Financial Risk Management

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Tutorial exercices #1

1 Quadratic Forms

1. Let us consider two assets whose volatility is respectively 10% and 20%. The correlation between these two assets is equal to 50%.
   (a) Write the covariance matrix.
   (b) Calculate the portfolio volatility composed by 50% of the first asset and 50% of the second asset.
   (c) Calculate the volatility of the portfolio being *long* in the first asset of 150% and *short* of 50% of the second asset.

2. Let us consider three assets with the following volatilities 10%, 20% and 30%. The correlation matrix among the three assets is:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 0\% & 100\% \end{pmatrix}$$

   (a) Write the covariance matrix.
   (b) Calculate the volatility of the portfolio being *long* in the first asset of 50% and *short* of 40% of the second asset.

3. Let us consider $X \sim \mathcal{N}(\mu, \Sigma)$ a Gaussian random vector of dimension n. Let us consider $Y = AX$ with A a deterministic matrix of dimension $m \times n$.
   (a) Prove that Y is a Gaussian vector.
   (b) Calculate $\mathbb{E}[Y]$.
   (c) Calculate $\operatorname{cov}[Y]$.

4. Let us consider $X \sim \mathcal{N}(\mu, \Sigma)$ a Gaussian random vector of dimension $n = 2$ where:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

   (a) Let us suppose that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Prove that $\Sigma = PP^\top$ where :

$$P = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

   How do we call this decomposition?
   (b) Deduce an algorithm to simulate a Gaussian vector X.
   (c) What happens to the previous algorithm if $\mu_1 \neq \mu_2$ and $\sigma_1 \neq \sigma_2$.

2 Statistical computations

Let L be the random variable representing the portfolio loss. We denote f and $\mathbf{F}$ the associated density and the distribution functions.

1. Express $\mathbf{F}(x)$ starting from f.
2. Write the definition of $\mathbf{F}^{-1}(\alpha)$.
3. Calculate the quantile $x = \mathbf{F}^{-1}(\alpha)$ while $L \sim \mathcal{N}(\mu, \sigma)$.
4. The same question if $L \sim \mathcal{LN}(\mu, \sigma)$.
5. The same question if $L \sim \mathcal{E}(\lambda)$.
6. Calculate $\mathbf{F}^{-1}(95\%)$ and $\mathbf{F}^{-1}(99\%)$ in the case of the following discrete distribution :

x	0	-5	6	3	8	10
$\Pr\{L = x\}$	10%	50%	23%	7%	6%	4%

3 Value-at-Risk of a long/short portfolio

Let us consider a « long/short » portfolio composed of a long (buying) position in asset A and a short (selling) position in the asset B. The current prices of the two actions are equal to 100 euros.

1. Using the historic prices of the last 250 trading days of asset A and B, we estimate that the annual volatilities $\hat{\sigma}_A$ and $\hat{\sigma}_B$ are both equal to 20%, and that the correlation is equal to 50%. Neglecting the mean effect, calculate the Gaussian VaR of the long/short portfolio for a time horizon of 1 day and for a confidence level of 99%.
2. How do we calculate the historical VaR? Using the historical chocks of the last 250 trading days, the five worst scenarios of the 250 simulated PnLs in a day of the portfolio are $-3{,}37$, $-3{,}09$, $-2{,}72$, $-2{,}67$ and $-2{,}61$. Calculate the historical VaR for a horizon of 1 day and a confidence level of 99%.
3. The long/short portfolio manager decides to sell a call option on action A. How do the previous calculations change if we consider the « *degraded* » method to measure the Value-at-Risk (we suppose that the delta of the option is equal to 50%)?

4 Value-at-Risk of an asset portfolio

Let us consider n securities. Let us denote x the vector of the weights of the portfolio (x_i is the weight of security i and we have $\sum x_i = 1$ and $x_i \geq 0$). Today the value of the portfolio is equal to 20 000 euros. The CSSF (Commission de Surveillance du Secteur Financier, Supervision commission of financial sector) asks you to communicate to him the VaR 99% of the portfolio (in euros) for a period of detension of 1 month.

1. Let us suppose that the risk factors are the yields on securities.
   (a) Let μ and Σ be the vector of the expected returns and the covariance matrix of returns (annual basis). How do we calculate the Gaussian VaR of the portfolio? Using the historic of equity returns of the last 250 trading days, we estimate that the annual portfolio expected return is -34% and the annual volatility associated is 40%. Calculate the VaR requested by CSSF. Discuss the relevance whether to introduce the mean effect in the calculation of VaR and show that the difference between the two measures is firstly of order 2 000 euros.

(b) How do we calculate the historic VaR? Using the historical shocks of the last 260 trading days, the five worst scenarios of the 260 portfolio simulated PnLs of a day are −1,254, −1,124, −1,012, −984 and −983 euros. Calculate the historical VaR of this portfolio requested by CSSF.

2. Let us suppose that the asset returns are explained by m common risk factors. We denote:

$$r_t^i = \sum_{j=1}^{m} \beta_j^i F_t^j + \varepsilon_t^i$$

where r_t^i is the return on asset i on date t, F_t^j is the value of factor j on date t, β_j^i is the exposition of the equity i toward factor j and ε_t^i the idiosyncratic factor of equity i. We suppose that all the idiosyncratic factors are independent among each other and that they are also independent to the common factors $F_t = \left(F_t^1, \ldots, F_t^m\right)$. We assume as well that $\varepsilon_t^i \sim \mathcal{N}(0, \sigma_i)$ et $F_t \sim \mathcal{N}(\mathbf{0}, \Omega)$.

(a) Write the expression of the analytical VaR assuming the previous factorial decomposition (neglect the mean effect).

(b) Let us assume that there exist only one common factor ($m = 1$) and that $\beta_1^i = \beta$ for each equity i (the equities have the same sensibility toward this factor). We denote $F_t^1 \sim \mathcal{N}(0, \sigma_F)$. Write the expression of analytical VaR (as always, neglect the mean effect). Deduce the portfolio corresponding to the minimal VaR in the case when all the equities have the same idiosyncratic volatility.

5 Building stress scenario with extreme values theory

1. Recall the theorem of Fisher-Tippet characterizing the asymptotic law $\max(X_1, \ldots, X_n)$ where X_i are random variables *iid*.

2. Let us denote a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution.

(a) Posing $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$, find the limit distribution $\mathbf{G}$ while $X \sim \mathcal{E}(\lambda)$.

(b) Same question if $X \sim \mathcal{U}_{[0,1]}$ ($a_n = n^{-1}$ et $b_n = 1 - n^{-1}$).

(c) Same question if X is a Pareto distribution :

$$\mathbf{F}(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

where $a_n = \theta \alpha^{-1} n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

3. We recall that the distribution probability function GEV is :

$$\mathbf{G}(x) = \exp\left\{-\left[1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

Could you mention the interest of this distribution? Write the log-likelihood associated to a sample of $\{x_1, \ldots, x_n\}$.

4. Show that , for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the following Gumbel distribution:

$$\mathbf{\Lambda}(x) = \exp\left(-\exp\left(-\left(\frac{x - \mu}{\sigma}\right)\right)\right)$$

5. Define the notion of payback. What is the payback associated with a one day VaR 99 %, a 1 year VaR 95 %?

6. Define the stress-testing. What is its usefulness in the risk management? How is it used in the regulation?

7. Let us consider a trading portfolio whose we modeled the return estimating the parameters of GEV distribution starting from the opposite of the minimum daily returns during n days of trading. We assume that ξ is equal to 1. Let $\check{t}$ be the payback (in trading days). Show that we can estimate the portfolio loss (in %) $r\left(\check{t}\right)$ associated to payback $\check{t}$ using the following approximation:

$$r\left(\check{t}\right) \simeq -\left(\hat{\mu} + \left(\frac{\check{t}}{n} - 1\right)\hat{\sigma}\right)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimated parameters of GEV distribution.

8. Let us consider two portfolios P_1 and P_2. We estimate the parameters of the GEV distribution using the corresponding data in the opposite minima of daily returns on one month of trading (or 20 trading days). Thus, we obtain the following results:

	$\hat{\mu}$	$\hat{\sigma}$	ξ
P_1	0,01	0,03	1,00
P_2	0,10	0,02	1,00

Calculate the stress scenario for each portfolio and for a return time equal to one year. Comment these results.

6 Stress-testing and order statistics

1. Define the stress-testing. What is its usefulness in the management risks? How is it used in the regulations? Give an example of stress testing in market risk and credit risk.

2. Let us consider a portfolio and denote its daily return X. Using the standard assumptions (in particular X is *i.i.d.*), which is the law $\mathbf{G}_n$ of daily returns maximum for a period of n days if we suppose that $X \sim \mathcal{N}(\mu, \sigma)$:

$$\max\left(X_1, \ldots, X_n\right) \sim \mathbf{G}_n$$

3. Using the maximum law, how could you test the hypothesis $X \sim \mathcal{N}(\mu, \sigma)$?

4. Define the notion of payback. Which is the which is the payback associated to a VaR of 1 day 99%, to a VaR 10 days 99%, to a VaR 1 year 99% ? What is the payback (return time) associated to quantiles $\mathbf{G}_1^{-1}(99\%)$, $\mathbf{G}_5^{-1}(99\%)$ and $\mathbf{G}_{22}^{-1}(99\%)$?

5. We consider the maximum law $\mathbf{G}_{20}$ for 20 days. Which is the confidence level α which permits to have a payback associated to the quantile $\mathbf{G}_{20}^{-1}(\alpha)$ equivalent to the one of a classic VaR of 1 day at 99,9%?

7 Risk measures

1. We denote L the distribution loss $\mathbf{F}$.

   (a) Write the definition of *expected shortfall* and of risk value.

   (b) Show that:

$$\mathrm{ES}(\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 \mathbf{F}^{-1}(t)\ \mathrm{d}t$$

(c) Let us consider that L follows a Pareto distribution $\mathcal{P}(\theta; x^-)$ defined by:

$$\Pr\{L \leq x\} = 1 - \left(\frac{x}{x_-}\right)^{-\theta}$$

where $x \geq x_-$ and $\theta > 1$. Calculate the moments of order one and two. Interpret the parameters x_- and θ. Calculate $\mathrm{ES}(\alpha)$ and show that:

$$\mathrm{ES}(\alpha) > \mathrm{VaR}(\alpha)$$

(d) Now calculate $\mathrm{ES}(\alpha)$ while L follows a Gaussian distribution $\mathcal{N}(\mu, \sigma)$. Show that:

$$\Phi(x) = -\frac{\phi(x)}{x^1} + \frac{\phi(x)}{x^3} + \ldots$$

Deduce that:

$$\mathrm{ES}(\alpha) \to \mathrm{VaR}(\alpha) \text{ while } \alpha \to 1$$

(e) Comment these results in a risk management prospective.

2. Let us define the risk measure $\mathcal{R}$ as follows:

$$\mathcal{R}(L) = \mathbb{E}[L]$$

$\mathcal{R}$ is it a coherent risk measure?

3. Same question with the following risk measure:

$$\mathcal{R}(L) = \mathbb{E}[L] + \sigma(L)$$

4. We assume that the probability distribution $\mathbf{F}$ of the loss L is:

ℓ_i	0	1	2	3	4	5	6	7	8
$\Pr\{L = \ell_i\}$	0,2	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1

(a) Calculate $\mathrm{ES}(\alpha)$ for $\alpha = 50\%$, $\alpha = 75\%$ and $\alpha = 90\%$.

(b) Let us consider two losses L_1 and L_2 of the same distribution $\mathbf{F}$. Construct a joint distribution of (L_1, L_2) which does not satisfy the property of the property of sub additivity if$\mathcal{R}(L) = \mathbf{F}^{-1}(\alpha)$.

5. Let us specify L as follows:

$$L = \sum_{i=1}^{n} x_i \times L_i$$

where L_i is the unitary loss of the i-th asset. We use the risk value as a measure of risk.

(a) Give the expression of the marginal value at risk.

(b) Calculate the marginal risk while:

$$(L_1, \ldots, L_n) \sim \mathcal{N}(\mu, \Sigma)$$

8 Value-at-Risk of non-grained credit risk portfolio

Let us consider a portfolio composed by a unique debt A of maturity 1 year and a notional of 1 000 euros.

1. If the default probability of A is equal to 50 bps/year and if the loss given default (LGD) is equal to 100%, which is the VaR 1 year at the 99% confidence level?

2. Same question if the loss given default is modeled by the following discrete distribution :

$$\begin{cases} \Pr\{\text{LGD} = 50\%\} = 25\% \\ \Pr\{\text{LGD} = 100\%\} = 75\% \end{cases}$$

3. Repeat the previous two questions with a probability of default of A equal to 10%/year.

4. Which is the VaR 1 year at 99% if the portfolio is composed by two debts (credits) A and B of maturity 1 year and notional 1 000 euros and if we assume that the default probabilities of A and B are both equal to 10%/year, that the loss given default is 100% for both debts and that the default times are independent.

5. Same question if we use the previous discrete distributions to model the loss given default.