Financial Risk Management
Lecture 7. Asset Liability Management Risk

Thierry Roncalli*

*University of Paris-Saclay

November 2020
General information

1. Overview
   The objective of this course is to understand the theoretical and practical aspects of risk management.

2. Prerequisites
   M1 Finance or equivalent

3. ECTS
   4

4. Keywords
   Finance, Risk Management, Applied Mathematics, Statistics

5. Hours
   Lectures: 36h, Training sessions: 15h, HomeWork: 30h

6. Evaluation
   There will be a final three-hour exam, which is made up of questions and exercises

7. Course website
The objective of the course is twofold:

1. knowing and understanding the financial regulation (banking and others) and the international standards (especially the Basel Accords)

2. being proficient in risk measurement, including the mathematical tools and risk models
Class schedule

Course sessions
- September 11 (6 hours, AM+PM)
- September 18 (6 hours, AM+PM)
- September 25 (6 hours, AM+PM)
- October 2 (6 hours, AM+PM)
- November 20 (6 hours, AM+PM)
- November 27 (6 hours, AM+PM)

Tutorial sessions
- October 10 (3 hours, AM)
- October 16 (3 hours, AM)
- November 13 (3 hours, AM)
- December 4 (6 hours, AM+PM)

Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry
General principles of the banking book risk management
Interest rate risk
Behavioral options

Agenda

- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models
Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:
  

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:
  
Agenda

- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models
• ALM risk ⇒ banking book
• Not only a risk management issue, but also concerns commercial choices and business models
• Several ALM risks: liquidity risk, interest rate risk, embedded option risk, currency risk

The ALM function is located in the finance department, not in the risk management department
### Table: A simplified balance sheet

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>Due to central banks</td>
</tr>
<tr>
<td>Loans and leases</td>
<td>Deposits</td>
</tr>
<tr>
<td>Mortgages</td>
<td>Deposit accounts</td>
</tr>
<tr>
<td>Consumer credit</td>
<td>Savings</td>
</tr>
<tr>
<td>Credit cards</td>
<td>Term deposits</td>
</tr>
<tr>
<td>Interbank loans</td>
<td>Interbank funding</td>
</tr>
<tr>
<td>Investment securities</td>
<td>Short-term debt</td>
</tr>
<tr>
<td>Sovereign bonds</td>
<td>Subordinated debt</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>Reserves</td>
</tr>
<tr>
<td>Other assets</td>
<td>Equity capital</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table: A simplified income statement

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest income</td>
<td></td>
</tr>
<tr>
<td>− Interest expenses</td>
<td></td>
</tr>
<tr>
<td>= Net interest income</td>
<td></td>
</tr>
<tr>
<td>+ Non-interest income</td>
<td></td>
</tr>
<tr>
<td>= Gross income</td>
<td></td>
</tr>
<tr>
<td>− Operating expenses</td>
<td></td>
</tr>
<tr>
<td>= Net income</td>
<td></td>
</tr>
<tr>
<td>− Provisions</td>
<td></td>
</tr>
<tr>
<td>= Earnings before tax</td>
<td></td>
</tr>
<tr>
<td>− Income tax</td>
<td></td>
</tr>
<tr>
<td>= Profit after tax</td>
<td></td>
</tr>
</tbody>
</table>
Accounting standards

Four main systems:

1. US GAAP
2. Japanese combined system
3. Chinese accounting standards

⇒ IFRS is implemented in European Union, Australia, Middle East, Russia, South Africa, etc.
Accounting standards

Before 2018

- financial assets at fair value through profit and loss (FVTPL)
- available-for-sale financial assets (AFS)
- loans and receivables (L&R);
- held-to-maturity investments (HTM)

After 2018

- amortized cost (AC)
- fair value (FV)

IAS 39

IFRS 9

Thierry Roncalli

Financial Risk Management (Lecture 7)
ALM committee (ALCO)

Figure: Internal and external funding transfer
Liquidity gap

- \( A(t) \) is the value of assets at time \( t \)
- \( L(t) \) is the value of liabilities at time \( t \)

- Funding ratio
  \[
  \mathcal{FR}(t) = \frac{A(t)}{L(t)}
  \]

- Funding gap
  \[
  \mathcal{FG}(t) = A(t) - L(t)
  \]

- Liquidity ratio
  \[
  \mathcal{LR}(t) = \frac{L(t)}{A(t)}
  \]

- Liquidity gap
  \[
  \mathcal{LG}(t) = L(t) - A(t)
  \]
Example

The assets $A(t)$ are composed of loans that are linearly amortized in a monthly basis during the next year. The amount is equal to 120. The liabilities $L(t)$ are composed of three short-term in fine debt instruments, and the capital. The corresponding debt notional is respectively equal to 65, 10 and 5 whereas the associated remaining maturity is equal to two, seven and twelve months. The amount of capital is stable for the next twelve months and is equal to 40.
### Liquidity gap

**Table: Computation of the liquidity gap**

<table>
<thead>
<tr>
<th>Period</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loans</td>
<td>120</td>
<td>110</td>
<td>100</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Assets</td>
<td>120</td>
<td>110</td>
<td>100</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Debt #1</td>
<td>65</td>
<td>65</td>
<td>65</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Debt #2</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Debt #3</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Debt (total)</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Equity</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Liabilities</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>55</td>
<td>55</td>
<td>55</td>
<td>55</td>
<td>55</td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>45</td>
</tr>
</tbody>
</table>

\[ LG(t) \]
Liquidity gap

**Figure:** An example of liquidity gap
Asset and liability amortization

General principles of debt amortization

- The annuity amount $A(t)$ at time $t$ is composed of the interest payment $I(t)$ and the principal payment $P(t)$:

\[ A(t) = I(t) + P(t) \]

- The interest payment at time $t$ is equal to the interest rate $i(t)$ times the outstanding principal balance $N(t - 1)$:

\[ I(t) = i(t) N(t - 1) \]

- The outstanding principal balance $N(t)$ equal to:

\[ N(t) = N(t - 1) - P(t) \]

- The outstanding principal balance $N(t)$ is equal to the present value $C(t)$ of forward annuity amounts: $N(t) = C(t)$
Asset and liability amortization

- Constant amortization debt (or linear amortization of the capital): $P(t)$ is constant over time (HFRM, page 379):

$$P(t) = \frac{1}{n} N_0$$

$$A(t) = I(t) + P(t) = \left(\frac{1}{n} + i \left(1 - \frac{t-1}{n}\right)\right) N_0$$

- Constant payment debt: the annuity amount $A(t)$ is constant

$$A(t) = A = \frac{i}{1 - (1+i)^{-n}} N_0$$

$$I(t) = \left(1 - \frac{1}{(1+i)^{n-t+1}}\right) A$$

- Bullet repayment debt: the notional is fully repaid at the time of maturity

$$I(t) = i N_0 \text{ and } P(t) = 1 \{t = n\} \cdot N_0$$
Asset and liability amortization

Example

We consider a 10-year mortgage, whose notional is equal to $100. The annual interest rate $i$ is equal to 5%, and we assume annual principal payments.
**Table**: Repayment schedule of the constant amortization mortgage

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C(t - 1)$</th>
<th>$A(t)$</th>
<th>$I(t)$</th>
<th>$P(t)$</th>
<th>$Q(t)$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.00</td>
<td>15.00</td>
<td>5.00</td>
<td>10.00</td>
<td>10.00</td>
<td>90.00</td>
</tr>
<tr>
<td>2</td>
<td>90.00</td>
<td>14.50</td>
<td>4.50</td>
<td>10.00</td>
<td>20.00</td>
<td>80.00</td>
</tr>
<tr>
<td>3</td>
<td>80.00</td>
<td>14.00</td>
<td>4.00</td>
<td>10.00</td>
<td>30.00</td>
<td>70.00</td>
</tr>
<tr>
<td>4</td>
<td>70.00</td>
<td>13.50</td>
<td>3.50</td>
<td>10.00</td>
<td>40.00</td>
<td>60.00</td>
</tr>
<tr>
<td>5</td>
<td>60.00</td>
<td>13.00</td>
<td>3.00</td>
<td>10.00</td>
<td>50.00</td>
<td>50.00</td>
</tr>
<tr>
<td>6</td>
<td>50.00</td>
<td>12.50</td>
<td>2.50</td>
<td>10.00</td>
<td>60.00</td>
<td>40.00</td>
</tr>
<tr>
<td>7</td>
<td>40.00</td>
<td>12.00</td>
<td>2.00</td>
<td>10.00</td>
<td>70.00</td>
<td>30.00</td>
</tr>
<tr>
<td>8</td>
<td>30.00</td>
<td>11.50</td>
<td>1.50</td>
<td>10.00</td>
<td>80.00</td>
<td>20.00</td>
</tr>
<tr>
<td>9</td>
<td>20.00</td>
<td>11.00</td>
<td>1.00</td>
<td>10.00</td>
<td>90.00</td>
<td>10.00</td>
</tr>
<tr>
<td>10</td>
<td>10.00</td>
<td>10.50</td>
<td>0.50</td>
<td>10.00</td>
<td>100.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
### Table: Repayment schedule of the constant payment mortgage

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C(t-1)$</th>
<th>$A(t)$</th>
<th>$I(t)$</th>
<th>$P(t)$</th>
<th>$Q(t)$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.00</td>
<td>12.95</td>
<td>5.00</td>
<td>7.95</td>
<td>7.95</td>
<td>92.05</td>
</tr>
<tr>
<td>2</td>
<td>92.05</td>
<td>12.95</td>
<td>4.60</td>
<td>8.35</td>
<td>16.30</td>
<td>83.70</td>
</tr>
<tr>
<td>3</td>
<td>83.70</td>
<td>12.95</td>
<td>4.19</td>
<td>8.77</td>
<td>25.06</td>
<td>74.94</td>
</tr>
<tr>
<td>4</td>
<td>74.94</td>
<td>12.95</td>
<td>3.75</td>
<td>9.20</td>
<td>34.27</td>
<td>65.73</td>
</tr>
<tr>
<td>5</td>
<td>65.73</td>
<td>12.95</td>
<td>3.29</td>
<td>9.66</td>
<td>43.93</td>
<td>56.07</td>
</tr>
<tr>
<td>6</td>
<td>56.07</td>
<td>12.95</td>
<td>2.80</td>
<td>10.15</td>
<td>54.08</td>
<td>45.92</td>
</tr>
<tr>
<td>7</td>
<td>45.92</td>
<td>12.95</td>
<td>2.30</td>
<td>10.65</td>
<td>64.73</td>
<td>35.27</td>
</tr>
<tr>
<td>8</td>
<td>35.27</td>
<td>12.95</td>
<td>1.76</td>
<td>11.19</td>
<td>75.92</td>
<td>24.08</td>
</tr>
<tr>
<td>9</td>
<td>24.08</td>
<td>12.95</td>
<td>1.20</td>
<td>11.75</td>
<td>87.67</td>
<td>12.33</td>
</tr>
<tr>
<td>10</td>
<td>12.33</td>
<td>12.95</td>
<td>0.62</td>
<td>12.33</td>
<td>100.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
### Table: Repayment schedule of the bullet repayment mortgage

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C(t-1)$</th>
<th>$A(t)$</th>
<th>$I(t)$</th>
<th>$P(t)$</th>
<th>$Q(t)$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>3</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>4</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>5</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>6</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>7</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>8</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>9</td>
<td>100.00</td>
<td>5.00</td>
<td>5.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>10</td>
<td>100.00</td>
<td>105.00</td>
<td>5.00</td>
<td>100.00</td>
<td>100.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Asset and liability amortization

**Figure:** Amortization schedule of the 30-year mortgage (monthly payments)
### Example

We consider the following balance sheet:

<table>
<thead>
<tr>
<th>Items</th>
<th>Notional</th>
<th>Rate</th>
<th>Mat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan #1</td>
<td>100</td>
<td>5%</td>
<td>10</td>
</tr>
<tr>
<td>Loan #2</td>
<td>50</td>
<td>8%</td>
<td>16</td>
</tr>
<tr>
<td>Loan #3</td>
<td>40</td>
<td>3%</td>
<td>8</td>
</tr>
<tr>
<td>Loan #4</td>
<td>110</td>
<td>2%</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Items</th>
<th>Notional</th>
<th>Rate</th>
<th>Mat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt #1</td>
<td>120</td>
<td>5%</td>
<td>10</td>
</tr>
<tr>
<td>Debt #2</td>
<td>80</td>
<td>3%</td>
<td>5</td>
</tr>
<tr>
<td>Debt #3</td>
<td>70</td>
<td>4%</td>
<td>10</td>
</tr>
<tr>
<td>Capital #4</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All the debt instruments are subject to monthly principal payments.

Mixed schedule = constant principal (loan #3 and debt #2), constant annuity (loan #1, loan #2 and debt #1) and bullet repayment (loan #4 and debt #2)
Asset and liability amortization

**Figure:** Impact of the amortization schedule on the liquidity gap
### Asset and liability amortization

**Table:** Computation of the liquidity gap (mixed schedule, **first twelve months**)

<table>
<thead>
<tr>
<th>$t$</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>$A_t$</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>$L_t$</th>
<th>$LG_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99.4</td>
<td>49.9</td>
<td>39.6</td>
<td>110</td>
<td>298.8</td>
<td>119.2</td>
<td>78.7</td>
<td>70</td>
<td>30</td>
<td>297.9</td>
<td>−0.92</td>
</tr>
<tr>
<td>2</td>
<td>98.7</td>
<td>49.7</td>
<td>39.2</td>
<td>110</td>
<td>297.6</td>
<td>118.5</td>
<td>77.3</td>
<td>70</td>
<td>30</td>
<td>295.8</td>
<td>−1.83</td>
</tr>
<tr>
<td>3</td>
<td>98.1</td>
<td>49.6</td>
<td>38.8</td>
<td>110</td>
<td>296.4</td>
<td>117.7</td>
<td>76.0</td>
<td>70</td>
<td>30</td>
<td>293.7</td>
<td>−2.75</td>
</tr>
<tr>
<td>4</td>
<td>97.4</td>
<td>49.5</td>
<td>38.3</td>
<td>110</td>
<td>295.2</td>
<td>116.9</td>
<td>74.7</td>
<td>70</td>
<td>30</td>
<td>291.6</td>
<td>−3.66</td>
</tr>
<tr>
<td>5</td>
<td>96.8</td>
<td>49.3</td>
<td>37.9</td>
<td>110</td>
<td>294.0</td>
<td>116.1</td>
<td>73.3</td>
<td>70</td>
<td>30</td>
<td>289.4</td>
<td>−4.58</td>
</tr>
<tr>
<td>6</td>
<td>96.1</td>
<td>49.2</td>
<td>37.5</td>
<td>110</td>
<td>292.8</td>
<td>115.3</td>
<td>72.0</td>
<td>70</td>
<td>30</td>
<td>287.3</td>
<td>−5.49</td>
</tr>
<tr>
<td>7</td>
<td>95.4</td>
<td>49.1</td>
<td>37.1</td>
<td>110</td>
<td>291.6</td>
<td>114.5</td>
<td>70.7</td>
<td>70</td>
<td>30</td>
<td>285.2</td>
<td>−6.41</td>
</tr>
<tr>
<td>8</td>
<td>94.8</td>
<td>48.9</td>
<td>36.7</td>
<td>110</td>
<td>290.4</td>
<td>113.7</td>
<td>69.3</td>
<td>70</td>
<td>30</td>
<td>283.1</td>
<td>−7.32</td>
</tr>
<tr>
<td>9</td>
<td>94.1</td>
<td>48.8</td>
<td>36.3</td>
<td>110</td>
<td>289.2</td>
<td>112.9</td>
<td>68.0</td>
<td>70</td>
<td>30</td>
<td>280.9</td>
<td>−8.24</td>
</tr>
<tr>
<td>10</td>
<td>93.4</td>
<td>48.7</td>
<td>35.8</td>
<td>110</td>
<td>287.9</td>
<td>112.1</td>
<td>66.7</td>
<td>70</td>
<td>30</td>
<td>278.8</td>
<td>−9.15</td>
</tr>
<tr>
<td>11</td>
<td>92.8</td>
<td>48.5</td>
<td>35.4</td>
<td>110</td>
<td>286.7</td>
<td>111.3</td>
<td>65.3</td>
<td>70</td>
<td>30</td>
<td>276.7</td>
<td>−10.06</td>
</tr>
<tr>
<td>12</td>
<td>92.1</td>
<td>48.4</td>
<td>35.0</td>
<td>110</td>
<td>285.5</td>
<td>110.5</td>
<td>64.0</td>
<td>70</td>
<td>30</td>
<td>274.5</td>
<td>−10.97</td>
</tr>
</tbody>
</table>
**Table:** Computation of the liquidity gap (mixed schedule, annual schedule)

<table>
<thead>
<tr>
<th></th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>$A_t$</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>$L_t$</th>
<th>$LG_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100.0</td>
<td>50.0</td>
<td>40.0</td>
<td>110</td>
<td>300.0</td>
<td>120.0</td>
<td>80.0</td>
<td>70</td>
<td>30</td>
<td>300.0</td>
<td>0.00</td>
</tr>
<tr>
<td>1</td>
<td>92.1</td>
<td>48.4</td>
<td>35.0</td>
<td>110</td>
<td>285.5</td>
<td>110.5</td>
<td>64.0</td>
<td>70</td>
<td>30</td>
<td>274.5</td>
<td>−10.97</td>
</tr>
<tr>
<td>2</td>
<td>83.8</td>
<td>46.7</td>
<td>30.0</td>
<td>110</td>
<td>270.4</td>
<td>100.5</td>
<td>48.0</td>
<td>70</td>
<td>30</td>
<td>248.5</td>
<td>−21.90</td>
</tr>
<tr>
<td>3</td>
<td>75.0</td>
<td>44.8</td>
<td>25.0</td>
<td>110</td>
<td>254.8</td>
<td>90.1</td>
<td>32.0</td>
<td>70</td>
<td>30</td>
<td>222.1</td>
<td>−32.76</td>
</tr>
<tr>
<td>4</td>
<td>65.9</td>
<td>42.7</td>
<td>20.0</td>
<td>110</td>
<td>238.6</td>
<td>79.0</td>
<td>16.0</td>
<td>70</td>
<td>30</td>
<td>195.0</td>
<td>−43.55</td>
</tr>
<tr>
<td>5</td>
<td>56.2</td>
<td>40.5</td>
<td>15.0</td>
<td>110</td>
<td>221.7</td>
<td>67.4</td>
<td>70</td>
<td>30</td>
<td>167.4</td>
<td>−54.27</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>46.1</td>
<td>38.1</td>
<td>10.0</td>
<td>110</td>
<td>204.2</td>
<td>55.3</td>
<td>70</td>
<td>30</td>
<td>155.3</td>
<td>−48.91</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>35.4</td>
<td>35.5</td>
<td>5.0</td>
<td>75.9</td>
<td>42.5</td>
<td>70</td>
<td>30</td>
<td>142.5</td>
<td>66.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>24.2</td>
<td>32.7</td>
<td></td>
<td>56.9</td>
<td>29.0</td>
<td>70</td>
<td>30</td>
<td>129.0</td>
<td>72.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>12.4</td>
<td>29.7</td>
<td></td>
<td>42.1</td>
<td>14.9</td>
<td>70</td>
<td>30</td>
<td>114.9</td>
<td>72.81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>26.4</td>
<td></td>
<td></td>
<td>26.4</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>3.62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>22.8</td>
<td></td>
<td></td>
<td>22.8</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>7.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>18.9</td>
<td></td>
<td></td>
<td>18.9</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>11.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>14.8</td>
<td></td>
<td></td>
<td>14.8</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>15.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>10.2</td>
<td></td>
<td></td>
<td>10.2</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>19.77</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>5.3</td>
<td></td>
<td></td>
<td>5.3</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>24.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.0</td>
<td></td>
<td>30</td>
<td>30</td>
<td>30.0</td>
<td>30.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Impact of prepayment

We have:
\[ N^c(t) = N(t) \cdot 1 \{ \tau > t \} \]

where:
- \( N^c(t) \) and \( N(t) \) are the outstanding principal balances with and without prepayment
- \( \tau \) is the prepayment time of the debt instrument

We deduce that:
\[ \mathbb{E}[N^c(t)] = S(t) \cdot N(t) \]

where \( S(t) = \mathbb{E}[1 \{ \tau > t \}] \) is the survival function of \( \tau \)

Remark

If \( \tau \sim \mathcal{E}(\lambda) \) where \( \lambda \) is the prepayment intensity, we obtain:
\[ \mathbb{E}[N^c(t)] = e^{-\lambda t} \cdot N(t) \leq N(t) \]
Impact of prepayment

Figure: Conventional amortization schedule with prepayment risk
Impact of new production

Accounting identity

\[ N(t) = N(t-1) - AM(t) + NP(t) \]

Figure: Impact of the new production on the outstanding amount
Dynamic analysis

- Run-off balance sheet
  A balance sheet where existing non-trading book positions amortize and are not replaced by any new business.

- Constant balance sheet
  A balance sheet in which the total size and composition are maintained by replacing maturing or repricing cash flows with new cash flows that have identical features.

- Dynamic balance sheet
  A balance sheet incorporating future business expectations, adjusted for the relevant scenario in a consistent manner.
Dynamic analysis

Notations

- \( \text{NP}(t) \): New production at time \( t \)
- \( \text{NP}(t, u) \): New production at time \( t \) that is present in the balance sheet at time \( u \geq t \)
- \( S(t, u) \): Survival function of the new production
- \( f(t, u) \) is the density function associated to the survival function \( S(t, u) \)
- \( N(t, u) \): Non-amortized outstanding amount at time \( t \) that is present in the balance sheet at time \( u \geq t \)
- \( S^*(t, u) \): Survival function of the outstanding amount
Dynamic analysis
Stock-flow analysis

- The amortization function \( S(t, u) \) is defined by:

\[
NP(t, u) = NP(t) \cdot S(t, u)
\]

It measures the proportion of $1 entering in the balance sheet at time \( t \) that remains present at time \( u \geq t \):

\[
N(t) = \int_{-\infty}^{t} NP(s) S(s, t) \, ds
\]

- The amortization function \( S^*(t, u) \) is defined by:

\[
N(t, u) = N(t) \cdot S^*(t, u)
\]

It measures the proportion of $1 of outstanding amount at time \( t \) that remains present at time \( u \geq t \):

\[
S^*(t, u) = \frac{\int_{-\infty}^{t} NP(s) S(s, u) \, ds}{\int_{-\infty}^{t} NP(s) S(s, t) \, ds}
\]
Dynamic analysis
Dynamics of the outstanding amount

We have:

\[
\frac{dN(t)}{dt} = -\int_{-\infty}^{t} NP(s) f(s, t) \, ds + NP(t)
\]

where \( f(t, u) = -\partial_u S(t, u) \) is the density function of the amortization
The dynamic liquidity gap at time $t$ for a future date $u \geq t$ is given by:

$$
\mathcal{L}G(t, u) = \sum_{k \in \text{Liabilities}} \left( N_k(t, u) + \int_t^u \NP_k(s) S_k(s, u) \, ds \right) - \sum_{k \in \text{Assets}} \left( N_k(t, u) + \int_t^u \NP_k(s) S_k(s, u) \, ds \right)
$$

In the case of the run-off balance sheet, we obtain:

$$
\mathcal{L}G(t, u) = \sum_{k \in \text{Liabilities}} N_k(t, u) - \sum_{k \in \text{Assets}} N_k(t, u)
$$
The liquidity duration is the weighted average life (WAL) of the principal repayments:

\[ D(t) = \int_{t}^{\infty} (u - t) f(t, u) \, du \]

where \( f(t, u) \) is the density function associated to the survival function \( S(t, u) \).

**Remark**

*All the previous formulas can be obtained in the discrete-time analysis (HFRM, Section 7.1.2.3, pages 385-392)*
## Dynamic analysis

### Mathematical formulas

**Table:** Survival function and liquidity duration of some amortization schemes  
(HFRM, Exercise 7.4.3, page 450; HFRM-CB, Section 7.4.3, pages 126-128)

<table>
<thead>
<tr>
<th>Amortization</th>
<th>( S(t,u) )</th>
<th>( D(t) )</th>
<th>( S^*(t,u) )</th>
<th>( D^*(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullet</td>
<td>( \mathbb{1} { t \leq u &lt; t + m } )</td>
<td>( m )</td>
<td>( \mathbb{1} { t \leq u &lt; t + m } \cdot \left( 1 - \frac{u - t}{m} \right) )</td>
<td>( m )</td>
</tr>
<tr>
<td>Constant</td>
<td>( \mathbb{1} { t \leq u &lt; t + m } \cdot \left( 1 - \frac{u - t}{m} \right) )</td>
<td>( \frac{m}{2} )</td>
<td>( \mathbb{1} { t \leq u &lt; t + m } \cdot \left( 1 - \frac{u - t}{m} \right)^2 )</td>
<td>( \frac{m}{3} )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( e^{-\lambda(u-t)} )</td>
<td>( \frac{1}{\lambda} )</td>
<td>( e^{-\lambda(u-t)} )</td>
<td>( \frac{1}{\lambda} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Amortization</th>
<th>( dN(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullet</td>
<td>( dN(t) = (NP(t) - NP(t - m)) , dt )</td>
</tr>
<tr>
<td>Constant</td>
<td>( dN(t) = \left( NP(t) - \frac{1}{m} \int_{t-m}^{t} NP(s) , ds \right) , dt )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( dN(t) = (NP(t) - \lambda N(t)) , dt )</td>
</tr>
</tbody>
</table>
Figure: Amortization functions $S(t, u)$ and $S^*(t, u)$
“IRRBB refers to the current or prospective risk to the bank’s capital and earnings arising from adverse movements in interest rates that affect the bank’s banking book positions. When interest rates change, the present value and timing of future cash flows change. This in turn changes the underlying value of a bank’s assets, liabilities and off-balance sheet items and hence its economic value. Changes in interest rates also affect a bank’s earnings by altering interest rate-sensitive income and expenses, affecting its net interest income” (BCBS, 2016)

1. Economic value (EV, EVE): changes in the net present value of the balance sheet
2. Earnings-based risk measures (EaR, NII): changes in the expected future profitability of the bank
Categories of IRR

3 main sources of interest rate risk

- Gap risk: mismatch risk arising from the term structure of banking book instruments
  - Repricing risk
  - Yield curve risk
- Basis risk: mismatch risk arising from different interest rate indices
  - Correlation risk of interest rate indices with the same maturity
- Option risk: option derivative positions and embedded options
  - Automatic option risk (caps, floors, swaptions and other interest rate derivatives)
  - Behavioral option risk
    - Prepayment risk
    - Early redemption risk (or withdrawal risk)
    - Non-maturity deposit (NMD)
Economic value (EV)

The economic value of a series of cash flows $CF = \{ CF(t_k), t_k \geq t \}$ is the present value of these cash flows:

$$
EV = \mathbb{E} \left[ \sum_{t_k \geq t} CF(t_k) \cdot e^{-\int_t^{t_k} r(s) ds} \right] = \sum_{t_k \geq t} CF(t_k) \cdot B(t, t_k)
$$

where $B(t, t_k)$ is the discount factor for the maturity date $t_k$
Application to the banking book

- We slot all notional repricing cash flows of assets and liabilities into a set of time buckets.
- We calculate the net cash flows:

\[ CF(t_k) = CF_A(t_k) - CF_L(t_k) \]

where \( CF_A(t_k) \) and \( CF_L(t_k) \) are the cash flows of assets and liabilities for the time bucket \( t_k \).

- The economic value is given by:

\[
EV = \sum_{t_k \geq t} CF(t_k) \cdot B(t, t_k) \\
= \sum_{t_k \geq t} CF_A(t_k) \cdot B(t, t_k) - \sum_{t_k \geq t} CF_L(t_k) \cdot B(t, t_k) \\
= EV_A - EV_L
\]
Stress testing of the economic value

- We note $EV_s$ the economic value corresponding to the stress scenario $s$.
- $EV_0$ is the base scenario and corresponds to the current term structure of interest rates.
- We have:

$$
\Delta EV_s = EV_0 - EV_s = \sum_{t_k \geq t} CF_0(t_k) \cdot B_0(t, t_k) - \sum_{t_k \geq t} CF_s(t_k) \cdot B_s(t, t_k)
$$
The economic value of equity (EVE or $EV_E$) is a specific form of EV where equity is excluded from the cash flows.

We have:

$$A(t) = L(t) = L^*(t) + E(t)$$
Since the value of the capital is equal to $E(t) = A(t) - L^*(t)$, we have:

$$EVE = EV_A - EV_{L^*}$$

and:

$$\Delta EVE_s = EVE_0 - EVE_s$$

**Remark**

*The economic value of equity is then equal to:*

$$EVE = \sum_{t_k \geq t} CF_A(t_k) \cdot B(t, t_k) - \sum_{t_k \geq t} CF_{L^*}(t_k) \cdot B(t, t_k)$$
The net interest income is the difference between the interest payments on assets and the interest payments of liabilities.

We have:

$$\Delta \text{NII}_s = \text{NII}_0 - \text{NII}_s$$

$$\Delta \text{NII}_s > 0$$ indicates a loss if the stress scenario $s$ occurs.
The risk measures are equal to the maximum of losses by considering the different scenarios:

\[ R(\text{EVE}) = \max_s (\Delta \text{EVE}_s, 0) \]

and:

\[ R(\text{NII}) = \max_s (\Delta \text{NII}_s, 0) \]

**IRRBB**
- No minimum capital requirements \( \mathcal{K} \)
- \( R(\text{EVE}) \leq 15\% \times \text{Tier 1} \)
- Pillar 2
Interest rate risk principles

- 9 IRR principles for banks (management, risk appetite, model governance process, capital adequacy policy, etc.)
- 3 IRR principles for supervisors (data collection, challenging the model assumptions, identification of outlier banks)

Some examples:

- To compute $\Delta \text{EVE}$, banks must consider a run-off balance sheet assumption
- To compute $\Delta \text{NII}$, banks must use a constant or dynamic balance sheet and a rolling 12-month period
- Banks must use:
  - Internal (historical and hypothetical) interest rate scenarios
  - 6 external interest rate scenarios
The standardized approach

5 steps for measuring the bank’s IRRBB

1. The first step consists in allocating the interest rate sensitivities of the banking book to three categories
   1. amenable to standardization
   2. less amenable to standardization
   3. not amenable to standardization

2. Then, the bank must slot cash flows into 19 predefined time buckets: overnight (O/N), O/N–1M, . . . , 10Y–15Y, 15Y–20Y, 20Y+

3. The bank determines $\Delta \text{EVE}_{s,c}$ for each shock $s$ and each currency $c$

4. The bank calculates the total measure for automatic interest rate option risk $\text{KAO}_{s,c}$

5. The bank calculates the $\text{EVE}$ for each shock $s$:

$$R(\text{EVE}_s) = \max \left( \sum_c (\Delta \text{EVE}_{s,c} + \text{KAO}_{s,c})^+ ; 0 \right)$$

The standardized $\text{EVE}$ risk measure is equal to:

$$R(\text{EVE}) = \max_s R(\text{EVE}_s)$$
Three shock sizes:

<table>
<thead>
<tr>
<th>Shock size</th>
<th>USD/CAD/SEK</th>
<th>EUR/HKD</th>
<th>GBP</th>
<th>JPY</th>
<th>EM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$ (parallel)</td>
<td>200</td>
<td>200</td>
<td>250</td>
<td>100</td>
<td>400</td>
</tr>
<tr>
<td>$S_1$ (short)</td>
<td>300</td>
<td>250</td>
<td>300</td>
<td>100</td>
<td>500</td>
</tr>
<tr>
<td>$S_2$ (long)</td>
<td>150</td>
<td>100</td>
<td>150</td>
<td>100</td>
<td>300</td>
</tr>
</tbody>
</table>

Given $S_0$, $S_1$ and $S_2$, we calculate the following generic shocks for a given maturity $t$:

\[
\begin{align*}
\Delta R^{\text{parallel}}(t) &= +S_0 \\
\Delta R^{\text{short}}(t) &= +S_1 \cdot e^{-t/\tau} \\
\Delta R^{\text{long}}(t) &= +S_2 \cdot \left(1 - e^{-t/\tau}\right)
\end{align*}
\]

\[
\begin{align*}
\Delta R^{\text{parallel}}(t) &= -S_0 \\
\Delta R^{\text{short}}(t) &= -S_1 \cdot e^{-t/\tau} \\
\Delta R^{\text{long}}(t) &= -S_2 \cdot \left(1 - e^{-t/\tau}\right)
\end{align*}
\]

where $\tau$ is equal to four years.
The six standardized interest rate shock scenarios are defined as follows:

1. Parallel shock up: $\Delta R^{(\text{parallel})} (t) = +S_0$
2. Parallel shock down: $\Delta R^{(\text{parallel})} (t) = -S_0$
3. Steepener shock (short rates down and long rates up):
   \[ \Delta R^{(\text{steepnener})} (t) = 0.90 \cdot |\Delta R^{(\text{long})} (t)| - 0.65 \cdot |\Delta R^{(\text{short})} (t)| \]
4. Flattener shock (short rates up and long rates down):
   \[ \Delta R^{(\text{flattener})} (t) = 0.80 \cdot |\Delta R^{(\text{short})} (t)| - 0.60 \cdot |\Delta R^{(\text{long})} (t)| \]
5. Short rates shock up:
   \[ \Delta R^{(\text{short})} (t) = +S_1 \cdot e^{-t/\tau} \]
6. Short rates shock down:
   \[ \Delta R^{(\text{short})} (t) = -S_1 \cdot e^{-t/\tau} \]
Example

We assume that $S_0 = 100$ bps, $S_1 = 150$ bps and $S_2 = 200$ bps. We would like to calculate the standardized shocks for the one-year maturity.

- The parallel shock up is equal to $+100$ bps, while the parallel shock down is equal to $-100$ bps.
- For the short rates shock, we obtain:
  \[
  \Delta R^{(\text{short})}(t) = 150 \times e^{-1/4} = 116.82 \text{ bps}
  \]
  for the up scenario and $-116.82$ bps for the down scenario.
- Since we have $|\Delta R^{(\text{short})}(t)| = 116.82$ and $|\Delta R^{(\text{long})}(t)| = 44.24$, the steepener shock is equal to:
  \[
  \Delta R^{(\text{steepnener})}(t) = 0.90 \times 44.24 - 0.65 \times 116.82 = -36.12 \text{ bps}
  \]
  For the flattener shock, we have:
  \[
  \Delta R^{(\text{flattener})}(t) = 0.80 \times 116.82 - 0.60 \times 44.24 = 66.91 \text{ bps}
  \]
The standardized approach
The supervisory interest rate shock scenarios

Figure: Interest rate shocks (in bps) with ($S_0 = 100, S_1 = 150, S_2 = 200$)
The standardized approach

The supervisory interest rate shock scenarios

**Figure:** Stressed yield curve (in %)
The standardized approach
Treatment of NMDs

- Retail transactional (RT)
- Retail non-transactional (RNT)
- Wholesale (W)

- Difference between stable and non-stable part of each category
- The stable part of NMDs must be split between:
  - Core deposits
    - Maximum proportion: 90% for RT, 70% for RNT and 50% for W
    - Maximum maturity: 5Y for RT, 4.5Y for RNT and 4Y for W
  - Non-core deposits (overnight maturity)
The bank estimates the baseline conditional prepayment rate (CPR) $CPR_0$ and calculates the stressed conditional prepayment rate as follows:

$$CPR_s = \min (1, \gamma_s \cdot CPR_0)$$

where $\gamma_s$ is the multiplier for the scenario $s$ and

- $\gamma_s = 0.8$ for the scenarios 1, 3 and 5 (parallel up, steepener and short rates up)
- $\gamma_s = 1.2$ for the scenarios 2, 4 and 6 (parallel down, flattener and short rates down)
The term deposit redemption ratio (TDRR) is stressed as follows:

\[ TDRR_s = \min (1, \gamma_s \cdot TDRR_0) \]

where \( \gamma_s \) is the multiplier for the scenario \( s \) and:

- \( \gamma_s = 1.2 \) for the scenarios 1, 4 and 5 (parallel up, flattener and short rates up)
- \( \gamma_s = 0.8 \) for the scenarios 2, 3 and 6 (parallel down, steepener and short rates down)
The computation of the automatic interest rate option risk $\text{KAO}_s$ is given by:

$$\text{KAO}_s = \sum_{i \in S} \Delta \text{FVAO}_{s,i} - \sum_{i \in B} \Delta \text{FVAO}_{s,i}$$

where:

- $i \in S$ (resp. $i \in B$) denotes an automatic interest rate option which is sold (resp. bought) by the bank
- $\text{FVAO}_{s,i}$ (resp. $\text{FVAO}_{0,i}$) is the fair value of the automatic option $i$ given the stressed (resp. current) yield curve and a relative increase in the implied volatility of 25% (resp. the current implied volatility)
- $\Delta \text{FVAO}_{s,i}$ is the change in the fair value of the option:

$$\Delta \text{FVAO}_{s,i} = \text{FVAO}_{s,i} - \text{FVAO}_{0,i}$$
We consider a USD-denominated balance sheet. The assets are composed of loans with the following cash flow slotting:

<table>
<thead>
<tr>
<th>Instruments</th>
<th>Loans</th>
<th>Loans</th>
<th>Loans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>1Y</td>
<td>5Y</td>
<td>13Y</td>
</tr>
<tr>
<td>Cash flows</td>
<td>200</td>
<td>700</td>
<td>200</td>
</tr>
</tbody>
</table>

The liabilities are composed of retail deposit accounts, term deposits, debt and tier-one equity capital:

<table>
<thead>
<tr>
<th>Instruments</th>
<th>Non-core deposits</th>
<th>Term deposits</th>
<th>Core deposits</th>
<th>Debt ST</th>
<th>Debt LT</th>
<th>Equity capital</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>O/N</td>
<td>7M</td>
<td>3Y</td>
<td>4Y</td>
<td>8Y</td>
<td>200</td>
</tr>
<tr>
<td>Cash flows</td>
<td>100</td>
<td>50</td>
<td>450</td>
<td>100</td>
<td>100</td>
<td>200</td>
</tr>
</tbody>
</table>
The standardized approach

Example

**Table:** Economic value of the assets

<table>
<thead>
<tr>
<th>Bucket</th>
<th>$t_k$</th>
<th>$CF_0(t_k)$</th>
<th>$R_0(t_k)$</th>
<th>$EV_0(t_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.875</td>
<td>200</td>
<td>1.55%</td>
<td>197.31</td>
</tr>
<tr>
<td>11</td>
<td>4.50</td>
<td>700</td>
<td>3.37%</td>
<td>601.53</td>
</tr>
<tr>
<td>17</td>
<td>12.50</td>
<td>100</td>
<td>5.71%</td>
<td>48.98</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$EV_0$ 847.82</td>
</tr>
</tbody>
</table>

**Table:** Economic value of the pure liabilities

<table>
<thead>
<tr>
<th>Bucket</th>
<th>$t_k$</th>
<th>$CF_0(t_k)$</th>
<th>$R_0(t_k)$</th>
<th>$EV_0(t_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0028</td>
<td>100</td>
<td>1.00%</td>
<td>100.00</td>
</tr>
<tr>
<td>5</td>
<td>0.625</td>
<td>50</td>
<td>1.39%</td>
<td>49.57</td>
</tr>
<tr>
<td>9</td>
<td>2.50</td>
<td>450</td>
<td>2.44%</td>
<td>423.35</td>
</tr>
<tr>
<td>10</td>
<td>3.50</td>
<td>100</td>
<td>2.93%</td>
<td>90.26</td>
</tr>
<tr>
<td>14</td>
<td>7.50</td>
<td>100</td>
<td>4.46%</td>
<td>71.56</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$EV_0$ 734.73</td>
</tr>
</tbody>
</table>
### Table: Stressed economic value of equity

<table>
<thead>
<tr>
<th>Bucket</th>
<th>$s = 1$</th>
<th>$s = 2$</th>
<th>$s = 3$</th>
<th>$s = 4$</th>
<th>$s = 5$</th>
<th>$s = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assets</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_s(t_6)$</td>
<td>3.55%</td>
<td>-0.45%</td>
<td>0.24%</td>
<td>3.30%</td>
<td>3.96%</td>
<td>-0.87%</td>
</tr>
<tr>
<td>$R_s(t_{11})$</td>
<td>5.37%</td>
<td>1.37%</td>
<td>3.65%</td>
<td>3.54%</td>
<td>4.34%</td>
<td>2.40%</td>
</tr>
<tr>
<td>$R_s(t_{17})$</td>
<td>7.71%</td>
<td>3.71%</td>
<td>6.92%</td>
<td>4.96%</td>
<td>5.84%</td>
<td>5.58%</td>
</tr>
<tr>
<td>$EV_s(t_6)$</td>
<td>193.89</td>
<td>200.80</td>
<td>199.57</td>
<td>194.31</td>
<td>193.20</td>
<td>201.52</td>
</tr>
<tr>
<td>$EV_s(t_{11})$</td>
<td>549.76</td>
<td>658.18</td>
<td>594.03</td>
<td>596.91</td>
<td>575.74</td>
<td>628.48</td>
</tr>
<tr>
<td>$EV_s(t_{17})$</td>
<td>38.15</td>
<td>62.89</td>
<td>42.13</td>
<td>53.83</td>
<td>48.18</td>
<td>49.79</td>
</tr>
<tr>
<td>$EV_s$</td>
<td>781.79</td>
<td>921.87</td>
<td>835.74</td>
<td>845.05</td>
<td>817.11</td>
<td>879.79</td>
</tr>
<tr>
<td><strong>Pure liabilities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_s(t_1)$</td>
<td>3.00%</td>
<td>-1.00%</td>
<td>-0.95%</td>
<td>3.40%</td>
<td>4.00%</td>
<td>-2.00%</td>
</tr>
<tr>
<td>$R_s(t_5)$</td>
<td>3.39%</td>
<td>-0.61%</td>
<td>-0.08%</td>
<td>3.32%</td>
<td>3.96%</td>
<td>-1.17%</td>
</tr>
<tr>
<td>$R_s(t_9)$</td>
<td>4.44%</td>
<td>0.44%</td>
<td>2.03%</td>
<td>3.31%</td>
<td>4.05%</td>
<td>0.84%</td>
</tr>
<tr>
<td>$R_s(t_{10})$</td>
<td>4.93%</td>
<td>0.93%</td>
<td>2.90%</td>
<td>3.40%</td>
<td>4.18%</td>
<td>1.68%</td>
</tr>
<tr>
<td>$R_s(t_{14})$</td>
<td>6.46%</td>
<td>2.46%</td>
<td>5.31%</td>
<td>4.07%</td>
<td>4.92%</td>
<td>4.00%</td>
</tr>
<tr>
<td>$EV_s(t_1)$</td>
<td>99.99</td>
<td>100.00</td>
<td>100.00</td>
<td>99.99</td>
<td>99.99</td>
<td>100.01</td>
</tr>
<tr>
<td>$EV_s(t_5)$</td>
<td>48.95</td>
<td>50.19</td>
<td>50.02</td>
<td>48.97</td>
<td>48.78</td>
<td>50.37</td>
</tr>
<tr>
<td>$EV_s(t_9)$</td>
<td>402.70</td>
<td>445.05</td>
<td>427.77</td>
<td>414.27</td>
<td>406.69</td>
<td>440.69</td>
</tr>
<tr>
<td>$EV_s(t_{10})$</td>
<td>84.16</td>
<td>96.80</td>
<td>90.34</td>
<td>88.77</td>
<td>86.39</td>
<td>94.30</td>
</tr>
<tr>
<td>$EV_s(t_{14})$</td>
<td>61.59</td>
<td>83.14</td>
<td>67.17</td>
<td>73.70</td>
<td>69.13</td>
<td>74.07</td>
</tr>
<tr>
<td>$EV_s$</td>
<td>697.39</td>
<td>775.18</td>
<td>735.31</td>
<td>725.71</td>
<td>710.98</td>
<td>759.43</td>
</tr>
<tr>
<td><strong>Equity</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$EVE_s$</td>
<td>84.41</td>
<td>146.68</td>
<td>100.43</td>
<td>119.34</td>
<td>106.13</td>
<td>120.37</td>
</tr>
<tr>
<td>$\Delta EVE_s$</td>
<td>28.69</td>
<td>-33.58</td>
<td>12.67</td>
<td>-6.24</td>
<td>6.97</td>
<td>-7.27</td>
</tr>
</tbody>
</table>
The current economic value of equity is equal to:

\[ \text{EVE}_0 = 847.82 - 734.73 = 113.09 \]

In the case of the first stress scenario, we have:

\[ \text{EVE}_1 = 781.79 - 697.39 = 84.41 \]

and:

\[ \Delta \text{EVE}_1 = 113.10 - 84.41 = 28.69 \]

EVE decreases for scenarios 1, 3 and 5.

The EVE risk measure is equal to:

\[ R(\text{EVE}) = \max_s (\Delta \text{EVE}_s, 0) = 28.69 \]

It represents 14.3% of the equity:

\[ \frac{28.69}{200} = 14.3\% \]

The materiality test is not satisfied.
Currency risk

- Currency hedging ⇒ also the equity capital?
- Dollar funding
- Multi-currency balance sheet
Credit spread risk

**Items at amortized cost**

- Administered rate
- Credit margin
- Funding rate
- Reference rate
- e.g. consumer loans
- e.g. corporate loans

**Items at fair value (MtM)**

- Idiosyncratic credit spread
- Market credit spread
- Market liquidity spread
- Market duration spread
- Risk-free rate
- e.g. bonds or interest-earning securities

**Figure:** Components of interest rates
Macaulay duration

The Macaulay duration $\mathcal{D}$ is the weighted average of the cash flow maturities:

$$\mathcal{D} = \sum_{t_k \geq t} w(t, t_k) \cdot (t_k - t)$$

We have:

$$\frac{\partial V}{\partial y} = -\frac{\mathcal{D}}{1 + y} \cdot V = -\mathcal{D} \cdot V$$

where $\mathcal{D}$ is the modified duration

Application to a portfolio

The market value of the portfolio is composed of $m$ cash flow streams:

$$V = \sum_{j=1}^{m} V_j$$

while the duration of the portfolio is the average of individual durations:

$$\mathcal{D} = \sum_{j=1}^{m} w_j \cdot D_j$$

where $w_j = \frac{V_j}{V}$
Since \( E(t) = A(t) - L^*(t) \) and \( EV_E = EV_A - EV_L^* \), the duration of equity is equal to:

\[
\mathcal{D}_E = \frac{EV_A}{EV_A - EV_L^*} \cdot \mathcal{D}_A - \frac{EV_L^*}{EV_A - EV_L^*} \cdot \mathcal{D}_L^* = \frac{EV_A}{EV_A - EV_L^*} \cdot \mathcal{D}_{Gap}
\]

where the duration gap (also called DGAP) is equal to

\[
\mathcal{D}_{Gap} = \mathcal{D}_A - \frac{EV_L^*}{EV_A} \cdot \mathcal{D}_L^*
\]
Duration gap risk

Another expression of the equity duration is:

\[ D_E = \frac{EV_A}{EV_E} \cdot D_{Gap} = \frac{L_{A/E}}{EV} \cdot D_{Gap} \]

where \( L_{A/E} \) is the leverage ratio

Relationship between EVE and duration gap

\[ \Delta EVE = \Delta EV_E \]
\[ \approx -D_E \cdot EV_E \cdot \frac{\Delta y}{1+y} \]
\[ \approx -D_{Gap} \cdot EV_A \cdot \frac{\Delta y}{1+y} \]
Illustration

We consider the following balance sheet:

<table>
<thead>
<tr>
<th>Assets</th>
<th>$V_j$</th>
<th>$D_j$</th>
<th>Liabilities</th>
<th>$V_j$</th>
<th>$D_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>5</td>
<td>0.0</td>
<td>Deposits</td>
<td>40</td>
<td>3.2</td>
</tr>
<tr>
<td>Loans</td>
<td>40</td>
<td>1.5</td>
<td>CDs</td>
<td>20</td>
<td>0.8</td>
</tr>
<tr>
<td>Mortgages</td>
<td>40</td>
<td>6.0</td>
<td>Debt</td>
<td>30</td>
<td>1.7</td>
</tr>
<tr>
<td>Securities</td>
<td>15</td>
<td>3.8</td>
<td>Equity capital</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>3.8</td>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

We have $EV_A = 100$, $EV_{L^*} = 90$, $EV_E = 10$ and:

$$\mathcal{L}_{A/E} = \frac{EV_A}{EV_E} = \frac{100}{10} = 10$$

The duration values are equal to:

$$D_A = \frac{5}{100} \times 0 + \frac{40}{100} \times 1.5 + \frac{40}{100} \times 6.0 + \frac{15}{100} \times 3.8 = 3.57 \text{ years}$$

$$D_{L^*} = \frac{40}{90} \times 3.2 + \frac{20}{90} \times 0.8 + \frac{30}{90} \times 1.7 = 2.17 \text{ years}$$
We deduce that:

\[ D_{\text{gap}} = 3.57 - \frac{90}{100} \times 2.17 = 1.62 \text{ years} \]

If we assume that the current yield to maturity is equal to 3%, we obtain:

<table>
<thead>
<tr>
<th>Δy</th>
<th>-2%</th>
<th>-1%</th>
<th>+1%</th>
<th>+2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔEVE</td>
<td>3.15</td>
<td>1.57</td>
<td>-1.57</td>
<td>-3.15</td>
</tr>
<tr>
<td>ΔEVE</td>
<td>31.46%</td>
<td>15.73%</td>
<td>-15.73%</td>
<td>-31.46%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ΔEVE</th>
<th>EVE</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.46%</td>
<td>15.73%</td>
</tr>
</tbody>
</table>
Immunization of the balance sheet

We must have:

$$
\Delta \text{ EVE} = 0 \iff \mathcal{D}_{\text{Gap}} = 0 \iff \mathcal{D}_A - \frac{\text{EV}_{L^*}}{\text{EV}_A} \cdot \mathcal{D}_{L^*} = 0
$$

Table: Bank balance sheet after immunization of the duration gap

<table>
<thead>
<tr>
<th>Assets</th>
<th>$V_j$</th>
<th>$D_j$</th>
<th>Liabilities</th>
<th>$V_j$</th>
<th>$D_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>5</td>
<td>0.0</td>
<td>Deposits</td>
<td>40</td>
<td>3.2</td>
</tr>
<tr>
<td>Loans</td>
<td>40</td>
<td>1.5</td>
<td>CDs</td>
<td>20</td>
<td>0.8</td>
</tr>
<tr>
<td>Mortgages</td>
<td>40</td>
<td>6.0</td>
<td>Debt</td>
<td>10.48</td>
<td>1.7</td>
</tr>
<tr>
<td>Securities</td>
<td>15</td>
<td>3.8</td>
<td>Zero-coupon bond</td>
<td>19.52</td>
<td>10.0</td>
</tr>
<tr>
<td>Equity capital</td>
<td></td>
<td></td>
<td>Equity capital</td>
<td>10</td>
<td>0.0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td></td>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td></td>
</tr>
</tbody>
</table>
If interest rates change, this induces a gap (or repricing) risk because the bank will have to reinvest assets and refinance liabilities at a different interest rate level in the future.

The gap is defined as the difference between rate sensitive assets (RSA) and rate sensitive liabilities (RSL):

\[ \text{GAP}(t, u) = \text{RSA}(t, u) - \text{RSL}(t, u) \]

where \( t \) is the current date and \( u \) is the time horizon of the gap.

We can show that:

\[ \Delta \text{NII}(t, u) \approx \text{GAP}(t, u) \cdot \Delta R \]

where \( \Delta R \) is the parallel shock of interest rates.
The net interest income of the bank is the difference between interest rate revenues of its assets and interest rate expenses of its liabilities:

$$\text{NII}(t, u) = \sum_{i \in \text{Assets}} N_i(t, u) \cdot R_i(t, u) - \sum_{j \in \text{Liabilities}} N_j(t, u) \cdot R_j(t, u)$$

where $\text{NII}(t, u)$ is the net interest income at time $t$ for the maturity date $u$

$\Rightarrow$ Mathematical formulation (HFRM, pages 412-418)
Modeling customer rates

Client rates ≠ market rates

Several issues:
- Correlation
- Next repricing date (known or unknown?)
- Sensitivity of the customer rate with respect to the market rate
Using a forward rate agreement, we can show that:

\[ NII_H(t, u) - NII(t, u) = GAP(t, u) \cdot \rho(t, u) \cdot (f(t, u) - r(u)) \]

We can draw several conclusions:

- When the interest rate gap is closed, the bank does not need to hedge the net interest income.
- When the correlation \( \rho(t, u) \) between the customer rate and the market rate is equal to one, the notional of the hedge is exactly equal to the interest rate gap (it is lower in the general case).
- If the bank hedges the net interest income and if the gap is positive, a decrease of interest rates is not favorable.

**Hedging the interest rate gap depends on the expectations of the bank \( \Rightarrow \) partial hedging and macro hedging.**
Hedging instruments

- Interest rate swaps (IRS)
- Forward rate agreements (FRA)
- Swaptions
All liquidity and interest rate risks are transferred to the ALM unit:

- Business units can then lend or borrow funding at a given internal price
- This price is called the funds transfer price (FTP) or the internal transfer rate
- The FTP charges interests to the business unit for client loans, whereas the FTP compensates the business unit for raising deposits
The net interest margin (NIM) is equal to:

\[
\text{NIM}(t,u) = \frac{\sum_{i \in \text{Assets}} N_i(t,u) \cdot R_i(t,u) - \sum_{j \in \text{Liabilities}} N_j(t,u) \cdot R_j(t,u)}{\sum_{i \in \text{Assets}} N_i(t,u)}
\]

\[
= \frac{RA(t,u) \cdot R_{RA}(t,u) - RL(t,u) \cdot R_{RL}(t,u)}{RA(t,u)}
\]

where \(R_{RA}\) and \(R_{RL}\) represent the weighted average interest rate of interest earning assets and interest bearing liabilities.

The net interest spread (NIS) is equal to:

\[
\text{NIS}(t,u) = \frac{\sum_{i \in \text{Assets}} N_i(t,u) \cdot R_i(t,u)}{\sum_{i \in \text{Assets}} N_i(t,u)} - \frac{\sum_{j \in \text{Liabilities}} N_j(t,u) \cdot R_j(t,u)}{\sum_{j \in \text{Liabilities}} N_j(t,u)}
\]

\[
= R_{RA}(t,u) - R_{RL}(t,u)
\]

NIM is the profitability ratio of the assets whereas NIS is the interest rate spread captured by the bank.
Net interest margin

Example

We consider the following interest earning and bearing items:

<table>
<thead>
<tr>
<th>Assets</th>
<th>( N_i(t, u) )</th>
<th>( R_i(t, u) )</th>
<th>Liabilities</th>
<th>( N_j(t, u) )</th>
<th>( R_j(t, u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loans</td>
<td>100</td>
<td>5%</td>
<td>Deposits</td>
<td>100</td>
<td>0.5%</td>
</tr>
<tr>
<td>Mortgages</td>
<td>100</td>
<td>4%</td>
<td>Debts</td>
<td>60</td>
<td>2.5%</td>
</tr>
</tbody>
</table>
Net interest margin

- The interest income is equal to \(100 \times 5\% + 100 \times 4\% = 9\) whereas the interest expense is equal to \(100 \times 0.5\% + 60 \times 2.5\% = 2\). We deduce that the net interest income is equal to:
  \[
  NII(t, u) = 9 - 2 = 7
  \]

- We have:
  \[
  R_{RA}(t, u) = \frac{100 \times 5\% + 100 \times 4\%}{100 + 100} = 4.5\%
  \]
  and:
  \[
  R_{RL}(t, u) = \frac{100 \times 0.5\% + 60 \times 2.5\%}{100 + 60} = 1.25\%
  \]
  Since \(RA(t, u) = 200\) and \(RL(t, u) = 160\), we deduce that:
  \[
  NIM(t, u) = \frac{200 \times 4.5\% - 160 \times 1.25\%}{200} = \frac{7}{200} = 3.5\%
  \]
  and:
  \[
  NIS(t, u) = 4.5\% - 1.25\% = 3.25\%
  \]
Net interest margin

Figure: Evolution of the net interest margin in the US
The commercial margin rate is the spread between the client rate $R_i(t, u)$ of the asset $i$ and the corresponding market rate $r(t, u)$:

$$m_i(t, u) = R_i(t, u) - r(t, u)$$

For the liability $j$, we have:

$$m_j(t, u) = r(t, u) - R_j(t, u)$$

In the case where we can perfectly match the asset $i$ with the liability $j$, the commercial margin rate is the net interest spread:

$$m(t, u) = m_i(t, u) + m_j(t, u) = R_i(t, u) - R_j(t, u)$$
Commercial margin

Introducing a funds transfer pricing system is equivalent to interpose the ALM unit between the business unit and the market

- For assets, we have:

\[
m_i(t, u) = \left( R_i(t, u) - FTP_i(t, u) \right) + \left( FTP_i(t, u) - r(t, u) \right) \\
\]

\[
= m_i^{(c)}(t, u) + m_i^{(t)}(t, u)
\]

where:
- \( m_i^{(c)}(t, u) \) is the commercial margin rate of the business unit
- \( m_i^{(t)}(t, u) \) is the transformation margin rate of the ALM unit

- For liabilities, we have:

\[
m_j(t, u) = \left( FTP_j(t, u) - R_j(t, u) \right) + \left( r(t, u) - FTP_j(t, u) \right) \\
\]

\[
= m_j^{(c)}(t, u) + m_j^{(t)}(t, u)
\]
Commercial margin

Commercial margins and funds transfer prices

The goal of FTP is to lock the commercial margin rates $m_i^{(c)}(t, u)$ and $m_j^{(c)}(t, u)$ over the lifetime of the product contract.
Commercial margin

Example

We consider the following interest earning and bearing items:

<table>
<thead>
<tr>
<th></th>
<th>(N_i(t,u))</th>
<th>(R_i(t,u))</th>
<th>(N_j(t,u))</th>
<th>(R_j(t,u))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loans</td>
<td>100</td>
<td>5%</td>
<td>100</td>
<td>0.5%</td>
</tr>
<tr>
<td>Mortgages</td>
<td>100</td>
<td>4%</td>
<td>60</td>
<td>2.5%</td>
</tr>
</tbody>
</table>

The FTP for the loans and the mortgages is equal to 3%, while the FTP for deposits is equal to 1.5% and the FTP for debts is equal to 2.5%. We assume that the market rate is equal to 2.5%.

Solution

We obtain the following results:

<table>
<thead>
<tr>
<th></th>
<th>(m_i^{(c)}(t,u))</th>
<th>(m_i^{(t)}(t,u))</th>
<th>(m_j^{(c)}(t,u))</th>
<th>(m_j^{(t)}(t,u))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loans</td>
<td>2%</td>
<td>0.5%</td>
<td>1.0%</td>
<td>1.0%</td>
</tr>
<tr>
<td>Mortgages</td>
<td>1%</td>
<td>0.5%</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>
The commercial margin of the bank is equal to:

\[ M^{(c)} = 100 \times 2\% + 100 \times 1\% + 100 \times 1\% + 60 \times 0\% = 4 \]

For the transformation margin, we have:

\[ M^{(t)} = 100 \times 0.5\% + 100 \times 0.5\% + 100 \times 1.0\% + 60 \times 0\% = 2 \]

We don’t have \( M^{(c)} + M^{(t)} = \text{NII} \) because assets and liabilities are not compensated:

\[ \text{NII} - \left( M^{(c)} + M^{(t)} \right) = (\text{RA} (t, u) - \text{RL} (t, u)) \cdot r (t, u) = 40 \times 2.5\% = 1 \]

The commercial margin of each product is:

- \( M^{(c)}_{\text{Loans}} = 2 \)
- \( M^{(c)}_{\text{Mortgages}} = 1 \)
- \( M^{(c)}_{\text{Deposits}} = 1 \)
Computing the internal transfer rate

The reference rate

- Since we have $m_i(t, u) = FTP_i(t, u) - r(t, u)$, internal prices are fair if the corresponding mark-to-market is equal to zero on average.
- For a contract with a bullet maturity, this implies that:

$$FTP_i(t, u) = \mathbb{E}[r(t, u)]$$

- The transformation margin can then be interpreted as an interest rate swap receiving a fixed leg $FTP_i(t, u)$ and paying a floating leg $r(t, u)$.

Remark

*It follows that the funds transfer price is equal to the market swap rate at the initial date $t$ with the same maturity than the asset item $i$.*
If we assume that the commercial margin rate of the business unit is constant:

\[ R(u) - FTP(t, u) = m \]

we can show that:

\[
FTP(t, u) = R(u) + \frac{\mathbb{E}_t \left[ \int_t^\infty B(t, u) S(t, u) (r(u) - R(u)) \, du \right]}{\mathbb{E}_t \left[ \int_t^\infty B(t, u) S(t, u) \, du \right]}
\]

We deduce that:

- for a loan with a fixed rate, the funds transfer price is exactly the swap rate with the same maturity than the loan and the same amortization scheme than the new production
- if the client rate \( R(u) \) is equal to the short-term market rate \( r(u) \), the funds transfer price \( FTP(t, u) \) is also equal to \( r(u) \)
Non-maturity deposits
What is the maturity of NMDs?

- The deposit balance of the client A is equal to $500 ⇒ the duration of this deposit is equal to zero day
- We consider 1 000 clients ⇒ the total amount that may be withdrawn today is then between $0 and $500 000
- We assume that the probability to withdraw $500 at once is equal to 50% ⇒ the probability that $500 000 are withdrawn is less than $10^{-300}$%
- Since we have $\Pr \{ S > 275000 \} < 0.1\%$, we can decide that 55% of the deposit balance has a duration of zero day, 24.75% has a duration of one day, 11.14% has a duration of two days, etc.

The statistical duration of NMDs is long (and not short)
In the case of non-maturity deposits, the hazard function rate $\lambda(t, u)$ of the amortization function $S(t, u)$ does not depend on the entry date $t$:

$$\lambda(t, u) = \lambda(u)$$

We can show that (HFRM, pages 428-428):

$$dN(t) = (NP(t) - \lambda(t)N(t)) \, dt$$

If we assume that the new production and the hazard rate are constant – $NP(t) = NP$ and $\lambda(t) = \lambda$, we obtain:

$$dN(t) = \lambda(N_\infty - N(t)) \, dt$$
Non-maturity deposits
Dynamic modeling

Two extensions:

- The Ornstein-Uhlenbeck model:

\[ dN(t) = \lambda (N_\infty - N(t)) \, dt + \sigma \, dW(t) \]

- The aggregate model:

\[
D(t) = D_\infty e^{g(t-s)} + (D_s - D_\infty) e^{(g-\lambda)(t-s)} + \varepsilon(t)
\]

where \( g \) is the growth rate of deposits
Non-maturity deposits
Stable vs non-stable deposits

Remark

We have:

\[ D(t) = \varphi D_\infty e^{g(t-s)} + (D_s - D_\infty) e^{(g-\lambda)(t-s)} + \varepsilon(t) + (1 - \varphi) D_\infty e^{g(t-s)} \]

Calibration of \( \varphi \)

We have:

\[ \Pr \{ D(t) \leq \varphi D_\infty \} = 1 - \alpha \]

If we consider the Ornstein-Uhlenbeck dynamics, we obtain:

\[ \varphi = 1 - \frac{\sigma \Phi^{-1}(1 - \alpha)}{D_\infty \sqrt{2\lambda}} \]
Non-maturity deposits
Stable vs non-stable deposits

Figure: Stable and non-stable deposits
The Hutchison-Pennacchi-Selvaggio framework

- The deposit rate \( i(t) \) is exogenous and the bank account holder modifies his current deposit balance \( D(t) \) to target a level \( D^*(t) \):
  \[
  \ln D^*(t) = \beta_0 + \beta_1 \ln i(t) + \beta_2 \ln Y(t)
  \]
  where \( Y(t) \) is the income of the account holder

- The behavior of the bank account holder can be represented by a mean-reverting AR(1) process:
  \[
  \ln D(t) - \ln D(t-1) = (1 - \phi) (\ln D^*(t) - \ln D(t-1)) + \varepsilon(t)
  \]

- The bank maximizes its profit \( i^*(t) = \arg \max \Pi(t) \) where the profit \( \Pi(t) \) is equal to the revenue minus the cost:
  \[
  \Pi(t) = r(t) \cdot D(t) - (i(t) + c(t)) \cdot D(t)
  \]
  \( r(t) \) is the market interest rate and \( c(t) \) is the cost of issuing deposits

- We can show that:
  \[
  i^*(t) = r(t) - s(t)
  \]
The IRS framework (Jarrow and van Deventer, 1998)

- The current market value of deposits is the net present value of the cash flow stream $D(t)$:

$$V(0) = \mathbb{E} \left[ \sum_{t=0}^{\infty} B(0, t+1) (r(t) - i(t)) D(t) \right]$$

$V(0)$ as an exotic interest rate swap, where the bank receives the market rate and pays the deposit rate.

- We have:

$$\ln D(t) = \ln D(t-1) + \beta_0 + \beta_1 r(t) + \beta_2 (r(t) - r(t-1)) + \beta_3 t$$

and:

$$i(t) = i(t) + \beta_0' + \beta_1' r(t) + \beta_2' (r(t) - r(t-1))$$
Asymmetric adjustment models

- O’Brien model:

\[ \Delta i(t) = \alpha(t) \cdot (\hat{i}(t) - i(t-1)) + \eta(t) \]

where \( \hat{i}(t) \) is the conditional equilibrium deposit rate and:

\[ \alpha(t) = \alpha^+ \cdot 1 \{ \hat{i}(t) > i(t-1) \} + \alpha^- \cdot 1 \{ \hat{i}(t) < i(t-1) \} \]

- Frachot model:

\[ \ln D(t) - \ln D(t-1) = (1 - \phi) (\ln D^*(t) - \ln D_{t-1}) + \delta_c (r(t), r^*) \]

where \( \delta_c (r(t), r^*) = \delta \cdot 1 \{ r(t) \leq r^* \} \) and \( r^* \) is the interest rate floor

- OTS model:

\[ d(t) = d(t-1) + \Delta \ln \left( \beta_0 + \beta_1 \arctan \left( \beta_2 + \beta_3 \frac{i(t)}{r(t)} \right) + \beta_4 i(t) \right) + \epsilon(t) \]
Non-maturity deposits
Behavioral modeling

Figure: Impact of the market rate on the growth rate of deposits
Prepayment risk

Definition

A prepayment is the settlement of a debt or the partial repayment of its outstanding amount before its maturity date.
Refinancing:

\[ \mathbb{P}(t) = \Pr \{ \tau \leq t \} = \vartheta (i_0 - i(t)) \]

“A household with a 30-year fixed-rate mortgage of $200,000 at an interest rate of 6.0% that refinances when rates fall to 4.5% (approximately the average rate decrease between 2008 and 2010 in the US) saves more than $60,000 in interest payments over the life of the loan, even after accounting for refinance transaction costs. Further, when mortgage rates reached all-time lows in late 2012, with rates of roughly 3.35% prevailing for three straight months, this household with a contract rate of 6.5% would save roughly $130,000 over the life of the loan by refinancing” (Keys, et al., 2016, pages 482-483).

Housing turnover (marriage, divorce, death, children leaving home or changing jobs)
The prepayment value is the premium of an American call option, meaning that we can derive the optimal option exercise. In this case, the prepayment strategy can be viewed as an arbitrage strategy between the market interest rate and the cost of refinancing.
Rate, coupon or maturity incentive?

We assume that the mortgage rate drops from $i_0$ to $i(t)$

- The absolute difference of the annuity is equal to:
  \[ \mathcal{D}_A (i_0, i(t)) = A(i_0, n) - A(i(t), n) \]

- The relative cumulative difference $\mathcal{C}(i_0, i(t))$ is equal to:
  \[ \mathcal{C}(i_0, i(t)) = \sum_{t=1}^{n} \mathcal{D}_A (i_0, i(t)) / N_0 \]

- By assuming that the borrower continues to pay the same annuity, the maturity reduction is given by:
  \[ \mathcal{M}(i_0, i(t)) = \{ x \in \mathbb{N} : A(i(t), x) \geq A(i(t), n), A(i(t), x + 1) < A(i(t), n) \} \]
## Prepayment risk

### Reduced-form models

**Table:** Impact of a new mortgage rate (100 KUSD, 5%, 10-year)

<table>
<thead>
<tr>
<th>(i) (in %)</th>
<th>(A) (in $)</th>
<th>(\mathcal{D}_A) (in $)</th>
<th>(\mathcal{D}_R) (in %)</th>
<th>(C) (in %)</th>
<th>(N) (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>1 061</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>1 036</td>
<td>24</td>
<td>291</td>
<td>2.3</td>
<td>2.9</td>
</tr>
<tr>
<td>4.0</td>
<td>1 012</td>
<td>48</td>
<td>578</td>
<td>4.5</td>
<td>5.8</td>
</tr>
<tr>
<td>3.5</td>
<td>989</td>
<td>72</td>
<td>862</td>
<td>6.8</td>
<td>8.6</td>
</tr>
<tr>
<td>3.0</td>
<td>966</td>
<td>95</td>
<td>1 141</td>
<td>9.0</td>
<td>11.4</td>
</tr>
<tr>
<td>2.5</td>
<td>943</td>
<td>118</td>
<td>1 415</td>
<td>11.1</td>
<td>14.2</td>
</tr>
<tr>
<td>2.0</td>
<td>920</td>
<td>141</td>
<td>1 686</td>
<td>13.2</td>
<td>16.9</td>
</tr>
<tr>
<td>1.5</td>
<td>898</td>
<td>163</td>
<td>1 953</td>
<td>15.3</td>
<td>19.5</td>
</tr>
<tr>
<td>1.0</td>
<td>876</td>
<td>185</td>
<td>2 215</td>
<td>17.4</td>
<td>22.2</td>
</tr>
<tr>
<td>0.5</td>
<td>855</td>
<td>206</td>
<td>2 474</td>
<td>19.4</td>
<td>24.7</td>
</tr>
</tbody>
</table>
## Table: Impact of a new mortgage rate (100 KUSD, 5%, 20-year)

<table>
<thead>
<tr>
<th>$i$ (in %)</th>
<th>$A$ (in $)</th>
<th>$D_A$ (in $)</th>
<th>$D_R$ (in %)</th>
<th>$C$ (in %)</th>
<th>$N$ (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>660</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>633</td>
<td>27</td>
<td>328</td>
<td>4.1</td>
<td>6.6</td>
</tr>
<tr>
<td>4.0</td>
<td>606</td>
<td>54</td>
<td>648</td>
<td>8.2</td>
<td>13.0</td>
</tr>
<tr>
<td>3.5</td>
<td>580</td>
<td>80</td>
<td>960</td>
<td>12.1</td>
<td>19.2</td>
</tr>
<tr>
<td>3.0</td>
<td>555</td>
<td>105</td>
<td>1,264</td>
<td>16.0</td>
<td>25.3</td>
</tr>
<tr>
<td>2.5</td>
<td>530</td>
<td>130</td>
<td>1,561</td>
<td>19.7</td>
<td>31.2</td>
</tr>
<tr>
<td>2.0</td>
<td>506</td>
<td>154</td>
<td>1,849</td>
<td>23.3</td>
<td>37.0</td>
</tr>
<tr>
<td>1.5</td>
<td>483</td>
<td>177</td>
<td>2,129</td>
<td>26.9</td>
<td>42.6</td>
</tr>
<tr>
<td>1.0</td>
<td>460</td>
<td>200</td>
<td>2,401</td>
<td>30.3</td>
<td>48.0</td>
</tr>
<tr>
<td>0.5</td>
<td>438</td>
<td>222</td>
<td>2,664</td>
<td>33.6</td>
<td>53.3</td>
</tr>
</tbody>
</table>
### Table: Impact of a new mortgage rate (100 KUSD, 10%, 10-year)

<table>
<thead>
<tr>
<th>$i$ (in %)</th>
<th>$A$ (in $)</th>
<th>$D_A$ (in $)_{Monthly}$</th>
<th>$D_A$ (in $)_{Anually}$</th>
<th>$D_R$ (in %)</th>
<th>$\mathcal{C}$ (in %)</th>
<th>$\mathcal{N}$ (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>1322</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.0</td>
<td>1267</td>
<td>55</td>
<td>657</td>
<td>4.1</td>
<td>6.6</td>
<td>9.33</td>
</tr>
<tr>
<td>8.0</td>
<td>1213</td>
<td>108</td>
<td>1299</td>
<td>8.2</td>
<td>13.0</td>
<td>8.75</td>
</tr>
<tr>
<td>7.0</td>
<td>1161</td>
<td>160</td>
<td>1925</td>
<td>12.1</td>
<td>19.3</td>
<td>8.33</td>
</tr>
<tr>
<td>6.0</td>
<td>1110</td>
<td>211</td>
<td>2536</td>
<td>16.0</td>
<td>25.4</td>
<td>7.92</td>
</tr>
<tr>
<td>5.0</td>
<td>1061</td>
<td>261</td>
<td>3130</td>
<td>19.7</td>
<td>31.3</td>
<td>7.58</td>
</tr>
<tr>
<td>4.0</td>
<td>1012</td>
<td>309</td>
<td>3709</td>
<td>23.3</td>
<td>37.1</td>
<td>7.25</td>
</tr>
<tr>
<td>3.0</td>
<td>966</td>
<td>356</td>
<td>4271</td>
<td>26.9</td>
<td>42.7</td>
<td>6.92</td>
</tr>
<tr>
<td>2.0</td>
<td>920</td>
<td>401</td>
<td>4816</td>
<td>30.4</td>
<td>48.2</td>
<td>6.67</td>
</tr>
<tr>
<td>1.0</td>
<td>876</td>
<td>445</td>
<td>5346</td>
<td>33.7</td>
<td>53.5</td>
<td>6.50</td>
</tr>
</tbody>
</table>
Prepayment risk

Reduced-form models

**Figure:** Evolution of 30-year and 15-year mortgage rates in the US

\[ \Delta i(t) = i_{30Y}(t) - i_{15Y}(t) \]
We have:

\[ S(t, u) = S_c(t, u) \cdot S_p(t, u) \]

where \( S_c(t, u) \) is the traditional amortization function (or the contract-based survival function) and \( S_p(t, u) \) is the prepayment-based survival function.
Prepayment risk
Survival function with prepayment risk

**Figure:** Survival function in the case of prepayment
Prepayment risk
Specification of the hazard function

- \( S_p (t, u) \) can be decomposed into the product of two survival functions:

\[
S_p (t, u) = S_{\text{refinancing}} (t, u) \cdot S_{\text{turnover}} (t, u)
\]

- OTC model:

\[
\lambda_p (t, u) = \lambda_{\text{age}} (u - t) \cdot \lambda_{\text{seasonality}} (u) \cdot \lambda_{\text{rate}} (u)
\]

where \( \lambda_{\text{age}} \) measures the impact of the loan age, \( \lambda_{\text{seasonality}} \) corresponds to the seasonality factor and \( \lambda_{\text{rate}} \) represents the influence of market rates.
Prepayment risk
Specification of the hazard function

Figure: Components of the OTC model
Prepayment risk
Statistical measure of prepayment

- Single monthly mortality:
  \[
  \text{SMM} = \frac{\text{prepayments during the month}}{\text{outstanding amount at the beginning of the month}}
  \]

- The constant prepayment rate (CPR) and the SMM are related by the following equation:
  \[
  \text{CPR} = (1 - (1 - \text{SMM}))^{12}
  \]

- In IRRBB, the CPR is also known as the conditional prepayment rate:
  \[
  \text{CPR}(u, t) = \Pr\{u < \tau \leq u + 1 \mid \tau \geq u\}
  = \frac{S_p(t, u) - S_p(t, u + 1)}{S_p(t, u)}
  = 1 - \exp\left(-\int_u^{u+1} \lambda_p(t, s) \, ds\right)
  \]
Prepayment risk
Statistical measure of prepayment

**Table:** Conditional prepayment rates in June 2018 by coupon rate and issuance date

<table>
<thead>
<tr>
<th>Year</th>
<th>2012</th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
<th>2016</th>
<th>2017</th>
<th>2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon = 3%</td>
<td>9.6%</td>
<td>10.2%</td>
<td>10.9%</td>
<td>10.0%</td>
<td>8.7%</td>
<td>5.3%</td>
<td>3.1%</td>
</tr>
<tr>
<td>Coupon = 4.5%</td>
<td>16.1%</td>
<td>15.8%</td>
<td>16.6%</td>
<td>17.9%</td>
<td>17.4%</td>
<td>12.8%</td>
<td>5.3%</td>
</tr>
<tr>
<td>Difference</td>
<td>6.5%</td>
<td>5.6%</td>
<td>5.7%</td>
<td>8.0%</td>
<td>8.7%</td>
<td>7.6%</td>
<td>2.2%</td>
</tr>
</tbody>
</table>
The funding risk of term deposits

- A term deposit, also known as time deposit or certificate of deposit (CD), is a fixed-term cash investment. The client deposits a minimum sum of money into a banking account in exchange for a fixed rate over a specified period.

- When buying a term deposit, the investor can withdraw their funds only after the term ends.

- Under some conditions, the investor may withdraw his term deposit before the maturity date if he pays early redemption costs and fees.
Early time deposit withdrawals may be motivated by two reasons:

1. Economic motivation: \( i(t) \gg i_0 \)

2. Negative liquidity shocks of depositors

The redemption-based survival function of time deposits can be decomposed as:

\[
S_r(t, u) = S_{\text{economic}}(t, u) \cdot S_{\text{liquidity}}(t, u)
\]
Redemption risk
Modeling the early withdrawal risk

Early withdrawals due to economic reasons

- We note $t$ the current date, $m$ the maturity of the time deposit and $N_0$ the initial investment at time 0
- The value of the time deposit at the maturity is equal to $V_0 = N_0 (1 + i_0)^m$
- The value of the investment for $\tau = t$ becomes:

$$V_r(t) = N_0 \cdot (1 + (1 - \varphi(t)) i_0)^t \cdot (1 + i(t))^{m-t} - C(t)$$

where $\varphi(t)$ is the penalty parameter applied to interest paid and $C(t)$ is the break fee
- The rational investor redeems the term deposit if the refinancing incentive is positive:

$$RI(t) = \frac{V_r(t) - V_0}{N_0} > 0$$
Redemption risk
Modeling the early withdrawal risk

Early withdrawals due to economic reasons

- We can assume that:

\[ \lambda_{\text{economic}} (t, u) = g (i (u) - i_0) \]

or:

\[ \lambda_{\text{economic}} (t, u) = g (r (u) - i_0) \]
Early withdrawals due to negative liquidity shocks

We can decompose the hazard function into two effects:

\[ \lambda_{\text{liquidity}}(t, u) = \lambda_{\text{structural}} + \lambda_{\text{cyclical}}(u) \]

where \( \lambda_{\text{structural}} \) is the structural rate of redemption and \( \lambda_{\text{cyclical}}(u) \) is the liquidity component due to the economic cycle. A simple way to model \( \lambda_{\text{cyclical}}(u) \) is to consider a linear function of the GDP growth.
Exercises

- Interest rate risk
  - Exercise 7.4.1 – Constant amortization of a loan
  - Exercise 7.4.2 – Computation of the amortization functions $S(t, u)$ and $S^*(t, u)$
  - Exercise 7.4.3 – Continuous-time analysis of the constant amortization mortgage (CAM)

- Non-maturity deposits (NMD)
  - Exercise 7.4.4 – Valuation of non-maturity deposits

- Prepayment risk
  - Exercise 7.4.5 – Impact of prepayment on the amortization scheme of the CAM
References


