Financial Risk Management

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Tutorial exercices #3

1 Maximum likelihood of the exponential distribution

1. We assume that $X \sim \mathcal{E}(\lambda)$. Write the density function of X.
2. Calculate the log-likelihood function of the sample $(x_1, \ldots, x_n)$.
3. Deduce the maximum likelihood estimator (MLE).
4. Compare this estimator with the one of the method of moments.

2 Exotic products and risk management

1. Give the definition of *mark-to-market* and *mark-to-model*. Why the back-testing of a VaR of a portfolio of exotic derivatives creates problems?
2. What do we call model risk?
3. Which are the different ways of calculating VaR of a portfolio of exotic derivatives?
4. Let us consider the sale of an exotic option on the underlying asset whose current price is 100. At this time t, the value of the option is equal to 6,78 euros.
   (a) At the time $t+1$, the value of the underlying is 97 while the implicit volatility has not changed. The trader discovers a PnL equal to 1,37 euros. Why the trader's PnL is it positive? Could you explain the PnL by the sensibilities knowing that the delta Δ_t is equal to 49%, that the gamma Γ_t is 2% and that the vega[1] υ_t is estimated at 40%?
   (b) At the time $t+2$, the value of the underlying is 100 while the implicit volatility goes from 20% to 22%. The trader discovers a negative PnL of -2,37 euros. Why the trader's PnL is it negative? Could you explain the PnL by the sensibilities knowing that the delta Δ_{t+1} is equal to 43%, that the gamma Γ_{t+1} is 2% and that the vega υ_{t+1} is estimated at 38% ?
   (c) What can we conclude in terms of model risk?

3 CDS valuation

1. Let us consider a CDS on the counterparty X with maturity 3 years and notional 1 million euros. The current spread of the CDS is equal to 200 bps. Write the CDS flow chart assuming that the rate of recovery is equal to 40%. What is the PnL of the protection seller A if the counterpart X does not default? Which is the PnL of the protection buyer B if the counterparty X defaults in 2 years and 2 months?
2. Seven months later, the spread of the counterparty X has increased and is now 1 000 bps. The buyer B returns its exposure in the market with C. How much has won or lost the buyer B in this operation? Estimate the new annual probability of default of the counterparty X.

[1] measured in volatility points.

4 Calibration of the parameter LGD

1. We consider a risk class $mathcalC$ corresponding to a customer segment and product specific to retail banking. A statistical analysis of 1 000 LGD data available for this class $mathcalC$ gives the following results:

LGD	0%	25%	50%	75%	100%
Effectives	100	100	600	100	100

   Let us consider a portfolio of 100 homogeneous credits (debts) of nominal 10 000 euros which belong to the class $\mathcal{C}$. The annual default probability is equal to 1%. Calculate the *expected loss* of a horizon of this credit portfolio of 1 year considering the previous empiric distribution to model the parameter LGD.

2. We assume that the margin profit is equal to 650 euros per credit. What is the impact on the bank's balance sheet if it is assumed that the requirement of capital covers the expected loss as well?What is the impact on the income statement when one considers that the *expected loss* is provisioned?

3. We want to model the parameter LGD by a Beta distribution $\mathcal{B}(a, b)$. We remind that the expression of the first two moments of this distribution is:

$$\begin{aligned}\mu &= \frac{a}{a+b} \\ \sigma^2 &= \frac{ab}{(a+b)^2(a+b+1)}\end{aligned}$$

   Calibrate the parameters a and b of the Beta distribution starting from the above empiric distribution.

4. We assume that the default is modeled by a unique factor of type Bâle II and that it is independent of loss in case of default. We consider the same portfolio as above and we search to measure the *unexpected loss*. What is the impact in this measure of risk if we use a uniform distribution instead of the calibrated Beta law? Why this result is it also verified in the case where we consider a model of two factors to model the default time?

5 Default time modeling

1. We model the default time τ as a generalized exponential distribution. We assume that the hazard rate $\lambda(t)$ is piecewise constant:

$$\lambda(t) = \lambda_m \qquad \text{if } t \in \left[t^\star_{m-1}, t^\star_m\right[$$

   Calculate the survival function τ as well as the density function.

2. We assume that $\lambda(t)$ is equal to 100 bps if t is inferior to 5 years and 200 bps if t is superior to 5 years. How do we simulate τ? Calculate the simulated values of τ if we if we pull uniforme random numbers which are respectively 0,04, 0,23 and 0,97.

3. Let us consider the rating system with 4 classes of risk: A, B, C and D. The notation D represents the default. An estimation of the transition probabilities for two years gives:

$$\pi = \begin{pmatrix} 94\% & 3\% & 2\% & 1\% \\ 10\% & 80\% & 5\% & 5\% \\ 10\% & 10\% & 60\% & 20\% \\ 0\% & 0\% & 0\% & 100\% \end{pmatrix}$$

We have as well the following:

$$\pi^2 = \begin{pmatrix} 88{,}86\% & 5{,}42\% & 3{,}23\% & 2{,}49\% \\ 17{,}90\% & 64{,}80\% & 7{,}20\% & 10{,}10\% \\ 16{,}40\% & 14{,}30\% & 36{,}70\% & 32{,}60\% \\ 0{,}00\% & 0{,}00\% & 0{,}00\% & 100{,}00\% \end{pmatrix}$$

et :

$$\pi^3 = \begin{pmatrix} 84{,}39\% & 7{,}32\% & 3{,}99\% & 4{,}30\% \\ 24{,}02\% & 53{,}10\% & 7{,}92\% & 14{,}96\% \\ 20{,}52\% & 15{,}60\% & 23{,}06\% & 40{,}82\% \\ 0{,}00\% & 0{,}00\% & 0{,}00\% & 100{,}00\% \end{pmatrix}$$

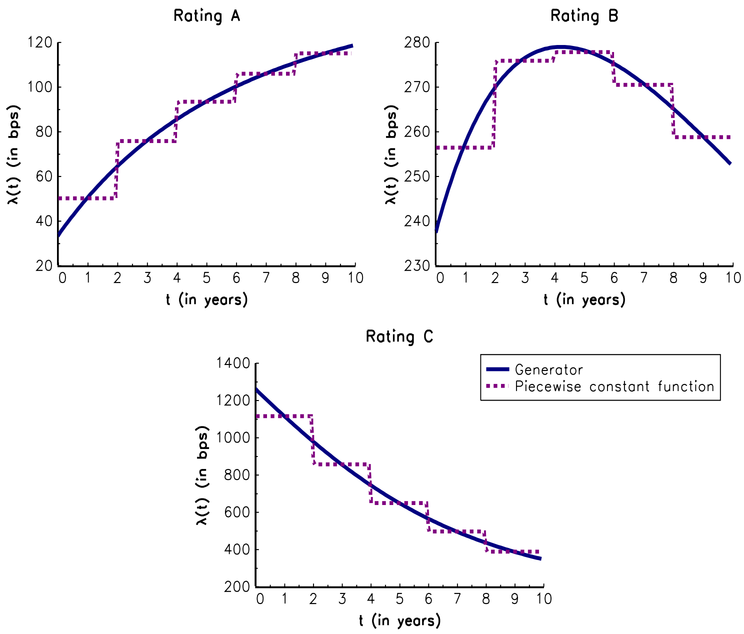
Let us denote $S_A(t)$, $S_B(t)$ and $S_C(t)$ the survival functions of each class of risk A, B and C. Assuming a generalized exponential model, calibrate the hazard rates for each of the classes $\lambda(t)$ pour $0 \leq t < 2$, $2 \leq t < 4$ and $4 \leq t < 6$.

4. Give the definition of the markovian generator. How do we estimate the generator Λ associated to the previous transition matrices? A numerical calculation shows that:

$$\hat{\Lambda} = \begin{pmatrix} -0{,}0325 & 0{,}0165 & 0{,}0126 & 0{,}0034 \\ 0{,}0558 & -0{,}1149 & 0{,}0353 & 0{,}0238 \\ 0{,}0622 & 0{,}0711 & -0{,}2592 & 0{,}1259 \\ 0{,}0000 & 0{,}0000 & 0{,}0000 & 0{,}0000 \end{pmatrix}$$

Is $\hat{\Lambda}$ a markovian generator?

Figure 1: Hazard rate $\lambda(t)$ for the ratings A, B and C



5. In Figure 1, we have represented the hazard risk $\lambda(t)$ considering the generalized exponential model and considering the markovian generator. Explain precisely how do we calculate $\lambda(t)$ in both cases. Why do we obtain an increasing curve for the A rating, a decreasing curve for the C rating and an inverse U shaped curve for the B rating?

6 Credit spreads

1. We model the payback τ as an exponential distribution $\mathcal{E}_\lambda$ of parameter λ. Write the distribution function F and the survival function S of τ. How do we simulate the default time τ?

2. Let us consider a CDS 3M of 1 year maturity. Write the Give the flow chart assuming that the CDS protection leg is paid at default and the recovery rate is fixed and equal to R.

3. What is the margin currency or the spread s of CDS? Which is the relation among s, R and λ?

4. We assume a recovery rate of 25 %. What is the probability 1 year implicit default of a counterpart whose CDS spread 3M 1 year is equal to 200 bps? What is the *relative value* strategy?

5. Give the definition of *credit curve* notion. Comment the following spread curves:

Maturity	Spread #1	Spread #2
6M	30 bps	80 bps
1Y	35 bps	70 bps
3Y	45 bps	60 bps
5Y	60 bps	60 bps

Find an example of strategy based on these credit curves.

7 Credit derivatives and default correlation

1. Let us consider a CDS 3M of a counterpart X of maturity 3 years and notional 1 million euros. The actual spread of CDS is equal to 200 bps. Write the flow chart of CDS assuming that the leg protection is payed at the time of default and the recovery rate is fixed and is equal to 60%.

2. Let us assume that the default time of X is exponential of parameter λ. We consider that the instantaneous interest rate is constant and equal to r. Write the theoretical expression of spread s for the CDS.

3. Using Merton model, you estimate that the λ parameter is equal to 250 bps. Which is the arbitrage position that you could consider?

4. Define the Normal copula of dimension n.

5. Let us consider a basket of n credits. What is a CDS *first-to-default* (F2D), a CDS *second-to-default* (S2D) and a *CDS last-to-default* (L2D)?

6. Define the notion of default correlation. Which is the impact of correlation of default with the three previous spreads of CDS?

7. Let us assume that $n = 3$. Show the following relation:

$$s_1^{\text{CDS}} + s_2^{\text{CDS}} + s_3^{\text{CDS}} = s^{\text{F2D}} + s^{\text{S2D}} + s^{\text{L2D}}$$

with s_i^{CDS} the spread CDS of the i-th credit.

8. Many professionals and academics believe that sub-prime crisis is due to the utilization of the Normal copula. Based on the results of the previous question, what can you conclude?

8 Loss given default modeling

1. What is the difference between the recovery rate and the loss given default?

2. Consider a bank that grants on average 250 000 credits per year. The average amount of a credit is equal to 50 000 euros. We estimated that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of litigation service is equal to 12,5 million euros. What is the loss in case of mean default?

3. We recall that the density expression of the density of a Beta distribution of parameters a and b is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathbf{B}(a,b)}$$

where $\mathbf{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$.

   (a) Why is Beta distribution a good candidate to model the parameter? Which kind of parameters (a, b) correspond to the uniform distribution?

   (b) Let us consider the sample $(x_1, \ldots, x_n)$ of n losses in case of default. Write the log-likelihood function. Deduce the first order conditions of the estimator of maximum likekihood (MLE).

   (c) We recall that the first two moments of Beta distribution are:

$$\begin{aligned} \mathbb{E}[X] &= \frac{a}{a+b} \\ \sigma^2(X) &= \frac{ab}{(a+b)^2(a+b+1)} \end{aligned}$$

   Deduce the expression of moments' estimator.

9 Value-at-risk of an option

1. Let us consider a long position on an option in an asset whose actual value is 100 euros. The delta and the gamma of the option are 50% and 4% respectively. We assume that the annual return of the asset is a Gaussian distribution of volatility 31,6%.

   (a) Calculate the Gaussian VaR considering the delta approximation.

   (b) Calculate the Gaussian VaR considering the delta-gamma approximation.

   (c) Calculate the Cornish-Fisher VaR[2].

   (d) Comment these results.

2. Explain how to implement the Monte Carlo method and the historic method delta-gamma for the previous case.

10 PnL calculation of optional products

1. Let C_t be the value of an option at date t. Define the coefficients delta, gamma, vega and theta of the option.

[2] We recall that if $X \sim \mathcal{N}(0,1)$, then the even moments are given by the following relation:

$$\mathbb{E}\left[X^{2n}\right] = (2n-1) \times \mathbb{E}\left[X^{2n-2}\right]$$

with $n \in \mathbb{N}$.

2. Give the value of these coefficients in the case of Black-Scholes while the underlying asset follows a geometric brownian motion.

3. Let us consider a European option to purchase residual maturity equal to 1 year. We denote K the strike price of this option. The actual value of the underlying is equal to 100 euros and the risk free interest rate is equal to 5%. The following table gives the value of the option C_0 and the coefficients Δ_0, Γ_0 and Θ_0 :

K	80	95	100	105	120
C_0	24,5888	13,3465	10,4506	8,0214	3,2475
Δ_0	0,9286	0,7279	0,6368	0,5422	0,2872
Γ_0	0,0068	0,0166	0,0188	0,0198	0,0170
Θ_0	−4,7765	−6,2914	−6,4140	−6,2771	−4,6810

   (a) Comment the numerical results.

   (b) One day later, the value of the underlying goes to 105. Calculating the new value of the price of the option, we obtain:

K	80	95	100	105	120
C_1	29,2889	17,1552	13,8324	10,9472	4,8798

   Explain how do we calculate the value C_1. Then write for each strike price K the PnL of a long position in this option.

   (c) For each strike price, calculate an approximation of the PnL considering the sensibilities Δ, $\Delta - \Gamma$, $\Delta - \Theta$ and $\Delta - \Gamma - \Theta$.

   (d) Fill in the following PnL table:

K	80	95	100	105	120
$C_1 - C_0$					
Δ					
$\Delta - \Gamma$					
$\Delta - \Theta$					
$\Delta - \Gamma - \Theta$					

   Comment these results.