2015

Credit risk hedging with a CDS contract

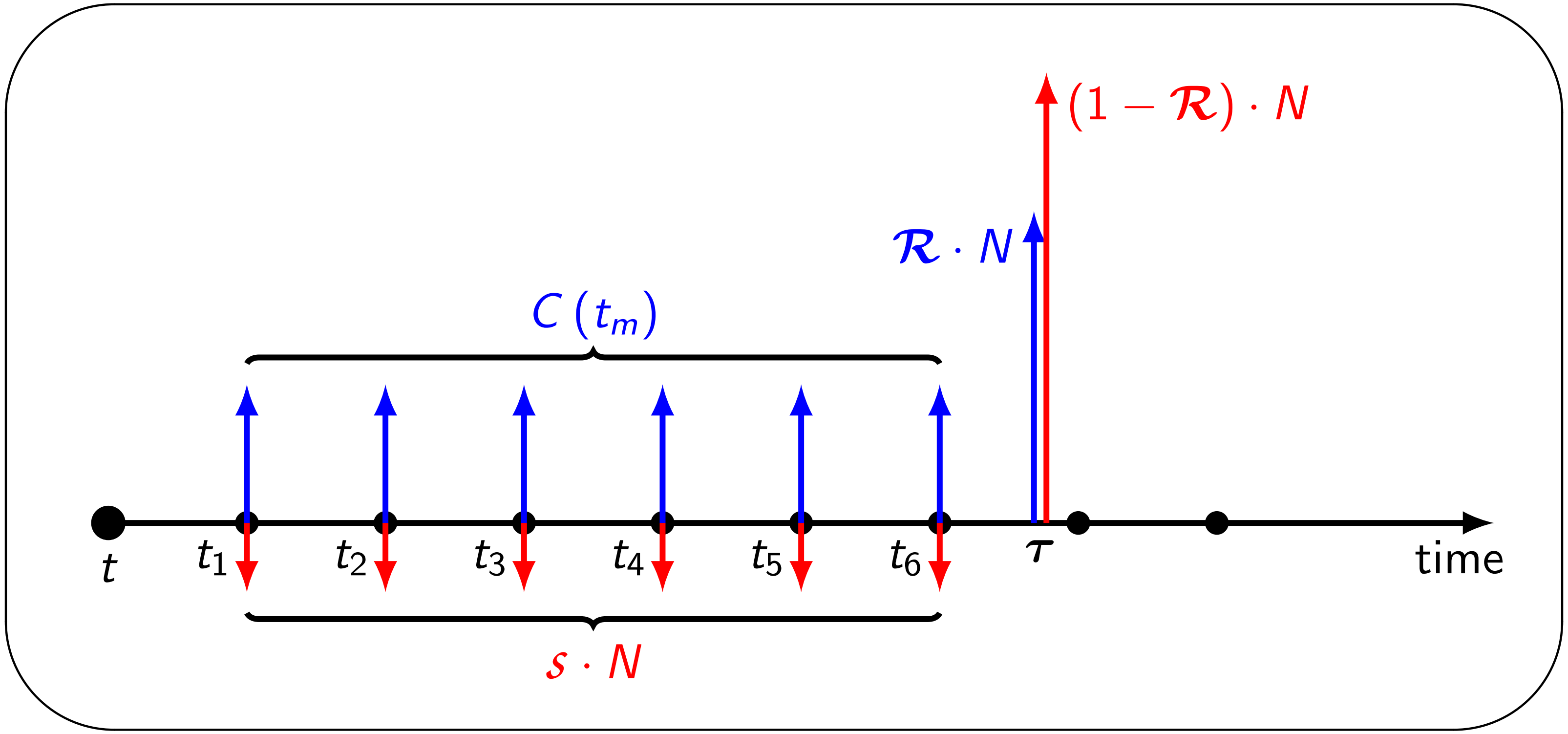


Figure: Hedging a defaultable bond with a credit default swap

$$y^{\star} = y - s \Rightarrow$$ **CDS spread = Credit spread**

Credit risk trading with a CDS contract

Two directional trading strategies:

- '*long credit*' refers to the position of the protection seller who is exposed to the credit risk
- '*short credit*' is the position of the protection buyer who sold the credit risk of the reference entity

$\Rightarrow$ A long exposure implies that the default results in a loss, whereas a short exposure implies that the default results in a gain

Credit risk trading with a CDS contract

Let $P_{t,t'}(T)$ be the mark-to-market of a CDS position whose inception date is t, valuation date is t' and maturity date is T. We have:

$$P_{t,t}^{\text{seller}}(T) = P_{t,t}^{\text{buyer}}(T) = 0$$

At date $t' > t$, the mark-to-market price of the CDS is:

$$P_{t,t'}^{\text{buyer}}(T) = N\left((1-\mathcal{R})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_{t}(T)\cdot \mathrm{RPV}_{01}\right)$$

whereas the value of the CDS spread satisfies the following relationship:

$$P_{t',t'}^{\text{buyer}}(T) = N\left((1-\mathcal{R})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_{t'}(T)\cdot \mathrm{RPV}_{01}\right) = 0$$

We deduce that the P&L of the protection buyer is:

$$\Pi^{\text{buyer}} = P_{t,t'}^{\text{buyer}}(T) - P_{t,t}^{\text{buyer}}(T) = P_{t,t'}^{\text{buyer}}(T)$$

Credit risk trading with a CDS contract

We know that $P_{t',t'}^{\mathrm{buyer}}(T) = 0$ and we obtain:

$$\begin{aligned} \Pi^{\mathrm{buyer}} &= P_{t,t'}^{\mathrm{buyer}}(T) - P_{t',t'}^{\mathrm{buyer}}(T) \\ &= N\left((1-\mathcal{R})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_t(T)\cdot \mathrm{RPV}_{01}\right) - \\ &\quad N\left((1-\mathcal{R})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_{t'}(T)\cdot \mathrm{RPV}_{01}\right) \\ &= N\cdot(\mathcal{s}_{t'}(T) - \mathcal{s}_t(T))\cdot \mathrm{RPV}_{01} \end{aligned}$$

Because $\Pi^{\mathrm{seller}} = -\Pi^{\mathrm{buyer}}$, we distinguish two cases:

- If $\mathcal{s}_{t'}(T) > \mathcal{s}_t(T)$, the protection buyer makes a profit, because this short credit exposure has benefited from the increase of the default risk.
- If $\mathcal{s}_{t'}(T) < \mathcal{s}_t(T)$, the protection seller makes a profit, because the default risk of the reference entity has decreased.

Credit risk trading with a CDS contract

Suppose that we are in the first case. To realize its P&L, the protection buyer has three options:

1. He could unwind the CDS exposure with the protection seller if the latter agrees. This implies that the protection seller pays the mark-to-market $P_{t,t'}^{\mathrm{buyer}}(T)$ to the protection buyer
2. He could hedge the mark-to-market value by selling a CDS on the same reference entity and the same maturity. In this situation, he continues to pay the spread $\mathcal{s}_t(T)$, but he now receives a premium, whose spread is equal to $\mathcal{s}_{t'}(T)$
3. He could reassign the CDS contract to another counterparty. The new counterparty (the protection buyer C in our case) will then pay the coupon rate $\mathcal{s}_t(T)$ to the protection seller. However, the spread is $\mathcal{s}_{t'}(T)$ at time t', which is higher than $\mathcal{s}_t(T)$. This is why the new counterparty also pays the mark-to-market $P_{t,t'}^{\mathrm{buyer}}(T)$ to the initial protection buyer

Credit risk trading with a CDS contract

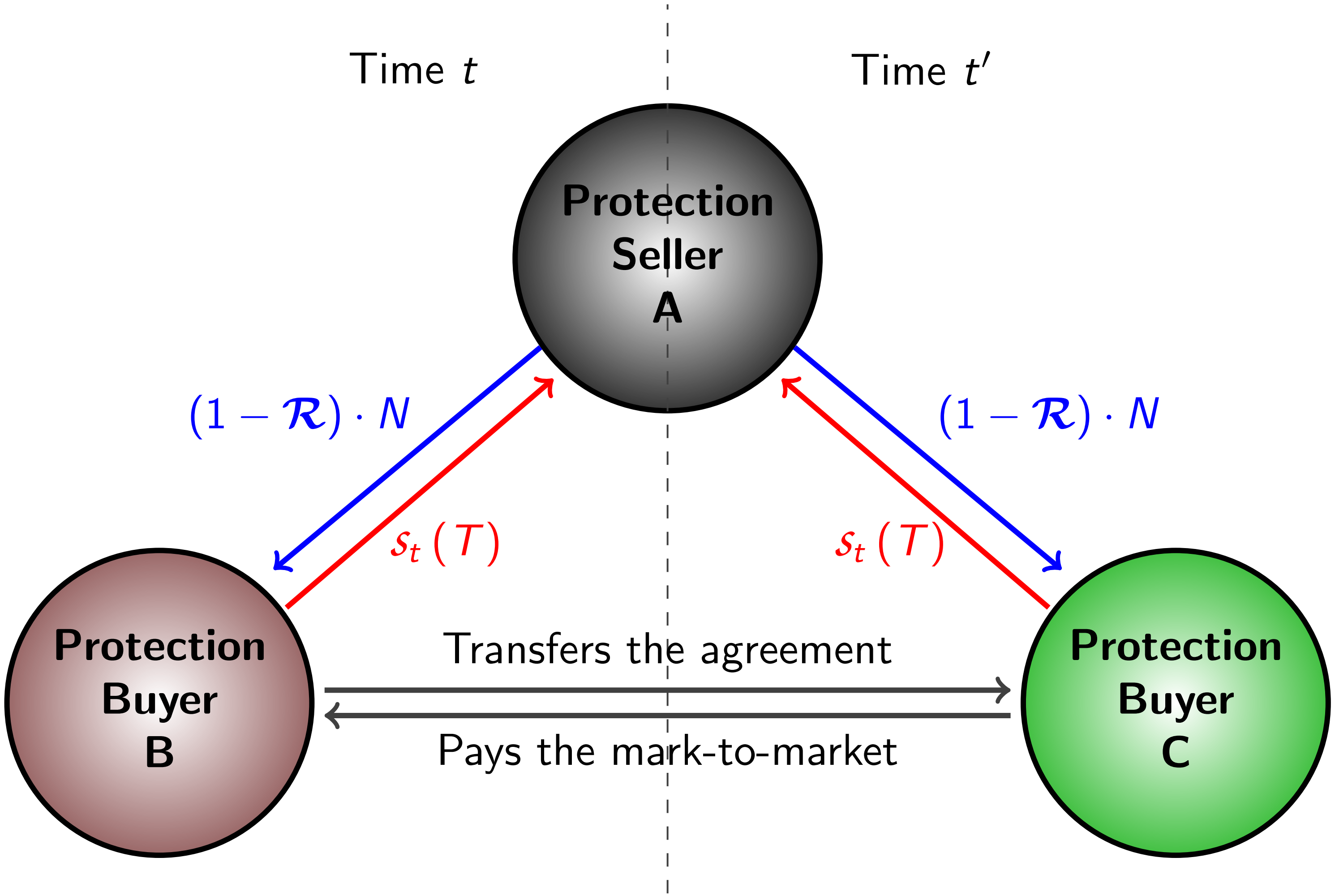


Figure: An example of CDS offsetting

Credit default swap

Example

The coupons are quarterly and the notional is equal to \$1 mn. The recovery rate $\mathcal{R}$ is set to 40% whereas the default time $\boldsymbol{\tau}$ is an exponential random variable, whose parameter λ is equal to 50 bps. We consider seven maturities (6M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y) and two coupon rates (10 and 100 bps).

Table: Price, spread and risky PV01 of CDS contracts

T	$P_t(T)$ $c = 10$	$P_t(T)$ $c = 100$	s	RPV_{01}
1/2	998	−3 492	30.01	0.499
1	1 992	−6 963	30.02	0.995
2	3 956	−13 811	30.04	1.974
3	5 874	−20 488	30.05	2.929
5	9 527	−33 173	30.08	4.744
7	12 884	−44 804	30.10	6.410
10	17 314	−60 121	30.12	8.604

Basket default swap

- First-to-default (FtD)
- Second-to-default (StD)
- k^{th}-to-default credit derivatives

$\Rightarrow$ Impact of the default correlation:

$$\max\left(s_1^{\mathrm{CDS}}, \ldots, s_n^{\mathrm{CDS}}\right) \leq s^{\mathrm{FtD}} \leq \sum_{i=1}^{n} s_i^{\mathrm{CDS}}$$

Credit default indices

Definition

A credit default index is a CDS on a basket of reference entities

Table: Historical spread of CDX/iTraxx indices (in bps)

Date	CDX			iTraxx		
	NA.IG	NA.HY	EM	Europe	Japan	Asia
Dec. 2012	94.1	484.4	208.6	117.0	159.1	108.8
Dec. 2013	62.3	305.6	272.4	70.1	67.5	129.0
Dec. 2014	66.3	357.2	341.0	62.8	67.0	106.0
Sep. 2015	93.6	505.3	381.2	90.6	82.2	160.5

Credit default indices

Table: List of Markit CDX main indices

Index name	Description	n	$\mathcal{R}$
CDX.NA.IG	Investment grade entities	125	40%
CDX.NA.IG.HVOL	High volatility IG entities	30	40%
CDX.NA.XO	Crossover entities	35	40%
CDX.NA.HY	High yield entities	100	30%
CDX.NA.HY.BB	High yield BB entities	37	30%
CDX.NA.HY.B	High yield B entities	46	30%
CDX.EM	EM sovereign issuers	14	25%
LCDX	Secured senior loans	100	70%
MCDX	Municipal bonds	50	80%

Credit default indices

Table: List of Markit iTraxx main indices

Index name	Description	n	$\mathcal{R}$
iTraxx Europe	European IG entities	125	40%
iTraxx Europe HiVol	European HVOL IG entities	30	40%
iTraxx Europe Crossover	European XO entities	40	40%
iTraxx Asia	Asian (ex-Japan) IG entities	50	40%
iTraxx Asia HY	Asian (ex-Japan) HY entities	20	25%
iTraxx Australia	Australian IG entities	25	40%
iTraxx Japan	Japanese IG entities	50	35%
iTraxx SovX G7	G7 governments	7	40%
iTraxx LevX	European leveraged loans	40	40%

Collateralized debt obligation (CDO)

A CDO is a pay-through ABS structure, whose securities are bonds linked to a series of tranches

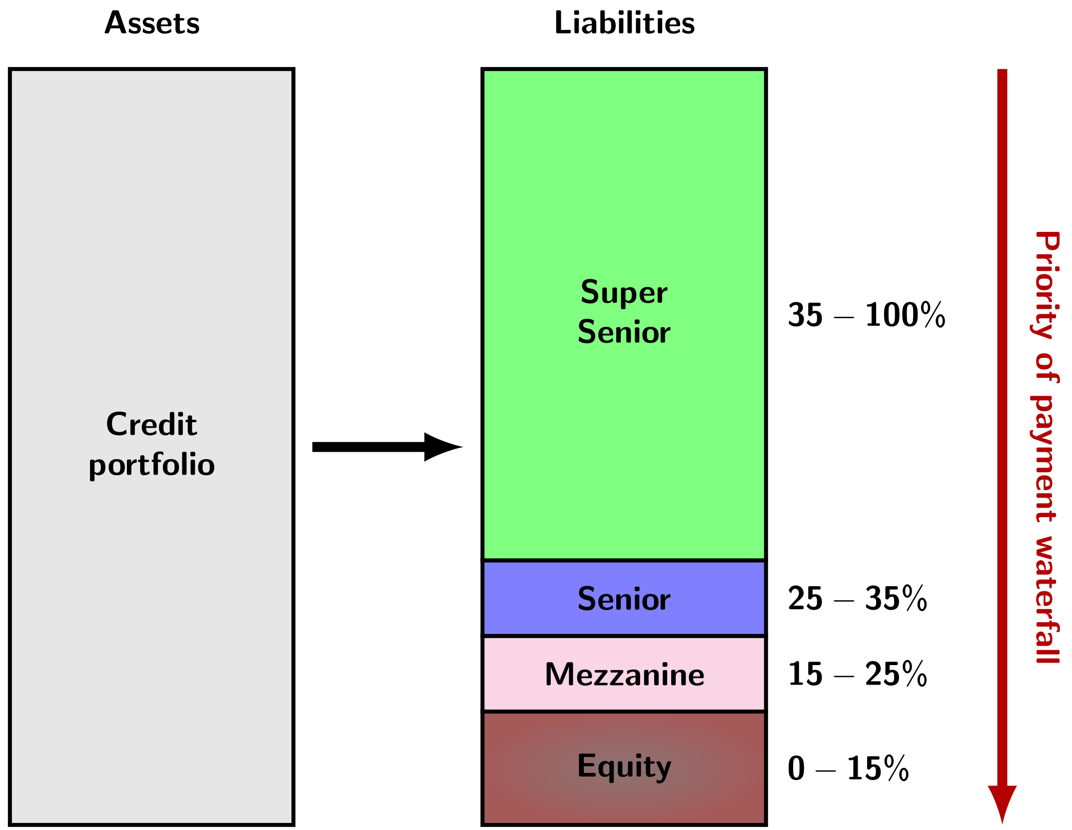


Figure: An example of a CDO structure

Collateralized debt obligation (CDO)

The returns of the 4 bonds depend on the loss of the corresponding tranche. Each tranche is characterized by an attachment point A and a detachment point D. In our example, we have:

Tranche	Equity	Mezzanine	Senior	Super senior
A	0%	15%	25%	35%
D	15%	25%	35%	100%

The protection buyer of the tranche $[A, D]$ pays a coupon rate $\mathbf{c}^{[A,D]}$ on the nominal outstanding amount of the tranche to the protection seller. In return, he receives the protection leg, which is the loss of the tranche $[A, D]$

CDO pricing

We have:

$$L_t(u) = \sum_{i=1}^{n} N_i \cdot (1 - \mathcal{R}_i) \cdot \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}$$

and:

$$L_t^{[A,D]}(u) = (L_t(u) - A) \cdot \mathbb{1}\{A \leq L_t(u) \leq D\} + (D - A) \cdot \mathbb{1}\{L_t(u) > D\}$$

The nominal outstanding amount of the tranche is therefore:

$$N_t^{[A,D]}(u) = (D - A) - L_t^{[A,D]}(u)$$

The spread of the CDO tranche is

$$\mathcal{s}^{[A,D]} = \frac{\mathbb{E}\left[\sum_{t_m \geq t} \Delta L_t^{[A,D]}(t_m) \cdot B_t(t_m)\right]}{\mathbb{E}\left[\sum_{t_m \geq t} \Delta t_m \cdot N_t^{[A,D]}(t_m) \cdot B_t(t_m)\right]}$$

We obviously have the following inequalities

$$\mathcal{s}^{\text{Equity}} > \mathcal{s}^{\text{Mezzanine}} > \mathcal{s}^{\text{Senior}} > \mathcal{s}^{\text{Super senior}}$$

Credit risk

It is the risk of loss on a debt instrument resulting from the failure of the borrower to make required payments: *default risk* $\neq$ *downgrading risk*

Definition (BCBS, 2006)

A default is considered to have occurred with regard to a particular obligor when either or both of the two following events have taken place

- The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realizing security (if held)
- The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered as being past due once the customer has breached an advised limit or been advised of a limit smaller than current outstandings

A fair game?

Table: World's largest banks in 1981 and 1988

	1981		1988	
	Bank	Assets	Bank	Assets
1	Bank of America (US)	115.6	Dai-Ichi Kangyo (JP)	352.5
2	Citicorp (US)	112.7	Sumitomo (JP)	334.7
3	BNP (FR)	106.7	Fuji (JP)	327.8
4	Crédit Agricole (FR)	97.8	Mitsubishi (JP)	317.8
5	Crédit Lyonnais (FR)	93.7	Sanwa (JP)	307.4
6	Barclays (UK)	93.0	Industrial Bank (JP)	261.5
7	Société Générale (FR)	87.0	Norinchukin (JP)	231.7
8	Dai-Ichi Kangyo (JP)	85.5	Crédit Agricole (FR)	214.4
9	Deutsche Bank (DE)	84.5	Tokai (JP)	213.5
10	National Westminster (UK)	82.6	Mitsubishi Trust (JP)	206.0

The Basel I framework

Table: Risk weights by category of on-balance sheet assets

RW	Instruments
0%	Cash Claims on central governments and central banks denominated in national currency and funded in that currency Other claims on OECD central governments and central banks Claims† collateralized by cash of OECD government securities
20%	Claims† on multilateral development banks Claims† on banks incorporated in the OECD and claims guaranteed by OECD incorporated banks Claims† on securities firms incorporated in the OECD subject to comparable supervisory and regulatory arrangements Claims† on banks incorporated in countries outside the OECD with a residual maturity of up to one year Claims† on non-domestic OECD public-sector entities Cash items in process of collection
50%	Loans fully secured by mortgage on residential property
100%	Claims on the private sector Claims on banks incorporated outside the OECD with a residual maturity of over one year Claims on central governments outside the OECD and non denominated in national currency All other assets

The Basel I framework

For off-balance sheet assets, the amount E of a credit line is converted to an exposure at default:

$$\mathrm{EAD} = E \cdot \mathrm{CCF}$$

where CCF is the credit conversion factor (100%, 50%, 20% and 0%)

The Basel I framework

Table: Illustration of capital requirement

Balance Sheet	Asset	E	CCF	EAD	RW	RWA
On-	US bonds			100	0%	0
	Mexico bonds			20	100%	20
	Argentine debt			20	0%	0
	Home mortgage			500	50%	250
	Corporate loans			500	100%	500
	Credit lines			40	100%	40
Off-	Standby facilities	20	100%	20	0%	0
	Credit lines ($>$ 1Y)	42	50%	21	100%	21
	Credit lines ($\leq$ 1Y)	18	0%	0	100%	0
	Total					831

The Basel II framework

- The standardized approach (SA)
- The internal ratings-based approach (IRB)

The Basel II standardized approach

Table: Risk weights of the SA approach (Basel II)

Rating		AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to B−	CCC+ to C	NR
Sovereigns		0%	20%	50%	100%	150%	100%
Banks	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
	2 ST	20%	20%	20%	50%	150%	20%
Corporates				BBB+ to BB−		B+ to C	
		20%	50%	100%		150%	100%
Retail				75%			
Residential mortgages				35%			
Commercial mortgages				100%			

The Basel II standardized approach

Table: Comparison of risk weights between Basel I and Basel II

Entity	Rating	Maturity	Basel I	Basel II
Sovereign (OECD)	AAA		0%	0%
Sovereign (OECD)	A-		0%	20%
Sovereign	BBB		100%	50%
Bank (OECD)	BBB	2Y	20%	50%
Bank	BBB	2M	100%	20%
Corporate	AA+		100%	20%
Corporate	BBB		100%	100%

Credit ratings

Table: Credit rating system of S&P, Moody's and Fitch

	Prime Maximum Safety	High Grade High Quality			Upper Medium Grade		
S&P/Fitch	AAA	AA+	AA	AA−	A+	A	A−
Moody's	Aaa	Aa1	Aa2	Aa3	A1	A2	A3

	Lower Medium Grade			Non Investment Grade Speculative		
S&P/Fitch	BBB+	BBB	BBB−	BB+	BB	BB−
Moody's	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3

	Highly Speculative			Substantial Risk	In Poor Standing		Extremely Speculative
S&P/Fitch	B+	B	B−	CCC+	CCC	CCC−	CC
Moody's	B1	B2	B3	Caa1	Caa2	Caa3	Ca

Credit ratings

Table: Examples of country's S&P rating

Country	Local currency		Foreign currency	
	Jun. 2009	Oct. 2015	Jun. 2009	Oct. 2015
Argentina	B-	CCC+	B-	SD
Brazil	BBB+	BBB-	BBB-	BB+
China	A+	AA-	A+	AA-
France	AAA	AA	AAA	AA
Italy	A+	BBB-	A+	BBB-
Japan	AA	A+	AA	A+
Russia	BBB+	BBB-	BBB	BB+
Spain	AA+	BBB+	AA+	BBB+
Ukraine	B-	CCC+	CCC+	SD
US	AAA	AA+	AA+	AA+

The Basel II standardized approach

CCF (Basel II $\approx$ Basel I)

Credit risk mitigation

1. Collateralized transactions
2. Guarantees and credit derivatives

Credit risk mitigation

Collateralized transactions

1. Cash and comparable instruments
2. Gold
3. Debt securities which are rated AAA to BB- when issued by sovereigns or AAA to BBB- when issued by other entities or at least A-3/P-3 for short-term debt instruments
4. Debt securities which are not rated but fulfill certain criteria (senior debt issued by banks, listed on a recognisee exchange and sufficiently liquid)
5. Equities that are included in a main index
6. UCITS and mutual funds, whose assets are eligible instruments and which offer a daily liquidity
7. Equities which are listed on a recognized exchange and UCITS/mutual funds which include such equities

Credit risk mitigation

Collateralized transactions

Simple approach

$$\mathrm{RWA} = (\mathrm{EAD} - C) \cdot \mathrm{RW} + C \cdot \max\left(\mathrm{RW}_C, 20\%\right)$$

where EAD is the exposure at default, C is the market value of the collateral, RW is the risk weight appropriate to the exposure and RW_C is the risk weight of the collateral

Comprehensive approach

The risk-weighted asset amount after risk mitigation is $\mathrm{RWA} = \mathrm{RW} \cdot \mathrm{EAD}^{\star}$ whereas $\mathrm{EAD}^{\star}$ is the modified exposure at default:

$$\mathrm{EAD}^{\star} = \max\left(0, (1 + H_E) \cdot \mathrm{EAD} - (1 - H_C - H_{FX}) \cdot C\right)$$

where H_E is the haircut applied to the exposure, H_C is the haircut applied to the collateral and H_{FX} is the haircut for currency risk

Credit risk mitigation

Collateralized transactions

Table: Standardized supervisory haircuts for collateralized transactions

Rating	Residual Maturity	Sovereigns	Others
AAA to AA−	0−1Y	0.5%	1%
	1−5Y	2%	4%
	5Y+	4%	8%
A+ to BBB−	0−1Y	1%	2%
	1−5Y	3%	6%
	5Y+	6%	12%
BB+ to BB−		15%	
Cash		0%	
Gold		15%	
Main index equities		15%	
Equities listed on a recognized exchange		25%	
FX risk		8%	

Credit risk mitigation

Guarantees and credit derivatives

Banks can use these credit protection instruments if they are direct, explicit, irrevocable and unconditional

Simple approach

$$\mathrm{RWA} = (\mathrm{EAD} - C) \cdot \mathrm{RW} + C \cdot \max\left(\mathrm{RW}_C, 20\%\right)$$

where EAD is the exposure at default, C is the market value of the collateral, RW is the risk weight appropriate to the exposure and RW_C is the risk weight of the collateral

The Basel II internal ratings-based approach

4 parameters:

- the exposure at default (EAD)
- the probability of default (PD)
- the loss given default (LGD)
- the effective maturity (M)

The credit risk measure is the sum of individual risk contributions:

$$\mathcal{R}(w) = \sum_{i=1}^{n} \mathcal{RC}_i$$

where $\mathcal{RC}_i$ is a function of the four risk components:

$$\mathcal{RC}_i = f_{\mathrm{IRB}}\left(\mathrm{EAD}_i, \mathrm{LGD}_i, \mathrm{PD}_i, \mathrm{M}_i\right)$$

and f_{IRB} is the IRB fomula

IRB is not an internal model, but an external model with internal parameters

The Basel II internal ratings-based approach

The mechanism of the IRB approach is the following:

- a classification of exposures (sovereigns, banks, corporates, retail portfolios, etc.)
- for each credit i, the bank estimates the probability of default
- it uses the standard regulatory values of the other risk components (EAD_i, LGD_i and M_i) or estimates them in the case of AIRB
- the bank calculate then the risk-weighted assets RWA_i of the credit by applying the right IRB formula f_{IRB} to the risk components

$\Rightarrow$ Distinction between FIRB (foundation IRB) and AIRB (advanced IRB)

$\Rightarrow$ **Internal ratings are central to the IRB approach**

The Basel II internal ratings-based approach

Table: An example of internal rating system

Rating		Degree of risk	Definition	Borrower category by self-assessment
1		No essential risk	Extremely high degree of certainty of repayment	Normal
2		Negligible risk	High degree of certainty of repayment	
3		Some risk	Sufficient certainty of repayment	
4	A B	Better than average	There is certainty of repayment but substantial changes in the environment in the future may have some impact on this uncertainty	
5	A B	Average	There are no problems foreseeable in the future, but a strong likelihood of impact from changes in the environment	
6	A B	Tolerable	There are no problems foreseeable in the future, but the future cannot be considered entirely safe	
7		Lower than average	There are no problems at the current time but the financial position of the borrower is relatively weak	
8	A B	Needs preventive management	There are problems with lending terms or fulfilment, or the borrower's business conditions are poor or unstable, or there are other factors requiring careful management	Needs attention
9		Needs serious management	There is a high likelihood of bankruptcy in the future	In danger of bankruptcy
10	I		The borrower is in serious financial straits and "effectively bankrupt"	Effectively bankruptcy
	II		The borrower is bankrupt	Bankrupt

The Basel II internal ratings-based approach

Another example of internal rating system

The rating system of Crédit Agricole is:

- A+, A,
- B+, B,
- C+, C, C-,
- D+, D, D-,
- E+, E and E-

Source: Crédit Agricole, Annual Financial Report 2014, page 201

The credit risk model of Basel II

Assumptions

- The portfolio loss is equal to:

$$L = \sum_{i=1}^{n} w_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

  where w_i and T_i are the exposure at default and the residual maturity of the i^{th} credit
- The loss given default LGD_i is a random variable
- The default time $\boldsymbol{\tau}_i$ depends on a set of risk factors X, whose probability distribution is denoted by $\mathbf{H}$
- Let $p_i(X)$ be the conditional default probability. The (unconditional or long-term) default probability is:

$$p_i = \mathbb{E}_X\left[\mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}\right] = \mathbb{E}_X\left[p_i(X)\right]$$

- Let $D_i = \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$ be the default indicator function. Conditionally to the risk factors X, D_i is a Bernoulli random variable with probability $p_i(X)$

The credit risk model of Basel II

Under the standard assumptions that the loss given default is independent from the default time and the default times are conditionally independent, we obtain:

$$\mathbb{E}[L \mid X] = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \mathbb{E}[D_i \mid X] = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(X)$$

The credit risk model of Basel II

We also have (HFRM, Exercise 3.4.8, page 255):

$$\sigma^2(L \mid X) = \sum_{i=1}^{n} w_i^2 \cdot \left(\mathbb{E}\left[\mathrm{LGD}_i^2\right] \cdot \mathbb{E}\left[D_i^2 \mid X\right] - \mathbb{E}^2\left[\mathrm{LGD}_i\right] \cdot p_i^2(X)\right)$$

Since we have:

$$\begin{aligned}
\mathbb{E}\left[D_i^2 \mid X\right] &= p_i(X) \\
\mathbb{E}\left[\mathrm{LGD}_i^2\right] &= \sigma^2(\mathrm{LGD}_i) + \mathbb{E}^2\left[\mathrm{LGD}_i\right]
\end{aligned}$$

we deduce that:

$$\sigma^2(L \mid X) = \sum_{i=1}^{n} w_i^2 \cdot A_i$$

where:

$$A_i = \mathbb{E}^2\left[\mathrm{LGD}_i\right] \cdot p_i(X) \cdot (1 - p_i(X)) + \sigma^2(\mathrm{LGD}_i) \cdot p_i(X)$$

The credit risk model of Basel II

The concept of granularity

Infinitely granular portfolio

The portfolio is infinitely fine-grained if there is no concentration risk:

$$\lim_{n\to\infty} \max \frac{w_i}{\sum_{j=1}^{n} w_j} = 0$$

$\Rightarrow$ the conditional distribution of L degenerates to its conditional expectation $\mathbb{E}[L \mid X]$

The intuition of this result is the following: We consider a fine-grained portfolio equivalent to the original portfolio by replacing the original credit i by m credits with the same default probability p_i, the same loss given default LGD_i but an exposure at default divided by m. Let L_m be the loss of the equivalent fine-grained portfolio. When m tends to ∞, we obtain the infinitely fine-grained portfolio. Conditionally to the risk factors X, the portfolio loss L_∞ is equal to the conditional mean $\mathbb{E}[L \mid X]$

The credit risk model of Basel II

Proof

We have:

$$\mathbb{E}\left[L_m \mid X\right] = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{w_i}{m} \right) \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \mathbb{E}\left[D_i \mid X\right] = \mathbb{E}\left[L \mid X\right]$$

and:

$$\sigma^2\left(L_m \mid X\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{w_i^2}{m^2} \right) \cdot A_i = \frac{1}{m} \sum_{i=1}^{n} w_i^2 \cdot A_i = \frac{1}{m} \sigma^2\left(L_m \mid X\right)$$

We note that $\mathbb{E}\left[L_\infty \mid X\right] = \mathbb{E}\left[L \mid X\right]$ and $\sigma^2\left(L_\infty \mid X\right) = 0$. Conditionally to the risk factors X, the portfolio loss L_∞ is equal to the conditional mean $\mathbb{E}\left[L \mid X\right]$

The credit risk model of Basel II

The associated probability distribution $\mathbf{F}$ is then:

$$\begin{aligned}
\mathbf{F}(\ell) &= \Pr\{L_\infty \leq \ell\} \\
&= \Pr\{\mathbb{E}[L \mid X] \leq \ell\} \\
&= \Pr\left\{\sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(X) \leq \ell\right\}
\end{aligned}$$

Let $g(x)$ be the function $\sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(x)$. We have:

$$\mathbf{F}(\ell) = \int \cdots \int \mathbb{1}\{g(x) \leq \ell\} \, \mathrm{d}\mathbf{H}(x)$$

$\Rightarrow$ Not possible to obtain a closed-form formula for the value-at-risk $\mathbf{F}^{-1}(\alpha)$:

$$\mathbf{F}^{-1}(\alpha) = \{\ell : \Pr\{g(X) \leq \ell\} = \alpha\}$$

The credit risk model of Basel II

The single risk factor case

If we consider a single risk factor and assume that $g(x)$ is an increasing function, we obtain:

$$\begin{aligned}\Pr\{g(X) \leq \ell\} = \alpha &\Leftrightarrow \Pr\left\{X \leq g^{-1}(\ell)\right\} = \alpha \\ &\Leftrightarrow \mathbf{H}\left(g^{-1}(\ell)\right) = \alpha \\ &\Leftrightarrow \ell = g\left(\mathbf{H}^{-1}(\alpha)\right)\end{aligned}$$

We finally deduce that the value-at-risk has the following expression:

$$\mathbf{F}^{-1}(\alpha) = g\left(\mathbf{H}^{-1}(\alpha)\right) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i\left(\mathbf{H}^{-1}(\alpha)\right)$$

The credit risk model of Basel II

Euler allocation principle

The value-at-risk satisfies the Euler allocation principle:

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{n} \mathcal{RC}_i$$

where the expression of the risk contribution is:

$$\mathcal{RC}_i = w_i \cdot \frac{\partial\, \mathbf{F}^{-1}(\alpha)}{\partial\, w_i} = w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i\left(\mathbf{H}^{-1}(\alpha)\right)$$

The credit risk model of Basel II

Remark

If $g(x)$ *is a decreasing function, we obtain* $\Pr\left\{X \geq g^{-1}(\ell)\right\} = \alpha$ *and:*

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(1-\alpha)\right)$$

The risk contribution becomes:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(1-\alpha)\right)$$

The credit risk model of Basel II

Summary

Under the assumptions:

- $\mathcal{H}_1$ The loss given default LGD_i is independent from the default time $\boldsymbol{\tau}_i$
- $\mathcal{H}_2$ The default times $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ depend on a single risk factor X and are conditionally independent with respect to X
- $\mathcal{H}_3$ The portfolio is infinitely fine-grained, meaning that there is no exposure concentration

we have:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i\left(\mathbf{H}^{-1}(\pi)\right)$$

where $\pi = \alpha$ if $p_i(X)$ is an increasing function of X or $\pi = 1 - \alpha$ if $p_i(X)$ is a decreasing function of X

The credit risk model of Basel II

Closed-form formula of the value-at-risk

$\Rightarrow$ Merton (1974) / Vasicek (1991)

Let Z_i be the normalized asset value of the entity i. In the Merton model, the default occurs when Z_i is below a given barrier B_i: $D_i = 1 \Leftrightarrow Z_i < B_i$. By assuming that Z_i is Gaussian, we deduce that:

$$p_i = \Pr\{D_i = 1\} = \Pr\{Z_i < B_i\} = \Phi(B_i)$$

and $B_i = \Phi^{-1}(p_i)$

We assume that the asset value Z_i depends on the common risk factor X and an idiosyncratic risk factor ε_i as follows:

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$$

X and ε_i are two independent standard normal random variables and we have:

$$\mathbb{E}[Z_i Z_j] = \mathbb{E}\left[\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i\right)\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_j\right)\right] = \rho$$

where ρ is the constant asset correlation

The credit risk model of Basel II

Closed-form formula of the value-at-risk

The conditional default probability is equal to:

$$\begin{aligned} p_i(X) := \Pr\{D_i = 1 \mid X\} &= \Pr\{Z_i < B_i \mid X\} \\ &= \Pr\left\{\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i < B_i\right\} \\ &= \Pr\left\{\varepsilon_i < \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right\} \\ &= \Phi\left(\frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \end{aligned}$$

We obtain:

$$g(x) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(x) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)$$

Since $g(x)$ is a decreasing function if $w_i \geq 0$, we have:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

The credit risk model of Basel II

Theorem (HFRM, Appendix A.2.2.5, page 1063)

$$\int_{-\infty}^{c} \Phi(a + bx)\,\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

p_i is the unconditional default probability

We verify that:

$$\begin{aligned}
\mathbb{E}_X\left[p_i(X)\right] &= \mathbb{E}_X\left[\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right)\right] \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)\phi(x)\,\mathrm{d}x \\
&= \Phi_2\left(\infty, \frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}} \cdot \left(\frac{1}{1-\rho}\right)^{-1/2}; \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\left(\frac{1}{1-\rho}\right)^{-1/2}\right) \\
&= \Phi_2\left(\infty, \Phi^{-1}(p_i); \sqrt{\rho}\right) = \Phi\left(\Phi^{-1}(p_i)\right) = p_i
\end{aligned}$$

The credit risk model of Basel II

Example

We consider a homogeneous portfolio with 100 credits. For each credit, the exposure at default, the expected LGD and the probability of default are set to \$1 mn, 50% and 5%

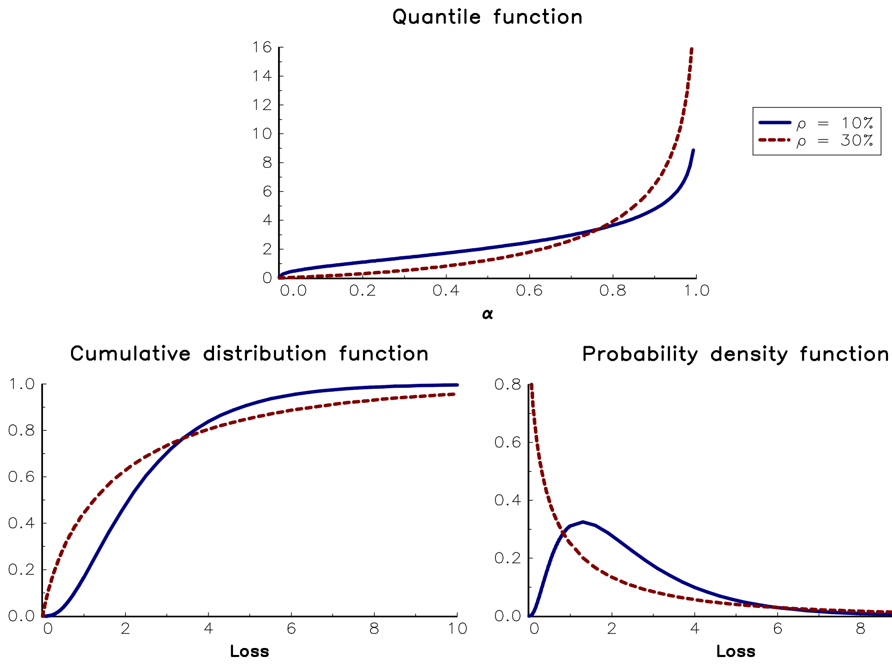


Figure: Probability functions of the credit portfolio loss

The credit risk model of Basel II

What is the impact of the maturity?

the maturity T_i is taken into account through the probability of default $\Rightarrow$ $p_i = \Pr\{\boldsymbol{\tau}_i \leq T_i\}$

Let us denote PD_i the annual default probability of the obligor. If we assume that the default time is Markovian, we have the following relationship:

$$p_i = 1 - \Pr\{\boldsymbol{\tau}_i > T_i\} = 1 - (1 - \mathrm{PD}_i)^{T_i}$$

We deduce that:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}\left(1 - (1 - \mathrm{PD}_i)^{T_i}\right) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

The credit risk model of Basel II

Maturity adjustment

The maturity adjustment is the function $\varphi(t)$ such that $\varphi(1) = 1$ and:

$$\mathcal{RC}_i \approx w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \cdot \varphi(T_i)$$

The IRB formulas

A long process to obtain the finalized formulas

- January 2001: $\alpha = 99.5\%$, $\rho = 20\%$ and a standard maturity of three years
- April 2001: **Quantitative Impact Study** (QIS)
- November 2001: Results of the QIS 2

Table: Percentage change in capital requirements under CP2 proposals

		SA	FIRB	AIRB
G10	Group 1	6%	14%	−5%
	Group 2	1%		
EU	Group 1	6%	10%	−1%
	Group 2	−1%		
Others		5%		

- July 2002: QIS 2.5
- May 2003: QIS 3
- June 2004: Basel II

The IRB formulas

If we use the notations of the Basel Committee, the risk contribution has the following expression:

$$\mathcal{RC} = \mathrm{EAD} \cdot \mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1}\left(1 - (1 - \mathrm{PD})^{\mathrm{M}}\right) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right)$$

where:

- EAD is the exposure at default
- LGD is the (expected) loss given default
- PD is the (one-year) probability of default
- M is the effective maturity

The IRB formulas

Because $\mathcal{RC}$ is directly the capital requirement ($\mathcal{RC} = 8\% \times \mathrm{RWA}$), we deduce that the risk-weighted asset amount is equal to:

$$\mathrm{RWA} = 12.50 \cdot \mathrm{EAD} \cdot \mathcal{K}^\star$$

where $\mathcal{K}^\star$ is the normalized required capital for a unit exposure:

$$\mathcal{K}^\star = \mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1}\left(1 - (1 - \mathrm{PD})^{\mathrm{M}}\right) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right)$$

The IRB formulas

In order to obtain the finalized formulas, the Basel Committee has introduced the following modifications:

- A maturity adjustment $\varphi(\mathrm{M})$ has been added:

$$\mathcal{K}^\star \approx \mathrm{LGD} \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \cdot \varphi(\mathrm{M})$$

- The confidence level is 99.9% instead of 99.5%
- The default correlation is a parametric function $\rho(\mathrm{PD})$ in order that low ratings are not too penalizing for capital requirements;
- The credit risk measure is the unexpected loss:

$$\mathrm{UL}_\alpha = \mathrm{VaR}_\alpha - \mathbb{E}[L]$$

Final supervisory formula

$$\mathcal{K}^\star = \left(\mathrm{LGD} \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \cdot \mathrm{PD}\right) \cdot \varphi(\mathrm{M})$$

The IRB formulas

Risk-weighted assets for corporate, sovereign, and bank exposures

The three asset classes use the same formula:

$$\mathcal{K}^{\star} = \left(\mathrm{LGD} \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \cdot \mathrm{PD}\right) \cdot \left(\frac{1 + (\mathrm{M} - 2.5) \cdot b(\mathrm{PD})}{1 - 1.5 \cdot b(\mathrm{PD})}\right)$$

with:

$$b(\mathrm{PD}) = (0.11852 - 0.05478 \cdot \ln(\mathrm{PD}))^2$$

and:

$$\rho(\mathrm{PD}) = 12\% \times \left(\frac{1 - e^{-50 \times \mathrm{PD}}}{1 - e^{-50}}\right) + 24\% \times \left(1 - \frac{1 - e^{-50 \times \mathrm{PD}}}{1 - e^{-50}}\right)$$

The IRB formulas

Risk-weighted assets for small and medium-sized enterprises

SMEs are defined as corporate entities where the reported sales for the consolidated group of which the firm is a part is less than 50 € mn

⇒ New parametric function for the default correlation:

$$\rho^{\mathrm{SME}}(\mathrm{PD}) = \rho(\mathrm{PD}) - 0.04 \cdot \left(1 - \frac{(\max(S, 5) - 5)}{45}\right)$$

where S is the reported sales expressed in € mn

⇒ This adjustment has the effect to reduce the default correlation and then the risk-weighted assets

The IRB formulas

Risk-weighted assets for corporate, sovereign, and bank exposures

Foundation IRB (FIRB)

- EAD is the amount of the claim
- For off-balance sheet items, the bank uses the CCF values of the SA approach.
- PD is estimated by the bank
- LGD is set to 45% for senior claims and 75% for subordinated claims
- M is set to 2.5 years

Advanced IRB (AIRB)

- For off-balance sheet items, the bank may estimate its own internal measures of CCF
- PD is estimated by the bank
- LGD may be estimated by the bank
- M is the weighted average time of the cash flows, with a one-year floor and a five-year cap

The IRB formulas

Risk-weighted assets for corporate, sovereign, and bank exposures

Example

We consider a senior debt of \$3 mn on a corporate firm. The residual maturity of the debt is equal to 2 years. We estimate the one-year probability of default at 5%

We first calculate the default correlation:

$$\rho(\mathrm{PD}) = 12\% \times \left(\frac{1 - e^{-50 \times 0.05}}{1 - e^{-50}}\right) + 24\% \times \left(1 - \frac{1 - e^{-50 \times 0.05}}{1 - e^{-50}}\right) = 12.985\%$$

We have:

$$b(\mathrm{PD}) = (0.11852 - 0.05478 \times \ln(0.05))^2 = 0.0799$$

It follows that the maturity adjustment is equal to:

$$\varphi(\mathrm{M}) = \frac{1 + (2 - 2.5) \times 0.0799}{1 - 1.5 \times 0.0799} = 1.0908$$

The IRB formulas

Risk-weighted assets for corporate, sovereign, and bank exposures

The normalized capital charge with a one-year maturity is:

$$\begin{aligned} \mathcal{K}^{\star} &= 45\% \times \Phi\left(\frac{\Phi^{-1}(5\%) + \sqrt{12.985\%}\Phi^{-1}(99.9\%)}{\sqrt{1 - 12.985\%}}\right) - 45\% \times 5\% \\ &= 0.1055 \end{aligned}$$

When the maturity is two years, we obtain:

$$\mathcal{K}^{\star} = 0.1055 \times 1.0908 = 0.1151$$

We deduce the value taken by the risk weight:

$$\mathrm{RW} = 12.5 \times 0.1151 = 143.87\%$$

It follows that the risk-weighted asset amount is equal to \$4.316 mn whereas the capital charge is \$345 287

The IRB formulas

Risk-weighted assets for corporate, sovereign, and bank exposures

Table: IRB risk weights (in %) for corporate exposures

Maturity		$M = 1$		$M = 2.5$		$M = 2.5$ (SME)	
LGD		45%	75%	45%	75%	45%	75%
PD (in %)	0.10	18.7	31.1	29.7	49.4	23.3	38.8
	0.50	52.2	86.9	69.6	116.0	54.9	91.5
	1.00	73.3	122.1	92.3	153.9	72.4	120.7
	2.00	95.8	159.6	114.9	191.4	88.5	147.6
	5.00	131.9	219.8	149.9	249.8	112.3	187.1
	10.00	175.8	292.9	193.1	321.8	146.5	244.2
	20.00	223.0	371.6	238.2	397.1	188.4	314.0

(*) For SME claims, sales are equal to 5 € mn

The IRB formulas

Risk-weighted assets for retail exposures

Claims can be included in the regulatory retail portfolio if they meet the following criteria:

1. The exposure must be to an individual person or to a small business
2. It satisfies the granularity criterion, meaning that no aggregate exposure to one counterpart can exceed 0.2% of the overall regulatory retail portfolio
3. The aggregated exposure to one counterparty cannot exceed 1 € mn

The IRB formulas

Risk-weighted assets for retail exposures

The maturity is set to one year:

$$\mathcal{K}^{\star} = \mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})} \Phi^{-1}(99.9\%)}{\sqrt{1 - \rho(\mathrm{PD})}} \right) - \mathrm{LGD} \cdot \mathrm{PD}$$

- Residential mortgage exposures:

$$\rho(\mathrm{PD}) = 15\%$$

- Qualifying revolving retail exposures:

$$\rho(\mathrm{PD}) = 4\%$$

- Other retail exposures:

$$\rho(\mathrm{PD}) = 3\% \times \left(\frac{1 - e^{-35 \times \mathrm{PD}}}{1 - e^{-35}} \right) + 16\% \times \left(1 - \frac{1 - e^{-35 \times \mathrm{PD}}}{1 - e^{-35}} \right)$$

The IRB formulas

Risk-weighted assets for retail exposures

Table: IRB risk weights (in %) for retail exposures

		Mortgage		Revolving		Other retail	
LGD		45%	25%	45%	85%	45%	85%
PD (in %)	0.10	10.7	5.9	2.7	5.1	11.2	21.1
	0.50	35.1	19.5	10.0	19.0	32.4	61.1
	1.00	56.4	31.3	17.2	32.5	45.8	86.5
	2.00	87.9	48.9	28.9	54.6	58.0	109.5
	5.00	148.2	82.3	54.7	103.4	66.4	125.5
	10.00	204.4	113.6	83.9	158.5	75.5	142.7
	20.00	253.1	140.6	118.0	222.9	100.3	189.4

Pillar 2 – Supervisory review process

Supervisory review process (SRP)

1. Supervisory review and evaluation process (SREP)
2. Internal capital adequacy assessment process (ICAAP)

$\Rightarrow$ SREP defines the regulatory response to the first pillar (validation processes of internal models), whereas ICAAP addresses risks that are not captured in Pillar 1 like:

- Concentration risk and non-granular portfolios
- Default correlation
- Stressed parameters (PD and LGD)
- *Point-in-time* (PIT) versus *through the-cycle* (TTC)

Pillar 3 – Market discipline

The third pillar requires banks to publish comprehensive information about their risk management process

Since 2015, standardized templates for quantitative disclosure with a fixed format in order to facilitate the comparison between banks

The Basel III revision

For credit risk capital requirements, Basel III is close to the Basel II framework with some adjustments, which mainly concern the parameters

Remark

SA and IRB methods continue to be the two approaches for computing the capital charge for credit risk

The Basel III revision
The standardized approach

Differences between Basel II et and Basel III:

- Two methods:
  1. External credit risk assessment approach (ECRA)
  2. Standardized credit risk approach (SCRA)
- Loan-to-value ratio (LTV)

The Basel III revision

The standardized approach (ECRA)

Table: Risk weights of the SA approach (ECRA, Basel III)

Rating		AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to B−	CCC+ to C	NR
Sovereigns		0%	20%	50%	100%	150%	100%
PSE	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
MDB		20%	30%	50%	100%	150%	50%
Banks	2	20%	30%	50%	100%	150%	SCRA
	2 ST	20%	20%	20%	50%	150%	SCRA
	Covered	10%	20%	20%	50%	100%	
Corporates		20%	50%	75%	100%	150%	100%
Retail*		75%					

(∗) The retail category includes revolving credits, credit cards, consumer credit loans, auto loans, student loans, etc., but not real estate exposures

The Basel III revision

The standardized approach (SCRA, banks)

The standardized credit risk approach (SCRA) must be used for all exposures to banks in two situations:

1. When the exposure is unrated
2. When external credit ratings are prohibited (e.g. in the US[2])

In this case, the bank must conduct a due diligence analysis in order to classify the exposures into three grades

- A Grade A refers to the most solid banks, whose capital exceeds the minimum regulatory capital requirements ($\mathrm{RW} = 40\%$ – 20% for short-term exposures)
- B Grade B refers to banks subject to substantial credit risk ($\mathrm{RW} = 75\%$ – 50% for short-term exposures)
- C Grade C refers to the most vulnerable banks ($\mathrm{RW} = 150\%$ – 150% for short-term exposures)

[2] The United States had abandoned in 2010 the use of commercial credit ratings after the Dodd-Frank reform

The Basel III revision

The standardized approach (SCRA, corporates)

When external credit ratings are prohibited, the risk weight of exposures to corporates is equal to 100% with two exceptions:

- A 65% risk weight is assigned to corporates, which can be considered investment grade (IG)
- For exposures to small and medium-sized enterprises, a 75% risk weight can be applied if the exposure can be classified in the retail category and 85% for the others

The Basel III revision

The standardized approach (ECRA, real estate)

Table: Risk weights of the SA approach (ECRA, Basel III)

Residential real estate			Commercial real estate		
Cash flows	ND	D	Cash flows	ND	D
$\mathrm{LTV} \leq 50$	20%	30%	$\mathrm{LTV} \leq 60$	$\min\left(60\%, \mathrm{RW}_C\right)$	70%
$50 < \mathrm{LTV} \leq 60$	25%	35%			
$60 < \mathrm{LTV} \leq 80$	30%	45%	$60 < \mathrm{LTV} \leq 80$	RW_C	90%
$80 < \mathrm{LTV} \leq 90$	40%	60%	$\mathrm{LTV} > 80$	RW_C	110%
$90 < \mathrm{LTV} \leq 100$	50%	75%			
$\mathrm{LTV} > 100$	70%	105%			

The Basel III revision

The standardized approach (ECRA, real estate)

Definition

The loan-to-value (LTV) ratio is the ratio of a loan to the value of an asset purchased

Example

If one borrows \$100 000 to purchase a house of \$150 000, the LTV ratio is 100 000/150 000 or 66.67%

This ratio is extensively used in English-speaking countries (e.g. the United States) to measure the risk of the loan

In continental Europe, the risk of home property loans is measured by the ability of the borrower to repay the capital and service his debt, meaning that the risk of the loan is generally related to the income of the borrower

The Basel III revision

The standardized approach

For off-balance sheet items, credit conversion factors (CCF) have been revised. They can take the values 10%, 20%, 40%, 50% and 100%. This is a more granular scale without the possibility to set the CCF to 0%

The Basel III revision

The internal ratings-based approach

The methodology of the IRB approach does not change with respect to Basel II, since the formulas are the same except the correlation parameter for bank exposures:

$$\rho(\mathrm{PD}) = 15\% \times \left(\frac{1 - e^{-50\times\mathrm{PD}}}{1 - e^{-50}}\right) + 30\% \times \left(\frac{1 - \left(1 - e^{-50\times\mathrm{PD}}\right)}{1 - e^{-50}}\right)$$

Other changes

- For banks and large corporates, only the FIRB approach can be used
- In the AIRB approach, the estimated parameters of PD and LGD are subject to some input floors[a]
- The default values of the LGD parameter are 75% for subordinated claims, 45% for senior claims on financial institutions and 40% for senior claims on corporates in the FIRB approach

[a] For example, the minimum PD is set to 5 bps for corporate and bank exposures

Exposure at default

Definition

The exposure at default "*for an on-balance sheet or off-balance sheet item is defined as the expected gross exposure of the facility upon default of the obligor*"

$\Rightarrow$ EAD corresponds to the gross notional in the case of a loan or a credit

The big issue concerns off-balance sheet items, such as revolving lines of credit, credit cards or home equity lines of credit (HELOC)

Exposure at default

At the default time $\boldsymbol{\tau}$, we have:

$$\mathrm{EAD}\left(\boldsymbol{\tau} \mid t\right) = B\left(t\right) + \mathrm{CCF} \cdot \left(L\left(t\right) - B\left(t\right)\right)$$

where:

- $B\left(t\right)$ is the outstanding balance (or current drawn) at time t
- $L\left(t\right)$ is the current undrawn limit of the credit facility
- CCF is the credit conversion factor
- $L\left(t\right) - B\left(t\right)$ is the current undrawn or the amount that the debtor is able to draw upon in addition to the current drawn $B\left(t\right)$

We deduce that:

$$\mathrm{CCF} = \frac{\mathrm{EAD}\left(\boldsymbol{\tau} \mid t\right) - B\left(t\right)}{L\left(t\right) - B\left(t\right)}$$

Exposure at default

Let us consider the off-balance sheet item i that has defaulted. We have:

$$\mathrm{CCF}_i(\boldsymbol{\tau}_i - t) = \frac{B_i(\boldsymbol{\tau}_i) - B_i(t)}{L_i(t) - B_i(t)}$$

At time $\boldsymbol{\tau}_i$, we observe the default of Asset i and the corresponding exposure at default, which is equal to the outstanding balance $B_i(\boldsymbol{\tau}_i)$

$\Rightarrow$ We have to choose a date $t < \boldsymbol{\tau}_i$ to observe $B_i(t)$ and $L_i(t)$ in order to calculate the CCF

Estimation of CCF is difficult because it is sensitive to the date t

Loss given default

Loss given default versus recovery rate

- The recovery is the percentage of the notional on the defaulted debt that can be recovered
- In the Basel framework, the recovery rate is not explicitly used, and the concept of loss given default is preferred for measuring the credit portfolio loss
- We have:

$$\mathrm{LGD} \geq 1 - \boldsymbol{\mathcal{R}}$$

Loss given default

Example

We consider a bank that is lending \$100 mn to a corporate firm. We assume that the firm defaults at one time and, the bank recovers \$60 mn and the litigation costs are equal to \$5 mn

We deduce that the recovery rate is equal to:

$$\mathcal{R} = \frac{60}{100} = 60\%$$

In order to recover \$60 mn, the bank has incurred some operational and litigation costs. In this case, the bank has lost \$40 mn plus \$5 mn, implying that the loss given default is equal to:

$$\mathrm{LGD} = \frac{40 + 5}{100} = 45\%$$

Loss given default

Relationship between $\mathcal{R}$ and LGD

We have:

$$\mathrm{LGD} = 1 - \mathcal{R} + c$$

where c is the litigation cost (expressed in %)

Loss given default

Two approaches for modeling LGD:

1. The first approach considers that LGD is a random variable, whose probability distribution has to be estimated:

$$\mathrm{LGD} \sim \mathbf{F}(x)$$

2. The second approach consists in estimating the conditional expectation:

$$\mathbb{E}[\mathrm{LGD}] = \mathbb{E}[\mathrm{LGD} \mid X_1 = x_1, \ldots, X_m = x_m] = g(x_1, \ldots, x_m)$$

where $(X_1, \ldots, X_m)$ are the risk factors that impact LGD

Remark

We recall that the loss given default in the Basel IRB formulas does not correspond to the random variable, but to its expectation $\mathbb{E}[\mathrm{LGD}]$. Therefore, only the mean $\mathbb{E}[\mathrm{LGD}]$ is important for Pillar 1

$\Rightarrow$ Pillar 2 uses the entire probability distribution $\mathbf{F}(x)$ and the condition expectation under stressed conditions

Loss given default

Stochastic modeling (parametric distribution)

Beta distribution

The beta distribution $\mathcal{B}(\alpha, \beta)$ has the following pdf:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathfrak{B}(\alpha, \beta)}$$

where $\mathfrak{B}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$. The mean and the variance are:

$$\mu(X) = \mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$$

and:

$$\sigma^2(X) = \operatorname{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

When α and β are greater than 1, the distribution has one mode $x_{\mathrm{mode}} = (\alpha-1)/(\alpha+\beta-2)$

Loss given default

Stochastic modeling (parametric distribution)

Several shapes:

- $\mathcal{B}(1,1) \sim \mathcal{U}_{[0,1]}$, $\mathcal{B}(\infty,\infty) \sim \delta_{0.5}([0,1])$, $\mathcal{B}(\alpha,0) \sim \mathcal{B}(1)$ and $\mathcal{B}(0,\beta) \sim \mathcal{B}(0)$
- If $\alpha = \beta$, the distribution is symmetric around $x = 0.5$; we have a bell curve when the two parameters α and β are higher than 1, and a **U**-shape curve when the two parameters α and β are lower than 1
- If $\alpha > \beta$, the skewness is negative and the distribution is left-skewed, if $\alpha < \beta$, the skewness is positive and the distribution is right-skewed

Loss given default

Stochastic modeling (parametric distribution)

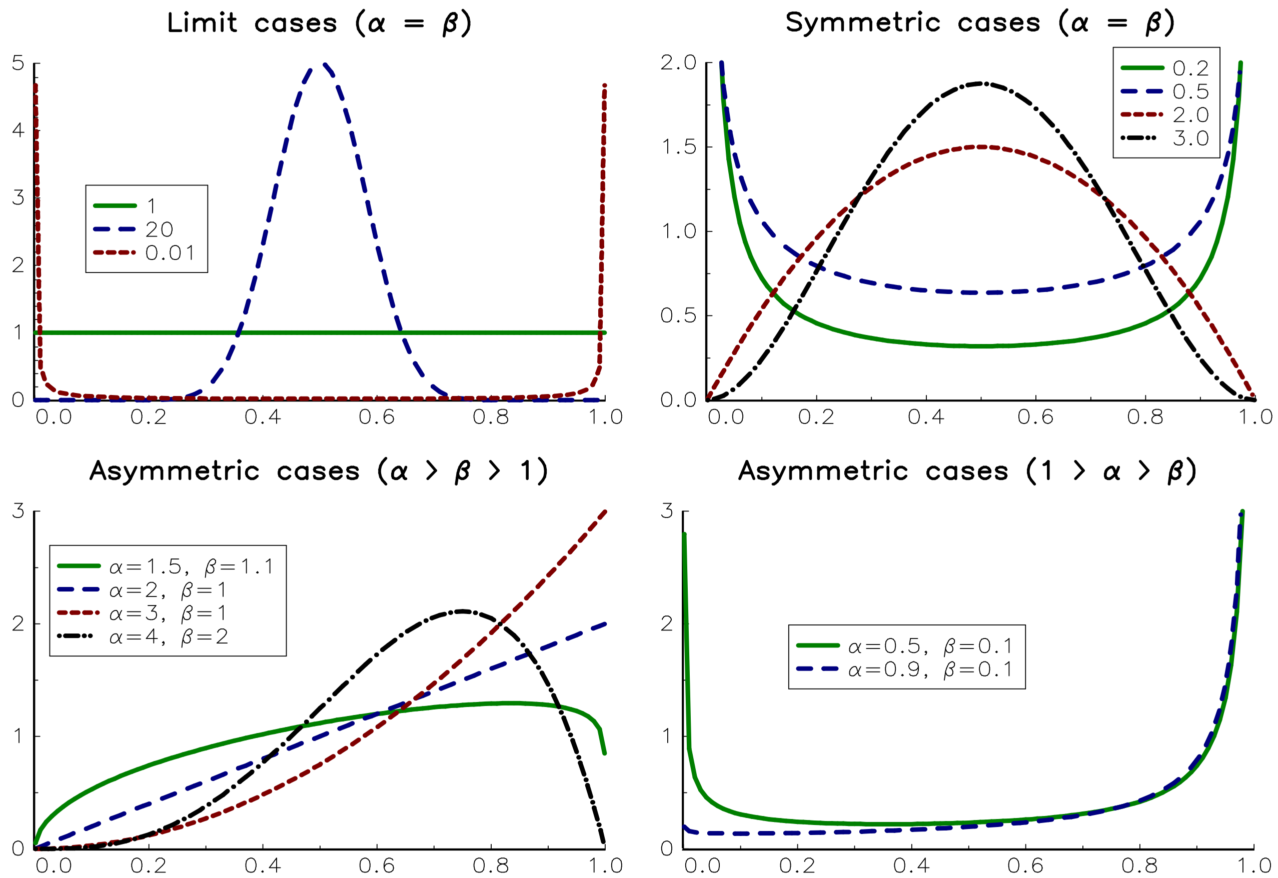


Figure: Probability density function of the beta distribution $\mathcal{B}(\alpha, \beta)$

Loss given default

Stochastic modeling (parametric distribution)

Method of moments (HFRM, Section 10.1.3, page 628)

We have:

$$\hat{\alpha}_{\mathrm{MM}} = \frac{\hat{\mu}_{\mathrm{LGD}}^2 \left(1 - \hat{\mu}_{\mathrm{LGD}}\right)}{\hat{\sigma}_{\mathrm{LGD}}^2} - \hat{\mu}_{\mathrm{LGD}}$$

and:

$$\hat{\beta}_{\mathrm{MM}} = \frac{\hat{\mu}_{\mathrm{LGD}} \left(1 - \hat{\mu}_{\mathrm{LGD}}\right)^2}{\hat{\sigma}_{\mathrm{LGD}}^2} - \left(1 - \hat{\mu}_{\mathrm{LGD}}\right)$$

Maximum likelihood estimation (HFRM, Section 10.1.2, page 614)

$$\begin{aligned}\left(\hat{\alpha}_{\mathrm{ML}}, \hat{\beta}_{\mathrm{ML}}\right) &= \arg\max \ell\left(\alpha, \beta\right) \\ &= \arg\max \left(\alpha - 1\right) \sum_{i=1}^{n} \ln y_i + \left(b - 1\right) \sum_{i=1}^{n} \ln\left(1 - y_i\right) - n \ln \mathfrak{B}\left(\alpha, \beta\right)\end{aligned}$$

Loss given default

Stochastic modeling (parametric distribution)

Example

We consider the following sample of losses given default: 68%, 90%, 22%, 45%, 17%, 25%, 89%, 65%, 75%, 56%, 87%, 92% and 46%

We obtain $\hat{\mu}_{\mathrm{LGD}} = 59.77\%$ and $\hat{\sigma}_{\mathrm{LGD}} = 27.02\%$. Using the method of moments, the estimated parameters are $\hat{\alpha}_{\mathrm{MM}} = 1.37$ and $\hat{\beta}_{\mathrm{MM}} = 0.92$

Using a **numerical optimization** method, we have $\hat{\alpha}_{\mathrm{ML}} = 1.84$ and $\hat{\beta}_{\mathrm{ML}} = 1.25$. See HFRM on page 619 for the statistical inference:

Table: Results of the maximum likelihood estimation

Parameter	Estimate	Standard error	*t*-statistic	*p*-value
α	1.8356	0.6990	2.6258	0.0236
β	1.2478	0.4483	2.7834	0.0178

Loss given default

Stochastic modeling (parametric distribution)

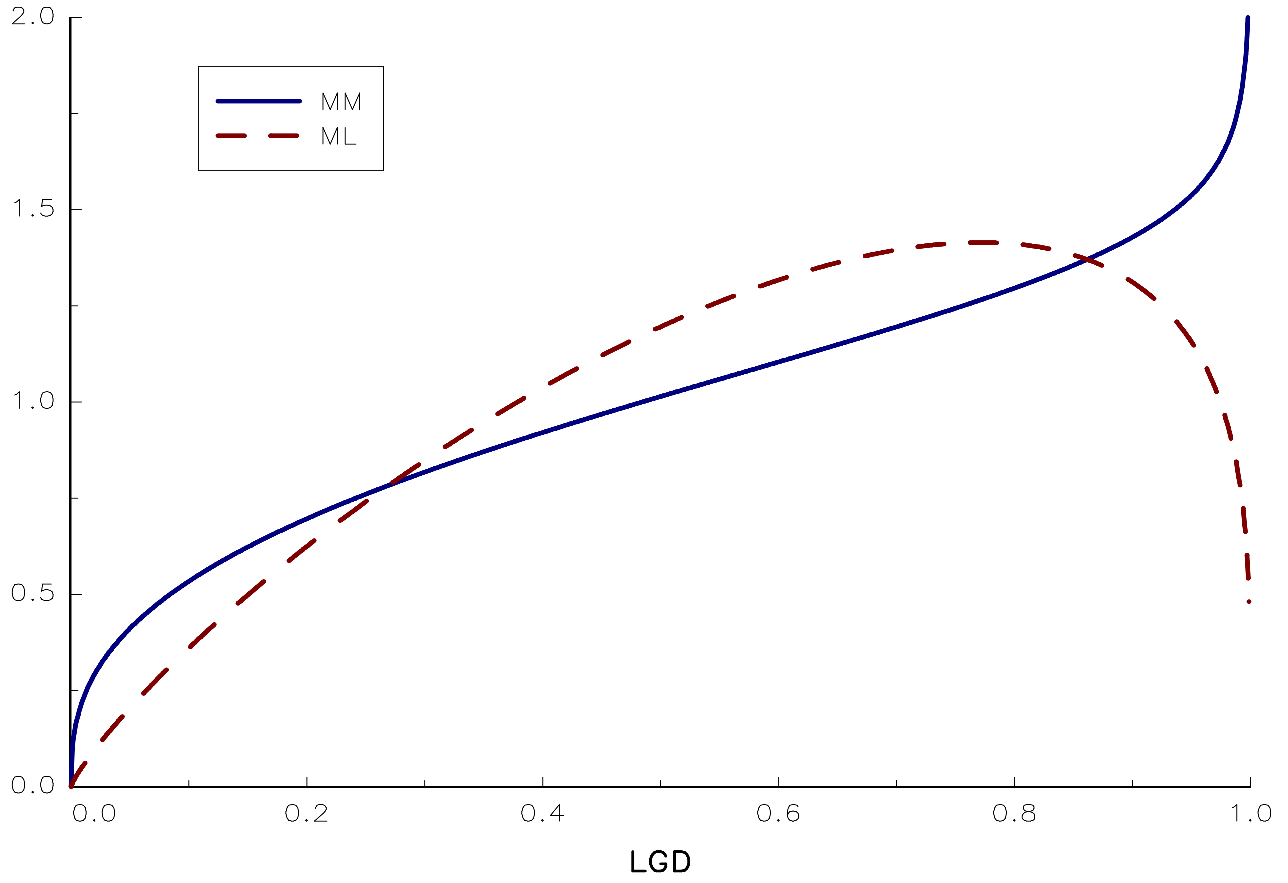


Figure: Calibration of the beta distribution

Loss given default

Stochastic modeling (non-parametric distribution)

The limit case of the beta distribution's **U**-shaped is the Bernoulli distribution:

LGD	0%	100%
Probability	$(1 - \mu_{\mathrm{LGD}})$	μ_{LGD}

$\Rightarrow$ Extension to the empirical distribution or histogram

Example

We consider the following empirical distribution of LGD:

LGD (in %)	0	10	20	25	30	40	50	60	70	75	80	90	100
$\hat{p}$ (in %)	1	2	10	25	10	2	0	2	10	25	10	2	1

Loss given default

Stochastic modeling (non-parametric distribution)

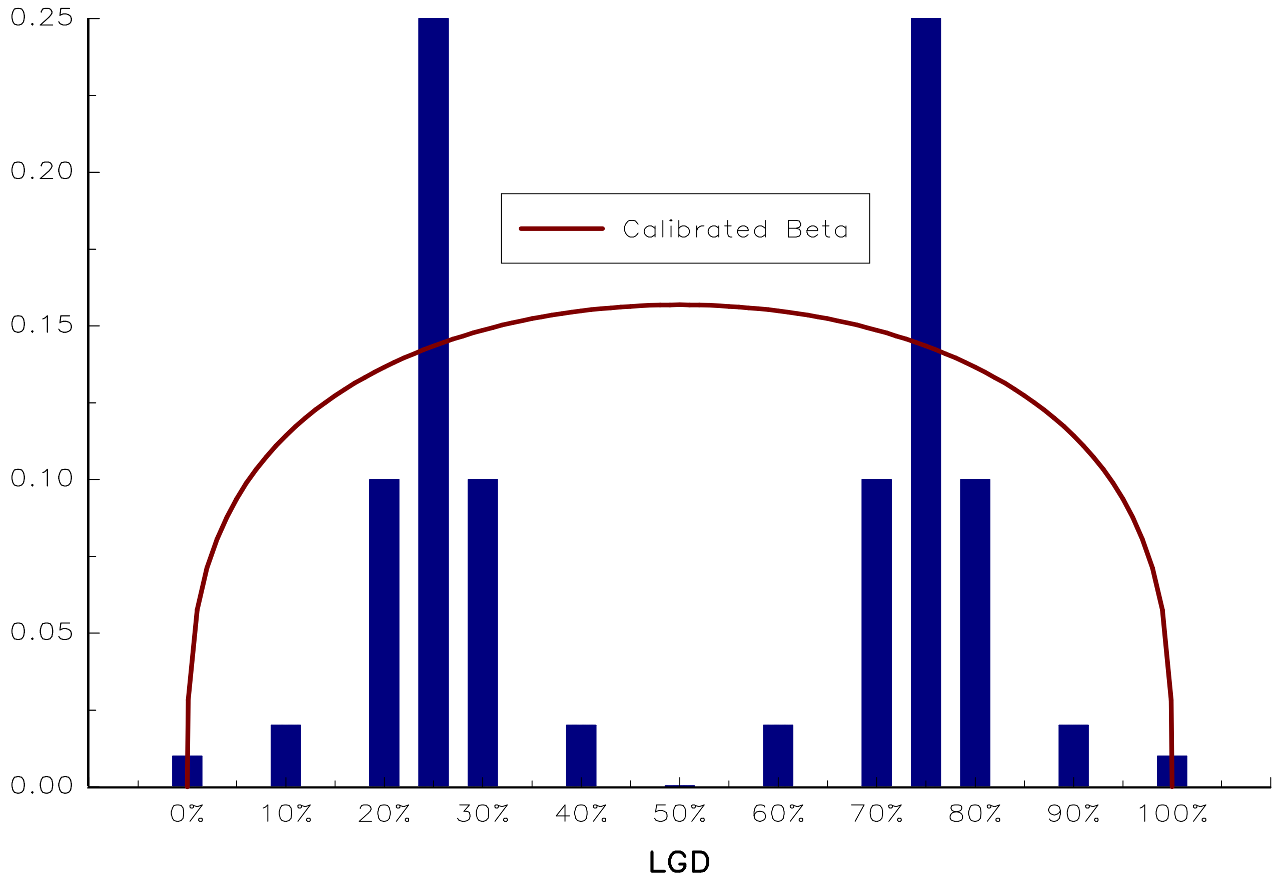


Figure: Calibration of a bimodal LGD distribution

Loss given default

The case of non-granular portfolios

Example

We consider a credit portfolio of 10 loans, whose loss is equal to:

$$L = \sum_{i=1}^{10} \mathrm{EaD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

where T_i is equal to 5 years, EaD_i is equal to \$1 000 and the default time $\boldsymbol{\tau}_i$ is exponential with the following intensity parameter λ_i:

i	1	2	3	4	5	6	7	8	9	10
λ_i (in bps)	10	10	25	25	50	100	250	500	500	1 000

The loss given default LGD_i is given by the previous empirical distribution:

LGD (in %)	0	10	20	25	30	40	50	60	70	75	80	90	100
$\hat{p}$ (in %)	1	2	10	25	10	2	0	2	10	25	10	2	1

Loss given default

The case of non-granular portfolios

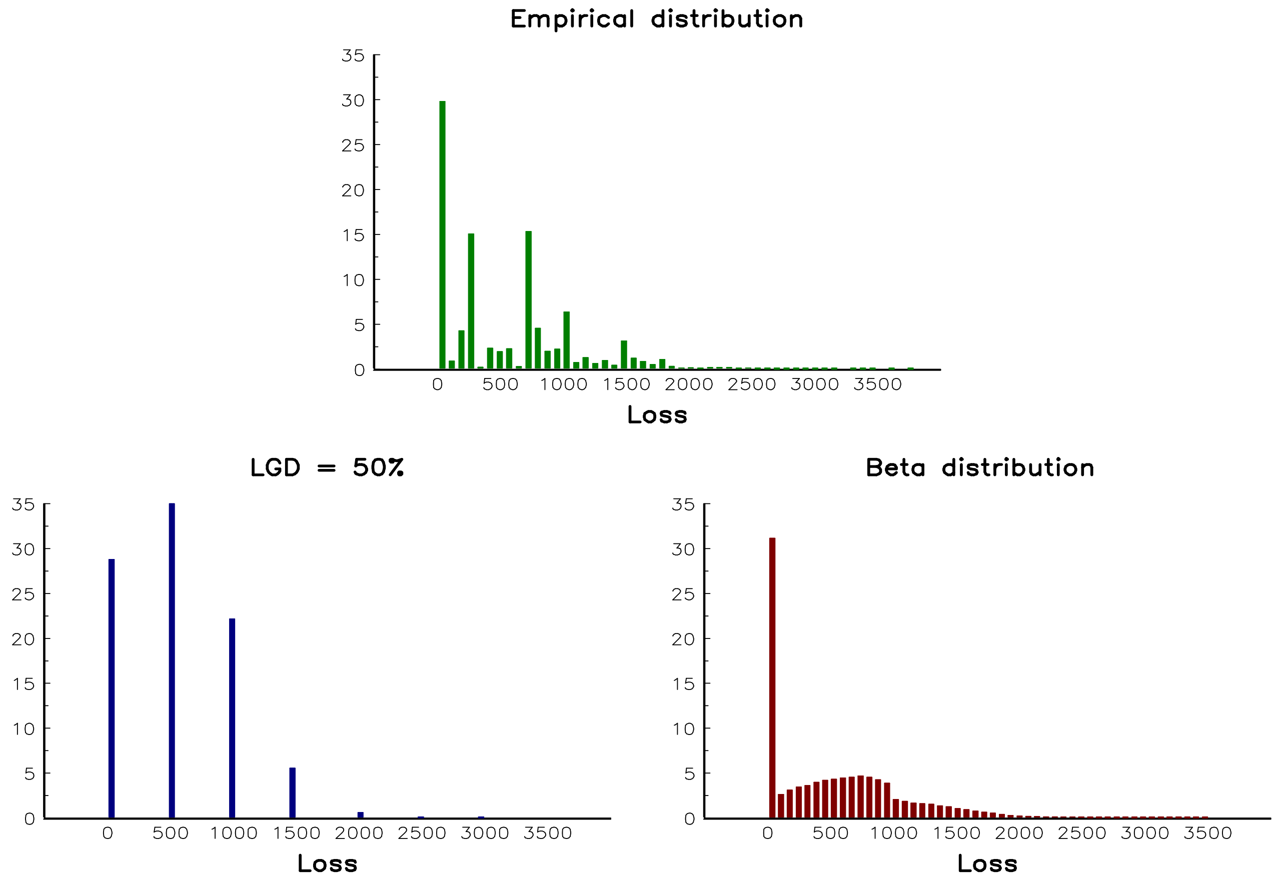


Figure: Loss frequency in % of the three LGD models

Loss given default

The case of non-granular portfolios

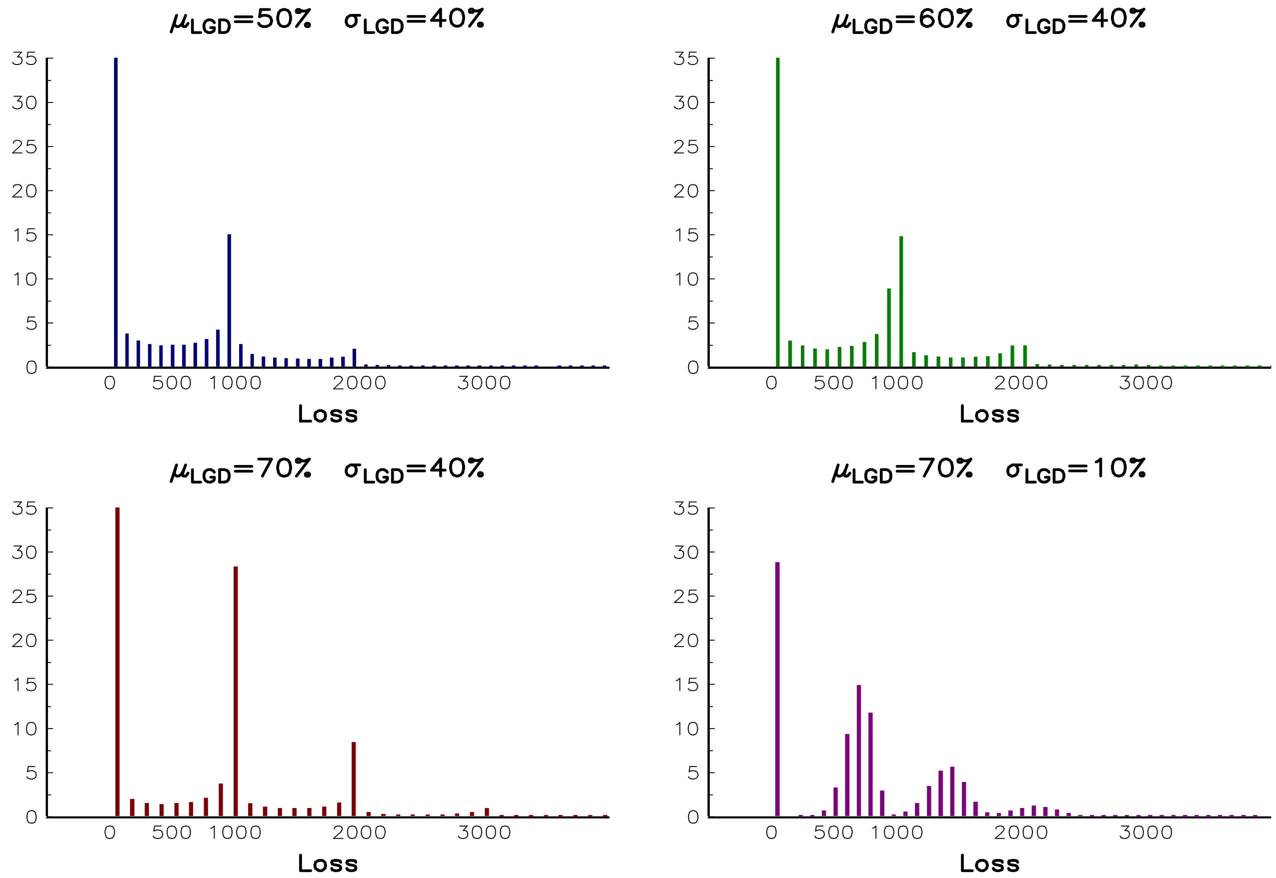


Figure: Loss frequency in % for different values of μ_{LGD} and σ_{LGD}

Loss given default

The case of granular portfolios

Expression of the portfolio loss

We recall that:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

If the portfolio is fined grained, we have:

$$\mathbb{E}\left[L \mid X\right] = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i\left(X\right)$$

We deduce that the distribution of the portfolio loss does not depend on the random variables LGD_i, but on their expected values $\mathbb{E}\left[\mathrm{LGD}_i\right]$. Therefore, we can replace the previous expression of the portfolio loss by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

Loss given default

Economic modeling

The third version of Moody's LossCalc considers seven factors that are grouped in three major categories:

1. factors external to the issuer: geography, industry, credit cycle stage
2. factors specific to the issuer: distance-to-default, probability of default (or leverage for private firms)
3. factors specific to the debt issuance: debt type, relative standing in capital structure, collateral

Once the factors are identified, we must estimate the LGD model:

$$\mathrm{LGD} = f\left(X_1, \ldots, X_m\right)$$

where $X_1, \ldots, X_m$ are the m factors, and f is a non-linear function

We apply a logit transformation and estimate the model using linear regression or quantile regression (see HFRM, Section 14.2.3, page 909) $\Rightarrow$ This approach will be studied in Lecture 11 dedicated to stress testing and scenario analysis

Probability of default

Three approaches:

- Survival function
- Transition probability matrix
- Structural models

Survival function

Let $\boldsymbol{\tau}$ be a default (or survival) time. The survival function is defined as follows:

$$\mathbf{S}(t) = \Pr\{\boldsymbol{\tau} > t\} = 1 - \mathbf{F}(t)$$

where $\mathbf{F}$ is the cumulative distribution function. We deduce that:

$$f(t) = -\frac{\partial\,\mathbf{S}(t)}{\partial\,t}$$

We define the hazard function $\lambda(t)$ as the instantaneous default rate given that the default has not occurred before t:

$$\lambda(t) = \lim_{\mathrm{d}t \to 0^+} \frac{\Pr\{t \leq \boldsymbol{\tau} \leq t + \mathrm{d}t \mid \boldsymbol{\tau} \geq t\}}{\mathrm{d}t}$$

We deduce that:

$$\begin{aligned}
\lambda(t) &= \lim_{\mathrm{d}t \to 0^+} \frac{\Pr\{t \leq \boldsymbol{\tau} \leq t + \mathrm{d}t\}}{\mathrm{d}t} \cdot \frac{1}{\Pr\{\boldsymbol{\tau} \geq t\}} \\
&= \frac{f(t)}{\mathbf{S}(t)} = -\frac{\partial_t\,\mathbf{S}(t)}{\mathbf{S}(t)} = -\frac{\partial\,\ln\mathbf{S}(t)}{\partial\,t}
\end{aligned}$$

Survival function

The survival function can then be rewritten with respect to the hazard function and we have:

$$\mathbf{S}(t) = e^{-\int_0^t \lambda(s)\,\mathrm{d}s}$$

Table: Common survival functions

Model	$\mathbf{S}(t)$	$\lambda(t)$
Exponential	$\exp(-\lambda t)$	λ
Weibull	$\exp(-\lambda t^{\gamma})$	$\lambda \gamma t^{\gamma-1}$
Log-normal	$1 - \Phi(\gamma \ln(\lambda t))$	$\gamma t^{-1} \phi(\gamma \ln(\lambda t)) / (1 - \Phi(\gamma \ln(\lambda t)))$
Log-logistic	$1 / \left(1 + \lambda t^{\frac{1}{\gamma}}\right)$	$\lambda \gamma^{-1} t^{\frac{1}{\gamma}} / \left(t + \lambda t^{1+\frac{1}{\gamma}}\right)$
Gompertz	$\exp(\lambda(1 - e^{\gamma t}))$	$\lambda \gamma \exp(\gamma t)$
Cox	$\mathbf{S}(t) = e^{-\exp\left(\beta^{\top} x\right) \int_0^t \lambda_0(s)\,\mathrm{d}s}$	$\lambda_0(t) \exp\left(\beta^{\top} x\right)$

Exponential survival time

We note $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and we have:

$$\mathbf{S}(t) = e^{-\lambda t}$$

Main properties

1. The mean residual life $\mathbb{E}[\boldsymbol{\tau} \mid \boldsymbol{\tau} \geq t]$ is constant
2. It satisfies the **lack of memory property** (LMP):

$$\Pr\{\boldsymbol{\tau} \geq t + u \mid \boldsymbol{\tau} \geq t\} = \Pr\{\boldsymbol{\tau} \geq u\}$$

   or equivalently $\mathbf{S}(t+u) = \mathbf{S}(t)\,\mathbf{S}(u)$
3. The probability distribution of $n \cdot \boldsymbol{\tau}_{1:n}$ is the same as probability distribution of $\boldsymbol{\tau}_i$

Piecewise exponential model

We have:

$$\lambda(t) = \sum_{m=1}^{M} \lambda_m \cdot \mathbb{1}\left\{t_{m-1}^{\star} < t \leq t_m^{\star}\right\} = \lambda_m \qquad \text{if } t \in \left]t_{m-1}^{\star}, t_m^{\star}\right]$$

where $t_m^{\star}$ are the knots of the function $\left(t_0^{\star} = 0,\ t_{M+1}^{\star} = \infty\right)$. For $t \in \left]t_{m-1}^{\star}, t_m^{\star}\right]$, the expression of the survival function becomes:

$$\mathbf{S}(t) = \exp\left(-\sum_{k=1}^{m-1} \lambda_k \left(t_k^{\star} - t_{k-1}^{\star}\right) - \lambda_m \left(t - t_{m-1}^{\star}\right)\right) = \mathbf{S}\left(t_{m-1}^{\star}\right) e^{-\lambda_m \left(t - t_{m-1}^{\star}\right)}$$

It follows that the density function is equal to:

$$f(t) = \lambda_m \exp\left(-\sum_{k=1}^{m-1} \lambda_k \left(t_k^{\star} - t_{k-1}^{\star}\right) - \lambda_m \left(t - t_{m-1}^{\star}\right)\right)$$

We verify that:

$$\frac{f(t)}{\mathbf{S}(t)} = \lambda_m \qquad \text{if } t \in \left]t_{m-1}^{\star}, t_m^{\star}\right]$$

Piecewise exponential model

Example

We consider three set of parameters $\{(t_m^\star, \lambda_m), m = 1, \ldots, M\}$:

$$
\begin{array}{rl}
\{(1, 1\%), (2, 1.5\%), (3, 2\%), (4, 2.5\%), (\infty, 3\%)\} & \text{for } \lambda_1(t) \\
\{(1, 10\%), (2, 7\%), (5, 5\%), (7, 4.5\%), (\infty, 6\%)\} & \text{for } \lambda_2(t) \\
\lambda_3(t) = 4\% & \text{for } \lambda_3(t)
\end{array}
$$

Piecewise exponential model

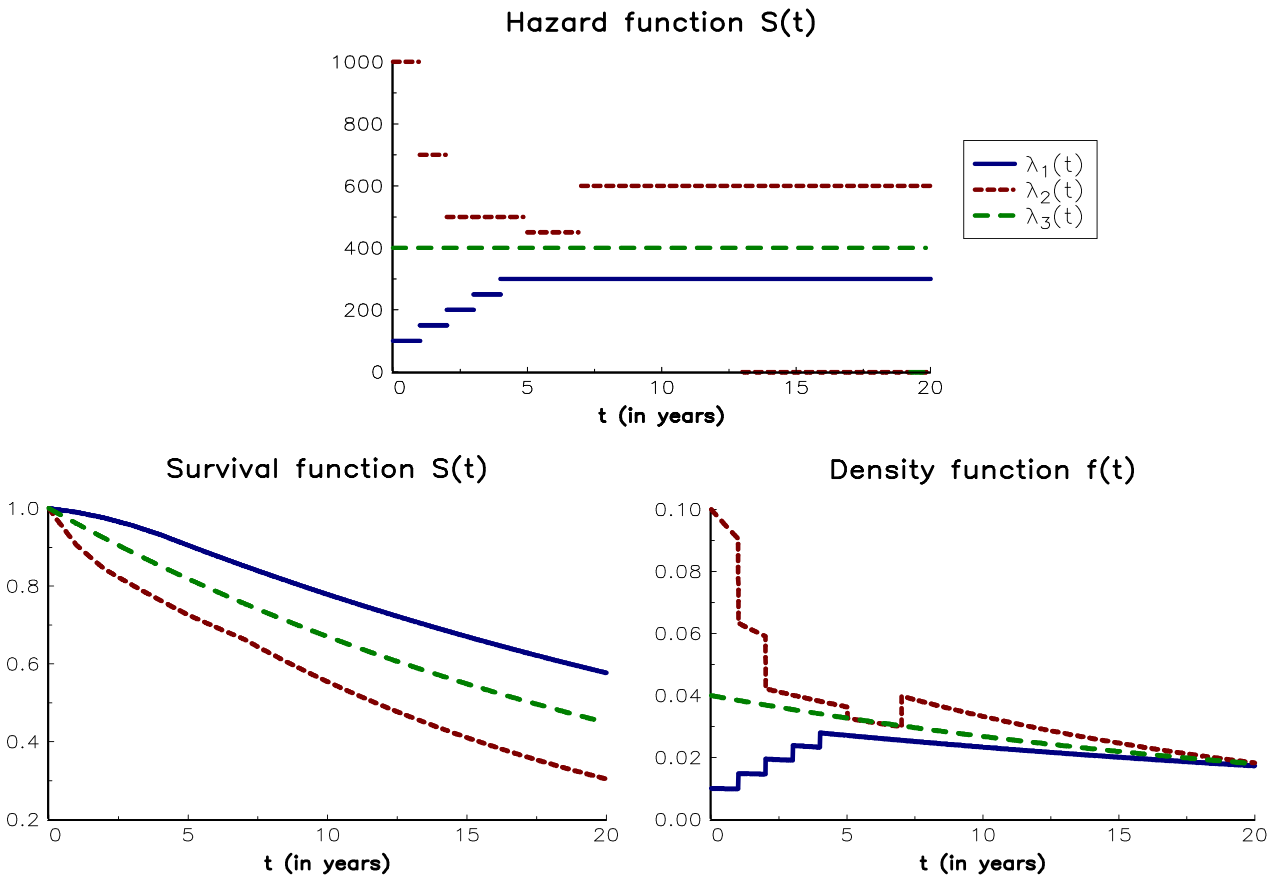


Figure: Example of the piecewise exponential model

Piecewise exponential model

Estimation methods:

- Non-linear least squares regression
- Kaplan-Meier estimation (non-parametric approach)
- Bootstrap

Bootstrap method

1. We first estimate the parameter λ_1 for the earliest maturity Δt_1
2. Assuming that $\left(\hat{\lambda}_1, \ldots, \hat{\lambda}_{i-1}\right)$ have been estimated, we calculate $\hat{\lambda}_i$ for the next maturity Δt_i
3. We iterate step 2 until the last maturity Δt_m

$\Rightarrow$ This algorithm is used for calibrating the credit curve of CDS spreads

Piecewise exponential model

Example

We consider three credit curves, whose CDS spreads expressed in bps are given in the table below. We assume that the recovery rate $\mathcal{R}$ is set to 40%

Table: Calibrated piecewise exponential model from CDS prices

	Credit curve			Bootstrap solution		
Maturity (in years)	#1	#2	#3	#1	#2	#3
1	50	50	350	83.3	83.3	582.9
3	60	60	370	110.1	110.1	637.5
5	70	90	390	140.3	235.0	702.0
7	80	115	385	182.1	289.6	589.4
10	90	125	370	194.1	241.9	498.5

Transition probability matrix

Definition

We consider a time-homogeneous Markov chain $\mathfrak{R}$, whose transition matrix is $P = (p_{i,j})$. We note $\mathcal{S} = \{1, 2, \ldots, K\}$ the state space of the chain and $p_{i,j}$ is the probability that the entity migrates from rating i to rating j. The matrix P satisfies the following properties:

- $\forall i, j \in \mathcal{S}$, $p_{i,j} \geq 0$;
- $\forall i \in \mathcal{S}$, $\sum_{j=1}^{K} p_{i,j} = 1$.

In credit risk, we generally assume that K is the absorbing state (or the default state), implying that any entity which has reached this state remains in this state ($p_{K,K} = 1$)

Transition probability matrix

Table: Example of credit migration matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.82	6.50	0.56	0.06	0.06	0.00	0.00	0.00
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.00
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.04
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.22
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.06
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Transition probability matrix

Let $\mathfrak{R}(t)$ be the value of the state at time t. We define $p(s,i;t,j)$ as the probability that the entity reaches the state j at time t given that it has reached the state i at time s:

$$p(s,i;t,j) = \Pr\{\mathfrak{R}(t) = j \mid \mathfrak{R}(s) = i\} = p_{i,j}^{(t-s)}$$

This is the Markov property

The n-step transition probability is defined as:

$$p_{i,j}^{(n)} = \Pr\{\mathfrak{R}(t+n) = j \mid \mathfrak{R}(t) = i\}$$

and we note $P^{(n)} = \left(p_{i,j}^{(n)}\right)$ the associated n-step transition matrix

Transition probability matrix

For $n = 2$, we obtain:

$$\begin{aligned}
p_{i,j}^{(2)} &= \Pr \left\{ \mathfrak{R}(t+2) = j \mid \mathfrak{R}(t) = i \right\} \\
&= \sum_{k=1}^{K} \Pr \left\{ \mathfrak{R}(t+2) = j, \mathfrak{R}(t+1) = k \mid \mathfrak{R}(t) = i \right\} \\
&= \sum_{k=1}^{K} \Pr \left\{ \mathfrak{R}(t+2) = j \mid \mathfrak{R}(t+1) = k \right\} \cdot \Pr \left\{ \mathfrak{R}(t+1) = k \mid \mathfrak{R}(t) = i \right\} \\
&= \sum_{k=1}^{K} p_{i,k} \cdot p_{k,j}
\end{aligned}$$

Transition probability matrix

Chapman-Kolmogorov (forward) equation

We have (scalar form):

$$p_{i,j}^{(n+m)} = \sum_{k=1}^{K} p_{i,k}^{(n)} \cdot p_{k,j}^{(m)} \qquad \forall n, m > 0$$

or (matrix form):

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

with the convention $P^{(0)} = I_K$

We deduce that:

$$P^{(n)} = P^n$$

and:

$$p\left(t, i; t+n, j\right) = p_{i,j}^{(n)} = \mathbf{e}_i^\top P^n \mathbf{e}_j$$

Transition probability matrix

$$
\begin{aligned}
p^{(2)}_{\mathrm{AAA},\mathrm{AAA}} &= p_{\mathrm{AAA},\mathrm{AAA}} \times p_{\mathrm{AAA},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{AA}} \times p_{\mathrm{AA},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{A}} \times p_{\mathrm{A},\mathrm{AAA}} + \\
& \quad p_{\mathrm{AAA},\mathrm{BBB}} \times p_{\mathrm{BBB},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{BB}} \times p_{\mathrm{BB},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{B}} \times p_{\mathrm{B},\mathrm{AAA}} + \\
& \quad p_{\mathrm{AAA},\mathrm{CCC}} \times p_{\mathrm{CCC},\mathrm{AAA}} \\
&= 0.9283^2 + 0.0650 \times 0.0063 + 0.0056 \times 0.0008 + \\
& \quad 0.0006 \times 0.0005 + 0.0006 \times 0.0004 \\
&= 86.1970\%
\end{aligned}
$$

Transition probability matrix

Table: Two-year transition probability matrix P^2 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	86.20	12.02	1.47	0.18	0.11	0.01	0.00	0.00
AA	1.17	84.59	12.23	1.51	0.18	0.22	0.07	0.02
A	0.16	4.17	84.47	9.23	1.31	0.51	0.04	0.11
BBB	0.10	0.63	10.53	77.66	8.11	2.10	0.32	0.56
BB	0.08	0.24	1.60	13.33	66.79	13.77	1.59	2.60
B	0.01	0.21	0.61	1.29	11.20	70.03	5.61	11.03
CCC	0.29	0.04	0.68	1.37	4.31	17.51	37.34	38.45
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Transition probability matrix

Table: Five-year transition probability matrix P^5 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	69.23	23.85	5.49	0.96	0.31	0.12	0.02	0.03
AA	2.35	66.96	24.14	4.76	0.86	0.62	0.13	0.19
A	0.43	8.26	68.17	17.34	3.53	1.55	0.18	0.55
BBB	0.24	1.96	19.69	56.62	13.19	5.32	0.75	2.22
BB	0.17	0.73	5.17	21.23	40.72	20.53	2.71	8.74
B	0.07	0.47	1.73	4.67	16.53	44.95	5.91	25.68
CCC	0.38	0.24	1.37	2.92	7.13	18.51	9.92	59.53
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Transition probability matrix

We note $\pi_i^{(n)}$ the probability of the state i at time n:

$$\pi_i^{(n)} = \Pr\left\{\mathfrak{R}(n) = i\right\}$$

and $\pi^{(n)} = \left(\pi_1^{(n)}, \ldots, \pi_K^{(n)}\right)$ the probability distribution. By construction, we have:

$$\pi^{(n+1)} = P^\top \pi^{(n)}$$

The Markov chain $\mathfrak{R}$ admits a stationary distribution $\pi^\star$ if $\pi^\star = P^\top \pi^\star$:

$$\lim_{n\to\infty} p_{k,i}^{(n)} = \pi_i^\star$$

We can interpret $\pi_i^\star$ as the average duration spent by the Markov chain $\mathfrak{R}$ in the state i

Transition probability matrix

Average return period of a Markov chain

Let $\mathcal{T}_i$ be the return period of state i:

$$\mathcal{T}_i = \inf \{n : \mathfrak{R}(n) = i \mid \mathfrak{R}(0) = i\}$$

The average return period is then equal to:

$$\mathbb{E}[\mathcal{T}_i] = \frac{1}{\pi_i^\star}$$

Transition probability matrix

Survival function

Survival function

Since K is the default state, the survival function $\mathbf{S}_i(t)$ of a firm whose initial rating is the state i is given by:

$$
\begin{aligned}
\mathbf{S}_i(t) &= 1 - \Pr\{\mathfrak{R}(t) = K \mid \mathfrak{R}(0) = i\} \\
&= 1 - \mathbf{e}_i^\top P^t \mathbf{e}_K
\end{aligned}
$$

Transition probability matrix

Survival function

Estimation of the piecewise exponential model

In the piecewise exponential model, the survival function is

$$\mathbf{S}(t) = \mathbf{S}\left(t^{\star}_{m-1}\right) e^{-\lambda_m \left(t - t^{\star}_{m-1}\right)}$$

for $t \in \left] t^{\star}_{m-1}, t^{\star}_{m} \right]$. We deduce that $\mathbf{S}\left(t^{\star}_{m}\right) = \mathbf{S}\left(t^{\star}_{m-1}\right) e^{-\lambda_m \left(t^{\star}_{m} - t^{\star}_{m-1}\right)}$, implying that:

$$\ln \mathbf{S}\left(t^{\star}_{m}\right) = \ln \mathbf{S}\left(t^{\star}_{m-1}\right) - \lambda_m \left(t^{\star}_{m} - t^{\star}_{m-1}\right)$$

and:

$$\lambda_m = \frac{\ln \mathbf{S}\left(t^{\star}_{m-1}\right) - \ln \mathbf{S}\left(t^{\star}_{m}\right)}{t^{\star}_{m} - t^{\star}_{m-1}}$$

Transition probability matrix

Survival function

Estimation of the piecewise exponential model

It is then straightforward to estimate the piecewise hazard function from a transition probability matrix:

- The knots of the piecewise function are the years $m \in \mathbb{N}^*$
- For each initial rating i, the hazard function $\lambda_i(t)$ is defined as:

$$\lambda_i(t) = \lambda_{i,m} \qquad \text{if } t \in \left]m-1, m\right]$$

where:

$$\begin{aligned} \lambda_{i,m} &= \frac{\ln \mathbf{S}_i(m-1) - \ln \mathbf{S}_i(m)}{m-(m-1)} \\ &= \ln \left(\frac{1 - \mathbf{e}_i^\top P^{m-1} \mathbf{e}_K}{1 - \mathbf{e}_i^\top P^m \mathbf{e}_K} \right) \end{aligned}$$

and $P^0 = I$

Transition probability matrix

Survival function

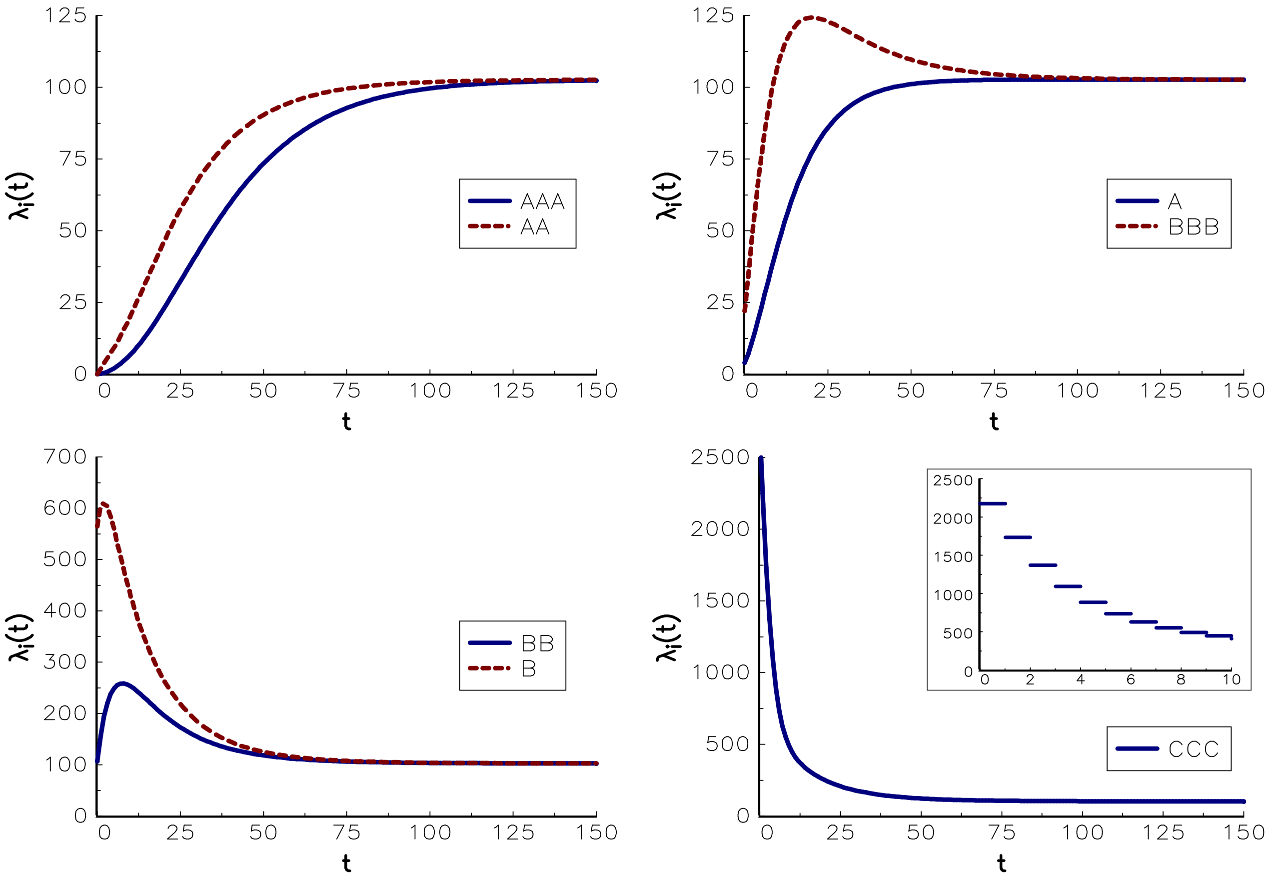


Figure: Estimated hazard function $\lambda_i(t)$ from the credit migration matrix

Transition probability matrix

Survival function

Why the hazard function of all the ratings converges to the same level, which is equal to 102.63 bps?

In the long run, the initial rating has no impact on the survival function:

Conditional probability distribution $\Rightarrow$ Unconditional probability distribution

We deduce that the annual default rate is exactly equal to 1.0263%

Transition probability matrix

Continuous-time modeling

Definition

The transition matrix $P(s;t)$ is defined as follows:

$$P_{i,j}(s;t) = p(s,i;t,j) = \Pr\{\mathfrak{R}(t) = j \mid \mathfrak{R}(s) = i\}$$

where $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$. Assuming that the Markov chain is time-homogenous, we have $P(t) = P(0;t)$

Markov generator

The Markov generator is defined by the matrix $\Lambda = (\lambda_{i,j})$ where $\lambda_{i,j} \geq 0$ for all $i \neq j$ and $\lambda_{i,i} = -\sum_{j\neq i}^{K} \lambda_{i,j}$. In this case, the transition matrix satisfies the following relationship:

$$P(t) = \exp(t\Lambda)$$

where $\exp(A)$ is the matrix exponential of A.

Transition probability matrix

Continuous-time modeling

Probabilistic interpretation of Λ

If we assume that the probability of jumping from rating i to rating j in a short time period Δt is proportional to Δt, we have:

$$p(t, i; t + \Delta t, j) = \lambda_{i,j} \Delta t$$

The matrix form of this equation is $P(t; t + \Delta t) = \Lambda \, \Delta t$. We deduce that:

$$P(t + \Delta t) = P(t) \, P(t; t + \Delta t) = P(t) \, \Lambda \, \Delta t$$

and:

$$\mathrm{d}P(t) = P(t) \, \Lambda \, \mathrm{d}t$$

Because we have $\exp(\mathbf{0}) = I$, we obtain the solution $P(t) = \exp(t\Lambda)$

$\lambda_{i,j}$ can be interpreted as the instantaneous transition rate of jumping from rating i to rating j

Transition probability matrix

Matrix exponential (HFRM, Appendix A.1.1.3, page 1034)

Let $f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. The matrix exponential of the matrix A is equal to:

$$B = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

whereas the matrix logarithm of A is the matrix B such that $e^B = A$ and we note $B = \ln A$

Let A and B be two $n \times n$ square matrices. Using the Taylor expansion, we can show that $f\left(A^\top\right) = f(A)^\top$, $Af(A) = f(A)A$ and $f\left(B^{-1}AB\right) = B^{-1}f(A)B$. It follows that $e^{A^\top} = \left(e^A\right)^\top$ and $e^{B^{-1}AB} = B^{-1}e^A B$. If $AB = BA$, we can also prove that $Ae^B = e^B A$ and $e^{A+B} = e^A e^B = e^B e^A$

Remark

Algorithms for computing matrix functions (e^A, $\ln A$, A^x, $\sqrt{A}$, $\cos A$, etc.) are available in programming languages (matlab, gauss, python, etc.)

Transition probability matrix

Continuous-time modeling

Example

We consider a rating system with three states: A (good rating), B (bad rating) and D (default). The Markov generator is equal to:

$$\Lambda = \begin{pmatrix} -0.30 & 0.20 & 0.10 \\ 0.15 & -0.40 & 0.25 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}$$

The one-year transition probability matrix is equal to:

$$P(1) = e^{\Lambda} = \begin{pmatrix} 75.16\% & 14.17\% & 10.67\% \\ 10.63\% & 68.07\% & 21.30\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

Transition probability matrix

Continuous-time modeling

For the two-year maturity, we get:

$$P(2) = e^{2\Lambda} = \begin{pmatrix} 58.00\% & 20.30\% & 21.71\% \\ 15.22\% & 47.85\% & 36.93\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

We verify that $P(2) = P(1)^2$. This derives from the property of the matrix exponential:

$$P(t) = e^{t\Lambda} = \left(e^{\Lambda}\right)^t = P(1)^t$$

Transition probability matrix

Continuous-time modeling

The one-month transition probability matrix is equal to:

$$P\left(\frac{1}{12}\right) = e^{\frac{1}{12}\Lambda} = \begin{pmatrix} 97.54\% & 1.62\% & 0.84\% \\ 1.21\% & 96.73\% & 2.05\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

Remark

Another way to compute the one-month transition probability matrix is to use the matrix exponent function:

$$P\left(\frac{1}{12}\right) = P(1)^{\frac{1}{12}}$$

Transition probability matrix

Continuous-time modeling

Let $\hat{P}(t)$ be the empirical transition matrix for a given t. We can estimate the Markov generator:

$$\hat{\Lambda} = \frac{1}{t} \ln \left(\hat{P}(t) \right)$$

Table: Markov generator $\hat{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.49	703.67	35.21	3.04	6.56	**-0.79**	**-0.22**	0.02
AA	67.94	−859.31	722.46	51.60	2.57	10.95	4.92	**-1.13**
A	7.69	245.59	−898.16	567.70	53.96	20.65	**-0.22**	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	**-0.84**	11.83	30.11	8.71	818.31	−1936.82	539.18	529.52
CCC	25.11	**-2.89**	44.11	84.87	272.05	1678.69	−5043.00	2941.06
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The matrix $\hat{\Lambda}$ does not verify the Markov conditions $\hat{\lambda}_{i,j} \geq 0$ for all $i \neq j$

Transition probability matrix

Continuous-time modeling

Israel *et al.* (2001) propose two estimators to obtain a valid generator:

1. The first approach consists in adding the negative values back into the diagonal values:

$$\left\{\begin{array}{ll} \bar{\lambda}_{i,j} = \max\left(\hat{\lambda}_{i,j}, 0\right) & i \neq j \\ \bar{\lambda}_{i,i} = \hat{\lambda}_{i,i} + \sum_{j \neq i} \min\left(\hat{\lambda}_{i,j}, 0\right) & \end{array}\right.$$

2. In the second method, we carry forward the negative values on the matrix entries which have the correct sign:

$$\left\{\begin{array}{l} G_i = \left|\hat{\lambda}_{i,i}\right| + \sum_{j \neq i} \max\left(\hat{\lambda}_{i,j}, 0\right) \\ B_i = \sum_{j \neq i} \max\left(-\hat{\lambda}_{i,j}, 0\right) \\ \tilde{\lambda}_{i,j} = \left\{\begin{array}{ll} 0 & \text{if } i \neq j \text{ and } \hat{\lambda}_{i,j} < 0 \\ \hat{\lambda}_{i,j} - B_i \left|\hat{\lambda}_{i,j}\right| / G_i & \text{if } G_i > 0 \\ \hat{\lambda}_{i,j} & \text{if } G_i = 0 \end{array}\right. \end{array}\right.$$

Transition probability matrix

Continuous-time modeling

Table: Markov generator $\tilde{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.99	703.19	35.19	3.04	6.55	0.00	0.00	0.02
AA	67.90	−859.88	721.98	51.57	2.57	10.94	4.92	0.00
A	7.69	245.56	−898.27	567.63	53.95	20.65	0.00	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	0.00	11.83	30.10	8.71	818.14	−1937.24	539.06	529.40
CCC	25.10	0.00	44.10	84.84	271.97	1678.21	−5044.45	2940.22
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table: 207-day transition probability matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.85	3.81	0.27	0.03	0.04	0.00	0.00	0.00
AA	0.37	95.28	3.90	0.34	0.03	0.06	0.02	0.00
A	0.04	1.33	95.12	3.03	0.33	0.12	0.00	0.02
BBB	0.03	0.14	3.47	92.75	2.88	0.53	0.09	0.11
BB	0.02	0.06	0.31	4.79	88.67	5.09	0.53	0.53
B	0.00	0.06	0.17	0.16	4.16	89.84	2.52	3.08
CCC	0.12	0.01	0.23	0.45	1.45	7.86	75.24	14.64
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Transition probability matrix

Continuous-time modeling

Remark

The continuous-time framework is more flexible when modeling credit risk. For instance, the expression of the survival function becomes:

$$\mathbf{S}_i(t) = \Pr\{\mathfrak{R}(t) = K \mid \mathfrak{R}(0) = i\} = 1 - \mathbf{e}_i^\top \exp(t\Lambda)\,\mathbf{e}_K$$

We can therefore calculate the probability density function in an easier way:

$$f_i(t) = -\partial_t\,\mathbf{S}_i(t) = \mathbf{e}_i^\top \Lambda \exp(t\Lambda)\,\mathbf{e}_K$$

Transition probability matrix

Continuous-time modeling

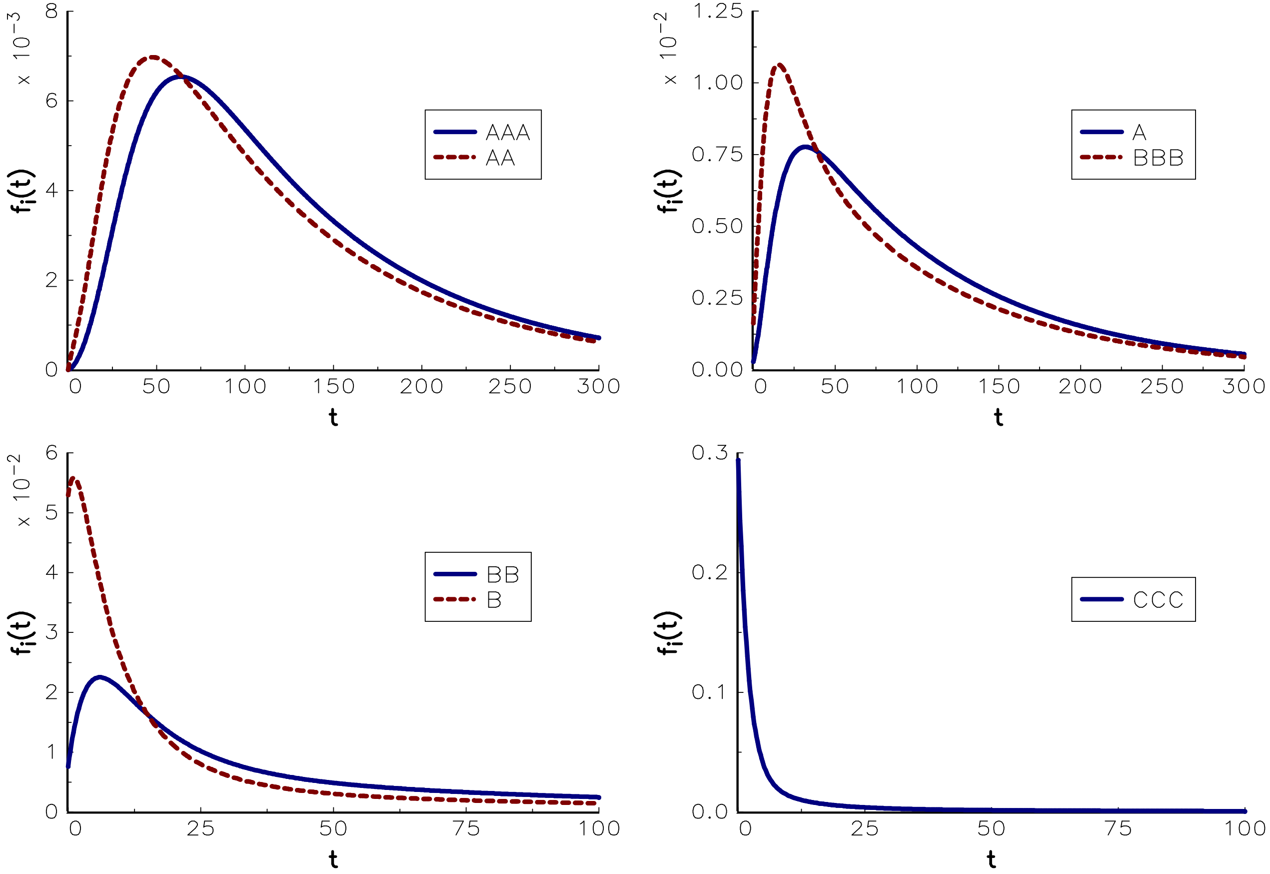


Figure: Probability density function $f_i(t)$ of S&P ratings

Structural models

Two main models:

- Merton (1974)
- Black and Cox (1976)

Two main implementations:

- KMV
- CreditGrades

Other topics

Pillar 1

- Exposure at default
- Expected loss given default
- Probability of default

Pillar 2

- Random loss given default
- Default correlation
- Granularity

Internal model

- Exposure at default
- Random loss given default
- Probability of default
- Default correlation
- Granularity

Default correlation

Two approaches:

- Copula models
- Factor models

$\Rightarrow$ Same concept

Default correlation
The copula model

Let $\mathbf{S}$ be the survival function of the random vector $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$, we can show that $\mathbf{S}$ admits a copula representation:

$$\mathbf{S}(t_1, \ldots, t_n) = \mathbf{C}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_n(t_n))$$

where $\mathbf{S}_i$ is the survival function of $\boldsymbol{\tau}_i$ and $\mathbf{C}$ is the survival copula associated to $\mathbf{S}$

Default correlation

The copula function of the Basel model

In the Basel mode, the (normalized) asset value of the i^{th} firm is $Z_i \sim \mathcal{N}(0,1)$ and the default occurs when Z_i is below a non-stochastic barrier B_i:

$$D_i = 1 \Leftrightarrow Z_i \leq B_i = \Phi^{-1}(p_i)$$

We recall that $Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$ where $X \sim \mathcal{N}(0,1)$ is the systematic risk factor and $\varepsilon_i \sim \mathcal{N}(0,1)$ is the specific risk factor, and the conditional default probability is equal to:

$$p_i(X) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right)$$

If we introduce the time dimension, we obtain:

$$p_i(t) = \Pr\{\boldsymbol{\tau}_i \leq t\} = 1 - S_i(t)$$

and:

$$p_i(t, X) = \Phi\left(\frac{\Phi^{-1}(1 - \mathbf{S}_i(t)) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right)$$

where $S_i(t)$ is the survival function of the i^{th} firm

Default correlation

The copula function of the Basel model

$Z = (Z_1, \ldots, Z_n) \sim \mathcal{N}(\mathbf{0}_n, \mathbb{C}_n(\rho))$ with:

$$\mathbb{C}_n(\rho) = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

It follows that the **joint default probability** is:

$$\begin{aligned} p_{1,\ldots,n} &= \Pr\{D_1 = 1, \ldots, D_n = 1\} = \Pr\{Z_1 \leq B_1, \ldots, Z_n \leq B_n\} \\ &= \Phi(B_1, \ldots, B_n; \mathbb{C}_n(\rho)) \end{aligned}$$

Since we have $B_i = \Phi^{-1}(p_i)$, we deduce that:

$$p_{1,\ldots,n} = \Phi\left(\Phi^{-1}(p_1), \ldots, \Phi^{-1}(p_n); \mathbb{C}_n(\rho)\right)$$

The Basel copula between default probabilities is the Normal copula with a constant correlation matrix

Default correlation

The copula function of the Basel model

If we consider the dependence between the survival times, we have:

$$\begin{aligned}
\mathbf{S}\left(t_1,\ldots,t_n\right) &= \Pr\left\{\boldsymbol{\tau}_1 > t_1,\ldots,\boldsymbol{\tau}_n > t_n\right\} \\
&= \Pr\left\{Z_1 > \Phi^{-1}\left(p_1\left(t_1\right)\right),\ldots,Z_n > \Phi^{-1}\left(p_n\left(t_n\right)\right)\right\} \\
&= \Pr\left\{\Phi\left(Z_1\right) > p_1\left(t_1\right),\ldots,\Phi\left(Z_n\right) > p_n\left(t_n\right)\right\} \\
&= \Pr\left\{\Phi\left(Z_1\right) \leq 1 - p_1\left(t_1\right),\ldots,\Phi\left(Z_n\right) \leq 1 - p_n\left(t_n\right)\right\} \\
&= \mathbf{C}\left(1 - p_1\left(t_1\right),\ldots,1 - p_n\left(t_n\right);\mathbb{C}_n\left(\rho\right)\right) \\
&= \mathbf{C}\left(\mathbf{S}_1\left(t_1\right),\ldots,\mathbf{S}_n\left(t_n\right);\mathbb{C}_n\left(\rho\right)\right)
\end{aligned}$$

The Basel copula between default times is the Normal copula with a constant correlation matrix

Default correlation

Extension to other copula functions

From an industrial point of view, only two copula functions are used and tractable:

1. The Normal copula
2. The Student t copula

with a general correlation matrix:

$$\mathbb{C} = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ & 1 & & \vdots \\ & & \ddots & \rho_{n-1,n} \\ & & & 1 \end{pmatrix}$$

$\Rightarrow$ In practice, we use a structural correlation matrix (HFRM, pages 221-225)

Default correlation

The factor model

One-factor model

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho} \varepsilon_i$$

$(m+1)$-factor model

$$Z_i = \sqrt{\rho} \cdot X + \sqrt{\rho_{\mathrm{map}(i)} - \rho} \cdot X_{\mathrm{map}(i)} + \sqrt{1 - \rho_{\mathrm{map}(i)}} \cdot \varepsilon_i$$

Default correlation

Jump-to-default

How default correlations affects default times

Let τ_1 and τ_2 be two default times, whose joint survival function is $\mathbf{S}(t_1, t_2) = \mathbf{C}(\mathbf{S}_1(t_1), \mathbf{S}_2(t_2))$. We have:

$$\begin{array}{rcl} \mathbf{S}_1(t \mid \tau_2 = t^\star) & = & \Pr\{\tau_1 > t \mid \tau_2 = t^\star\} \\ & = & \partial_2 \mathbf{C}(\mathbf{S}_1(t), \mathbf{S}_2(t^\star)) \\ & = & \mathbf{C}_{2|1}(\mathbf{S}_1(t), \mathbf{S}_2(t^\star)) \\ & \neq & \mathbf{S}_1(t) \qquad \text{except if } \mathbf{C} = \mathbf{C}^\perp \end{array}$$

where $\mathbf{C}_{2|1}$ is the conditional copula function

$\Rightarrow$ This phenomenon is called jump-to-defaut (JTD) or spread jump

Default correlation

Jump-to-default of credit ratings

The hazard function is equal to:

$$\lambda_i(t) = \frac{f_i(t)}{\mathbf{S}_i(t)} = \frac{\mathbf{e}_i^\top \Lambda \exp(t\Lambda)\,\mathbf{e}_K}{1 - \mathbf{e}_i^\top \exp(t\Lambda)\,\mathbf{e}_K}$$

We deduce that:

$$\lambda_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^\star) = \frac{f_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^\star)}{\mathbf{S}_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^\star)}$$

With the Basel copula, we have:

$$\mathbf{S}_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^\star) = \Phi\left(\frac{\Phi^{-1}(\mathbf{S}_{i_1}(t)) - \rho\Phi^{-1}(\mathbf{S}_{i_2}(t^\star))}{\sqrt{1-\rho^2}}\right)$$

and:

$$f_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^\star) = \phi\left(\frac{\Phi^{-1}(\mathbf{S}_{i_1}(t)) - \rho\Phi^{-1}(\mathbf{S}_{i_2}(t^\star))}{\sqrt{1-\rho^2}}\right)\frac{f_{i_1}(t)}{\sqrt{1-\rho^2}\,\phi(\Phi^{-1}(\mathbf{S}_{i_1}(t)))}$$

Default correlation
Jump-to-default of credit ratings

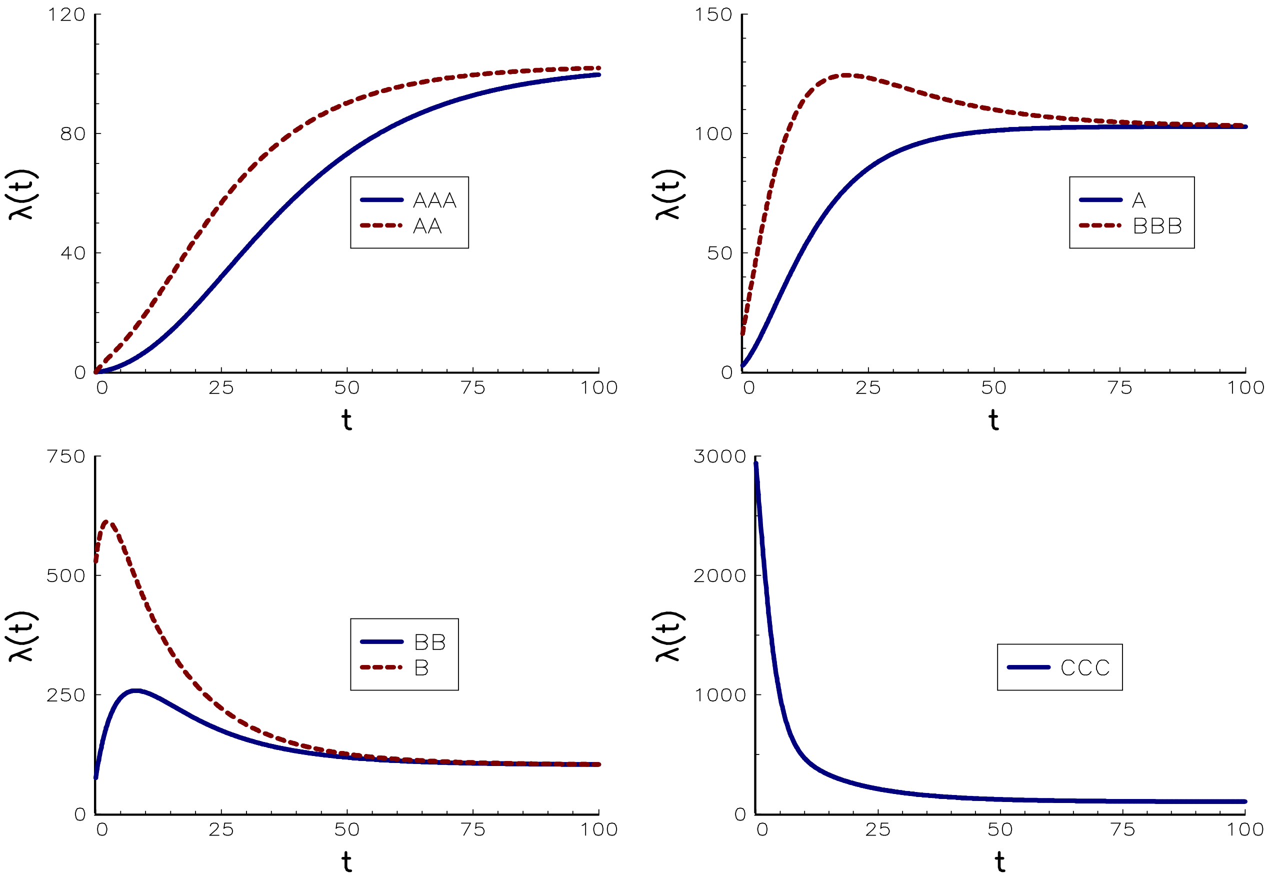


Figure: Hazard function $\lambda_i(t)$ (in bps)

Default correlation

Jump-to-default of credit ratings

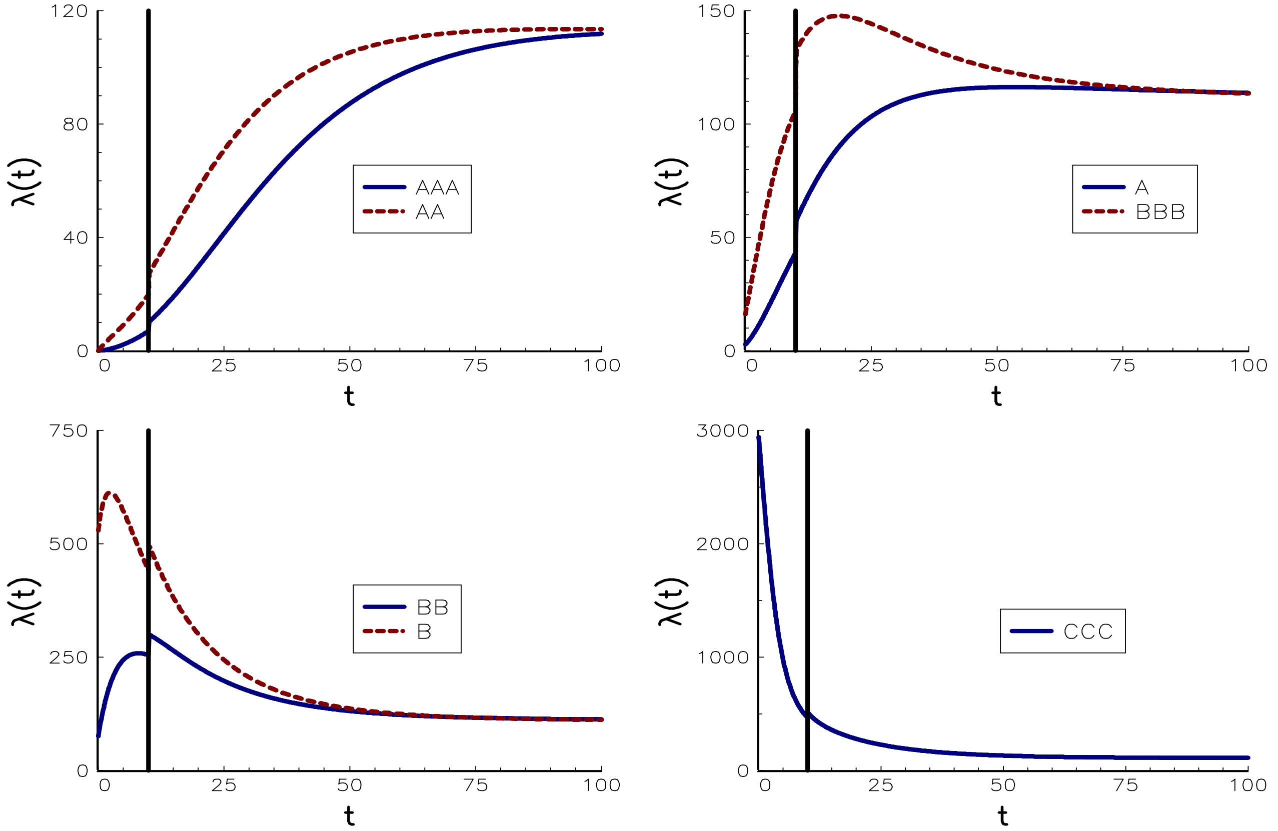


Figure: Hazard function $\lambda_i(t)$ (in bps) when a AAA-rated company defaults after 10 years ($\rho = 5\%$)

Default correlation

Jump-to-default of credit ratings

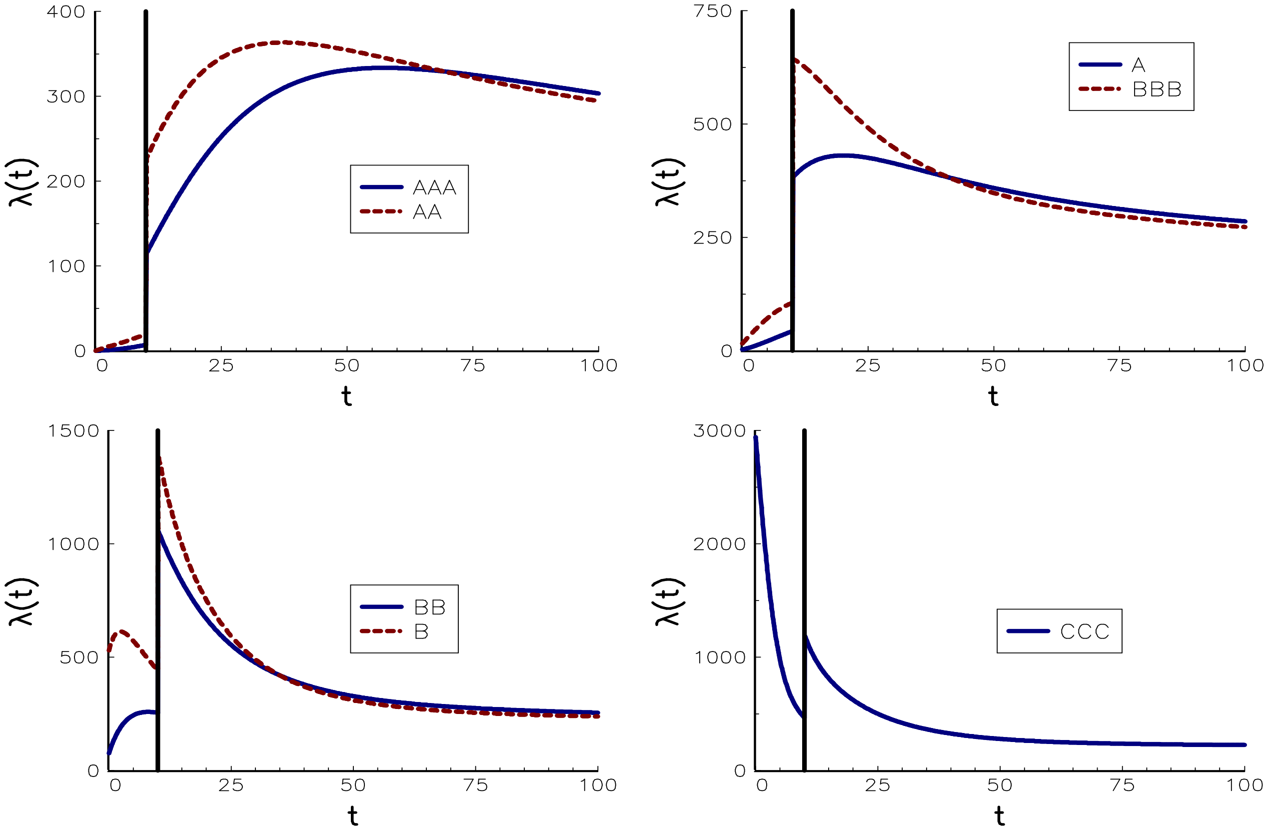


Figure: Hazard function $\lambda_i(t)$ (in bps) when a AAA-rated company defaults after 10 years ($\rho = 50\%$)

Default correlation

Jump-to-default of credit ratings

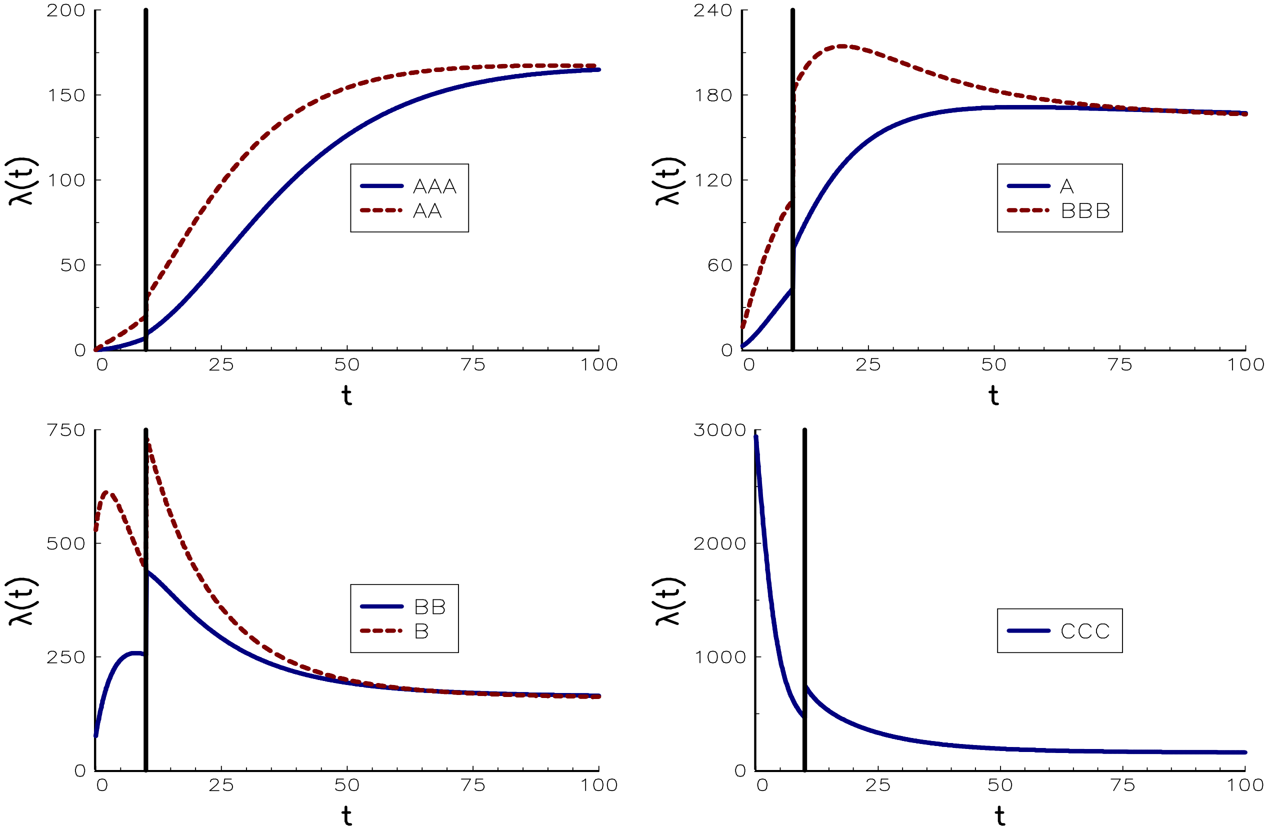


Figure: Hazard function $\lambda_i(t)$ (in bps) when a BB-rated company defaults after 10 years ($\rho = 50\%$)

Default correlation

Jump-to-default of credit ratings

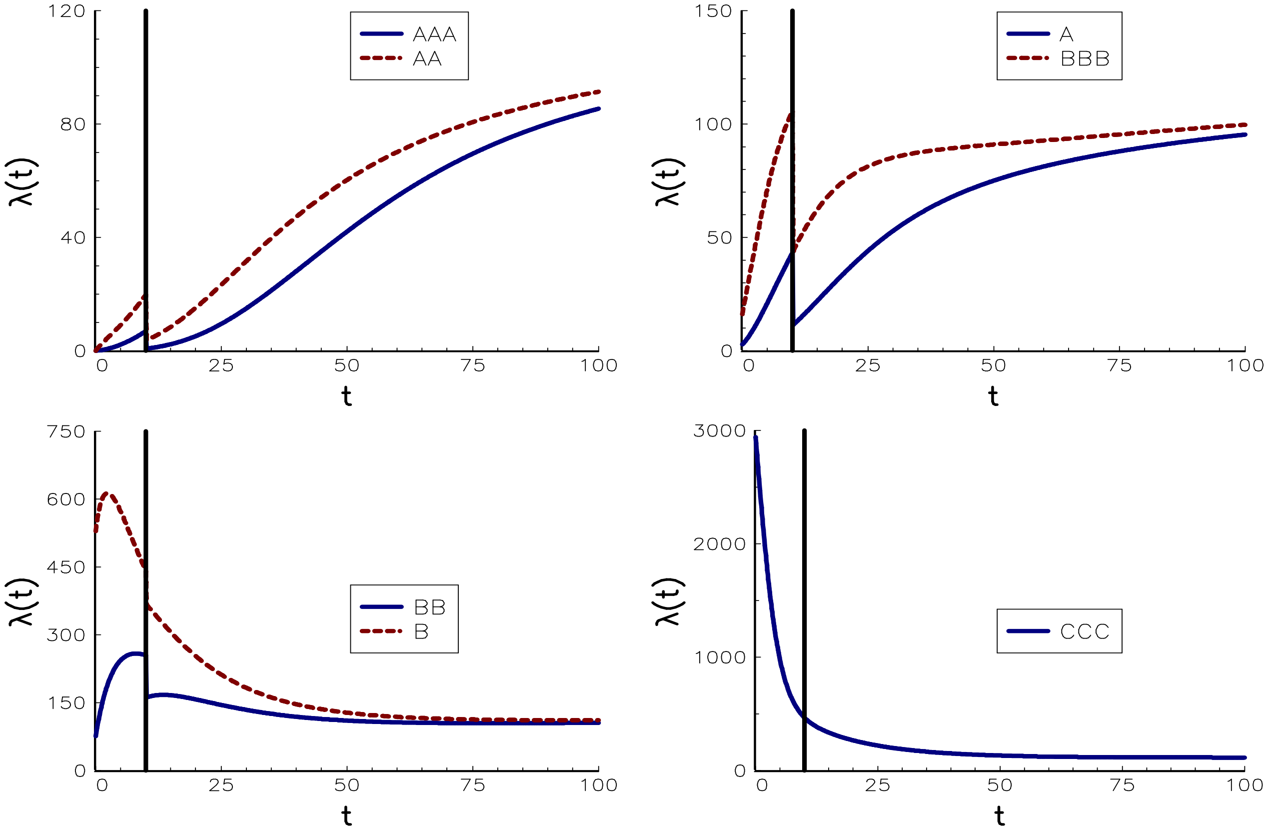


Figure: Hazard function $\lambda_i(t)$ (in bps) when a CCC-rated company defaults after 10 years ($\rho = 50\%$)

Granularity and concentration

Definition of the granularity adjustment

We recall that the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

For an infinitely fine-grained (IFG) portfolio, we have:

$$\mathrm{VaR}_\alpha\left(w_{\mathrm{IFG}}\right) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \Phi\left(\frac{\Phi^{-1}\left(\mathrm{PD}_i\right) + \sqrt{\rho}\Phi^{-1}\left(\mathrm{PD}_i\right)}{\sqrt{1-\rho}}\right)$$

However, the portfolio w cannot be fine-grained and present some concentration issues, implying that the value-at-risk is equal to the quantile α of the loss distribution:

$$\mathrm{VaR}_\alpha\left(w\right) = \mathbf{F}_L^{-1}\left(\alpha\right)$$

The granularity adjustment GA is the difference between the two risk measures:

$$\mathrm{GA} = \mathrm{VaR}_\alpha\left(w\right) - \mathrm{VaR}_\alpha\left(w_{\mathrm{IFG}}\right)$$

Granularity and concentration

The case of a perfectly concentrated portfolio

Let us consider a portfolio that is made up of one credit:

$$L = \mathrm{EAD} \cdot \mathrm{LGD} \cdot \mathbb{1} \{ \boldsymbol{\tau} \leq T \}$$

It follows that:

$$\mathbf{F}_L(\ell) = \Pr \{ \mathrm{EAD} \cdot \mathrm{LGD} \cdot \mathbb{1} \{ \boldsymbol{\tau} \leq T \} \leq \ell \}$$

Since we have $\ell = 0 \Leftrightarrow \boldsymbol{\tau} > T$, we deduce that $\mathbf{F}_L(0) = \Pr \{ \boldsymbol{\tau} > T \} = 1 - \mathrm{PD}$. If $\ell \neq 0$, we have:

$$\begin{aligned} \mathbf{F}_L(\ell) &= \mathbf{F}_L(0) + \Pr \{ \mathrm{EAD} \cdot \mathrm{LGD} \leq \ell \mid \boldsymbol{\tau} \leq T \} \\ &= (1 - \mathrm{PD}) + \mathrm{PD} \cdot \mathbf{G}\left(\frac{\ell}{\mathrm{EAD}} \right) \end{aligned}$$

where $\mathbf{G}$ is the distribution function of the loss given default. The value-at-risk of this portfolio is then equal to:

$$\mathrm{VaR}_\alpha(w) = \begin{cases} \mathrm{EAD} \cdot \mathbf{G}^{-1}\left(\dfrac{\alpha + \mathrm{PD} - 1}{\mathrm{PD}} \right) & \text{if } \alpha \geq 1 - \mathrm{PD} \\ 0 & \text{otherwise} \end{cases}$$

Granularity and concentration

The case of a perfectly concentrated portfolio

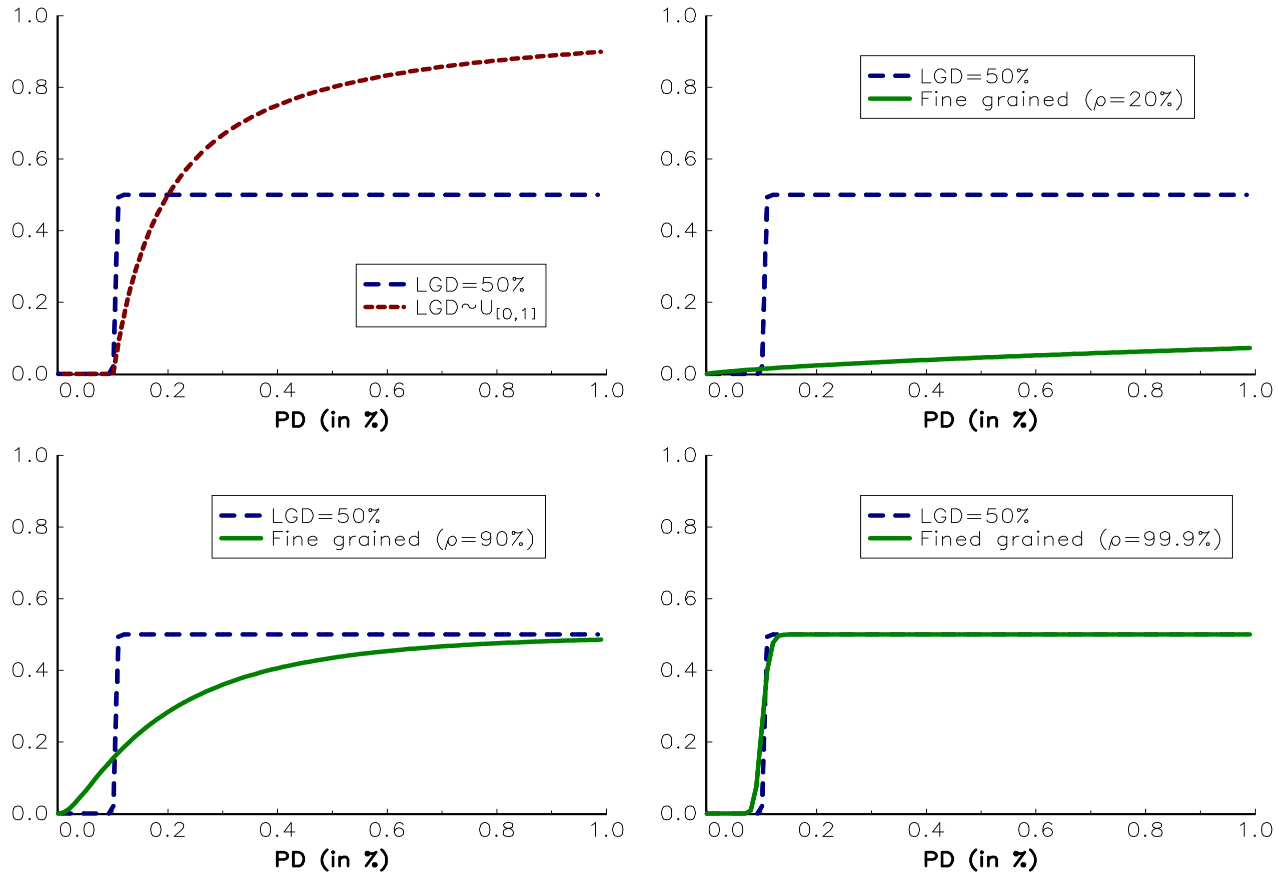


Figure: Comparison between the 99.9% value-at-risk of a loan and its risk contribution in an IFG portfolio

Granularity and concentration

IFG versus non-IFG portfolios

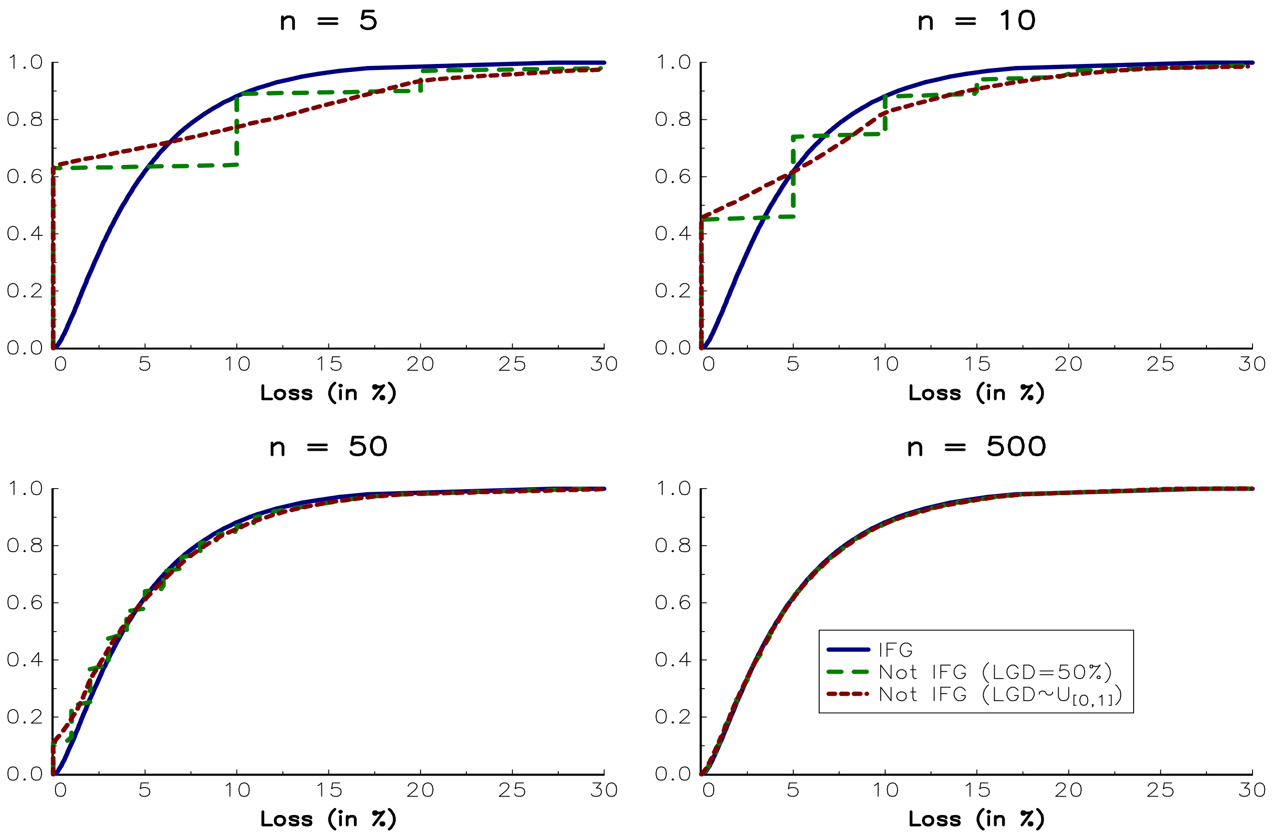


Figure: Comparison of the loss distribution of non-IFG and IFG portfolios

Exercises

- Credit derivatives
  - Exercise 3.4.1 – Single- and multi-name credit default swaps
- Basel II model
  - Exercise 3.4.8 – Variance of the conditional portfolio loss
  - Exercise 3.4.2 – Risk contribution in the Basel II model
  - Exercise 3.4.7 – Derivation of the original Basel granularity adjustement
- Parameter modeling
  - Exercise 3.4.3 – Calibration of the piecewise exponential model
  - Exercise 3.4.4 – Modeling loss given default
  - Exercise 3.4.5 – Modeling default times with a Markov chain
  - Exercise 3.4.6 – Continuous-time modeling of default risk

References

Basel Committee on Banking Supervision (2001)
The New Basel Capital Accord, Second consultative paper on Basel II, January 2001.

Basel Committee on Banking Supervision (2004)
International Convergence of Capital Measurement and Capital Standards — A Revised Framework, June 2004.

Basel Committee on Banking Supervision (2015)
Revisions to the Standardized Approach for Credit Risk, Consultative Document, December 2015.

Basel Committee on Banking Supervision (2017)
Basel III: Finalising Post-crisis Reforms, December 2017.

Roncalli, T. (2020)
Handbook of Financial Risk Management, Chapman and Hall/CRC Financial Mathematics Series, Chapter 3.