Financial Risk Management

Professor: Thierry Roncalli
Evry University

Assistant: Enareta Kurtbegu
Evry University

Tutorial exercices #4

1 Correlation and copulas

1. The bivariate Gaussian copula is given by the following expression:

$$\mathbf{C}(u_1, u_2) = \Phi_2\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho\right)$$

where $\Phi(x)$ is the distribution function of univariate standard normal distribution while $\Phi_2(x, y; \rho)$ is the distribution function of the bivariate standard normal distribution with correlation ρ. Let (X_1, X_2) be a random Gaussian vector of distribution Φ_2. Show that the copula of (X_1, X_2) is the same with the one of the random vector $(\Phi(X_1), \Phi(X_2))$. Deduce an algorithm to simulate the Gaussian copula of parameter ρ.

2. Let us consider a random vector (R_1, R_2) corresponding to the returns of two assets. We assume that $R_1 = \sigma_1 X_1$ and $R_2 = \sigma_2 X_2$ with (X_1, X_2) the random Gaussian vector of distribution Φ_2 and of parameter ρ. Show that the linear correlation of (R_1, R_2) is equal to ρ. Let S_1 and S_2 be the normalized asset prices. We have $S_1 = e^{R_1}$ and $S_2 = e^{R_2}$. Show that the linear correlation of (S_1, S_2) is equal to 1 if and only if the linear correlation of (R_1, R_2) is equal to 1 and if the volatility σ_1 is equal to the volatility σ_2. Comment this result in the case of Black-Scholes model.

3. Let it be $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$. We consider that the copula $\mathbf{C}_{\langle X_1, X_2 \rangle}$ is an ordinal sum of copulas $\mathbf{C}^-$ and $\mathbf{C}^+$ of parameter θ.

   (a) We assume that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Specify the copula $\mathbf{C}_{\langle X_1, X_2 \rangle}$ such that the linear correlation of X_1 and X_2 is null. Show that there exists a function f such that $X_1 = f(X_2)$. Comment this result.

   (b) Calculate the linear correlation of X_1 and X_2 as a function of the parameters μ_1, μ_2, σ_1, σ_2 and θ.

   (c) Propose a method of moments to estimate θ.

2 The exponential distribution

1. Let it be $\tau \sim \mathcal{E}(\lambda)$. Show that:

$$\Pr\{\tau > t \mid \tau > s\} = \Pr\{\tau > t - s\}$$

with $t > s$. Interpret this result. Which is its interest in modeling the credit risk?

2. Let $\tau_1, \ldots, \tau_n$ n be the random variables of distribution $\mathcal{E}(\lambda_i)$.

   (a) Calculate the distribution of $\min(\tau_1, \ldots, \tau_n)$ et $\max(\tau_1, \ldots, \tau_n)$ in the independent case.

   (b) Same question if the random variables $\tau_1, \ldots, \tau_n$ are co-monotone.

(c) We place ourselves in the independent case. Let us consider $\tau = \min(\tau_1, \ldots, \tau_n)$. Show that:

$$\Pr\{\tau = \tau_i\} = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

3. Let us consider two exponential default times τ_1 and τ_2 of parameter λ_1 and λ_2.

   (a) Show that if the dependence function (τ_1, τ_2) is $\mathbf{C}^+$, then the following relation is true:

   $$\tau_1 = \frac{\lambda_2}{\lambda_1} \tau_2$$

   (b) Show that there exists a function f such that $\tau_1 = f(\tau_2)$ is the dependence function of (τ_1, τ_2) is $\mathbf{C}^-$.

   (c) We denote ρ the coefficient of linear correlation of (τ_1, τ_2). Show that:

   $$-1 < \rho \leq 1$$

   (d) In a more general case, show that the correlation coefficient ρ of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

3 Copula functions

1. Let us consider the function:

   $$\mathbf{C}(u_1, u_2) = u_1 u_2 (1 + \theta (1 - u_1)(1 - u_2))$$

   (a) Show that $\mathbf{C}$ is a copula[1] for $\theta \in [-1, 1]$.

   (b) Calculate the tail dependence coefficient λ of this copula, the Kendall tau and the Spearman rho as well.

   (c) Let us consider $X_1 \sim \mathcal{N}(\mu, \sigma)$ and $X_2 \sim \mathcal{E}(\lambda)$. We assume that the copula of the random vector (X_1, X_2) is the previous function. Propose an algorithm to simulate the random vector (X_1, X_2).

   (d) Calculate the log-likelihood of the sample $\left\{ (x_1^i, x_2^i)_{i=1}^{i=n} \right\}$.

2. Let $\mathbf{S}$ be the bivariate function defined by:

   $$\mathbf{S}(x_1, x_2) = \exp\left(-\left(x_1 + x_2 - \theta \frac{x_1 x_2}{x_1 + x_2}\right)\right)$$

   with $\theta \in [0, 1]$, $x_1 \geq 0$ et $x_2 \geq 0$.

   (a) Show that $\mathbf{S}$ is a survival distribution.

   (b) Define the survival copula associated to$\mathbf{S}$.

3. We recall that an Archimedean copula has the following expression:

   $$\mathbf{C}(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$$

   where φ is a function called generator.

   (a) Under which conditions for φ should we be, in a way $\mathbf{C}$ to be a copula?

[1] This copula is known with the name of Farlie-Gumbel-Morgenstern (see Nelsen, 1999, page 68).

(b) Are the following copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ Archimedean? If yes, give the corresponding generators.

(c) Which is the copula associated to generator $\varphi(u) = (-\ln u)^{\theta}$ where $\theta \geq 1$? Show that thos copula is a copula of extreme values.

(d) Calculate the conditional distribution $\mathbf{C}_{2|1}$ associated to the Archimedean copula. Deduce an algorithm to simulate the Archimedean copula.

4. Let (X_1, X_2) be a standard Gaussian vector of correlation ρ.

   (a) Calculate the distribution of X_2 knowing that $X_1 = x_1$.

   (b) Deduce an expression of the copula $\mathbf{C}$ associated to (X_1, X_2). Calculate the conditional distribution $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the copula $\mathbf{C}$.

5. Which is the property of a copula of extreme values? Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are copulas of extreme values, while that is not the case for $\mathbf{C}^{-}$.

4 Calculation of the upper/lower bounds of correlation

1. Give the mathematical definitions of the copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$.

2. Define the Normal bivariate copula $\mathbf{C}_{(\rho)}$ of correlation ρ.

3. Which are the probabilistic interpretations of the three copulas defined in question 1? Deduce that $\mathbf{C}_{(\rho=-1)} = \mathbf{C}^{-}$, $\mathbf{C}_{(\rho=0)} = \mathbf{C}^{\perp}$ and $\mathbf{C}_{(\rho=1)} = \mathbf{C}^{+}$.

4. Let us consider the random vector (τ, LGD) which model the joint law of default τ and the loss given default LGD of a counterpart. We assume that $\tau \sim \mathcal{E}_{\lambda}$ and $\mathrm{LGD} \sim \mathcal{U}_{[0,1]}$.

   (a) Show that the dependence of (τ, LGD) is maximal while:

   $$\mathrm{LGD} + e^{-\lambda\tau} - 1 = 0$$

   (b) Show that the correlation $\rho(\tau, \mathrm{LGD})$ verifies the following inequality:

   $$|\rho(\tau, \mathrm{LGD})| \leq \frac{\sqrt{3}}{2}$$

   (c) Comment these results.

5 The generalized exponential model

1. We denote $\mathbf{F}$ and $\mathbf{S}$ the distribution function and the survival function of the random variable τ. Define the function $\mathbf{S}(t)$ and deduce the expression of the associated density function $f(t)$.

2. Give the definition of hazard rate $\lambda(t)$. Deduce that the exponential model corresponds to a particular case $\lambda(t) = \lambda$.

3. How could we simulate the random variable τ in the case when $\lambda(t) = \lambda$.

4. Let us assume now that:

   $$\lambda(t) = \begin{cases} \lambda_1 & \text{if } t \leq 3 \\ \lambda_2 & \text{if } 3 < t \leq 5 \\ \lambda_3 & \text{if } t > 5 \end{cases}$$

   Give the expression of the survival distribution $\mathbf{S}(t)$. Deduce the expression of the density function $f(t)$. Verify that:

   $$\frac{f(t)}{\mathbf{S}(t)} = \text{constant}$$

5. We assume that the interest rate is constant r. We recall that in the case of CDS whose margin is payed in ongoing basis, the spread of CDS is equal to:

$$s = \frac{(1-R)\int_0^T e^{-rt} \cdot f(t)\ \mathrm{d}t}{\int_0^T e^{-rt} \cdot \mathbf{S}(t)\ \mathrm{d}t}$$

where $\mathbf{S}$ and f are the survival and density functions of default time τ associated to CDS, R is the recovery rate and T is the CDS maturity. Give the triangular equality while $\tau \sim \mathcal{E}_\lambda$.

6. Let us consider that τ is the generalized exponential model of question 4. We assume that there exist three market spreads CDS s_1, s_2 and s_3 whose respective maturity are 3 years, 5 years and 7 years. Show that the calibration of the generalized exponential model implies to solve a system of 3 equations and 3 unknowns λ_1, λ_2 and λ_3. Which is the name of this calibration method?

6 Risk contributions

We denote L the loss of a portfolio of n credits, and x_i the exposure to default of the i-th credit. We have:

$$L = \mathbf{x}^\top \mathbf{e} = \sum_{i=1}^n x_i \times e_i$$

with e_i the unitary loss of the i-th credit. We denote $\mathbf{F}$ the distribution function of L.

1. Let us assume that $\mathbf{e} = (e_1, \ldots, e_n) \sim \mathcal{N}(0, \Sigma)$. Calculate the value at risk of confidence level α.

2. Deduce the marginal value-at-risk of the i-th credit. Define the risk contribution of the i-th credit.

3. Verify that the value at risk is equal to:

$$\frac{\partial\ \mathrm{VaR}}{\partial\, x_i} = \mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

Interpret this result.

4. Let us consider the Bâle II model of credit risk.. We have:

$$e_i = \mathrm{LGD}_i \times D_i$$

where $D_i = 1\{\tau_i < M_i\}$ is the default indicator and the M_i is the maturity of the i-th credit. Which are the necessary conditions to be satisfied to obtain the following result:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

5. We assume that the default occurs before the maturity M_i if the latent variable Z_i falls below a certain level B_i:

$$\tau_i \leq M_i \Leftrightarrow Z_i \leq B_i$$

We model $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$ with Z_i, X and ε_i three independent standard normal random variables. X is the factor (or the systemic risk) and ε_i is the individual. Calculate the conditional probability of default.

6. Show that in the Bâle II model, we have:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid X = \Phi^{-1}(1-\alpha)\right]$$

7. Deduce the expression of the risk contribution of the i-th credit in the Bâle II model.

8. We assume that the portfolio is homogeneous, which means that the credits have the same exposure to default, the same distribution of loss given default and the same default probability. Using the following result:

$$\int_{-\infty}^{c} \Phi(a+bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

with $\Phi_2(x, y; \rho)$ the distribution function of the bivariate Gaussian distribution of correlation ρ in the domain $[-\infty, x] \times [-\infty, y]$, calculate the expected shortfall in the case of Bâle II model. Comment this result.