Agenda

- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- **Lecture 9: Copulas and Extreme Value Theory**
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models
Sklar’s theorem

A bi-dimensional copula is a function $C$ which satisfies the following properties:

1. $\text{Dom } C = [0, 1] \times [0, 1]$
2. $C(0, u) = C(u, 0) = 0$ and $C(1, u) = C(u, 1) = u$ for all $u$ in $[0, 1]$
3. $C$ is 2-increasing:

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0$$

for all $(u_1, u_2) \in [0, 1]^2$, $(v_1, v_2) \in [0, 1]^2$ such that $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$

Remark

This means that $C$ is a cumulative distribution function with uniform marginals:

$$C(u_1, u_2) = \Pr \{U_1 \leq u_1, U_2 \leq u_2\}$$

where $U_1$ and $U_2$ are two uniform random variables
We consider the function \( C^\perp (u_1, u_2) = u_1 u_2 \). We have:

- \( C^\perp (0, u) = C^\perp (u, 0) = 0 \)
- \( C^\perp (1, u) = C^\perp (u, 1) = u \)

Since we have \( v_2 - u_2 \geq 0 \) and \( v_1 \geq u_1 \), it follows that \( v_1 (v_2 - u_2) \geq u_1 (v_2 - u_2) \) and:

\[
v_1 v_2 + u_1 u_2 - u_1 v_2 - v_1 u_2 \geq 0
\]

\( \Rightarrow C^\perp \) is a copula function and is called the product copula
Let $F_1$ and $F_2$ be two univariate distributions. 
$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$ is a probability distribution with marginals $F_1$ and $F_2$:

- $u_i = F_i(x_i)$ defines a uniform transformation ($u_i \in [0, 1]$)
- $C(F_1(x_1), F_2(\infty)) = C(F_1(x_1), 1) = F_1(x_1)$

Sklar also shows that:

- Any bivariate distribution $F$ admits a copula representation:

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

- The copula $C$ is unique if the marginals are continuous
The Gumbel logistic distribution is equal to:

\[ F(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2}\right)^{-1} \]

We have:

\[ F_1(x_1) \equiv F(x_1, \infty) = (1 + e^{-x_1})^{-1} \]

and \[ F_2(x_2) \equiv (1 + e^{-x_2})^{-1} \]. The quantile functions are then:

\[ F_1^{-1}(u_1) = \ln u_1 - \ln (1 - u_1) \]

and \[ F_2^{-1}(u_2) = \ln u_2 - \ln (1 - u_2) \]. We finally deduce that:

\[ C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} \]

is the Gumbel logistic copula
Expression of the copula density function

If the joint distribution function $F(x_1, x_2)$ is absolutely continuous, we obtain:

$$f(x_1, x_2) = \partial_{1,2} F(x_1, x_2)$$
$$= \partial_{1,2} C(F_1(x_1), F_2(x_2))$$
$$= c(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2)$$

where $f(x_1, x_2)$ is the joint probability density function, $f_1$ and $f_2$ are the marginal densities and $c$ is the copula density:

$$c(u_1, u_2) = \partial_{1,2} C(u_1, u_2)$$

Remark

*The condition $C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0$ is equivalent to $\partial_{1,2} C(u_1, u_2) \geq 0$ when the copula density exists.*
In the case of the Gumbel logistic copula, we have:

\[ C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2} \]

and:

\[ c(u_1, u_2) = \frac{2u_1 u_2}{(u_1 + u_2 - u_1 u_2)^3} \]
Expression of the copula density function

We deduce that:

\[ c(u_1, u_2) = \frac{f(F_1^{-1}(u_1), F_2^{-1}(u_2))}{f_1(F_1^{-1}(u_1)) \cdot f_2(F_2^{-1}(u_2))} \]

If we consider the Normal copula, we have:

\[ C(u_1, u_2; \rho) = \Phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho) \]

and:

\[ c(u_1, u_2; \rho) = \frac{2\pi (1 - \rho^2)^{-1/2} \exp \left( -\frac{1}{2(1-\rho^2)} (x_1^2 + x_2^2 - 2\rho x_1 x_2) \right)}{(2\pi)^{-1/2} \exp(-\frac{1}{2}x_1^2) \cdot (2\pi)^{-1/2} \exp(-\frac{1}{2}x_2^2)} \]

\[ = \frac{1}{\sqrt{1 - \rho^2}} \exp \left( -\frac{1}{2} \frac{(x_1^2 + x_2^2 - 2\rho x_1 x_2)}{(1 - \rho^2)} + \frac{1}{2} (x_1^2 + x_2^2) \right) \]

where \( x_1 = \Phi_1^{-1}(u_1) \) and \( x_2 = \Phi_2^{-1}(u_2) \)
Expression of the copula density function

**Figure:** Construction of a bivariate probability distribution with given marginals and the Normal copula
Concordance ordering

Let $C_1$ and $C_2$ be two copula functions. We say that the copula $C_1$ is smaller than the copula $C_2$ and we note $C_1 \prec C_2$ if we have:

$$C_1(u_1, u_2) \leq C_2(u_1, u_2)$$

for all $(u_1, u_2) \in [0, 1]^2$

Let $C_\theta (u_1, u_2) = C(u_1, u_2; \theta)$ be a family of copula functions that depends on the parameter $\theta$. The copula family $\{C_\theta\}$ is totally ordered if, for all $\theta_2 \geq \theta_1$, $C_{\theta_2} \succeq C_{\theta_1}$ (positively ordered) or $C_{\theta_2} \prec C_{\theta_1}$ (negatively ordered)

Remark

*The Normal copula family is positively ordered*
Fréchet bounds

We have:

\[ C^- \prec C \prec C^+ \]

where:

\[ C^- (u_1, u_2) = \max (u_1 + u_2 - 1, 0) \]

and:

\[ C^+ (u_1, u_2) = \min (u_1, u_2) \]
The multivariate case

The canonical decomposition of a multivariate distribution function is:

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

We have:

$$C^- \prec C \prec C^+$$

where:

$$C^- (u_1, \ldots, u_n) = \max \left( \sum_{i=1}^{n} u_i - n + 1, 0 \right)$$

and:

$$C^+ (u_1, \ldots, u_n) = \min (u_1, \ldots, u_n)$$

**Remark**

$C^-$ is not a copula when $n \geq 3$
Countermonotonicity and comonotonicity

Let $X = (X_1, X_2)$ be a random vector with distribution $F$. We define the copula of $(X_1, X_2)$ by the copula of $F$:

$$F(x_1, x_2) = C\langle X_1, X_2 \rangle (F_1(x_1), F_2(x_2))$$

**Definition**

- $X_1$ and $X_2$ are countermonotonic – or $C\langle X_1, X_2 \rangle = C^-$ – if there exists a random variable $X$ such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where $f_1$ and $f_2$ are respectively decreasing and increasing functions. In this case, $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is a decreasing function.

- $X_1$ and $X_2$ are independent if the dependence function is the product copula $C^\perp$.

- $X_1$ are $X_2$ are comonotonic – or $C\langle X_1, X_2 \rangle = C^+$ – if there exists a random variable $X$ such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where $f_1$ and $f_2$ are both increasing functions. In this case, $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is an increasing function.
We consider a uniform random vector \((U_1, U_2)\):

\[
C(U_1, U_2) = C^- \iff U_2 = 1 - U_1
\]

\[
C(U_1, U_2) = C^+ \iff U_2 = U_1
\]

We consider a standardized Gaussian random vector \((X_1, X_2)\). We have \(U_1 = \Phi(X_1)\) and \(U_2 = \Phi(X_2)\). We deduce that:

\[
C(X_1, X_2) = C^- \iff \Phi(X_2) = 1 - \Phi(X_1) \iff X_2 = -X_1
\]

\[
C(X_1, X_2) = C^+ \iff \Phi(X_2) = \Phi(X_1) \iff X_2 = X_1
\]
Countermonotonicity and comonotonicity

We consider a random vector \((X_1, X_2)\) where \(X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)\). We have

\[
U_i = \Phi \left( \frac{X_i - \mu_i}{\sigma_i} \right)
\]

We deduce that:

\[
\mathbf{C} \langle X_1, X_2 \rangle = \mathbf{C}^- \iff \Phi \left( \frac{X_2 - \mu_2}{\sigma_2} \right) = 1 - \Phi \left( \frac{X_1 - \mu_1}{\sigma_1} \right)
\]

\[
\iff \Phi \left( \frac{X_2 - \mu_2}{\sigma_2} \right) = \Phi \left( - \frac{X_1 - \mu_1}{\sigma_1} \right)
\]

\[
\iff X_2 = \left( \mu_2 + \frac{\sigma_2}{\sigma_1} \mu_1 \right) - \frac{\sigma_2}{\sigma_1} X_1
\]

and:

\[
\mathbf{C} \langle X_1, X_2 \rangle = \mathbf{C}^+ \iff X_2 = \left( \mu_2 - \frac{\sigma_2}{\sigma_1} \mu_1 \right) + \frac{\sigma_2}{\sigma_1} X_1
\]
Countermonotonicity and comonotonicity

- We consider a random vector \((X_1, X_2)\) where \(X_i \sim \mathcal{LN}(\mu_i, \sigma_i^2)\). We have:

\[
U_i = \Phi\left(\frac{\ln X_i - \mu_i}{\sigma_i}\right)
\]

We deduce that:

\[
\mathbf{C} \langle X_1, X_2 \rangle = \mathbf{C}^- \iff \ln X_2 = \left(\mu_2 + \frac{\sigma_2}{\sigma_1} \mu_1\right) - \frac{\sigma_2}{\sigma_1} \ln X_1
\]

\[
\iff X_2 = e^{\left(\mu_2 + \frac{\sigma_2}{\sigma_1} \mu_1\right)} e^{-\frac{\sigma_2}{\sigma_1} \ln X_1}
\]

\[
\iff X_2 = e^{\left(\mu_2 + \frac{\sigma_2}{\sigma_1} \mu_1\right)} X_1^{-\frac{\sigma_2}{\sigma_1}}
\]

and:

\[
\mathbf{C} \langle X_1, X_2 \rangle = \mathbf{C}^+ \iff \ln X_2 = \left(\mu_2 - \frac{\sigma_2}{\sigma_1} \mu_1\right) + \frac{\sigma_2}{\sigma_1} \ln X_1
\]

\[
\iff X_2 = e^{\left(\mu_2 - \frac{\sigma_2}{\sigma_1} \mu_1\right)} X_1^{\frac{\sigma_2}{\sigma_1}}
\]
Countermonotonicity and comonotonicity

- If $X_1 \sim \mathcal{L}N(0, 1)$ and $X_2 \sim \mathcal{L}N(0, 1)$, we have:
  \[ C(X_1, X_2) = C^- \iff X_2 = \frac{1}{X_1} \]

- If $X_1 \sim \mathcal{L}N(0, 2^2)$ and $X_2 \sim \mathcal{L}N(0, 1)$, we have:
  \[ C(X_1, X_2) = C^+ \iff X_2 = \sqrt{X_1} \]

Linear dependence vs non-linear dependence

The concepts of counter- and comonotonicity concepts generalize the cases where the linear correlation of a Gaussian vector is equal to $-1$ or $+1$. 
Non-linear stochastic dependence

**Scale invariance property**

If $h_1$ and $h_2$ are two increasing functions on $\text{Im} X_1$ and $\text{Im} X_2$, then we have:

$$C \langle h_1 (X_1), h_2 (X_2) \rangle = C \langle X_1, X_2 \rangle$$
Proof (marginals)

We note $F$ and $G$ the probability distributions of the random vectors $(X_1, X_2)$ and $(Y_1, Y_2) = (h_1(X_1), h_2(X_2))$. The marginals of $G$ are:

$$G_1(y_1) = \Pr \{ Y_1 \leq y_1 \}$$
$$= \Pr \{ h_1(X_1) \leq y_1 \}$$
$$= \Pr \{ X_1 \leq h_1^{-1}(y_1) \} \quad \text{(because $h_1$ is strictly increasing)}$$
$$= F_1(h_1^{-1}(y_1))$$

and $G_2(y_2) = F_2(h_2^{-1}(y_2))$. We deduce that $G_1^{-1}(u_1) = h_1(F_1^{-1}(u_1))$ and $G_2^{-1}(u_2) = h_2(F_2^{-1}(u_2))$.
Proof (copula)

By definition, we have:

\[ C(Y_1, Y_2)(u_1, u_2) = G(G_1^{-1}(u_1), G_2^{-1}(u_2)) \]

Moreover, it follows that:

\[
G(G_1^{-1}(u_1), G_2^{-1}(u_2)) = \Pr \{ Y_1 \leq G_1^{-1}(u_1), Y_2 \leq G_2^{-1}(u_2) \} \\
= \Pr \{ h_1(X_1) \leq G_1^{-1}(u_1), h_2(X_2) \leq G_2^{-1}(u_2) \} \\
= \Pr \{ X_1 \leq h_1^{-1}(G_1^{-1}(u_1)), X_2 \leq h_2^{-1}(G_2^{-1}(u_2)) \} \\
= \Pr \{ X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2) \} \\
= F(F_1^{-1}(u_1), F_2^{-1}(u_2))
\]

Because we have \( C(X_1, X_2)(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)) \), we deduce that:

\[ C(Y_1, Y_2) = C(X_1, X_2) \]
We have:

\[ G(y_1, y_2) = C \langle X_1, X_2 \rangle (G_1(y_1), G_2(y_1)) = C \langle X_1, X_2 \rangle (F_1(h_1^{-1}(y_1)), F_2(h_2^{-1}(y_2))) \]

Applying an increasing transformation does not change the copula function, only the marginals.

The copula function is the minimum exhaustive statistic of the dependence.
If $X_1$ and $X_2$ are two positive random variables, the previous theorem implies that:

\[
C \langle X_1, X_2 \rangle = C \langle \ln X_1, X_2 \rangle = C \langle \ln X_1, \ln X_2 \rangle = C \langle X_1, \exp X_2 \rangle = C \langle \sqrt{X_1}, \exp X_2 \rangle
\]
A numeric measure $m$ of association between $X_1$ and $X_2$ is a measure of concordance if it satisfies the following properties:

1. $-1 = m\langle X, -X \rangle \leq m\langle C \rangle \leq m\langle X, X \rangle = 1$;
2. $m\langle C^\perp \rangle = 0$;
3. $m\langle -X_1, X_2 \rangle = m\langle X_1, -X_2 \rangle = -m\langle X_1, X_2 \rangle$;
4. If $C_1 \prec C_2$, then $m\langle C_1 \rangle \leq m\langle C_2 \rangle$;

We have:

$C \prec C^\perp \Rightarrow m\langle C \rangle < 0$

and:

$C \succ C^\perp \Rightarrow m\langle C \rangle > 0$
Kendall’s tau and Spearman’s rho

- Kendall’s tau is the probability of concordance minus the probability of discordance:

\[
\tau = \Pr \{(X_i - X_j) \cdot (Y_i - Y_j) > 0\} - \Pr \{(X_i - X_j) \cdot (Y_i - Y_j) < 0\} \\
= 4 \int_0^1 \int_0^1 C(u_1, u_2) \, dC(u_1, u_2) - 1
\]

- Spearman’s rho is the linear correlation of the rank statistics:

\[
\rho = \frac{\text{cov}(F_X(X), F_Y(Y))}{\sigma(F_X(X)) \cdot \sigma(F_Y(Y))} \\
= 12 \int_0^1 \int_0^1 u_1 u_2 \, dC(u_1, u_2) - 3
\]

- For the normal copula, we have:

\[
\tau = \frac{2}{\pi} \arcsin \rho \quad \text{and} \quad \rho = \frac{6}{\pi} \arcsin \frac{\rho}{2}
\]
Exhaustive vs non-exhaustive statistics of stochastic dependence

**Figure:** Contour lines of bivariate densities (Normal copula with $\tau = 50\%$)
The linear correlation (or Pearson’s correlation) is defined as follows:

\[
\rho \langle X_1, X_2 \rangle = \frac{\mathbb{E} [X_1 \cdot X_2] - \mathbb{E} [X_1] \cdot \mathbb{E} [X_2]}{\sigma (X_1) \cdot \sigma (X_2)}
\]

It satisfies the following properties:

- if \( C \langle X_1, X_2 \rangle = C^\perp \), then \( \rho \langle X_1, X_2 \rangle = 0 \)
- \( \rho \) is an increasing function with respect to the concordance measure:
  \[
  C_1 \succ C_2 \Rightarrow \rho_1 \langle X_1, X_2 \rangle \geq \rho_2 \langle X_1, X_2 \rangle
  \]
- \( \rho \langle X_1, X_2 \rangle \) is bounded:
  \[
  \rho^- \langle X_1, X_2 \rangle \leq \rho \langle X_1, X_2 \rangle \leq \rho^+ \langle X_1, X_2 \rangle
  \]
  and the bounds are reached for the Fréchet copulas \( C^- \) and \( C^+ \)
However, we don’t have $\rho \langle C^- \rangle = -1$ and $\rho \langle C^+ \rangle = +1$. If we use the stochastic representation of Fréchet bounds, we have:

$$
\rho^- \langle X_1, X_2 \rangle = \rho^+ \langle X_1, X_2 \rangle = \frac{E[f_1(X) \cdot f_2(X)] - E[f_1(X)] \cdot E[f_2(X)]}{\sigma(f_1(X)) \cdot \sigma(f_2(X))}
$$

The solution of the equation $\rho^- \langle X_1, X_2 \rangle = -1$ is $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ where $a_1 a_2 < 0$. For the equation $\rho^+ \langle X_1, X_2 \rangle = +1$, the condition becomes $a_1 a_2 > 0$.

Moreover, we have:

$$
\rho \langle X_1, X_2 \rangle = \rho \langle f_1(X_1), f_2(X_2) \rangle \iff \begin{cases} 
  f_1(x) = a_1 x + b_1 \\
  f_2(x) = a_2 x + b_2 \\
  a_1 a_2 > 0
\end{cases}
$$

**Remark**

*The linear correlation is only valid for a linear (or Gaussian) world. The copula function generalizes the concept of linear correlation in a non-Gaussian non-linear world.*
We consider the bivariate log-normal random vector \((X_1, X_2)\) where \(X_1 \sim \mathcal{LN}(\mu_1, \sigma^2_1)\), \(X_2 \sim \mathcal{LN}(\mu_2, \sigma^2_2)\) and \(\rho = \rho(\ln X_1, \ln X_2)\).

We can show that:

\[
\mathbb{E}[X_1^{p_1} \cdot X_2^{p_2}] = \exp\left(p_1\mu_1 + p_2\mu_2 + \frac{p_1^2\sigma^2_1 + p_2^2\sigma^2_2}{2} + p_1p_2\rho\sigma_1\sigma_2\right)
\]

and:

\[
\rho(X_1, X_2) = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{\exp(\sigma^2_1) - 1} \cdot \sqrt{\exp(\sigma^2_2) - 1}}
\]
If $\sigma_1 = 1$ and $\sigma_2 = 3$, we obtain the following results:

<table>
<thead>
<tr>
<th>Copula</th>
<th>$\rho \langle X_1, X_2 \rangle$</th>
<th>$\tau \langle X_1, X_2 \rangle$</th>
<th>$\varrho \langle X_1, X_2 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^-$</td>
<td>$-0.008$</td>
<td>$-1.000$</td>
<td>$-1.000$</td>
</tr>
<tr>
<td>$\rho = -0.7$</td>
<td>$-0.007$</td>
<td>$-0.494$</td>
<td>$-0.683$</td>
</tr>
<tr>
<td>$C^\perp$</td>
<td>$0.000$</td>
<td>$0.000$</td>
<td>$0.000$</td>
</tr>
<tr>
<td>$\rho = 0.7$</td>
<td>$0.061$</td>
<td>$0.494$</td>
<td>$0.683$</td>
</tr>
<tr>
<td>$C^+$</td>
<td>$0.162$</td>
<td>$1.000$</td>
<td>$1.000$</td>
</tr>
</tbody>
</table>
Tail dependence

Definition

We consider the following statistic:

\[ \lambda^+ = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \]

We say that \( C \) has an upper tail dependence when \( \lambda^+ \in (0, 1] \) and \( C \) has no upper tail dependence when \( \lambda^+ = 0 \).

- For the lower tail dependence \( \lambda^- \), the limit becomes:

\[ \lambda^- = \lim_{u \to 0^+} \frac{C(u, u)}{u} \]

- We notice that \( \lambda^+ \) and \( \lambda^- \) can also be defined as follows:

\[ \lambda^+ = \lim_{u \to 1^-} \Pr \{ U_2 > u \mid U_1 > u \} \]

and:

\[ \lambda^- = \lim_{u \to 0^+} \Pr \{ U_2 < u \mid U_1 < u \} \]
Tail dependence

- For the copula functions \( C^- \) and \( C^\perp \), we have \( \lambda^- = \lambda^+ = 0 \)
- For the copula \( C^+ \), we obtain \( \lambda^- = \lambda^+ = 1 \)
- In the case of the Gumbel copula:
  \[
  C(u_1, u_2; \theta) = \exp \left( - \left[ (-\ln u_1)^{\theta} + (-\ln u_2)^{\theta} \right]^{1/\theta} \right)
  \]
  we obtain \( \lambda^- = 0 \) and \( \lambda^+ = 2 - 2^{1/\theta} \)
- In the case of the Clayton copula:
  \[
  C(u_1, u_2; \theta) = \left( u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-1/\theta}
  \]
  we obtain \( \lambda^- = 2^{-1/\theta} \) and \( \lambda^+ = 0 \)
The quantile-quantile dependence function is equal to:

\[
\lambda^+ (\alpha) = \frac{\Pr \{ X_2 > F_2^{-1} (\alpha) \mid X_1 > F_1^{-1} (\alpha) \}}{\Pr \{ X_1 > F_1^{-1} (\alpha) \}}
\]

\[
= 1 - \Pr \{ X_1 \leq F_1^{-1} (\alpha) \} - \Pr \{ X_2 \leq F_2^{-1} (\alpha) \} + \frac{\Pr \{ X_2 \leq F_2^{-1} (\alpha), X_1 \leq F_1^{-1} (\alpha) \}}{1 - \Pr \{ F_1 (X_1) \leq \alpha \}}
\]

\[
= 1 - 2\alpha + \mathbf{C} (\alpha, \alpha)
\]

\[
= \frac{1 - 2\alpha + \mathbf{C} (\alpha, \alpha)}{1 - \alpha}
\]
Tail dependence

Figure: Quantile-quantile dependence measures $\lambda^+ (\alpha)$ and $\lambda^- (\alpha)$
We consider two portfolios, whose losses correspond to the random variables $L_1$ and $L_2$ with probability distributions $F_1$ and $F_2$. We have:

$$\lambda^+ (\alpha) = \Pr \{ L_2 > F_2^{-1} (\alpha) \mid L_1 > F_1^{-1} (\alpha) \}$$

$$= \Pr \{ L_2 > \text{VaR}_\alpha (L_2) \mid L_1 > \text{VaR}_\alpha (L_1) \}$$
Archimedean copulas

Definition

An Archimedean copula is defined by:

\[
C(u_1, u_2) = \begin{cases} 
\varphi^{-1}(\varphi(u_1) + \varphi(u_2)) & \text{if } \varphi(u_1) + \varphi(u_2) \leq \varphi(0) \\
0 & \text{otherwise}
\end{cases}
\]

where \( \varphi \) a \( C^2 \) is a function which satisfies \( \varphi(1) = 0 \), \( \varphi'(u) < 0 \) and \( \varphi''(u) > 0 \) for all \( u \in [0, 1] \)

\( \Rightarrow \) \( \varphi(u) \) is called the generator of the copula function
Archimedean copulas

Example

If \( \varphi(u) = u^{-1} - 1 \), we have \( \varphi^{-1}(u) = (1 + u)^{-1} \) and:

\[
C(u_1, u_2) = \left(1 + (u_1^{-1} - 1 + u_2^{-1} - 1)\right)^{-1} = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}
\]

The Gumbel logistic copula is then an Archimedean copula.

Remark

- **The product copula** \( C^\perp \) **is Archimedean and the associated generator is** \( \varphi(u) = -\ln u \)
- **Concerning Fréchet copulas, only** \( C^- \) **is Archimedean with** \( \varphi(u) = 1 - u \)
## Archimedean copulas

<table>
<thead>
<tr>
<th>Copula</th>
<th>( \phi(u) )</th>
<th>( C(u_1, u_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cl^{-}</td>
<td>(- \ln u)</td>
<td>( u_1 u_2 )</td>
</tr>
<tr>
<td>Clayton</td>
<td>( u^{-\theta} - 1 )</td>
<td>( \left( u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-1/\theta} )</td>
</tr>
<tr>
<td>Frank</td>
<td>(- \ln \frac{e^{-\theta u} - 1}{e^{-\theta} - 1} )</td>
<td>(- \frac{1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u_1} - 1) (e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right) )</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( (-\ln u)^{\theta} )</td>
<td>( \exp \left( - \left( \tilde{u}_1^{\theta} + \tilde{u}_2^{\theta} \right)^{1/\theta} \right) )</td>
</tr>
<tr>
<td>Joe</td>
<td>(- \ln \left( 1 - (1 - u)^{\theta} \right) )</td>
<td>( 1 - \left( \tilde{u}_1^{\theta} + \tilde{u}_2^{\theta} - \tilde{u}_1^{\theta} \tilde{u}_2^{\theta} \right)^{1/\theta} )</td>
</tr>
</tbody>
</table>

*We use the notations \( \tilde{u} = 1 - u \) and \( \tilde{u} = -\ln u \)*
The Normal copula is the dependence function of the multivariate normal distribution with a correlation matrix $\rho$:

$$C(u_1,\ldots,u_n;\rho) = \Phi_n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\rho)$$

By using the canonical decomposition of the multivariate density function:

$$f(x_1,\ldots,x_n) = c(F_1(x_1),\ldots,F_n(x_n)) \prod_{i=1}^{n} f_i(x_i)$$

we deduce that the probability density function of the Normal copula is:

$$c(u_1,\ldots,u_n;\rho) = \frac{1}{|\rho|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} x^\top (\rho^{-1} - I_n) x \right)$$

where $x_i = \Phi^{-1}(u_i)$
In the bivariate case, we obtain:

\[ c(u_1, u_2; \rho) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left( -\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1 - \rho^2)} + \frac{x_1^2 + x_2^2}{2} \right) \]

It follows that the expression of the bivariate Normal copula function is also equal to:

\[ C(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \phi_2(x_1, x_2; \rho) \, dx_1 \, dx_2 \]

where \( \phi_2(x_1, x_2; \rho) \) is the bivariate normal density:

\[ \phi_2(x_1, x_2; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( -\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1 - \rho^2)} \right) \]
Copulas
Extreme value theory
Definition and properties
Parametric copula functions
Estimation

Bivariate Normal copula

Remark

Let \((X_1, X_2)\) be a standardized Gaussian random vector, whose cross-correlation is \(\rho\). Using the Cholesky decomposition, we write \(X_2\) as follows: \(X_2 = \rho X_1 + \sqrt{1 - \rho^2} X_3\) where \(X_3 \sim \mathcal{N}(0, 1)\) is independent from \(X_1\) and \(X_2\). We have:

\[
\Phi_2 (x_1, x_2; \rho) = \Pr \{ X_1 \leq x_1, X_2 \leq x_2 \} = \mathbb{E} \left[ \Pr \left\{ X_1 \leq x_1, \rho X_1 + \sqrt{1 - \rho^2} X_3 \leq x_2 \mid X_1 \right\} \right] = \int_{-\infty}^{x_1} \Phi \left( \frac{x_2 - \rho x}{\sqrt{1 - \rho^2}} \right) \phi(x) \, dx
\]

It follows that:

\[
C(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi \left( \frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1 - \rho^2}} \right) \phi(x) \, dx
\]
Bivariate Normal copula

- We deduce that:

\[
C(u_1, u_2; \rho) = \int_0^{u_1} \Phi \left( \frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1 - \rho^2}} \right) \, du
\]

- We have:

\[
\tau = \frac{2}{\pi} \arcsin \rho
\]

and:

\[
\varrho = \frac{6}{\pi} \arcsin \frac{\rho}{2}
\]

- We can show that:

\[
\lambda^+ = \lambda^- = \begin{cases} 
0 & \text{if } \rho < 1 \\
1 & \text{if } \rho = 1
\end{cases}
\]
Bivariate Normal copula

Figure: Tail dependence $\lambda^+ (\alpha)$ for the Normal copula
Multivariate Student’s $t$ copula

We have:

$$C(u_1, \ldots, u_n; \rho, \nu) = T_n(T_{\nu}^{-1}(u_1), \ldots, T_{\nu}^{-1}(u_n); \rho, \nu)$$

By using the definition of the cumulative distribution function:

$$T_n(x_1, \ldots, x_n; \rho, \nu) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{\Gamma\left(\frac{\nu+n}{2}\right) |\rho|^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right) (\nu\pi)^{n/2}} \left(1 + \frac{1}{\nu} x^\top \rho^{-1} x\right)^{-\frac{\nu+n}{2}} dx$$

we can show that the copula density function is then:

$$c(u_1, \ldots, u_n; \rho, \nu) = |\rho|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\nu+n}{2}\right) \left[\Gamma\left(\frac{\nu}{2}\right)\right]^n}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n \Gamma\left(\frac{\nu}{2}\right)} \frac{(1 + \frac{1}{\nu} x^\top \rho^{-1} x)^{-\frac{\nu+n}{2}}}{\prod_{i=1}^{n} \left(1 + \frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

where $x_i = T_{\nu}^{-1}(u_i)$
Bivariate Student’s $t$ copula

- We have:
  \[
  C(u_1, u_2; \rho, \nu) = \int_0^{u_1} C_{2|1}(u, u_2; \rho, \nu) \, du
  \]
  where:
  \[
  C_{2|1}(u_1, u_2; \rho, \nu) = T_{\nu+1} \left( \left( \frac{\nu + 1}{\nu + \left[ T_{\nu}^{-1}(u_1) \right]^2} \right)^{1/2} \frac{T_{\nu}^{-1}(u_2) - \rho T_{\nu}^{-1}(u_1)}{\sqrt{1 - \rho^2}} \right)
  \]
- We have:
  \[
  \lambda^+ = 2 - 2 \cdot T_{\nu+1} \left( \left( \frac{(\nu + 1)(1 - \rho)}{(1 + \rho)} \right)^{1/2} \right) = \begin{cases} 
  0 & \text{if } \rho = -1 \\
  > 0 & \text{if } \rho > -1
  \end{cases}
  \]
Bivariate Student’s $t$ copula

Figure: Tail dependence $\lambda^+(\alpha)$ for the Student’s $t$ copula ($\nu = 1$)
Figure: Tail dependence $\lambda^+(\alpha)$ for the Student’s $t$ copula ($\nu = 4$)
Dependogram

The dependogram is the scatter plot between $u_{t,1}$ and $u_{t,2}$ where:

$$u_{t,i} = \frac{1}{T + 1} \mathcal{R}_{t,i}$$

and $\mathcal{R}_{t,i}$ is the rank statistic ($T$ is the sample size)

### Example

<table>
<thead>
<tr>
<th>$x_{t,1}$</th>
<th>-3</th>
<th>4</th>
<th>1</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{t,2}$</td>
<td>105</td>
<td>65</td>
<td>17</td>
<td>9</td>
</tr>
<tr>
<td>$\mathcal{R}_{t,1}$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$\mathcal{R}_{t,2}$</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$u_{t,1}$</td>
<td>0.20</td>
<td>0.60</td>
<td>0.40</td>
<td>0.80</td>
</tr>
<tr>
<td>$u_{t,2}$</td>
<td>0.80</td>
<td>0.60</td>
<td>0.40</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Figure: Dependogram of EU and US equity returns ($\rho = 57.8\%$)
Figure: Dependogram of simulated Gaussian returns ($\rho = 57.8\%$)
The method of moments

If \( \tau = f_\tau(\theta) \) is the relationship between \( \theta \) and Kendall’s tau, the MM estimator is simply the inverse of this relationship:

\[
\hat{\theta} = f^{-1}_\tau(\hat{\tau})
\]

where \( \hat{\tau} \) is the estimate of Kendall’s tau based on the sample

Remark

We have:

\[
\hat{\tau} = \frac{c - d}{c + d}
\]

where \( c \) and \( d \) are the number of concordant and discordant pairs

For instance, in the case of the Gumbel copula, we have:

\[
\tau = \frac{\theta - 1}{\theta}
\]

and:

\[
\hat{\theta} = \frac{1}{1 - \hat{\tau}}
\]
The method of maximum likelihood

We have:

\[ F(x_1, \ldots, x_n) = C(F_1(x_1; \theta_1), \ldots, F_n(x_n; \theta_n); \theta_c) \]

with two types of parameters:

- the parameters \((\theta_1, \ldots, \theta_n)\) of univariate distribution functions
- the parameters \(\theta_c\) of the copula function

The expression of the log-likelihood function is:

\[
\ell(\theta_1, \ldots, \theta_n, \theta_c) = \sum_{t=1}^{T} \ln c(F_1(x_{t,1}; \theta_1), \ldots, F_n(x_{t,n}; \theta_n); \theta_c) + \\
\sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i(x_{t,i}; \theta_i)
\]

The ML estimator is then defined as follows:

\[
(\hat{\theta}_1, \ldots, \hat{\theta}_n, \hat{\theta}_c) = \arg \max \ell(\theta_1, \ldots, \theta_n, \theta_c)
\]
The IFM method is a two-stage parametric method:

1. the first stage involves maximum likelihood from univariate marginals
2. the second stage involves maximum likelihood of the copula parameters $\theta_c$ with the univariate parameters $\hat{\theta}_1, \ldots, \hat{\theta}_n$ held fixed from the first stage:

$$\hat{\theta}_c = \arg \max \sum_{t=1}^{T} \ln c\left( F_1\left( x_{t,1}; \hat{\theta}_1 \right), \ldots, F_n\left( x_{t,n}; \hat{\theta}_n \right); \theta_c \right)$$
The omnibus method replaces the marginals \( F_1, \ldots, F_n \) by their non-parametric estimates:

\[
\hat{\theta}_c = \arg \max \sum_{t=1}^{T} \ln c \left( \hat{F}_1 (x_{t,1}), \ldots, \hat{F}_n (x_{t,n}) ; \theta_c \right)
\]

where:

\[
\hat{F}_i (x_{t,i}) = u_{t,i} = \frac{1}{T+1} \mathcal{R}_{t,i}
\]
In the case of the Normal copula, the matrix $\rho$ of the parameters is estimated with the following algorithm:

1. We first transform the uniform variates $u_{t,i}$ into Gaussian variates:
   \[ n_{t,i} = \Phi^{-1}(u_{t,i}) \]
2. We then calculate the correlation matrix $\hat{\rho}$ of the Gaussian variates $n_{t,i}$. 
Definition

- Let $X_1, \ldots, X_n$ be iid random variables, whose probability distribution is denoted by $F$.
- We rank these random variables by increasing order:

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n-1:n} \leq X_{n:n}$$

- $X_{i:n}$ is called the $i^{th}$ order statistic in the sample of size $n$.
- We note $x_{i:n}$ the corresponding random variate or the value taken by $X_{i:n}$.
Order statistics

We have:

\[ F_{i:n}(x) = \Pr \{ X_{i:n} \leq x \} \]

= \Pr \{ \text{at least} \ i \ \text{variables among} \ X_1, \ldots, X_n \ \text{are less or equal to} \ x \} 

= \sum_{k=i}^{n} \Pr \{ \text{k variables among} \ X_1, \ldots, X_n \ \text{are less or equal to} \ x \} 

= \sum_{k=i}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} 

and:

\[ f_{i:n}(x) = \frac{\partial F_{i:n}(x)}{\partial x} \]
Example

If $X_1, \ldots, X_n$ follow a uniform distribution $\mathcal{U}_{[0,1]}$, we obtain:

$$
F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} x^k (1-x)^{n-k} = IB(x; i, n-i+1)
$$

where $IB(x; \alpha, \beta)$ is the regularized incomplete beta function:

$$
IB(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_{0}^{x} t^{\alpha-1} (1-t)^{\beta-1} \, dt
$$

We deduce that $X_{i:n} \sim \mathcal{B}(i, n-i+1)$ and\footnote{We recall that $\mathbb{E}[\mathcal{B}(\alpha, \beta)] = \alpha / (\alpha + \beta)$}: 

$$
\mathbb{E}[X_{i:n}] = \mathbb{E}[\mathcal{B}(i, n-i+1)] = \frac{i}{n+1}
$$
Extreme order statistics

The extreme order statistics are:

\[ X_{1:n} = \min (X_1, \ldots, X_n) \]

and:

\[ X_{n:n} = \max (X_1, \ldots, X_n) \]

We have:

\[
F_{1:n} (x) = \sum_{k=1}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} = 1 - \binom{n}{0} F(x)^0 (1 - F(x))^n \\
= 1 - (1 - F(x))^n
\]

and:

\[
F_{i:n} (x) = \sum_{k=n}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k} = \binom{n}{n} F(x)^n (1 - F(x))^{n-n} \\
= F(x)^n
\]
Alternative proof

We have:

\[ F_{1:n}(x) = \Pr \{ \min(X_1, \ldots, X_n) \leq x \} = 1 - \Pr \{ \min(X_1, \ldots, X_n) \geq x \} = 1 - \Pr \{ X_1 \geq x, X_2 \geq x, \ldots, X_n \geq x \} = 1 - \prod_{i=1}^{n} \Pr \{ X_i \geq x \} = 1 - \prod_{i=1}^{n} (1 - \Pr \{ X_i \leq x \}) = 1 - (1 - F(x))^n \]

and:

\[ F_{n:n}(x) = \Pr \{ \max(X_1, \ldots, X_n) \leq x \} = \Pr \{ X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x \} = \prod_{i=1}^{n} \Pr \{ X_i \leq x \} = F(x)^n \]
We deduce that the density functions are equal to:

\[ f_{1:n}(x) = n (1 - F(x))^{n-1} f(x) \]

and

\[ f_{n:n}(x) = nF(x)^{n-1} f(x) \]
We consider the daily returns of the MSCI USA index from 1995 to 2015

$H_1$ Daily returns are Gaussian, meaning that:

$$R_t = \hat{\mu} + \hat{\sigma}X_t$$

where $X_t \sim \mathcal{N}(0,1)$, $\hat{\mu}$ is the empirical mean of daily returns and $\hat{\sigma}$ is the daily standard deviation

$H_2$ Daily returns follow a Student’s $t$ distribution\(^1\):

$$R_t = \hat{\mu} + \hat{\sigma} \sqrt{\frac{\nu - 2}{\nu}} X_t$$

where $X_t \sim t_\nu$. We consider two alternative assumptions: $H_{2a} : \nu = 3$ and $H_{2b} : \nu = 6$

\(^1\)We add the factor $\sqrt{(\nu - 2)/\nu}$ in order to verify that $\text{var}(R_t) = \hat{\sigma}^2$
Extreme order statistics

Figure: Density function of the maximum order statistic (daily return of the MSCI USA index, 1995-2015)
Extreme order statistics

Remark

The limit distributions of minima and maxima are degenerate:

\[ \lim_{n \to \infty} F_{1:n}^{1:n}(x) = \lim_{n \to \infty} 1 - (1 - F(x))^n = \begin{cases} 0 & \text{if } F(x) = 0 \\ 1 & \text{if } F(x) > 0 \end{cases} \]

and:

\[ \lim_{n \to \infty} F_{n:n}^{1:n}(x) = \lim_{n \to \infty} F(x)^n = \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases} \]

Remark

We only consider the largest order statistic \( X_{n:n} \) because the minimum order statistic \( X_{1:n} \) is equal to \( Y_{n:n} \) by setting \( Y_i = -X_i \).
Fisher-Tippett theorem

Let $X_1, \ldots, X_n$ be a sequence of iid random variables, whose distribution function is $F$. If there exist two constants $a_n$ and $b_n$ and a non-degenerate distribution function $G$ such that:

$$\lim_{n \to \infty} \Pr \left\{ \frac{X_{n:n} - b_n}{a_n} \leq x \right\} = G(x)$$

then $G$ can be classified as one of the following three types:

<table>
<thead>
<tr>
<th>Type</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(Gumbel) $\Lambda(x) = \exp(-e^{-x})$</td>
</tr>
<tr>
<td>II</td>
<td>(Fréchet) $\Phi_\alpha(x) = \mathbb{1}(x \geq 0) \cdot \exp(-x^{-\alpha})$</td>
</tr>
<tr>
<td>III</td>
<td>(Weibull) $\Psi_\alpha(x) = \mathbb{1}(x \leq 0) \cdot \exp(-(-x)^{\alpha})$</td>
</tr>
</tbody>
</table>

$\Lambda$, $\Phi_\alpha$ and $\Psi_\alpha$ are called extreme value distributions.

Fisher-Tippett theorem $\approx$ an extreme value analog of the central limit theorem
We recall that:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \exp(x)$$
We consider the exponential distribution: $F(x) = 1 - \exp(-\lambda x)$. We have:

$$\lim_{n \to \infty} F_{n:n}(x) = \lim_{n \to \infty} (1 - e^{-\lambda x})^n = \lim_{n \to \infty} \left(1 - \frac{ne^{-\lambda x}}{n}\right)^n = \lim_{n \to \infty} \exp\left(-ne^{-\lambda x}\right) = 0$$

We verify that the limit distribution is degenerate.

If we consider the affine transformation with $a_n = 1/\lambda$ et $b_n = (\ln n) / \lambda$, we obtain:

$$\Pr \left\{ \frac{X_{n:n} - b_n}{a_n} \leq x \right\} = \Pr \{ X_{n:n} \leq a_n x + b_n \} = \left(1 - e^{-\lambda(a_n x + b_n)}\right)^n = (1 - e^{-x - \ln n})^n = \left(1 - \frac{e^{-x}}{n}\right)^n$$

and:

$$G(x) = \lim_{n \to \infty} \left(1 - \frac{e^{-x}}{n}\right)^n = \exp\left(-e^{-x}\right) = \Lambda(x)$$
We combine the three distributions $\Lambda$, $\Phi_\alpha$ et $\Psi_\alpha$ into a single distribution function $GEV (\mu, \sigma, \xi)$:

$$G(x) = \exp \left( - \left( 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right)^{-1/\xi} \right)$$

defined on the support $\Delta = \{x : 1 + \xi \sigma^{-1} (x - \mu) > 0\}$

- the limit case $\xi \to 0$ corresponds to the Gumbel distribution $\Lambda$
- $\xi = -\alpha^{-1} > 0$ defines the Fréchet distribution $\Phi_\alpha$
- the Weibull distribution $\Psi_\alpha$ is obtained by considering $\xi = -\alpha^{-1} < 0$
Generalized extreme value distribution

The density function is equal to:

\[ g(x) = \frac{1}{\sigma} \left( 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right)^{-\frac{1+\xi}{\xi}} \exp \left( - \left( 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right)^{-\frac{1}{\xi}} \right) \]

Block maxima approach

The log-likelihood function is equal to:

\[ \ell_t = -\ln \sigma - \left( \frac{1 + \xi}{\xi} \right) \ln \left( 1 + \xi \left( \frac{x_t - \mu}{\sigma} \right) \right) - \left( 1 + \xi \left( \frac{x_t - \mu}{\sigma} \right) \right)^{-\frac{1}{\xi}} \]

where \( x_t \) is the observed maximum for the \( t^{th} \) period (or block maximum)
We consider the example of the MSCI USA index.

Using daily returns, we calculate the block maximum for each period of 22 trading days and estimate the GEV distribution using the method of maximum likelihood.

We compare the estimated GEV distribution with the distribution function $F_{22:22}(x)$ when we assume that daily returns are Gaussian:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>90%</th>
<th>95%</th>
<th>96%</th>
<th>97%</th>
<th>98%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>3.26%</td>
<td>3.56%</td>
<td>3.65%</td>
<td>3.76%</td>
<td>3.92%</td>
<td>4.17%</td>
</tr>
<tr>
<td>GEV</td>
<td>3.66%</td>
<td>4.84%</td>
<td>5.28%</td>
<td>5.91%</td>
<td>6.92%</td>
<td>9.03%</td>
</tr>
</tbody>
</table>
Figure: Probability density function of the maximum return $R_{22:22}$
Value-at-risk estimation

We recall that the P&L between $t$ and $t + 1$ is equal to:

$$\Pi (w) = P_{t+1} (w) - P_t (w) = P_t (w) \cdot R (w)$$

We have:

$$\text{VaR}_\alpha (w) = -P_t (w) \cdot \hat{F}^{-1} (1 - \alpha)$$

We now estimate the GEV distribution $\hat{G}$ of the maximum of $-R (w)$ for a period of $n$ trading days. The confidence level must be adjusted in order to obtain the same return time:

$$\frac{1}{1 - \alpha} \times 1 \text{ day} = \frac{1}{1 - \alpha_{GEV}} \times n \text{ days} \iff \alpha_{GEV} = 1 - (1 - \alpha) \cdot n$$

It follows that the value-at-risk is equal to:

$$\text{VaR}_\alpha (w) = P (t) \cdot \hat{G}^{-1} (\alpha_{GEV}) = P (t) \cdot \left( \hat{\mu} - \frac{\hat{\sigma}}{\hat{\xi}} \left( 1 - (-\ln \alpha_{GEV})^{-\hat{\xi}} \right) \right)$$

because we have $G^{-1} (\alpha) = \mu - \frac{\sigma}{\xi} \left( 1 - (-\ln \alpha)^{-\xi} \right)$.
## Value-at-risk estimation

**Table:** Comparing Gaussian, historical and GEV value-at-risk measures

<table>
<thead>
<tr>
<th>VaR</th>
<th>α</th>
<th>Long US</th>
<th>Long EM</th>
<th>Long US</th>
<th>Long EM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>99.0%</td>
<td>2.88%</td>
<td>2.83%</td>
<td>3.06%</td>
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Expected shortfall estimation

We use the peak over threshold approach (HFRM, pages 773-777)
Extreme value copulas

Definition

An extreme value (EV) copula satisfies the following relationship:

\[ C \left( u_1^t, \ldots, u_n^t \right) = C^t \left( u_1, \ldots, u_n \right) \]

for all \( t > 0 \)
The Gumbel copula is an EV copula:

\[
C (u_1^t, u_2^t) = \exp \left( - \left( (- \ln u_1^t)^\theta + (- \ln u_2^t)^\theta \right)^{1/\theta} \right) \\
= \exp \left( - \left( t^\theta \left( (- \ln u_1)^\theta + (- \ln u_2)^\theta \right) \right)^{1/\theta} \right) \\
= \left( \exp \left( - \left( (- \ln u_1)^\theta + (- \ln u_2)^\theta \right)^{1/\theta} \right) \right)^{t} \\
= C^t (u_1, u_2)
\]
The Farlie-Gumbel-Morgenstern copula is not an EV copula:

\[
C(u_1^t, u_2^t) = u_1^t u_2^t + \theta u_1^t u_2^t (1 - u_1^t) (1 - u_2^t)
\]
\[
= u_1^t u_2^t (1 + \theta - \theta u_1^t - \theta u_2^t + \theta u_1^t u_2^t)
\]
\[
\neq u_1^t u_2^t (1 + \theta - \theta u_1 - \theta u_2 + \theta u_1 u_2)^t
\]
\[
\neq C^t(u_1, u_2)
\]
Show that:

- $C^+$ is an EV copula
- $C^\perp$ is an EV copula
- $C^-$ is not an EV copula
Let $X = (X_1, \ldots, X_n)$ be a random vector of dimension $n$. We note $X_{m:m}$ the random vector of maxima:

$$X_{m:m} = \left( \begin{array}{c} X_{m:m, 1} \\ \vdots \\ X_{m:m, n} \end{array} \right)$$

and $F_{m:m}$ the corresponding distribution function:

$$F_{m:m}(x_1, \ldots, x_n) = \Pr \{ X_{m:m, 1} \leq x_1, \ldots, X_{m:m, n} \leq x_n \}$$

The multivariate extreme value (MEV) theory considers the asymptotic behavior of the non-degenerate distribution function $G$ such that:

$$\lim_{m \to \infty} \Pr \left( \frac{X_{m:m, 1} - b_{m, 1}}{a_{m, 1}} \leq x_1, \ldots, \frac{X_{m:m, n} - b_{m, n}}{a_{m, n}} \leq x_n \right) = G(x_1, \ldots, x_n)$$
Using Sklar’s theorem, there exists a copula function $C\langle G \rangle$ such that:

$$G(x_1, \ldots, x_n) = C\langle G \rangle(G_1(x_1), \ldots, G_n(x_n))$$

We have:

- The marginals $G_1, \ldots, G_n$ satisfy the Fisher-Tippett theorem
- $C\langle G \rangle$ is an extreme value copula

**Remark**

An extreme value copula satisfies the PQD property:

$$C^{-} \prec C \prec C^{+}$$
We can show that the (upper) tail dependence of $C \langle G \rangle$ is equal to the (upper) tail dependence of $C \langle F \rangle$:

$$\lambda^+ (C \langle G \rangle) = \lambda^+ (C \langle F \rangle)$$

$\Rightarrow$ Extreme values are independent if the copula function $C \langle F \rangle$ has no (upper) tail dependence
Advanced topics

- Maximum domain of attraction
  - Univariate extreme value theory (HFRM, pages 765-770)
  - Multivariate extreme value theory (HFRM, pages 779 and 781-782)
- Deheuvels-Pickands representation (HFRM, pages 779-781)
- Generalized Pareto distribution $\mathcal{GPD} (\sigma, \xi)$ (HFRM, pages 773-777)
Exercises

- **Copulas**
  - Exercise 11.5.5 – Correlated loss given default rates
  - Exercise 11.5.6 – Calculation of correlation bounds
  - Exercise 11.5.7 – The bivariate Pareto copula

- **Extreme value theory**
  - Exercise 12.4.2 – Order statistics and return period
  - Exercise 12.4.4 – Extreme value theory in the bivariate case
  - Exercise 12.4.5 – Maximum domain of attraction in the bivariate case
References


RONCALLI, T. (2020)

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