Chapter 3

Credit Risk

In this chapter, we give an overview of the credit market. It concerns loans and bonds, but also credit derivatives whose development was impressive during the 2000s. A thorough knowledge of the products is necessary to understand the regulatory framework for computing the capital requirements for credit risk. In this second section, we will therefore compare Basel I, Basel II and Basel III approaches. The case of counterparty credit risk will be treated in the next chapter, which focuses on collateral risk. Finally, the last section is dedicated to the modeling of credit risk. We will develop the statistical methods for modeling and estimating the main parameters (probability of default, loss given default and default correlations) and we will show the tools of credit risk management. Concerning credit scoring models, we refer to Chapter 15, which is fully dedicated on this topic.

3.1 The market of credit risk

3.1.1 The loan market

In this section, we present the traditional debt market of loans based on banking intermediation, as opposed to the financial market of debt securities (money market instruments, bonds and notes). We generally distinguish this credit market along two main lines: counterparties and products.

Counterparties are divided into 4 main categories: sovereign, financial, corporate and retail. Banking groups have adopted this customer-oriented approach by differentiating retail banking and corporate and investment banking (CIB) businesses. Retail banking refers to individuals. It may also include micro-sized firms and small and medium-sized enterprises (SME). CIBs concern middle market firms, corporates, financial institutions and public entities. In retail banking, the bank pursues a client segmentation, meaning that all the clients that belongs to the same segment have the same conditions in terms of financing and financial investments. This also implies that the pricing of the loan is the same for two individuals of the same segment. The issue for the bank is then to propose or not a loan offer to his client. For that, the bank uses statistical decision-making methods, which are called credit scoring models. Contrary to this binary approach (yes or no), CIBs have a personalized approach to their clients. They estimate their probability of default and changes the pricing condition of the loan on the basis of the results. A client with a low default probability will have a lower rate or credit spread than a client with a higher default probability for the same loan.

The household credit market is organized as follows: mortgage and housing debt, consumer credit and student loans. A mortgage is a debt instrument secured by the collateral of a real estate property. In the case where the borrower defaults on the loan, the lender can take possession and sell the secured property. For instance, the home buyer pledges his house to the bank in a residential mortgage. This type of credit is very frequent in

English-speaking countries, notably England and the United States. In continental Europe, home loans are generally not collateralized for a primary home. This is not always the case for buy-to-let investments and second-home loans. Consumer credit is used for equipment financing or leasing. We usually make the distinction between auto loans, credit cards, revolving credit and other loans (personal loans and sales financing). Auto loans are personal loans to purchase a car. Credit cards and revolving credit are two forms of personal lines of credit. Revolving credit facilities for individuals are very popular in the US. It can be secured, as in the case of a home equity line of credit (HELOC). Student loans are used to finance educational expenses, for instance post-graduate studies at the university. The corporate credit market is organized differently, because large corporates have access to the financial market for long-term financing. This explains that revolving credit facilities are essential to provide liquidity for the firm's day-to-day operations. The average maturity is then lower for corporates than for individuals.

Credit statistics for the private non-financial sector (households and non-financial corporations) are reported in Figures 3.1 and 3.2. These statistics include loan instruments, but also debt securities. In the case of the United States[1], we notice that the credit amount for households[2] is close to the figure for non-financial business. We also observe the significant share of consumer credit and the strong growth of student loans. Figure 3.2 illustrates the evolution of debt outstanding[3] for different countries: China, United Kingdom, Japan, United States and the Euro area. In China, the annual growth rate is larger than 20% these last five years. Even if credit for households develops much faster than credit for corporations, it only represents 24% of the total credit market of the private non-financial sector. The Chinese market contrasts with developed markets where the share of household credit is larger[4] and growth rates are almost flat since the 2008 financial crisis. The Japanese case is also very specific, because this country experienced a strong financial crisis after the bursting of a bubble in the 1990s. At that time, the Japanese market was the world's leading market followed by the United States.

3.1.2 The bond market

Contrary to loan instruments, bonds are debt securities that are traded in a financial market. The primary market concerns the issuance of bonds whereas bond trading is organized through the secondary market. The bond issuance market is dominated by two sectors: central and local governments (including public entities) and corporates. This is the principal financing source for government projects and public budget deficits. Large corporates also use extensively the bond market for investments, business expansions and external growth. The distinction government bonds/corporate bonds was crucial before the 2008 Global Financial Crisis. Indeed, it was traditionally believed that government bonds (in developed countries) were not risky because the probability of default was very low. In this case, the main risk was the interest rate risk, which is a market risk. Conversely, corporate bonds were supposed to be risky because the probability of default was higher. Besides the interest rate risk, it was important to take into account the credit risk. Bonds issued from the financial and banking sector were considered as low risk investments. Since 2008,

[1] Data are from the statistical release Z.1 "*Financial Accounts of the United States*". They are available from the website of the Federal Reserve System: `https://www.federalreserve.gov/releases/z1` or more easily with the database of the Federal Reserve Bank of St. Louis: `https://fred.stlouisfed.org`.

[2] Data for households include non-profit institutions serving households (NPISH).

[3] Data are collected by the Bank for International Settlements and are available in the website of the BIS: `https://www.bis.org/statistics`. The series are adjusted for breaks (Dembiermont *et al.*, 2013) and we use the average exchange rate from 2000 to 2014 in order to obtain credit amounts in USD.

[4] This is especially true in the UK and the US.

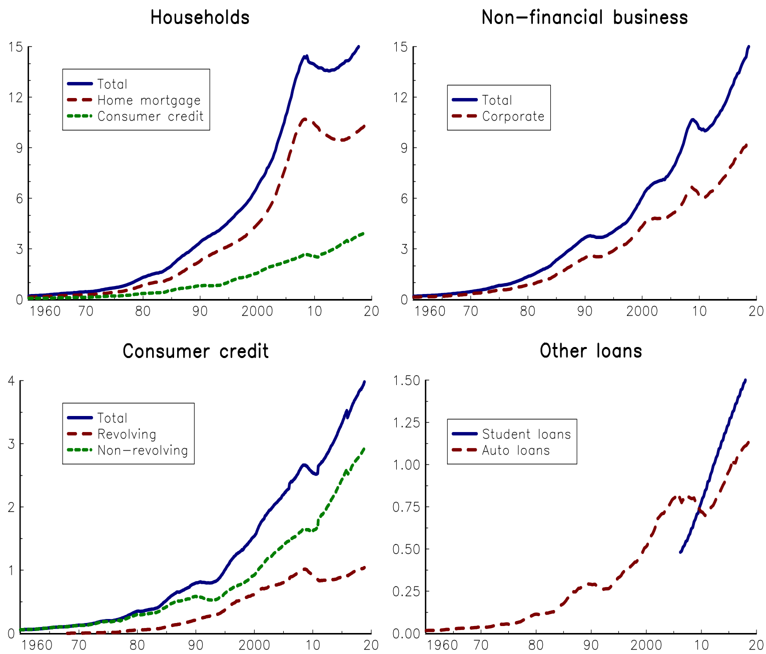

FIGURE 3.1: Credit debt outstanding in the United States (in $ tn)

Source: Board of Governors of the Federal Reserve System (2019).

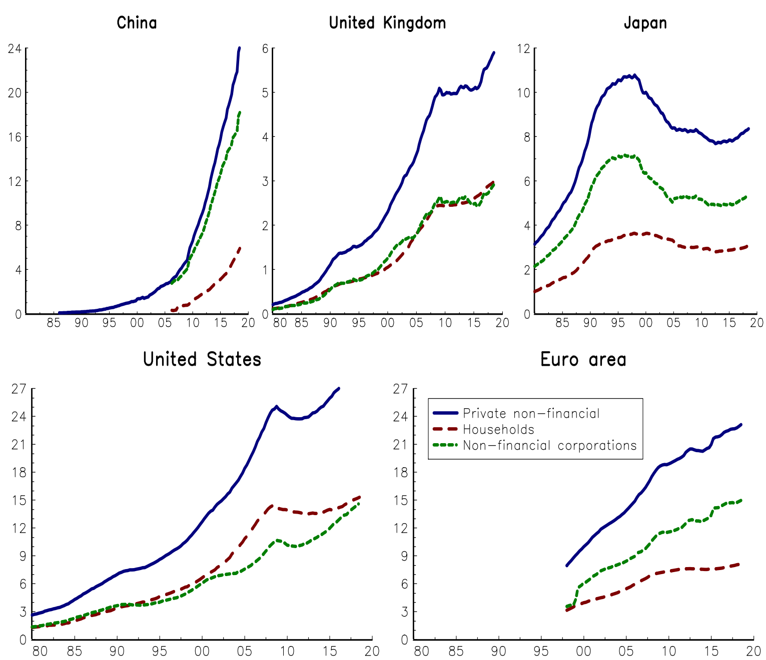

FIGURE 3.2: Credit to the private non-financial sector (in $ tn)

Source: Bank for International Settlements (2019) and author's calculations.

TABLE 3.1: Debt securities by residence of issuer (in $ bn)

		Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2017
Canada	Gov.	682	841	1 149	1 264
	Fin.	283	450	384	655
	Corp.	212	248	326	477
	Total	1 180	1 544	1 863	2 400
France	Gov.	1 236	1 514	1 838	2 258
	Fin.	968	1 619	1 817	1 618
	Corp.	373	382	483	722
	Total	2 576	3 515	4 138	4 597
Germany	Gov.	1 380	1 717	2 040	1 939
	Fin.	2 296	2 766	2 283	1 550
	Corp.	133	174	168	222
	Total	3 809	4 657	4 491	3 712
Italy	Gov.	1 637	1 928	2 069	2 292
	Fin.	772	1 156	1 403	834
	Corp.	68	95	121	174
	Total	2 477	3 178	3 593	3 299
Japan	Gov.	6 336	6 315	10 173	9 477
	Fin.	2 548	2 775	3 451	2 475
	Corp.	1 012	762	980	742
	Total	9 896	9 852	14 604	12 694
Spain	Gov.	462	498	796	1 186
	Fin.	434	1 385	1 442	785
	Corp.	15	19	19	44
	Total	910	1 901	2 256	2 015
UK	Gov.	798	1 070	1 674	2 785
	Fin.	1 775	3 127	3 061	2 689
	Corp.	452	506	473	533
	Total	3 027	4 706	5 210	6 011
US	Gov.	6 459	7 487	12 072	17 592
	Fin.	12 706	17 604	15 666	15 557
	Corp.	3 004	3 348	3 951	6 137
	Total	22 371	28 695	31 960	39 504

Source: Bank for International Settlements (2019).

this difference between non-risky and risky bonds has disappeared, meaning that all issuers are risky. The 2008 GFC had also another important consequence on the bond market. It is today less liquid even for sovereign bonds. Liquidity risk is then a concern when measuring and managing the risk of a bond portfolio. This point is developed in Chapter 6.

3.1.2.1 Statistics of the bond market

In Table 3.1, we indicate the outstanding amount of debt securities by residence of issuer[5]. The total is split into three sectors: general governments (Gov.), financial corporations (Fin.) and non-financial corporations (Corp.). In most countries, debt securities issued by general governments largely dominate, except in the UK and US where debt securities

[5] The data are available in the website of the BIS: https://www.bis.org/statistics.

issued by financial corporations (banks and other financial institutions) are more important. The share of non-financial business varies considerably from one country to another. For instance, it represents less than 10% in Germany, Italy, Japan and Spain, whereas it is equal to 20% in Canada. The total amount of debt securities tends to rise, with the notable exception of Germany, Japan and Spain.

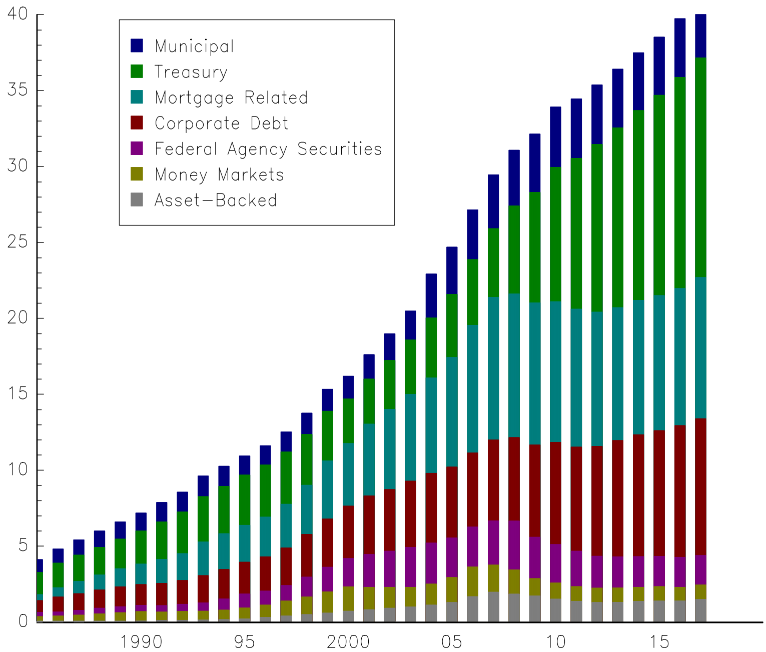

FIGURE 3.3: US bond market outstanding (in $ tn)

Source: Securities Industry and Financial Markets Association (2019a).

The analysis of the US market is particularly interesting and relevant. Using the data collected by the Securities Industry and Financial Markets Association[6] (SIFMA), we have reported in Figure 3.3 the evolution of outstanding amount for the following sectors: municipal bonds, treasury bonds, mortgage-related bonds, corporate related debt, federal agency securities, money markets and asset-backed securities. We notice an important growth during the beginning of the 2000s (see also Figure 3.4), followed by a slowdown after 2008. However, the debt outstanding continues to grow because the average maturity of new issuance increases. Another remarkable fact is the fall of the liquidity, which can be measured by the average daily volume (ADV). Figure 3.5 shows that the ADV of treasury bonds remains constant since 2000 whereas the outstanding amount has been multiplied by four during the same period. We also notice that the turnover of US bonds mainly concerns treasury and agency MBS bonds. The liquidity on the other sectors is very poor. For instance, according to SIFMA (2019a), the ADV of US corporate bonds is less than $30 bn in 2014, which is 22 times lower than the ADV for treasury bonds[7].

[6] Data are available in the website of the SIFMA: `https://www.sifma.org/resources/archive/research/`.

[7] However, the ratio between their outstanding amount is only 1.6.

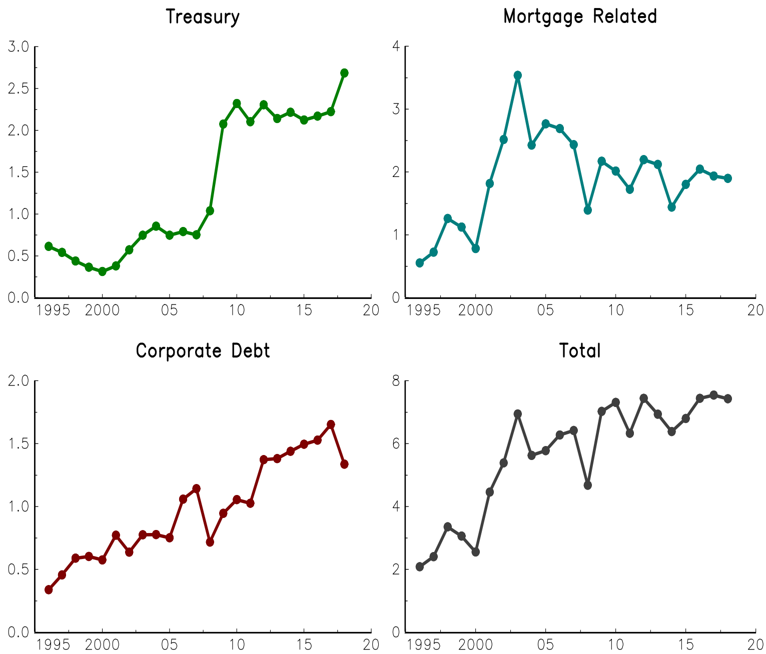

FIGURE 3.4: US bond market issuance (in $ tn)

Source: Securities Industry and Financial Markets Association (2019a).

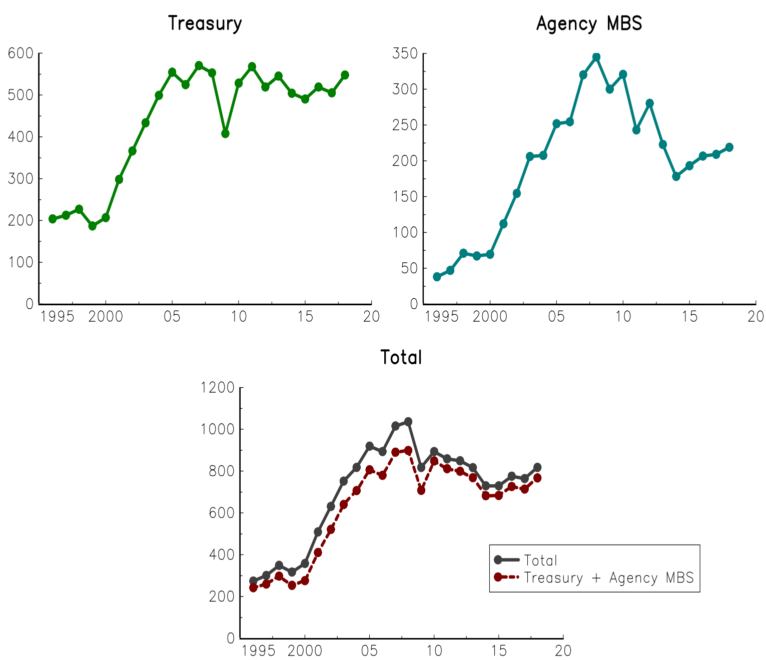

FIGURE 3.5: Average daily trading volume in US bond markets (in $ bn)

Source: Securities Industry and Financial Markets Association (2019a).

3.1.2.2 Bond pricing

We first explain how to price a bond by only considering the interest rate risk. Then, we introduce the default risk and define the concept of credit spread, which is key in credit risk modeling.

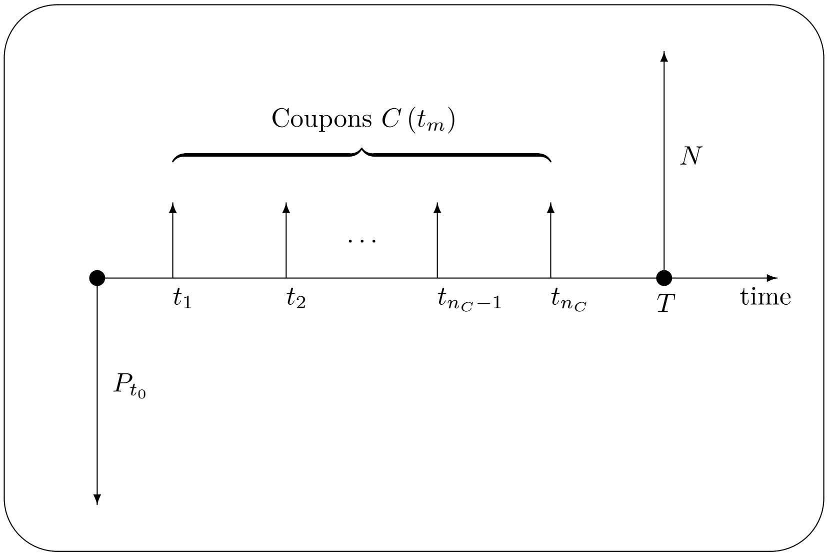

FIGURE 3.6: Cash flows of a bond with a fixed coupon rate

Without default risk We consider that the bond pays coupons $C(t_m)$ with fixing dates t_m and the notional N (or the par value) at the maturity date T. We have reported an example of a cash flows scheme in Figure 3.6. Knowing the yield curve[8], the price of the bond at the inception date t_0 is the sum of the present values of all the expected coupon payments and the par value:

$$P_{t_0} = \sum_{m=1}^{n_C} C(t_m) \cdot B_{t_0}(t_m) + N \cdot B_{t_0}(T)$$

where $B_t(t_m)$ is the discount factor at time t for the maturity date t_m. When the valuation date is not the issuance date, the previous formula remains valid if we take into account the accrued interests. In this case, the buyer of the bond has the benefit of the next coupon. The price of the bond then satisfies:

$$P_t + AC_t = \sum_{t_m \geq t} C(t_m) \cdot B_t(t_m) + N \cdot B_t(T) \tag{3.2}$$

[8] A convenient way to define the yield curve is to use a parametric model for the zero-coupon rates $R_t(T)$. The most famous model is the parsimonious functional form proposed by Nelson and Siegel (1987):

$$\begin{aligned} R_t(T) &= \theta_1 + \theta_2 \left(\frac{1 - \exp(-(T-t)/\theta_4)}{(T-t)/\theta_4} \right) + \\ & \quad \theta_3 \left(\frac{1 - \exp(-(T-t)/\theta_4)}{(T-t)/\theta_4} - \exp(-(T-t)/\theta_4) \right) \end{aligned} \tag{3.1}$$

This is a model with four parameters: θ_1 is a parameter of level, θ_2 is a parameter of rotation, θ_3 controls the shape of the curve and θ_4 permits to localize the break of the curve. We also note that the short-term and long-term interest rates $R_t(t)$ and $R_t(\infty)$ are respectively equal to $\theta_1 + \theta_2$ and θ_1.

Here, AC_t is the accrued coupon:

$$AC_t = C(t_c) \cdot \frac{t - t_c}{365}$$

and t_c is the last coupon payment date with $c = \{m : t_{m+1} > t, t_m \leq t\}$. $P_t + AC_t$ is called the '*dirty price*' whereas P_t refers to the '*clean price*'. The term structure of interest rates impacts the bond price. We generally distinguish three movements:

1. The movement of level corresponds to a parallel shift of interest rates.
2. A twist in the slope of the yield curve indicates how the spread between long and short interest rates moves.
3. A change in the curvature of the yield curve affects the convexity of the term structure.

All these movements are illustrated in Figure 3.7.

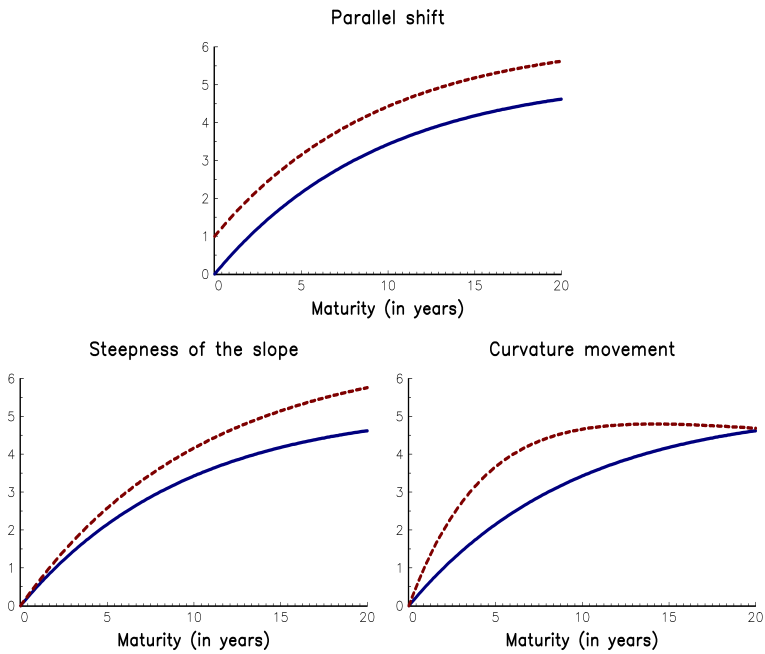

FIGURE 3.7: Movements of the yield curve

The yield to maturity y of a bond is the constant discount rate which returns its market price:

$$\sum_{t_m \geq t} C(t_m) e^{-(t_m - t)y} + N e^{-(T-t)y} = P_t + AC_t$$

We also define the sensitivity[9] S of the bond price as the derivative of the clean price P_t with respect to the yield to maturity y:

$$\begin{aligned} S &= \frac{\partial P_t}{\partial y} \\ &= -\sum_{t_m \geq t} (t_m - t) C(t_m) e^{-(t_m - t)y} - (T - t) N e^{-(T-t)y} \end{aligned}$$

[9]This sensitivity is also called the \$-duration or DV01.

It indicates how the P&L of a long position on the bond moves when the yield to maturity changes:

$$\Pi \approx S \cdot \Delta y$$

Because $S < 0$, the bond price is a decreasing function with respect to interest rates. This implies that an increase of interest rates reduces the value of the bond portfolio.

Example 21 *We assume that the term structure of interest rates is generated by the Nelson-Siegel model with $\theta_1 = 5\%$, $\theta_2 = -5\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. We consider a bond with a constant annual coupon of 5%. The nominal of the bond is \$100. We would like to price the bond when the maturity T ranges from 1 to 5 years.*

TABLE 3.2: Price, yield to maturity and sensitivity of bonds

T	$R_t(T)$	$B_t(T)$	P_t	y	S
1	0.52%	99.48	104.45	0.52%	−104.45
2	0.99%	98.03	107.91	0.98%	−210.86
3	1.42%	95.83	110.50	1.39%	−316.77
4	1.80%	93.04	112.36	1.76%	−420.32
5	2.15%	89.82	113.63	2.08%	−520.16

TABLE 3.3: Impact of a parallel shift of the yield curve on the bond with five-year maturity

ΔR (in bps)	$\breve{P}_t$	ΔP_t	$\hat{P}_t$	ΔP_t	$S \times \Delta y$
−50	116.26	2.63	116.26	2.63	2.60
−30	115.20	1.57	115.20	1.57	1.56
−10	114.15	0.52	114.15	0.52	0.52
0	113.63	0.00	113.63	0.00	0.00
10	113.11	−0.52	113.11	−0.52	−0.52
30	112.08	−1.55	112.08	−1.55	−1.56
50	111.06	−2.57	111.06	−2.57	−2.60

Using the Nelson-Siegel yield curve, we report in Table 3.2 the price of the bond with maturity T (expressed in years) with a 5% annual coupon. For instance, the price of the four-year bond is calculated in the following way:

$$P_t = \frac{5}{(1+0.52\%)} + \frac{5}{(1+0.99\%)^2} + \frac{5}{(1+1.42\%)^3} + \frac{105}{(1+1.80\%)^4} = \$112.36$$

We also indicate the yield to maturity y (in %) and the corresponding sensitivity S. Let $\breve{P}_t$ (resp. $\hat{P}_t$) be the bond price by taking into account a parallel shift ΔR (in bps) directly on the zero-coupon rates (resp. on the yield to maturity). The results are given in Table 3.3 in the case of the bond with a five-year maturity[10]. We verify that the computation based on

[10] We have:

$$\breve{P}_t = \sum_{t_m \geq t} C(t_m) e^{-(t_m - t)(R_t(t_m) + \Delta R)} + N e^{-(T-t)(R_t(T)+\Delta R)}$$

and:

$$\hat{P}_t = \sum_{t_m \geq t} C(t_m) e^{-(t_m - t)(y + \Delta R)} + N e^{-(T-t)(y+\Delta R)}$$

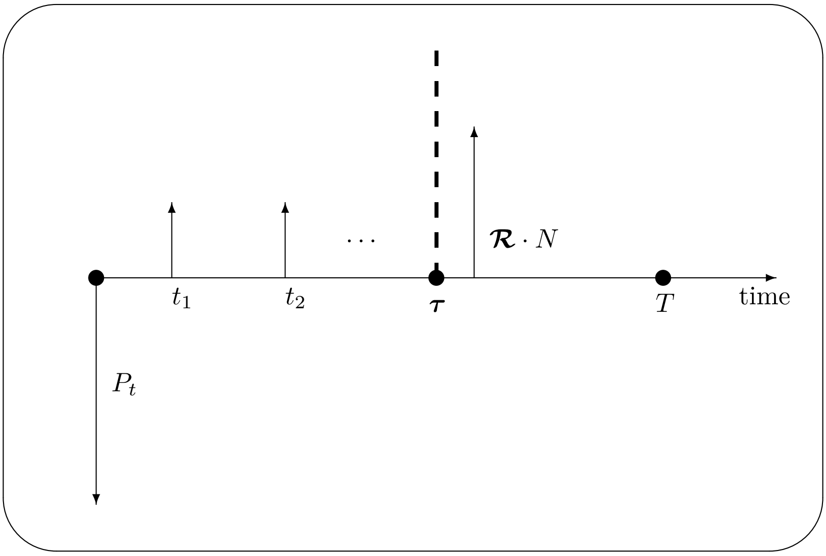

FIGURE 3.8: Cash flows of a bond with default risk

the sensitivity provides a good approximation. This method has been already used in the previous chapter on page 77 to calculate the value-at-risk of bonds.

With default risk In the previous paragraph, we assume that there is no default risk. However, if the issuer defaults at time $\boldsymbol{\tau}$ before the bond maturity T, some coupons and the notional are not paid. In this case, the buyer of the bond recovers part of the notional after the default time. An illustration is given in Figure 3.8. In terms of cash flows, we have therefore:

- the coupons $C(t_m)$ if the bond issuer does not default before the coupon date t_m:

$$\sum_{t_m \geq t} C(t_m) \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\}$$

- the notional if the bond issuer does not default before the maturity date:

$$N \cdot \mathbb{1}\{\boldsymbol{\tau} > T\}$$

- the recovery part if the bond issuer defaults before the maturity date:

$$\boldsymbol{\mathcal{R}} \cdot N \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\}$$

  where $\boldsymbol{\mathcal{R}}$ is the corresponding recovery rate.

If we assume that the recovery part is exactly paid at the default time $\boldsymbol{\tau}$, we deduce that the stochastic discounted value of the cash flow leg is:

$$\begin{aligned} SV_t &= \sum_{t_m \geq t} C(t_m) \cdot e^{-\int_t^{t_m} r_s \, \mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} + \\ & \quad N \cdot e^{-\int_t^T r_s \, \mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} > T\} + \boldsymbol{\mathcal{R}} \cdot N \cdot e^{-\int_t^{\boldsymbol{\tau}} r_s \, \mathrm{d}s} \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\} \end{aligned}$$

The price of the bond is the expected value of the stochastic discounted value[11]: $P_t + AC_t = \mathbb{E}\left[SV_t \mid \mathcal{F}_t\right]$. If we assume that $(\mathcal{H}_1)$ the default time and the interest rates are independent and $(\mathcal{H}_2)$ the recovery rate is known and not stochastic, we obtain the following closed-form formula:

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) \mathbf{S}_t(t_m) + N B_t(T) \mathbf{S}_t(T) + \\ & \quad \boldsymbol{\mathcal{R}} N \int_t^T B_t(u) f_t(u) \, \mathrm{d}u \end{aligned} \tag{3.3}$$

where $\mathbf{S}_t(u)$ is the survival function at time u and $f_t(u)$ the associated density function[12].

Remark 20 *If the issuer is not risky, we have* $\mathbf{S}_t(u) = 1$ *and* $f_t(u) = 0$. *In this case, Equation (3.3) reduces to Equation (3.2).*

Remark 21 *If we consider an exponential default time with parameter* λ *–* $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$, *we have* $\mathbf{S}_t(u) = e^{-\lambda(u-t)}$, $f_t(u) = \lambda e^{-\lambda(u-t)}$ *and:*

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) e^{-\lambda(t_m - t)} + N B_t(T) e^{-\lambda(T-t)} + \\ & \quad \lambda \boldsymbol{\mathcal{R}} N \int_t^T B_t(u) e^{-\lambda(u-t)} \, \mathrm{d}u \end{aligned}$$

If we assume a flat yield curve – $R_t(u) = r$, *we obtain:*

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) e^{-(r+\lambda)(t_m - t)} + N e^{-(r+\lambda)(T-t)} + \\ & \quad \lambda \boldsymbol{\mathcal{R}} N \left(\frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda} \right) \end{aligned}$$

Example 22 *We consider a bond with ten-year maturity. The notional is* \$100 *whereas the annual coupon rate is equal to 4.5%.*

If we consider that $r = 0$, the price of the non-risky bond is \$145. With $r = 5\%$, the price becomes \$95.19. Let us now take into account the default risk. We assume that the recovery rate $\boldsymbol{\mathcal{R}}$ is 40%. If $\lambda = 2\%$ (resp. 10%), the price of the risky bond is \$86.65 (resp. \$64.63). If the yield curve is not flat, we must use the general formula (3.3) to compute the price of the bond. In this case, the integral is evaluated with a numerical integration procedure, typically a Gauss-Legendre quadrature[13]. For instance, if we consider the yield curve defined in Example 21, the bond price is equal to \$110.13 if there is no default risk, \$99.91 if $\lambda = 2\%$ and \$73.34 if $\lambda = 10\%$.

The yield to maturity of the defaultable bond is computed exactly in the same way as without default risk. The credit spread $\mathcal{s}$ is then defined as the difference of the yield to maturity with default risk $\mathcal{y}$ and the yield to maturity without default risk $\mathcal{y}^\star$:

$$\mathcal{s} = \mathcal{y} - \mathcal{y}^\star \tag{3.4}$$

[11] It is also called the present value.

[12] We have:

$$\mathbf{S}_t(u) = \mathbb{E}\left[\mathbb{1}\left\{\boldsymbol{\tau} > u \mid \boldsymbol{\tau} > t\right\}\right] = \Pr\left\{\boldsymbol{\tau} > u \mid \boldsymbol{\tau} > t\right\}$$

The density function is then given by $f_t(u) = -\partial_u \mathbf{S}_t(u)$.

[13] See Appendix A.1.2.3 on page 1037 for a primer on numerical integration.

This spread is a credit risk measure and is an increasing function of the default risk. Reconsider the simple model with a flat yield curve and an exponential default time. If the recovery rate $\mathcal{R}$ is equal to zero, we deduce that the yield to maturity of the defaultable bond is $y = r + \lambda$. It follows that the credit spread is equal to the parameter λ of the exponential distribution. Moreover, if λ is relatively small (less than 20%), the annual probability of default is:

$$\mathrm{PD} = \mathbf{S}_t(t+1) = 1 - e^{-\lambda} \approx \lambda$$

In this case, the credit spread is approximately equal to the annual default probability ($s \approx \mathrm{PD}$).

If we reuse our previous example with the yield curve specified in Example 21, we obtain the results reported in Table 3.4. For instance, the yield to maturity of the bond is equal to 3.24% without default risk. If λ and $\mathcal{R}$ are set to 200 bps and 0%, the yield to maturity becomes 5.22% which implies a credit spread of 198.1 bps. If the recovery rate is higher, the credit spread decreases. Indeed, with λ equal to 200 bps, the credit spread is equal to 117.1 bps if $\mathcal{R} = 40\%$ and only 41.7 bps if $\mathcal{R} = 80\%$.

TABLE 3.4: Computation of the credit spread s

$\mathcal{R}$ (in %)	λ (in bps)	PD (in bps)	P_t (in \$)	y (in %)	s (in bps)
0	0	0.0	110.1	3.24	0.0
	10	10.0	109.2	3.34	9.9
	200	198.0	93.5	5.22	198.1
	1000	951.6	50.4	13.13	988.9
40	0	0.0	110.1	3.24	0.0
	10	10.0	109.6	3.30	6.0
	200	198.0	99.9	4.41	117.1
	1000	951.6	73.3	8.23	498.8
80	0	0.0	110.1	3.24	0.0
	10	10.0	109.9	3.26	2.2
	200	198.0	106.4	3.66	41.7
	1000	951.6	96.3	4.85	161.4

Remark 22 *In the case of loans, we do not calculate a capital requirement for market risk, only a capital requirement for credit risk. The reason is that there is no market price of the loan, because it cannot be traded in an exchange. For bonds, we calculate a capital requirement for both market and credit risks. In the case of the market risk, risk factors are the yield curve rates, but also the parameters associated to the credit risk, for instance the default probabilities and the recovery rate. In this context, market risk has a credit component. To illustrate this property, we consider the previous example and we assume that λ_t varies across time whereas the recovery rate $\mathcal{R}$ is equal to 40%. In Figure 3.9, we show the evolution of the process λ_t for the next 10 years (top panel) and the clean price[14] P_t (bottom/left panel). If we suppose now that the issuer defaults suddenly at time $t = 6.25$, we observe a jump in the clean price (bottom/right panel). It is obvious that the market risk takes into account the short-term evolution of the credit component (or the smooth part), but does not incorporate the jump risk (or the discontinuous part) and also the large uncertainty on the recovery price. This is why these risks are covered by credit risk capital requirements.*

[14] We assume that the yield curve remains constant.

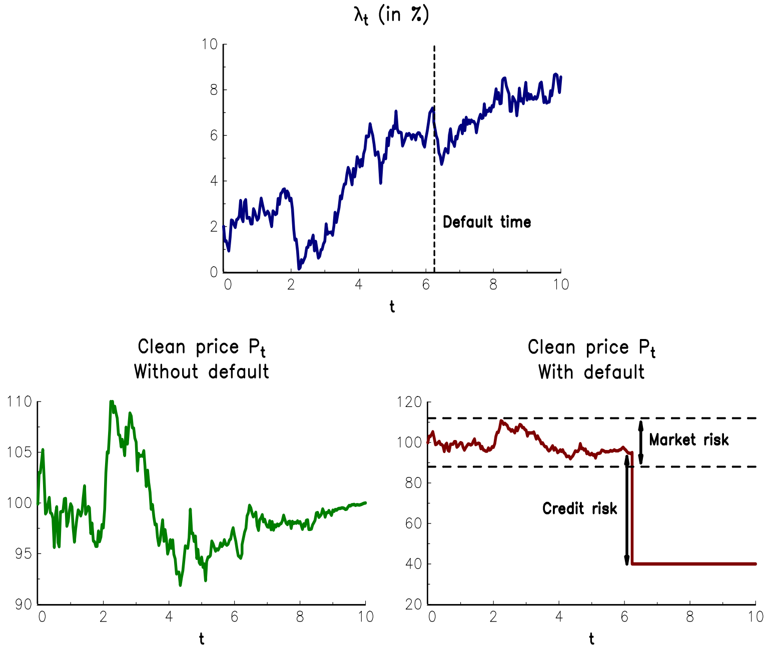

FIGURE 3.9: Difference between market and credit risks for a bond

3.1.3 Securitization and credit derivatives

Since the 1990s, banks have developed credit transfer instruments in two directions: credit securitization and credit derivatives. The term securitization refers to the process of transforming illiquid and non-tradable assets into tradable securities. Credit derivatives are financial instruments whose payoff explicitly depends on credit events like the default of an issuer. These two topics are highly connected because credit securities can be used as underlying assets of credit derivatives.

3.1.3.1 Credit securitization

According to AFME (2019), outstanding amount of securitization is close to €9 tn. Figure 3.10 shows the evolution of issuance in Europe and US since 2000. We observe that the financial crisis had a negative impact on the growth of credit securitization, especially in Europe that represents less than 20% of this market. This market is therefore dominated by the US, followed by UK, France, Spain, the Netherlands and Germany.

Credit securities are better known as asset-backed securities (ABS), even if this term is generally reserved to assets that are not mortgage, loans or corporate bonds. In its simplest form, an ABS is a bond whose coupons are derived from a collateral pool of assets. We generally make the following distinction with respect to the type of collateral assets:

- Mortgage-backed securities (MBS)
  - Residential mortgage-backed securities (RMBS)
  - Commercial mortgage-backed securities (CMBS)
- Collateralized debt obligations (CDO)
  - Collateralized loan obligations (CLO)

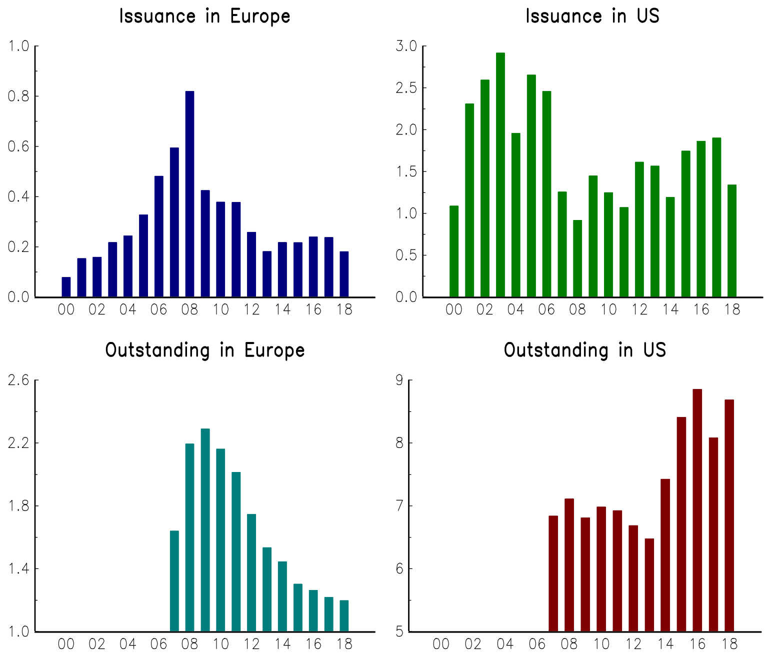

FIGURE 3.10: Securitization in Europe and US (in € tn)

Source: Association for Financial Markets in Europe (2019).

  - Collateralized bond obligations (CBO)

- Asset-backed securities (ABS)
  - Auto loans
  - Credit cards and revolving credit
  - Student loans

MBS are securities that are backed by residential and commercial mortgage loans. The most basic structure is a pass-through security, where the coupons are the same for all the investors and are proportional to the revenue of the collateral pool. Such structure is shown in Figure 3.11. The originator (e.g. a bank) sells a pool of debt to a special purpose vehicle (SPV). The SPV is an ad-hoc legal entity[15] whose sole function is to hold the loans as assets and issue the securities for investors. In the pass-through structure, the securities are all the same and the cash flows paid to investors are directly proportional to interests and principals of collateral assets. More complex structures are possible with several classes of bonds (see Figure 3.12). In this case, the cash flows differ from one type of securities to another one. The most famous example is the collateralized debt obligation, where the securities are divided into tranches. This category includes also collateralized mortgage obligations (CMO), which are both MBS and CDO. The two other categories of CDOs are CLOs, which are backed by corporate bank debt (e.g. SME loans) and CBOs, which are backed by bonds (e.g. high yield bonds). Finally, pure ABS principally concerns consumer credit such as auto loans, credit cards and student loans.

[15]It may be a subsidiary of the originator.

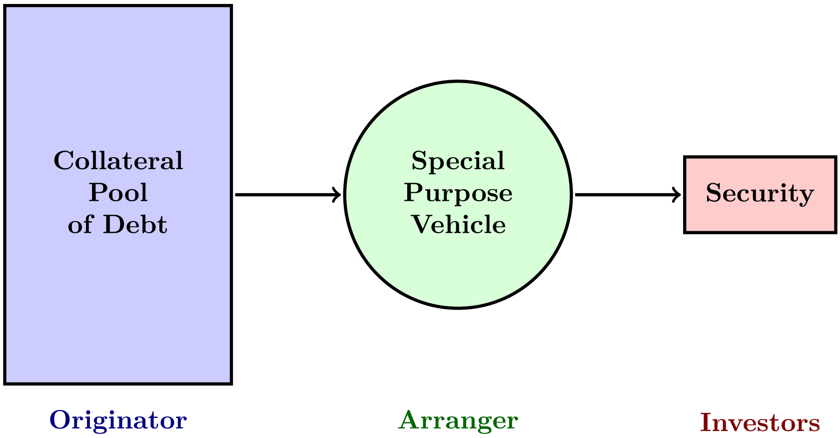

FIGURE 3.11: Structure of pass-through securities

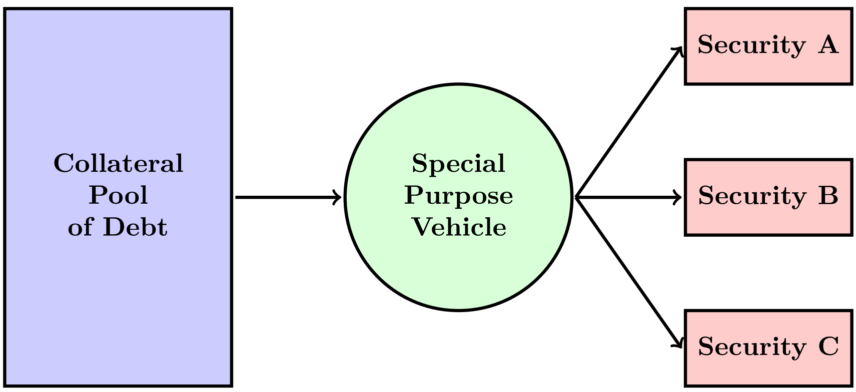

FIGURE 3.12: Structure of pay-through securities

In Table 3.5, we report some statistics about US mortgage-backed securities. SIFMA (2019b) makes the distinction between agency MBS and non-agency MBS. After the Great Depression, the US government created three public entities to promote home ownership and provide insurance of mortgage loans: the Federal National Mortgage Association (FNMA or Fannie Mae), the Federal Home Loan Mortgage Corporation (FHLMC or Freddie Mac) and the Government National Mortgage Association (GNMA or Ginnie Mae). Agency MBS refer to securities guaranteed by these three public entities and represent the main part of the US MBS market. This is especially true since the 2008 financial crisis. Indeed, non-agency MBS represent 53.5% of the issuance in 2006 and only 3.5% in 2012. Because agency MBS are principally based on home mortgage loans, the RMBS market is ten times more larger than the CMBS market. CDO and ABS markets are smaller and represent together about \$1.5 tn (see Table 3.6). The CDO market strongly suffered from the subprime crisis[16]. During the same period, the structure of the ABS market changed with an increasing proportion of ABS backed by auto loans and a fall of ABS backed by credit cards and student loans.

Remark 23 *Even if credit securities may be viewed as bonds, their pricing is not straightforward. Indeed, the measure of the default probability and the recovery depends on the*

[16]For instance, the issuance of US CDO was less than \$10 bn in 2010.

TABLE 3.5: US mortgage-backed securities

Year	Agency		Non-agency		Total
	MBS	CMO	CMBS	RMBS	(in $ bn)
			Issuance		
2002	57.5%	23.6%	2.2%	16.7%	2 515
2006	33.6%	11.0%	7.9%	47.5%	2 691
2008	84.2%	10.8%	1.2%	3.8%	1 394
2010	71.0%	24.5%	1.2%	3.3%	2 013
2012	80.1%	16.4%	2.2%	1.3%	2 195
2014	68.7%	19.2%	7.0%	5.1%	1 440
2016	76.3%	15.7%	3.8%	4.2%	2 044
2018	69.2%	16.6%	4.7%	9.5%	1 899
			Outstanding amount		
2002	59.7%	17.4%	5.6%	17.2%	5 289
2006	45.7%	14.9%	8.3%	31.0%	8 390
2008	52.4%	14.0%	8.8%	24.9%	9 467
2010	59.2%	14.6%	8.1%	18.1%	9 258
2012	64.0%	14.8%	7.2%	14.0%	8 838
2014	68.0%	13.7%	7.1%	11.2%	8 842
2016	72.4%	12.3%	5.9%	9.5%	9 023
2018	74.7%	11.3%	5.6%	8.4%	9 732

Source: Securities Industry and Financial Markets Association (2019b,c) and author's calculations.

TABLE 3.6: US asset-backed securities

Year	Auto Loans	CDO & CLO	Credit Cards	Equip-ement	Other	Student Loans	Total (in $ bn)
				Issuance			
2002	34.9%	21.0%	25.2%	2.6%	6.8%	9.5%	269
2006	13.5%	60.1%	9.3%	2.2%	4.6%	10.3%	658
2008	16.5%	37.8%	25.9%	1.3%	5.4%	13.1%	215
2010	46.9%	6.4%	5.2%	7.0%	22.3%	12.3%	126
2012	33.9%	23.1%	12.5%	7.1%	13.7%	9.8%	259
2014	25.2%	35.6%	13.1%	5.2%	17.0%	4.0%	393
2016	28.3%	36.8%	8.3%	4.6%	16.9%	5.1%	325
2018	20.8%	54.3%	6.1%	5.1%	10.1%	3.7%	517
				Outstanding amount			
2002	20.7%	28.6%	32.5%	4.1%	7.5%	6.6%	905
2006	11.8%	49.3%	17.6%	3.1%	6.0%	12.1%	1 657
2008	7.7%	53.5%	17.3%	2.4%	6.2%	13.0%	1 830
2010	7.6%	52.4%	14.4%	2.4%	7.1%	16.1%	1 508
2012	11.0%	48.7%	10.0%	3.3%	8.7%	18.4%	1 280
2014	13.2%	46.8%	10.1%	3.9%	9.8%	16.2%	1 349
2016	13.9%	48.0%	9.3%	3.7%	11.6%	13.5%	1 397
2018	13.3%	48.2%	7.4%	5.0%	16.0%	10.2%	1 677

Source: Securities Industry and Financial Markets Association (2019b,c) and author's calculations.

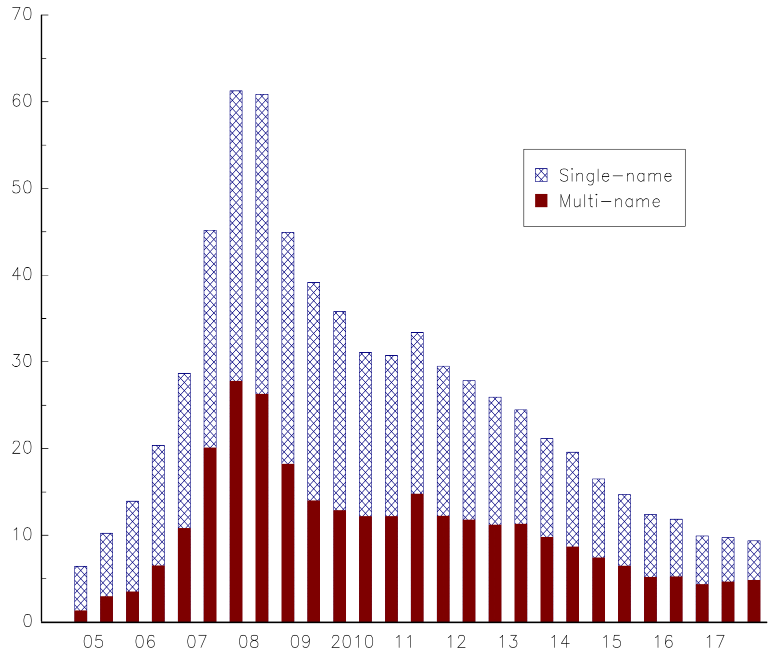

FIGURE 3.13: Outstanding amount of credit default swaps (in $ tn)

Source: Bank for International Settlements (2019).

characteristics of the collateral assets (individual default probabilities and recovery rates), but also on the correlation between these risk factors. Measuring credit risk of such securities is then a challenge. Another issue concerns design and liquidity problems faced when packaging and investing in these assets[17] *(Duffie and Rahi, 1995; DeMarzo and Duffie, 1999). This explains that credit securities suffered a lot during the 2008 financial crisis, even if some of them were not linked to subprime mortgages. In fact, securitization markets pose a potential risk to financial stability (Segoviano* et al.*, 2013). This is a topic we will return to in Chapter 8, which deals with systemic risk.*

3.1.3.2 Credit default swap

A credit default swap (CDS) may be defined as an insurance derivative, whose goal is to transfer the credit risk from one party to another. In a standard contract, the protection buyer makes periodic payments (known as the premium leg) to the protection seller. In return, the protection seller pays a compensation (known as the default leg) to the protection buyer in the case of a credit event, which can be a bankruptcy, a failure to pay or a debt restructuring. In its most basic form, the credit event refers to an issuer (sovereign or corporate) and this corresponds to a single-name CDS. If the credit event relates to a universe of different entities, we speak about a multi-name CDS. In Figure 3.13, we report the evolution of outstanding amount of CDS since 2007. The growth of this market was very strong before 2008 with a peak close to $60 tn. The situation today is different, because the market of single-name CDS stabilized whereas the market of basket default swaps continues to fall significantly. Nevertheless, it remains an important OTC market with a total outstanding around $9 tn.

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[17] The liquidity issue is treated in Chapter 6.

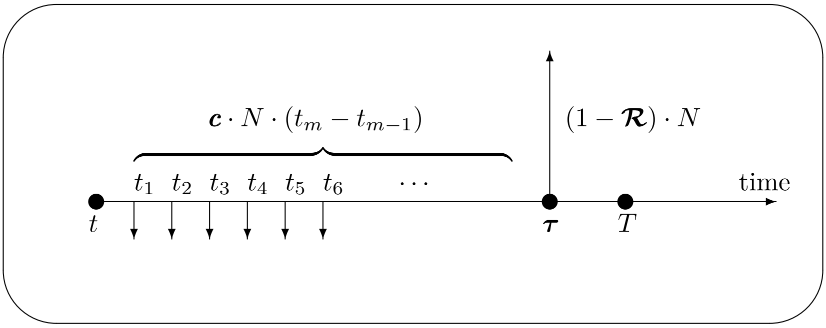

FIGURE 3.14: Cash flows of a single-name credit default swap

In Figure 3.14, we report the mechanisms of a single-name CDS. The contract is defined by a reference entity (the name), a notional principal N, a maturity or tenor T, a payment frequency, a recovery rate $\boldsymbol{\mathcal{R}}$ and a coupon rate[18] $\boldsymbol{c}$. From the inception date t to the maturity date T or the default time $\boldsymbol{\tau}$, the protection buyer pays a fixed payment, which is equal to $\boldsymbol{c} \cdot N \cdot \Delta t_m$ at the fixing date t_m with $\Delta t_m = t_m - t_{m-1}$. This means that the annual premium leg is equal to $\boldsymbol{c} \cdot N$. If there is no credit event, the protection buyer will also pay a total of $\boldsymbol{c} \cdot N \cdot (T - t)$. In case of credit event before the maturity, the protection seller will compensate the protection buyer and will pay $(1 - \boldsymbol{\mathcal{R}}) \cdot N$.

Example 23 *We consider a credit default swap, whose notional principal is* \$10 *mn, maturity is* 5 *years and payment frequency is quarterly. The credit event is the bankruptcy of the corporate entity* A*. We assume that the recovery rate is set to* 40% *and the coupon rate is equal to* 2%*.*

Because the payment frequency is quarterly, there are 20 fixing dates, which are 3M, 6M, 9M, 1Y, $\ldots$, 5Y. Each quarter, if the corporate A does not default, the protection buyer pays a premium, which is approximately equal to $\$10\text{mn} \times 2\% \times 0.25 = \$50\,000$. If there is no default during the next five years, the protection buyer will pay a total of $\$50\,000 \times 20 = \$1$ mn whereas the protection seller will pay nothing. Suppose now that the corporate defaults two years and four months after the CDS inception date. In this case, the protection buyer will pay \$50 000 during 9 quarters and will receive the protection leg from the protection seller at the default time. This protection leg is equal to $(1 - 40\%) \times \$10\,\text{mn} = \$6\,\text{mn}$.

To compute the mark-to-market value of a CDS, we use the reduced-form approach as in the case of bond pricing. If we assume that the premium is not paid after the default time $\boldsymbol{\tau}$, the stochastic discounted value of the premium leg is[19]:

$$SV_t(\mathcal{PL}) = \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot (t_m - t_{m-1}) \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s}$$

[18] We will see that the coupon rate $\boldsymbol{c}$ is in fact the CDS spread $\boldsymbol{s}$ for par swaps.

[19] In order to obtain a simple formula, we do not deal with the accrued premium (see Remark 26 on page 149).

Using the standard assumptions that the default time is independent of interest rates and the recovery rate, we deduce the present value of the premium leg as follows:

$$\begin{aligned}
PV_t(\mathcal{PL}) &= \mathbb{E}\left[\sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot \Delta t_m \cdot \mathbb{1}\{\boldsymbol{\tau} > t_m\} \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s} \,\middle|\, \mathcal{F}_t\right] \\
&= \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot \Delta t_m \cdot \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} > t_m\}\right] \cdot \mathbb{E}\left[e^{-\int_t^{t_m} r_s \,\mathrm{d}s}\right] \\
&= \boldsymbol{c} \cdot N \cdot \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)
\end{aligned}$$

where $\mathbf{S}_t(u)$ is the survival function at time u. If we assume that the default leg is exactly paid at the default time $\boldsymbol{\tau}$, the stochastic discount value of the default (or protection) leg is[20]:

$$SV_t(\mathcal{DL}) = (1 - \boldsymbol{\mathcal{R}}) \cdot N \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\} \cdot e^{-\int_t^{\boldsymbol{\tau}} r(s) \,\mathrm{d}s}$$

It follows that its present value is:

$$\begin{aligned}
PV_t(\mathcal{DL}) &= \mathbb{E}\left[(1 - \boldsymbol{\mathcal{R}}) \cdot N \cdot \mathbb{1}\{\boldsymbol{\tau} \leq T\} \cdot e^{-\int_t^{\boldsymbol{\tau}} r_s \,\mathrm{d}s} \,\middle|\, \mathcal{F}_t\right] \\
&= (1 - \boldsymbol{\mathcal{R}}) \cdot N \cdot \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} \leq T\} \cdot B_t(\boldsymbol{\tau})\right] \\
&= (1 - \boldsymbol{\mathcal{R}}) \cdot N \cdot \int_t^T B_t(u) f_t(u) \,\mathrm{d}u
\end{aligned}$$

where $f_t(u)$ is the density function associated to the survival function $\mathbf{S}_t(u)$. We deduce that the mark-to-market of the swap is[21]:

$$\begin{aligned}
P_t(T) &= PV_t(\mathcal{DL}) - PV_t(\mathcal{PL}) \\
&= (1 - \boldsymbol{\mathcal{R}}) N \int_t^T B_t(u) f_t(u) \,\mathrm{d}u - \boldsymbol{c} N \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m) \\
&= N \left((1 - \boldsymbol{\mathcal{R}}) \int_t^T B_t(u) f_t(u) \,\mathrm{d}u - \boldsymbol{c} \cdot \mathrm{RPV}_{01} \right) \qquad (3.5)
\end{aligned}$$

where $\mathrm{RPV}_{01} = \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)$ is called the risky PV01 and corresponds to the present value of 1 bp paid on the premium leg. The CDS price is then inversely related to the spread. At the inception date, the present value of the premium leg is equal to the present value of the default leg meaning that the CDS spread corresponds to the coupon rate such that $P_t^{\text{buyer}} = 0$. We obtain the following expression:

$$\boldsymbol{\mathcal{s}} = \frac{(1 - \boldsymbol{\mathcal{R}}) \int_t^T B_t(u) f_t(u) \,\mathrm{d}u}{\sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)} \qquad (3.6)$$

The spread $\boldsymbol{\mathcal{s}}$ is in fact the fair value coupon rate $\boldsymbol{c}$ in such a way that the initial value of the credit default swap is equal to zero.

---

[20] Here the recovery rate $\boldsymbol{\mathcal{R}}$ is assumed to be deterministic.

[21] P_t is the swap price for the protection buyer. We have then $P_t^{\text{buyer}}(T) = P_t(T)$ and $P_t^{\text{seller}}(T) = -P_t(T)$.

We notice that if there is no default risk, this implies that $\mathbf{S}_t(u) = 1$ and we get $\mathcal{s} = 0$. In the same way, the spread is also equal to zero if the recovery rate is set to 100%. If we assume that the premium leg is paid continuously, the formula (3.6) becomes:

$$\mathcal{s} = \frac{(1 - \boldsymbol{\mathcal{R}}) \int_t^T B_t(u) f_t(u) \, \mathrm{d}u}{\int_t^T B_t(u) \mathbf{S}_t(u) \, \mathrm{d}u}$$

If the interest rates are equal to zero ($B_t(u) = 1$) and the default times is exponential with parameter λ – $\mathbf{S}_t(u) = e^{-\lambda(u-t)}$ and $f_t(u) = \lambda e^{-\lambda(u-t)}$, we get:

$$\begin{aligned} \mathcal{s} &= \frac{(1 - \boldsymbol{\mathcal{R}}) \cdot \lambda \cdot \int_t^T e^{-\lambda(u-t)} \, \mathrm{d}u}{\int_t^T e^{-\lambda(u-t)} \, \mathrm{d}u} \\ &= (1 - \boldsymbol{\mathcal{R}}) \cdot \lambda \end{aligned}$$

If λ is relatively small, we also notice that this relationship can be written as follows:

$$\mathcal{s} \approx (1 - \boldsymbol{\mathcal{R}}) \cdot \mathrm{PD}$$

where PD is the one-year default probability[22]. This relationship is known as the '*credit triangle*' because it is a relationship between three variables where knowledge of any two is sufficient to calculate the third (O'Kane, 2008). It basically states that the CDS spread is approximatively equal to the one-year loss. The spread contains also the same information than the survival function and is an increasing function of the default probability. It can then be interpreted as a credit risk measure of the reference entity.

We recall that the first CDS was traded by J.P. Morgan in 1994 (Augustin *et al.*, 2014). The CDS market structure has been organized since then, especially the standardization of the CDS contract. Today, CDS agreements are governed by 2003 and 2014 ISDA credit derivatives definitions. For instance, the settlement of the CDS contract can be either physical or in cash. In the case of cash settlement, there is a monetary exchange from the protection seller to the protection buyer[23]. In the case of physical settlement, the protection buyer delivers a bond to the protection seller and receives the notional principal amount. Because the price of the defaulted bond is equal to $\boldsymbol{\mathcal{R}} \cdot N$, this means that the implied mark-to-market of this operation is $N - \boldsymbol{\mathcal{R}} \cdot N$ or equivalently $(1 - \boldsymbol{\mathcal{R}}) \cdot N$. Or course, physical settlement is only possible if the reference entity is a bond or if the credit event is based on the bond default. Whereas physical settlement was prevailing in the 1990s, most of the settlements are today in cash. Another standardization concerns the price of CDS. With the exception of very specific cases[24], CDS contracts are quoted in (fair) spread expressed in bps. In Figures 3.15 and 3.16, we show the evolution of some CDS spreads for a five-year maturity. We notice the increase of credit spreads since the 2008 financial turmoil and the

[22] We have:

$$\begin{aligned} \mathrm{PD} &= \Pr\{\boldsymbol{\tau} \le t + 1 \mid \boldsymbol{\tau} \le t\} \\ &= 1 - \mathbf{S}_t(t+1) \\ &= 1 - e^{-\lambda} \\ &\simeq \lambda \end{aligned}$$

For instance, if λ is equal respectively to 1%, 5%, 10% and 20% , the one-year default probability takes the values 1.00%, 4.88%, 9.52% and 18.13%.

[23] This monetary exchange is equal to $(1 - \boldsymbol{\mathcal{R}}) \cdot N$.

[24] When the default probability is high (larger than 20%), CDS contracts can be quoted with an upfront meaning that the protection seller is asking an initial amount to enter into the swap. For instance, it was the case of CDS on Greece in spring 2013.

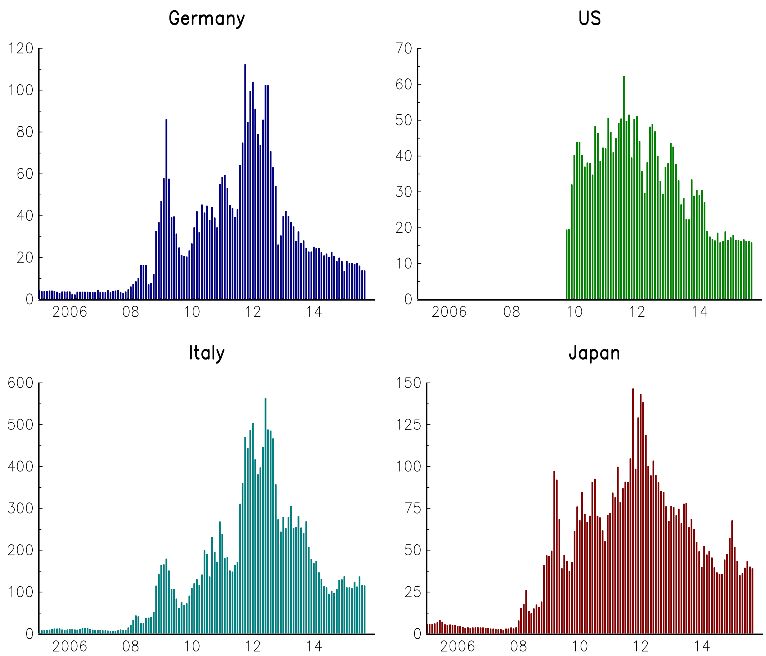

FIGURE 3.15: Evolution of some sovereign CDS spreads

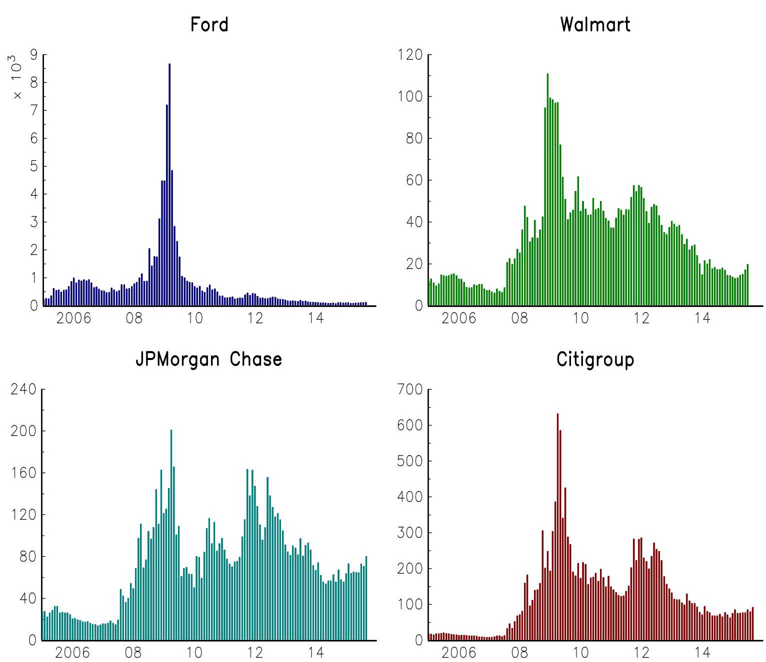

FIGURE 3.16: Evolution of some financial and corporate CDS spreads

default of Lehman Brothers bankruptcy, the sensitivity of German and Italian spreads with respect to the Eurozone crisis and also the difference in level between the different countries. Indeed, the spread is globally lower for US than for Germany or Japan. In the case of Italy, the spread is high and has reached 600 bps in 2012. We observe that the spread of some corporate entities may be lower than the spread of many developed countries (see Figure 3.16). This is the case of Walmart, whose spread is lower than 20 bps since 2014. When a company (or a country) is in great difficulty, the CDS spread explodes as in the case of Ford in February 2009. CDS spreads can be used to compare the default risk of two entities in the same sector. For instance, Figure 3.16 shows than the default risk of Citigroup is higher that this of JPMorgan Chase.

The CDS spread changes over time, but depends also on the maturity or tenor. This implies that we have a term structure of credit spreads for a given date t. This term structure is known as the credit spread curve and is noted $s_t(T)$ where T is the maturity time. Figure 3.17 shows the credit curve for different entities as of 17 September 2015. We notice that the CDS spread increases with the maturity. This is the most common case for investment grade (IG) entities, whose short-term default risk is low, but long-term default risk is higher. Nevertheless, we observe some distinguishing patterns between these credit curves. For instance, the credit risk of Germany is lower than the credit risk of US if the maturity is less than five years, but it is higher in the long run. There is a difference of 4 bps between Google and Apple on average when the time-to-maturity is less than 5 years. In the case of 10Y CDS, the spread of Apple is 90.8 bps whereas it is only 45.75 bps for Google.

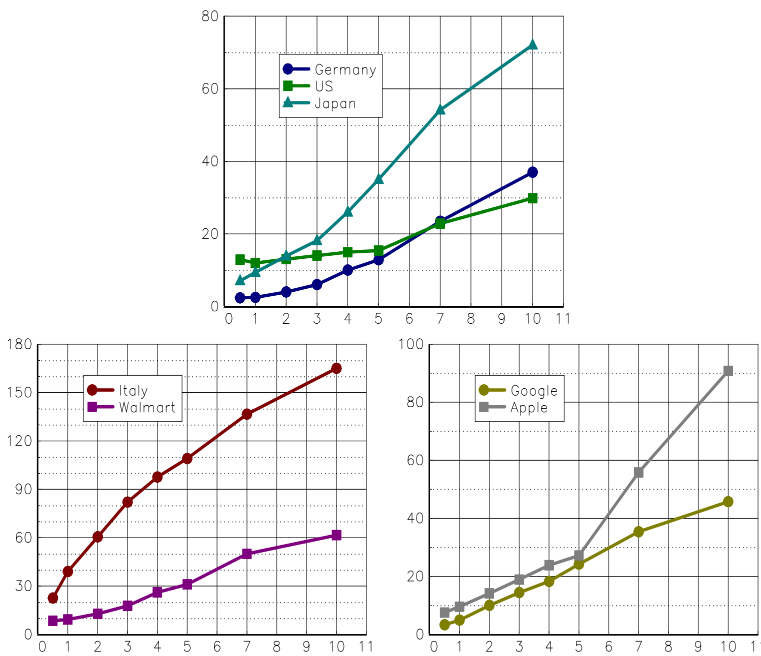

FIGURE 3.17: Example of CDS spread curves as of 17 September 2015

Remark 24 *In other cases, the credit curve may be decreasing (for some high yield corporates) or have a complex curvature (bell-shaped or U-shaped). In fact, Longstaff et al. (2005) showed that the dynamics of credit default swaps also depends on the liquidity risk. For instance, the most liquid CDS contract is generally the 5Y CDS. The liquidity on the other maturities depends on the reference entity and other characteristics such as the bond* *market liquidity. For example, the liquidity may be higher for short maturities when the credit risk of the reference entity is very high.*

Initially, CDS were used to hedge the credit risk of corporate bonds by banks and insurance companies. This hedging mechanism is illustrated in Figure 3.18. We assume that the bond holder buys a protection using a CDS, whose fixing dates of the premium leg are exactly the same as the coupon dates of the bond. We also assume that the credit even is the bond default and the notional of the CDS is equal to the notional of the bond. At each fixing date t_m, the bond holder receives the coupon $C(t_m)$ of the bond and pays to the protection seller the premium $\mathcal{s} \cdot N$. This implies that the net cash flow is $C(t_m) - \mathcal{s} \cdot N$. If the default occurs, the value of the bond becomes $\boldsymbol{\mathcal{R}} \cdot N$, but the protection seller pays to the bond holder the default leg $(1 - \boldsymbol{\mathcal{R}}) \cdot N$. In case of default, the net cash flow is then equal to $\boldsymbol{\mathcal{R}} \cdot N + (1 - \boldsymbol{\mathcal{R}}) \cdot N = N$, meaning that the exposure on the defaultable bond is perfectly hedged. We deduce that the annualized return R of this hedged portfolio is the difference between the yield to maturity $\mathcal{y}$ of the bond and the annual cost $\mathcal{s}$ of the protection:

$$R = \mathcal{y} - \mathcal{s} \tag{3.7}$$

We recognize a new formulation of Equation (3.4) on page 135. In theory, R is then equal to the yield to maturity $\mathcal{y}^\star$ of the bond without credit risk.

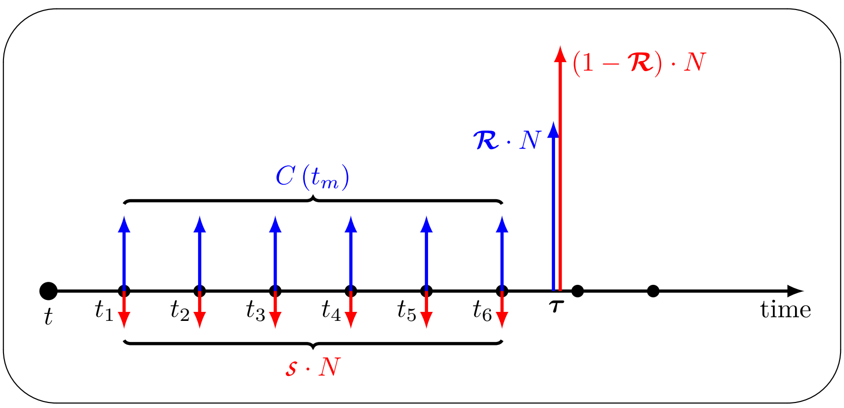

FIGURE 3.18: Hedging a defaultable bond with a credit default swap

Since the 2000s, end-users of CDS are banks and securities firms, insurance firms including pension funds, hedge funds and mutual funds. They continue to be used as hedging instruments, but they have also become financial instruments to express views about credit risk. In this case, '*long credit*' refers to the position of the protection seller who is exposed to the credit risk, whereas '*short credit*' is the position of the protection buyer who sold the credit risk of the reference entity[25]. To understand the mark-to-market of such positions, we consider the initial position at the inception date t of the CDS contract. In this case, the CDS spread $\mathcal{s}_t(T)$ verifies that the face value of the swap is equal to zero. Let us introduce the notation $P_{t,t'}(T)$, which defines the mark-to-market of a CDS position whose inception date is t, valuation date is t' and maturity date is T. We have:

$$P_{t,t}^{\text{seller}}(T) = P_{t,t}^{\text{buyer}}(T) = 0$$

[25]Said differently, a long exposure implies that the default results in a loss, whereas a short exposure implies that the default results in a gain.

At date $t' > t$, the mark-to-market price of the CDS is:

$$P_{t,t'}^{\text{buyer}}(T) = N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u)\, f_{t'}(u)\, \mathrm{d}u - \mathcal{s}_t(T)\cdot \mathrm{RPV}_{01}\right)$$

whereas the value of the CDS spread satisfies the following relationship:

$$P_{t',t'}^{\text{buyer}}(T) = N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u)\, f_{t'}(u)\, \mathrm{d}u - \mathcal{s}_{t'}(T)\cdot \mathrm{RPV}_{01}\right) = 0$$

We deduce that the P&L of the protection buyer is:

$$\Pi^{\text{buyer}} = P_{t,t'}^{\text{buyer}}(T) - P_{t,t}^{\text{buyer}}(T) = P_{t,t'}^{\text{buyer}}(T)$$

Using Equation (3.8), we know that $P_{t',t'}^{\text{buyer}}(T) = 0$ and we obtain:

$$\begin{aligned}
\Pi^{\text{buyer}} &= P_{t,t'}^{\text{buyer}}(T) - P_{t',t'}^{\text{buyer}}(T) \\
&= N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u)\, f_{t'}(u)\, \mathrm{d}u - \mathcal{s}_t(T)\cdot \mathrm{RPV}_{01}\right) - \\
&\quad\; N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u)\, f_{t'}(u)\, \mathrm{d}u - \mathcal{s}_{t'}(T)\cdot \mathrm{RPV}_{01}\right) \\
&= N\cdot\left(\mathcal{s}_{t'}(T) - \mathcal{s}_t(T)\right)\cdot \mathrm{RPV}_{01}
\end{aligned} \tag{3.8}$$

This equation highlights the role of the term RPV_{01} when calculating the P&L of the CDS position. Because $\Pi^{\text{seller}} = -\Pi^{\text{buyer}}$, we distinguish two cases:

- If $\mathcal{s}_{t'}(T) > \mathcal{s}_t(T)$, the protection buyer makes a profit, because this short credit exposure has benefited from the increase of the default risk.
- If $\mathcal{s}_{t'}(T) < \mathcal{s}_t(T)$, the protection seller makes a profit, because the default risk of the reference entity has decreased.

Suppose that we are in the first case. To realize its P&L, the protection buyer has three options (O'Kane, 2008):

1. He could unwind the CDS exposure with the protection seller if the latter agrees. This implies that the protection seller pays the mark-to-market $P_{t,t'}^{\text{buyer}}(T)$ to the protection buyer.
2. He could hedge the mark-to-market value by selling a CDS on the same reference entity and the same maturity. In this situation, he continues to pay the spread $\mathcal{s}_t(T)$, but he now receives a premium, whose spread is equal to $\mathcal{s}_{t'}(T)$.
3. He could reassign the CDS contract to another counterparty as illustrated in Figure 3.19. The new counterparty (the protection buyer C in our case) will then pay the coupon rate $\mathcal{s}_t(T)$ to the protection seller. However, the spread is $\mathcal{s}_{t'}(T)$ at time t', which is higher than $\mathcal{s}_t(T)$. This is why the new counterparty also pays the mark-to-market $P_{t,t'}^{\text{buyer}}(T)$ to the initial protection buyer.

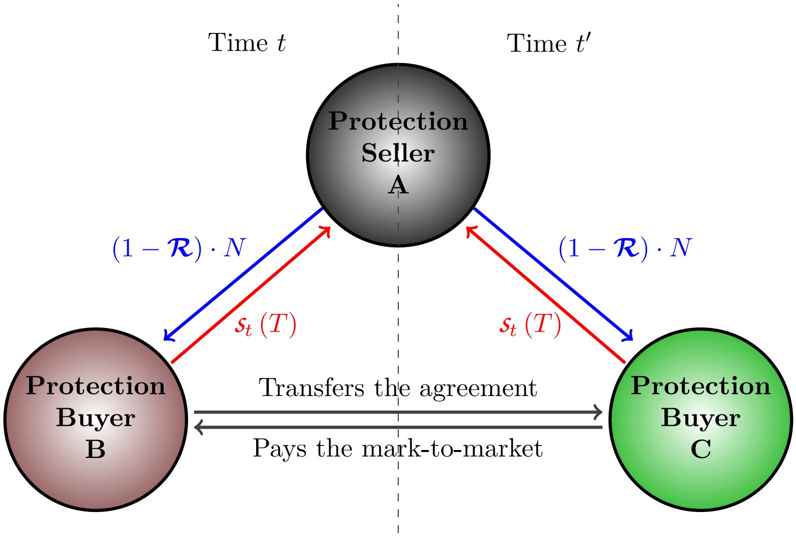

FIGURE 3.19: An example of CDS offsetting

Remark 25 *When the default risk is very high, CDS are quoted with an upfront*[26]*. In this case, the annual premium leg is equal to $\boldsymbol{c}^\star \cdot N$ where $\boldsymbol{c}^\star$ is a standard value*[27]*, and the protection buyer has to pay an upfront UF_t to the protection seller defined as follows:*

$$\mathrm{UF}_t = N\left((1-\mathcal{R})\int_t^T B_t(u)\, f_t(u)\,\mathrm{d}u - \boldsymbol{c}^\star \cdot \mathrm{RPV}_{01}\right)$$

Remark 26 *Until now, we have simplified the pricing of the premium leg in order to avoid complicated calculations. Indeed, if the default occurs between two fixing dates, the protection buyer has to pay the premium accrual. For instance, if $\boldsymbol{\tau} \in \left]t_{m-1}, t_m\right[$, the accrued premium is equal to $\boldsymbol{c} \cdot N \cdot (\boldsymbol{\tau} - t_{m-1})$ or equivalently to:*

$$\mathcal{AP} = \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot (\boldsymbol{\tau} - t_{m-1}) \cdot \mathbb{1}\{t_{m-1} \leq \boldsymbol{\tau} \leq t_m\}$$

We deduce that the stochastic discount value of the accrued premium is:

$$SV_t(\mathcal{AP}) = \sum_{t_m \geq t} \boldsymbol{c} \cdot N \cdot (\boldsymbol{\tau} - t_{m-1}) \cdot \mathbb{1}\{t_{m-1} \leq \boldsymbol{\tau} \leq t_m\} \cdot e^{-\int_t^{\boldsymbol{\tau}} r_s\,\mathrm{d}s}$$

It follows that:

$$PV_t(\mathcal{AP}) = \boldsymbol{c} \cdot N \cdot \sum_{t_m \geq t} \int_{t_{m-1}}^{t_m} (u - t_{m-1})\, B_t(u)\, f_t(u)\,\mathrm{d}u$$

All the previous formulas remain valid by replacing the expression of the risky PV01 by the following term:

$$\mathrm{RPV}_{01} = \sum_{t_m \geq t}\left(\Delta t_m \mathbf{S}_t(t_m)\, B_t(t_m) + \int_{t_{m-1}}^{t_m} (u - t_{m-1})\, B_t(u)\, f_t(u)\,\mathrm{d}u\right) \tag{3.9}$$

[26] It was the case several times for CDS on Greece.
[27] For distressed names, the default coupon rate $\boldsymbol{c}^\star$ is typically equal to 500 bps.

Example 24 *We assume that the yield curve is generated by the Nelson-Siegel model with the following parameters: $\theta_1 = 5\%$, $\theta_2 = -5\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. We consider several credit default swaps on the same entity with quarterly coupons and a notional of* \$1 *mn. The recovery rate $\boldsymbol{\mathcal{R}}$ is set to* 40% *whereas the default time $\boldsymbol{\tau}$ is an exponential random variable, whose parameter λ is equal to* 50 *bps. We consider seven maturities (6M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y) and two coupon rates (*10 *and* 100 *bps).*

To calculate the prices of these CDS, we use Equation (3.5) with $N = 10^6$, $\boldsymbol{c} = 10$ (or 100) $\times 10^{-4}$, $\Delta t_m = 1/4$, $\lambda = 50 \times 10^{-4} = 0.005$, $\boldsymbol{\mathcal{R}} = 0.40$, $\mathbf{S}_t(u) = e^{-0.005(u-t)}$, $f_t(u) = 0.005 \cdot e^{-0.005(u-t)}$ and $B_t(u) = e^{-(u-t)R_t(u)}$ where the zero-coupon rate is given by Equation (3.1). To evaluate the integral, we consider a Gauss-Legendre quadrature of 128$^{\text{th}}$ order. By including the accrued premium[28], we obtain results reported in Table 3.7. For instance, the price of the 5Y CDS is equal to \$9 527 if $\boldsymbol{c} = 10 \times 10^{-4}$ and $-$\$33 173 if $\boldsymbol{c} = 100 \times 10^{-4}$. In the first case, the protection buyer has to pay an upfront to the protection seller because the coupon rate is too low. In the second case, the protection buyer receives the upfront because the coupon rate is too high. We also indicate the spread $\boldsymbol{s}$ and the risky PV01. We notice that the CDS spread is almost constant. This is normal since the default rate is constant. This is why the CDS spread is approximatively equal to $(1 - 40\%) \times 50$ bps or 30 bps. The difference between the several maturities is due to the yield curve. The risky PV01 is a useful statistic to compute the mark-to-market. Suppose for instance that the two parties entered in a 7Y credit default swap of 10 bps spread two years ago. Now, the residual maturity of the swap is five years, meaning that the mark-to-market of the protection buyer is equal to:

$$\begin{aligned} \Pi^{\text{buyer}} &= 10^6 \times \left(30.08 \times 10^{-4} - 10 \times 10^{-4}\right) \times 4.744 \\ &= \$9\,526 \end{aligned}$$

We retrieve the 5Y CDS price (subject to rounding error).

TABLE 3.7: Price, spread and risky PV01 of CDS contracts

T	$P_t(T)$ $\boldsymbol{c} = 10$	$P_t(T)$ $\boldsymbol{c} = 100$	$\boldsymbol{s}$	RPV$_{01}$
1/2	998	−3 492	30.01	0.499
1	1 992	−6 963	30.02	0.995
2	3 956	−13 811	30.04	1.974
3	5 874	−20 488	30.05	2.929
5	9 527	−33 173	30.08	4.744
7	12 884	−44 804	30.10	6.410
10	17 314	−60 121	30.12	8.604

Example 25 *We consider a variant of Example 24 by assuming that the default time follows a Gompertz distribution:*

$$\mathbf{S}_t(u) = \exp\left(\phi\left(1 - e^{\gamma(u-t)}\right)\right)$$

The parameters ϕ and γ are set to 5% *and* 10%.

[28] This means that the risky PV01 corresponds to Equation (3.9). We also report results without taking into account the accrued premium in Table 3.8. We notice that its impact is limited.

TABLE 3.8: Price, spread and risky PV01 of CDS contracts (without the accrued premium)

T	$P_t(T)$ $c = 10$	$P_t(T)$ $c = 100$	s	RPV_{01}
1/2	999	−3 489	30.03	0.499
1	1 993	−6 957	30.04	0.994
2	3 957	−13 799	30.06	1.973
3	5 876	−20 470	30.07	2.927
5	9 530	−33 144	30.10	4.742
7	12 888	−44 764	30.12	6.406
10	17 319	−60 067	30.14	8.598

Results are reported in Table 3.9. In this example, the spread is increasing with the maturity of the CDS. Until now, we have assumed that we know the survival function $\mathbf{S}_t(u)$ in order to calculate the CDS spread. However, in practice, the CDS spread s is a market price and $\mathbf{S}_t(u)$ has to be determined thanks to a calibration procedure. Suppose for instance that we postulate that $\boldsymbol{\tau}$ is an exponential default time with parameter λ. We can calibrate the estimated value $\hat{\lambda}$ such that the theoretical price is equal to the market price. For instance, Table 3.9 shows the parameter $\hat{\lambda}$ for each CDS. We found that $\hat{\lambda}$ is equal to 51.28 bps for the six-month maturity and 82.92 bps for the ten-year maturity. We face here an issue, because the parameter $\hat{\lambda}$ is not constant, meaning that we cannot use an exponential distribution to represent the default time of the reference entity. This is why we generally consider a more flexible survival function to calibrate the default probabilities from a set of CDS spreads[29].

TABLE 3.9: Calibration of the CDS spread curve using the exponential model

T	$P_t(T)$ $c = 10$	$P_t(T)$ $c = 100$	s	RPV_{01}	$\hat{\lambda}$
1/2	1 037	−3 454	30.77	0.499	51.28
1	2 146	−6 808	31.57	0.995	52.59
2	4 585	−13 175	33.24	1.973	55.34
3	7 316	−19 026	35.00	2.927	58.25
5	13 631	−28 972	38.80	4.734	64.54
7	21 034	−36 391	42.97	6.380	71.44
10	33 999	−42 691	49.90	8.521	82.92

3.1.3.3 Basket default swap

A basket default swap is similar to a credit default swap except that the underlying asset is a basket of reference entities rather than one single reference entity. These products are part of multi-name credit default swaps with collateralized debt obligations.

First-to-default and k^{th}-to-default credit derivatives Let us consider a credit portfolio with n reference entities, which are referenced by the index i. With a first-to-default (FtD) credit swap, the credit event corresponds to the first time that a reference entity of the

[29] This problem will be solved later in Section 3.3.3.1 on page 203.

credit portfolio defaults. We deduce that the stochastic discounted values of the premium and default legs are[30]:

$$SV_t(\mathcal{PL}) = \boldsymbol{c} \cdot N \cdot \sum_{t_m \geq t} \Delta t_m \cdot \mathbb{1}\{\boldsymbol{\tau}_{1:n} > t_m\} \cdot e^{-\int_t^{t_m} r(s)\,\mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = \mathfrak{X} \cdot \mathbb{1}\{\boldsymbol{\tau}_{1:n} \leq T\} \cdot e^{-\int_t^{\boldsymbol{\tau}_{1:n}} r_s\,\mathrm{d}s}$$

where $\boldsymbol{\tau}_i$ is the default time of the i^{th} reference entity, $\boldsymbol{\tau}_{1:n} = \min(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ is the first default time in the portfolio and $\mathfrak{X}$ is the payout of the protection leg:

$$\begin{aligned} \mathfrak{X} &= \sum_{i=1}^{n} \mathbb{1}\{\boldsymbol{\tau}_{1:n} = \boldsymbol{\tau}_i\} \cdot (1 - \boldsymbol{\mathcal{R}}_i) \cdot N_i \\ &= (1 - \boldsymbol{\mathcal{R}}_{i^\star}) \cdot N_{i^\star} \end{aligned}$$

In this formula, $\boldsymbol{\mathcal{R}}_i$ and N_i are respectively the recovery and the notional of the i^{th} reference entity whereas the index $i^\star = \{i : \boldsymbol{\tau}_i = \boldsymbol{\tau}_{1:n}\}$ corresponds to the first reference entity that defaults. For instance, if the portfolio is composed by 10 names and the third name is the first default, the value of the protection leg will be equal to $(1 - \boldsymbol{\mathcal{R}}_3) \cdot N_3$. Using the same assumptions than previously, we deduce that the FtD spread is:

$$\mathcal{s}^{\mathrm{FtD}} = \frac{\mathbb{E}\left[\mathfrak{X} \cdot \mathbb{1}\{\boldsymbol{\tau}_{1:n} \leq T\} \cdot B_t(\boldsymbol{\tau}_{1:n})\right]}{N \sum_{t_m \geq t} \Delta t_m \cdot \mathbf{S}_{1:n,t}(t_m) \cdot B_t(t_m)}$$

where $\mathbf{S}_{1:n,t}(u)$ is the survival function of $\boldsymbol{\tau}_{1:n}$. If we assume a homogenous basket (same recovery $\boldsymbol{\mathcal{R}}_i = \boldsymbol{\mathcal{R}}$ and same notional $N_i = N$), the previous formula becomes:

$$\mathcal{s}^{\mathrm{FtD}} = \frac{(1 - \boldsymbol{\mathcal{R}}) \int_t^T B_t(u)\, f_{1:n,t}(u)\,\mathrm{d}u}{\sum_{t_m \geq t} \Delta t_m \mathbf{S}_{1:n,t}(t_m)\, B_t(t_m)} \tag{3.10}$$

where $f_{1:n,t}(u)$ is the survival function of $\boldsymbol{\tau}_{1:n}$.

To compute the spread[31], we use Monte Carlo simulation (or numerical integration when the number of entities is small). In fact, the survival function of $\boldsymbol{\tau}_{1:n}$ is related to the individual survival functions, but also to the dependence between the default times $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$. The spread of the FtD is then a function of default correlations[32]. If we denote by $\mathcal{s}_i^{\mathrm{CDS}}$ the CDS spread of the i^{th} reference, we can show that:

$$\max\left(\mathcal{s}_1^{\mathrm{CDS}}, \ldots, \mathcal{s}_n^{\mathrm{CDS}}\right) \leq \mathcal{s}^{\mathrm{FtD}} \leq \sum_{i=1}^{n} \mathcal{s}_i^{\mathrm{CDS}} \tag{3.11}$$

When the default times are uncorrelated, the FtD is equivalent to buy the basket of all the credit defaults swaps. In the case of a perfect correlation, one default is immediately followed by the other $n-1$ defaults, implying that the FtD is equivalent to the CDS with the worst spread. In practice, the FtD spread is therefore located between these two bounds as expressed in Equation (3.11). From the viewpoint of the protection buyer, a FtD is seen as a hedging method of the credit portfolio with a lower cost than buying the protection

[30] In order to simplify the notations, we do not take into account the accrued premium.

[31] Laurent and Gregory (2005) provide semi-explicit formulas that are useful for pricing basket default swaps.

[32] This point is developed in Section 3.3.4 on page 220 and in Chapter 11 dedicated to copula functions.

for all the credits. For example, suppose that the protection buyer would like to be hedged to the default of the automobile sector. He can buy a FtD on the basket of the largest car manufacturers in the world, e.g. Volkswagen, Toyota, Hyundai, General Motors, Fiat Chrysler and Renault. If there is only one default, the protection buyer is hedged. However, the protection buyer keeps the risk of multiple defaults, which is a worst-case scenario.

Remark 27 *The previous analysis can be extended to k^{th}-to-default swaps. In this case, the default leg is paid if the k^{th} default occurs before the maturity date. We then obtain a similar expression as Equation (3.10) by considering the order statistic $\boldsymbol{\tau}_{k:n}$ in place of $\boldsymbol{\tau}_{1:n}$.*

From a theoretical point of view, it is equivalent to buy the CDS protection for all the components of the credit basket or to buy all the k^{th}-to-default swaps. We have therefore the following relationship:

$$\sum_{i=1}^{n} \mathcal{s}_i^{\text{CDS}} = \sum_{i=1}^{n} \mathcal{s}^{i:n} \tag{3.12}$$

We see that the default correlation highly impacts the distribution of the k^{th}-to-default spreads[33].

Credit default indices Credit derivative indices[34] have been first developed by J.P. Morgan, Morgan Stanley and iBoxx between 2001 and 2003. A credit default index (or CDX) is in fact a credit default swap on a basket of reference entities. As previously, we consider a portfolio with n credit entities. The protection buyer pays a premium leg with a coupon rate $\boldsymbol{c}$. Every time a reference entity defaults, the notional is reduced by a factor, which is equal to $1/n$. At the same time, the protection buyer receives the portfolio loss between two fixing dates. The expression of the notional outstanding is then given by:

$$N_t(u) = N \cdot \left(1 - \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}\right)$$

At the inception date, we verify that $N_t(t) = N$. After the first default, the notional outstanding is equal to $N(1 - 1/n)$. After the k^{th} default, its value is $N(1 - k/n)$. At time $u \geq t$, the cumulative loss of the credit portfolio is:

$$L_t(u) = \frac{1}{n}\sum_{i=1}^{n} N \cdot (1 - \boldsymbol{\mathcal{R}}_i) \cdot \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}$$

meaning that the incremental loss between two fixing dates is:

$$\Delta L_t(t_m) = L_t(t_m) - L_t(t_{m-1})$$

We deduce that the stochastic discounted value of the premium and default legs is:

$$SV_t(\mathcal{PL}) = \boldsymbol{c} \cdot \sum_{t_m \geq t} \Delta t_m \cdot N_t(t_m) \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = \sum_{t_m \geq t} \Delta L_t(t_m) \cdot e^{-\int_t^{t_m} r_s \,\mathrm{d}s}$$

[33] See page 762 for an illustration.

[34] They are also known as synthetic credit indices, credit default swap indices or credit default indices.

We deduce that the spread of the CDX is:

$$\mathcal{s}^{\mathrm{CDX}} = \frac{\mathbb{E}\left[\sum_{t_m \geq t} \Delta L_t(t_m) \cdot B_t(t_m)\right]}{\mathbb{E}\left[\sum_{t_m \geq t} \Delta t_m \cdot N_t(t_m) \cdot B_t(t_m)\right]} \tag{3.13}$$

Remark 28 *A CDX is then equivalent to a portfolio of CDS whose each principal notional is equal to N/n. Indeed, when a default occurs, the protection buyer receives $N/n \cdot (1 - \boldsymbol{\mathcal{R}}_i)$ and stops to pay the premium leg of the defaulted reference entity. At the inception date, the annual premium of the CDX is then equal to the annual premium of the CDS portfolio:*

$$\mathcal{s}^{\mathrm{CDX}} \cdot N = \sum_{i=1}^{n} \mathcal{s}_i^{\mathrm{CDS}} \cdot \frac{N}{n}$$

We deduce that the spread of the CDX is an average of the credit spreads that compose the portfolio[35]*:*

$$\mathcal{s}^{\mathrm{CDX}} = \frac{1}{n} \sum_{i=1}^{n} \mathcal{s}_i^{\mathrm{CDS}} \tag{3.14}$$

Today, credit default indices are all managed by Markit and have been standardized. For instance, coupon payments are made on a quarterly basis (March 20, June 20, September 20, December 20) whereas indices roll every six months with an updated portfolio[36]. With respect to the original credit indices, Markit continues to produces two families:

- Markit CDX
  It focuses on North America and Emerging Markets credit default indices. The three major sub-indices are IG (investment grade), HY (high yield) and EM (emerging markets). A more comprehensive list is provided in Table 3.10. Besides these credit default indices, Markit CDX produces also four other important indices: ABX (basket of ABS), CMBX (basket of CMBS), LCDX (portfolio of 100 US secured senior loans) and MCDX (basket of 50 municipal bonds).

- Markit iTraxx
  It focuses on Europe, Japan, Asia ex-Japan and Australia (see the list in Table 3.11). Markit iTraxx also produces LevX (portfolio of 40 European secured loans), sector indices (e.g. European financials and industrials) and SovX, which corresponds to a portfolio of sovereign issuers. There are 7 SovX indices: Asia Pacific, BRIC, CEEMEA[37], G7, Latin America, Western Europe and Global Liquid IG.

In Table 3.12, we report the spread of some CDX/iTraxx indices. We note that the spread of the CDX.NA.HY index is on average four times larger than the spread of the CDX.NA.IG index. While spreads of credit default indices have generally decreased between December 2012 and December 2014, we observe a reversal in 2015. For instance, the spread of the CDX.NA.IG index is equal to 93.6 bps in September 2015 whereas it was only equal to 66.3 bps nine months ago. We observe a similar increase of 30 bps for the iTraxx Europe index. For the CDX.NA.HY index, it is more impressive with a variation of +150 bps in nine months.

[35] In fact, this is an approximation because the payment of the default leg does not exactly match between the CDX index and the CDS portfolio.
[36] See Markit (2014) for a detailed explanation of the indices' construction.
[37] Central and Eastern Europe, Middle East and Africa.

TABLE 3.10: List of Markit CDX main indices

Index name	Description	n	$\mathcal{R}$
CDX.NA.IG	Investment grade entities	125	40%
CDX.NA.IG.HVOL	High volatility IG entities	30	40%
CDX.NA.XO	Crossover entities	35	40%
CDX.NA.HY	High yield entities	100	30%
CDX.NA.HY.BB	High yield BB entities	37	30%
CDX.NA.HY.B	High yield B entities	46	30%
CDX.EM	EM sovereign issuers	14	25%
LCDX	Secured senior loans	100	70%
MCDX	Municipal bonds	50	80%

Source: Markit (2014).

TABLE 3.11: List of Markit iTraxx main indices

Index name	Description	n	$\mathcal{R}$
iTraxx Europe	European IG entities	125	40%
iTraxx Europe HiVol	European HVOL IG entities	30	40%
iTraxx Europe Crossover	European XO entities	40	40%
iTraxx Asia	Asian (ex-Japan) IG entities	50	40%
iTraxx Asia HY	Asian (ex-Japan) HY entities	20	25%
iTraxx Australia	Australian IG entities	25	40%
iTraxx Japan	Japanese IG entities	50	35%
iTraxx SovX G7	G7 governments	7	40%
iTraxx LevX	European leveraged loans	40	40%

Source: Markit (2014).

TABLE 3.12: Historical spread of CDX/iTraxx indices (in bps)

Date	CDX			iTraxx		
	NA.IG	NA.HY	EM	Europe	Japan	Asia
Dec. 2012	94.1	484.4	208.6	117.0	159.1	108.8
Dec. 2013	62.3	305.6	272.4	70.1	67.5	129.0
Dec. 2014	66.3	357.2	341.0	62.8	67.0	106.0
Sep. 2015	93.6	505.3	381.2	90.6	82.2	160.5

3.1.3.4 Collateralized debt obligations

A collateralized debt obligation (CDO) is another form of multi-name credit default swaps. It corresponds to a pay-through ABS structure[38], whose securities are bonds linked to a series of tranches. If we consider the example given in Figure 3.20, they are 4 types of bonds, whose returns depend on the loss of the corresponding tranche (equity, mezzanine, senior and super senior). Each tranche is characterized by an attachment point A and a

[38]See Figure 3.12 on page 139.

detachment point D. In our example, we have:

Tranche	Equity	Mezzanine	Senior	Super senior
A	0%	15%	25%	35%
D	15%	25%	35%	100%

The protection buyer of the tranche $[A, D]$ pays a coupon rate $\mathbf{c}^{[A,D]}$ on the nominal outstanding amount of the tranche to the protection seller. In return, he receives the protection leg, which is the loss of the tranche $[A, D]$. However, the losses satisfy a payment priority which is the following:

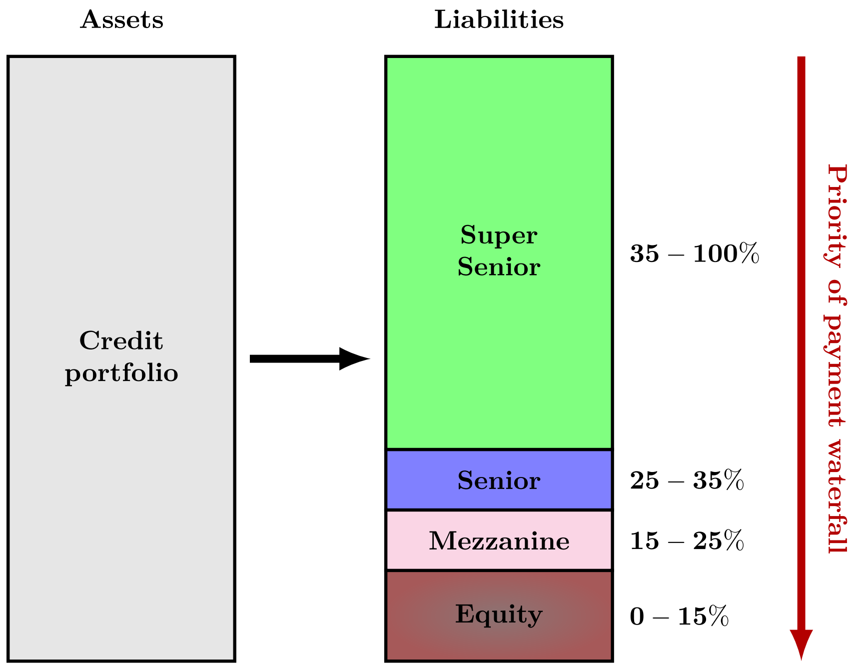

FIGURE 3.20: Structure of a collateralized debt obligation

- the equity tranche is the most risky security, meaning that the first losses hit this tranche alone until the cumulative loss reaches the detachment point;
- from the time the portfolio loss is larger than the detachment point of the equity tranche, the equity tranche no longer exists and this is the protection seller of the mezzanine tranche, who will pay the next losses to the protection buyer of the mezzanine tranche;
- the protection buyer of a tranche pays the coupon from the inception of the CDO until the death of the tranche, *i.e.*, when the cumulative loss is larger than the detachment point of the tranche; moreover, the premium payments are made on the reduced notional after each credit event of the tranche.

Each CDO tranche can then be viewed as a CDS with a time-varying notional principal to define the premium leg and a protection leg, which is paid if the portfolio loss is between the attachment and detachment points of the tranche. We can therefore interpret a CDO as a basket default swap, where the equity, mezzanine, senior and super senior tranches correspond respectively to a first-to-default, second-to-default, third-to-default and last-to-default swaps.

Let us now see the mathematical framework to price a CDO tranche. Assuming a portfolio of n credits, the cumulative loss is equal to:

$$L_t(u) = \sum_{i=1}^{n} N_i \cdot (1 - \boldsymbol{\mathcal{R}}_i) \cdot \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}$$

whereas the loss of the tranche $[A, D]$ is given by[39]:

$$\begin{array}{rcl} L_t^{[A,D]}(u) & = & (L_t(u) - A) \cdot \mathbb{1}\{A \leq L_t(u) \leq D\} + \\ & & (D - A) \cdot \mathbb{1}\{L_t(u) > D\} \end{array}$$

where A and D are the attachment and detachment points expressed in \$. The nominal outstanding amount of the tranche is therefore:

$$N_t^{[A,D]}(u) = (D - A) - L_t^{[A,D]}(u)$$

This notional principal decreases then by the loss of the tranche. At the inception of the CDO, $N_t^{[A,D]}(t)$ is equal to the tranche thickness: $(D - A)$. At the maturity date T, we have:

$$\begin{array}{rcl} N_t^{[A,D]}(T) & = & (D - A) - L_t^{[A,D]}(T) \\ & = & \left\{ \begin{array}{ll} (D - A) & \text{if } L_t(T) \leq A \\ (D - L_t(T)) & \text{if } A < L_t(T) \leq D \\ 0 & \text{if } L_t(T) > D \end{array} \right. \end{array}$$

We deduce that the stochastic discounted value of the premium and default legs is:

$$SV_t(\mathcal{PL}) = \boldsymbol{c}^{[A,D]} \cdot \sum_{t_m \geq t} \Delta t_m \cdot N_t^{[A,D]}(t_m) \cdot e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = \sum_{t_m \geq t} \Delta L_t^{[A,D]}(t_m) \cdot e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

Therefore, the spread of the CDO tranche is[40]:

$$\boldsymbol{s}^{[A,D]} = \frac{\mathbb{E}\left[\sum_{t_m \geq t} \Delta L_t^{[A,D]}(t_m) \cdot B_t(t_m)\right]}{\mathbb{E}\left[\sum_{t_m \geq t} \Delta t_m \cdot N_t^{[A,D]}(t_m) \cdot B_t(t_m)\right]} \tag{3.15}$$

We obviously have the following inequalities:

$$\boldsymbol{s}^{\text{Equity}} > \boldsymbol{s}^{\text{Mezzanine}} > \boldsymbol{s}^{\text{Senior}} > \boldsymbol{s}^{\text{Super senior}}$$

[39] Another expression is:

$$L_t^{[A,D]}(u) = \min\left(D - A, (L_t(u) - A)^+\right)$$

[40] This formula is obtained by assuming no upfront and accrued interests.

As in the case of k^{th}-to-default swaps, the distribution of these tranche spreads highly depends on the default correlation[41]. Depending on the model and the parameters, we can therefore promote the protection buyer/seller of one specific tranche with respect to the other tranches.

When collateralized debt obligations emerged in the 1990s, they were used to transfer credit risk from the balance sheet of banks to investors (e.g. insurance companies). They were principally portfolios of loans (CLO) or asset-backed securities (ABS CDO). With these balanced-sheet CDOs, banks could recover regulatory capital in order to issue new credits. In the 2000s, a new type of CDOs was created by considering CDS portfolios as underlying assets. These synthetic CDOs are also called arbitrage CDOs, because they have used by investors to express their market views on credit.

The impressive success of CDOs with investors before the 2008 Global Financial Crisis is due to the rating mechanism of tranches. Suppose that the underlying portfolio is composed of BB rated credits. It is obvious that the senior and super senior tranches will be rated higher than BB, because the probability that these tranches will be impacted is very low. The slicing approach of CDOs enables then to create high-rated securities from medium or low-rated debts. Since the appetite of investors for AAA and AA rated bonds was very important, CDOs were solutions to meet this demand. Moreover, this lead to the development of rating methods in order to provide an attractive spread. This explains that most of AAA-rated CDO tranches promised a return higher than AAA-rated sovereign and corporate bonds. In fact, the 2008 GFC has demonstrated that many CDO tranches were more risky than expected, because the riskiness of the assets were underestimated[42].

TABLE 3.13: List of Markit credit default tranches

Index name	Tranche				
CDX.NA.IG	$0-3$	$3-7$	$7-15$	$15-100$	
CDX.NA.HY	$0-10$	$10-15$	$15-25$	$25-35$	$35-100$
LCDX	$0-5$	$5-8$	$8-12$	$12-15$	$15-100$
iTraxx Europe	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$	$22-100$
iTraxx Europe XO	$0-10$	$10-15$	$15-25$	$25-35$	$35-100$
iTraxx Asia	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$
iTraxx Australia	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$
iTraxx Japan	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$

Source: Markit (2014).

For some years now, CDOs have been created using credit default indices as the underlying portfolio. For instance, Table 3.13 provides the list of available tranches on Markit indices[43]. We notice that attachment and detachment points differ from one index to another index. The first tranche always indicates the equity tranche. For IG underlying assets, the notional corresponds to the first 3% losses of the portfolio, whereas the detachment point is higher for crossover or high yield assets. We also notice that some senior tranches are not traded (Asia, Australia and Japan). These products are mainly used in correlation trading activities and also served as benchmarks for all the other OTC credit debt obligations.

[41]See Section 3.3.4 on page 220.

[42]More details of the impact of the securitization market on the 2008 Global Financial Crisis are developed in Chapter 8 dedicated to systemic risk.

[43]They are also called credit default tranches (CDT).

3.2 Capital requirement

This section deals with regulatory aspects of credit risk. From a historical point of view, this is the first risk which has requested regulatory capital before market risk. Nevertheless, the development of credit risk management is more recent and was accelerated with the Basel II Accord. Before presenting the different approaches for calculating capital requirements, we need to define more precisely what credit risk is.

It is the risk of loss on a debt instrument resulting from the failure of the borrower to make required payments. We generally distinguish two types of credit risk. The first one is the '*default risk*', which arises when the borrower is unable to pay the principal or interests. An example is a student loan or a mortgage loan. The second type is the '*downgrading risk*', which concerns debt securities. In this case, the debt holder may face a loss, because the price of the debt security is directly related to the credit risk of the borrower. For instance, the price of the bond may go down because the credit risk of the issuer increases and even if the borrower does not default. Of course, default risk and downgrading risk are highly correlated, because it is rare that a counterparty suddenly defaults without downgrading of its credit rating.

To measure credit risk, we first need to define the default of the obligor. BCBS (2006) provides the following standard definition:

> "*A default is considered to have occurred with regard to a particular obligor when either or both of the two following events have taken place.*
>
> - *The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realizing security (if held).*
> - *The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered as being past due once the customer has breached an advised limit or been advised of a limit smaller than current outstandings*" (BCBS, 2006, page 100).

This definition contains both objective elements (when a payment has been missed or delayed) and subjective elements (when a loss becomes highly probable). This last case generally corresponds to an extreme situation (specific provision, distressed restructuring, etc.). The Basel definition of default covers then two types of credit: debts under litigation and doubtful debts.

Downgrading risk is more difficult to define. If the counterparty is rated by an agency, it can be measured by a single or multi-notch downgrade. However, it is not always the case in practice, because the credit quality decreases before the downgrade announcement. A second measure is to consider a market-based approach by using CDS spreads. However, we notice that the two methods concern counterparties, which are able to issue debt securities, in particular bonds. For instance, the concept of downgrading risk is difficult to apply for retail assets.

The distinction between default risk and downgrading risk has an impact on the credit risk measure. For loans and debt-like instruments that cannot be traded in a market, the time horizon for managing credit risk is the maturity of the credit. Contrary to this held-to-maturity approach, the time horizon for managing debt securities is shorter, typically one year. In this case, the big issue is not to manage the default, but the mark-to-market of the credit exposure.

3.2.1 The Basel I framework

According to Tarullo (2008), two explanatory factors were behind the Basel I Accord. The first motivation was to increase capital levels of international banks, which were very low at that time and had continuously decreased for many years. For instance, the ratio of equity capital to total assets[44] was 5.15% in 1970 and only 3.83% in 1981 for the 17 largest US banks. In 1988, this capital ratio was equal to 2.55% on average for the five largest bank in the world. The second motivation concerned the distortion risk of competition resulting from heterogeneous national capital requirements. One point that was made repeatedly, especially by US bankers, was the growth of Japanese banks. In Table 3.14, we report the ranking of the 10 world's largest banks in terms of assets ($ bn) between 2001 and 2008. While there is only one Japanese bank in the top 10 in 1981, nine Japanese banks are included in the ranking seven years later. In this context, the underlying idea of the Basel I Accord was then to increase capital requirements and harmonize national regulations for international banks.

TABLE 3.14: World's largest banks in 1981 and 1988

	1981		1988	
	Bank	Assets	Bank	Assets
1	Bank of America (US)	115.6	Dai-Ichi Kangyo (JP)	352.5
2	Citicorp (US)	112.7	Sumitomo (JP)	334.7
3	BNP (FR)	106.7	Fuji (JP)	327.8
4	Crédit Agricole (FR)	97.8	Mitsubishi (JP)	317.8
5	Crédit Lyonnais (FR)	93.7	Sanwa (JP)	307.4
6	Barclays (UK)	93.0	Industrial Bank (JP)	261.5
7	Société Générale (FR)	87.0	Norinchukin (JP)	231.7
8	Dai-Ichi Kangyo (JP)	85.5	Crédit Agricole (FR)	214.4
9	Deutsche Bank (DE)	84.5	Tokai (JP)	213.5
10	National Westminster (UK)	82.6	Mitsubishi Trust (JP)	206.0

Source: Tarullo (2008).

The Basel I Accord provides a detailed definition of bank capital C and risk-weighted assets RWA. We reiterate that tier one (T1) capital consists mainly of common stock and disclosed reserves, whereas tier two (T2) capital includes undisclosed reserves, general provisions, hybrid debt capital instruments and subordinated term debt. Risk-weighted assets are simply calculated as the product of the asset notional (the exposure at default or EAD) by a risk weight (RW). Table 3.15 shows the different values of RW with respect to the category of the asset. For off-balance sheet assets, BCBS (1988) defines credit conversion factor (CCF) for converting the amount E of a credit line or off-balance sheet asset to an exposure at default:

$$\mathrm{EAD} = E \cdot \mathrm{CCF}$$

The CCF values are 100% for direct credit substitutes (standby letters of credit), sale and repurchase agreements, forward asset purchases, 50% for standby facilities and credit lines with an original maturity of over one year, note issuance facilities and revolving underwriting facilities, 20% for short-term self-liquidating trade-related contingencies and 0% for standby facilities and credit lines with an original maturity of up to one year. The above framework is used to calculate the Cooke ratio, which is in fact a set of two capital ratios. The core

[44] All the statistics of this section comes from Chapters 2 and 3 of Tarullo (2008).

TABLE 3.15: Risk weights by category of on-balance sheet assets

RW	Instruments
0%	Cash Claims on central governments and central banks denominated in national currency and funded in that currency Other claims on OECD central governments and central banks Claims† collateralized by cash of OECD government securities
20%	Claims† on multilateral development banks Claims† on banks incorporated in the OECD and claims guaranteed by OECD incorporated banks Claims† on securities firms incorporated in the OECD subject to comparable supervisory and regulatory arrangements Claims† on banks incorporated in countries outside the OECD with a residual maturity of up to one year Claims† on non-domestic OECD public-sector entities Cash items in process of collection
50%	Loans fully secured by mortgage on residential property
100%	Claims on the private sector Claims on banks incorporated outside the OECD with a residual maturity of over one year Claims on central governments outside the OECD and non denominated in national currency All other assets

†or guaranteed by these entities.

Source: BCBS (1988).

capital ratio includes only tier one capital whereas the total capital ratio considers both tier one C_1 and tier two C_2 capital:

$$\begin{aligned}\text{Tier 1 ratio} &= \frac{C_1}{\text{RWA}} \geq 4\% \\ \text{Tier 2 ratio} &= \frac{C_1 + C_2}{\text{RWA}} \geq 8\%\end{aligned}$$

Example 26 *The assets of the bank are composed of* \$100 *mn of US treasury bonds,* \$20 *mn of Mexico government bonds denominated in US dollar,* \$20 *mn of Argentine debt denominated in Argentine peso,* \$500 *mn of residential mortgage,* \$500 *mn of corporate loans,* \$20 *mn of non-used standby facilities for OECD governments and* \$100 *mn of retail credit lines, which are decomposed as follows:* \$40 *mn are used and* 70% *of non-used credit lines have a maturity greater than one year.*

For each asset, we calculate RWA by choosing the right risk weight and credit conversion factor for off-balance sheet items. We obtain the results below. The risk-weighted assets of the bank are then equal to \$831 mn. We deduce that the required capital $\mathcal{K}$ is \$33.24 mn for tier one.

Balance Sheet	Asset	E	CCF	EAD	RW	RWA
On-	US bonds			100	0%	0
	Mexico bonds			20	100%	20
	Argentine debt			20	0%	0
	Home mortgage			500	50%	250
	Corporate loans			500	100%	500
	Credit lines			40	100%	40
Off-	Standby facilities	20	100%	20	0%	0
	Credit lines (> 1Y)	42	50%	21	100%	21
	Credit lines (≤ 1Y)	18	0%	0	100%	0
	Total					831

3.2.2 The Basel II standardized approach

The main criticism of the Cooke ratio is the lack of economic rationale with respect to risk weights. Indeed, most of the claims have a 100% risk weight and do not reflect the real credit risk of the borrower. Other reasons have been given to justify a reformulation of capital requirements for credit risk with the goal to:

- obtain a better credit risk measure by taking into account the default probability of the counterparty;
- avoid regulatory arbitrage, in particular by using credit derivatives;
- have a more coherent framework that supports credit risk mitigation.

3.2.2.1 Standardized risk weights

In Basel II, the probability of default is the key parameter to define risk weights. For the standardized approach (SA), they depend directly on external ratings whereas they are based on internal rating for the IRB approach. Table 3.16 shows the new matrix of risk weights, when we consider the Standard & Poor's rating system[45]. We notice that there are four main categories of claims[46]: sovereigns, banks, corporates and retail portfolios.

The sovereign exposure category include central governments and central banks, whereas non-central public sector entities are treated with the bank exposure category. We note that there are two options for the latter, whose choice is left to the discretion of the national supervisors[47]. Under the first option, the risk weight depends on the rating of the country where the bank is located. Under the second option, it is the rating of the bank that determines the risk weight, which is more favorable for short-term claims (three months or less). The risk weight of a corporate is calculated with respect to the rating of the entity, but uses a slightly different breakdown of ratings than the second option of the bank category. Finally, the Basel Committee uses lower levels for retail portfolios than those provided in the Basel I Accord. Indeed, residential mortgages and retail loans are now risk-weighted at 35% and 75% instead of 50% and 100% previously. Other comparisons between Basel I and Basel II (with the second option for banks) are shown in Table 3.17.

[45] NR stands for non-rated entities.

[46] The regulatory framework is more comprehensive by considering three other categories (public sector entities, multilateral development banks and securities firms), which are treated as banks. For all other assets, the standard risk weight is 100%.

[47] The second option is more frequent and was implemented in Europe, US and Japan for instance.

TABLE 3.16: Risk weights of the SA approach (Basel II)

Rating		AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to B−	CCC+ to C	NR
Sovereigns		0%	20%	50%	100%	150%	100%
Banks	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
	2 ST	20%	20%	20%	50%	150%	20%
Corporates				BBB+ to BB−		B+ to C	
		20%	50%	100%		150%	100%
Retail					75%		
Residential mortgages					35%		
Commercial mortgages					100%		

TABLE 3.17: Comparison of risk weights between Basel I and Basel II

Entity	Rating	Maturity	Basel I	Basel II
Sovereign (OECD)	AAA		0%	0%
Sovereign (OECD)	A-		0%	20%
Sovereign	BBB		100%	50%
Bank (OECD)	BBB	2Y	20%	50%
Bank	BBB	2M	100%	20%
Corporate	AA+		100%	20%
Corporate	BBB		100%	100%

The SA approach is based on external ratings and then depends on credit rating agencies. The most famous are Standard & Poor's, Moody's and Fitch. However, they cover only large companies. This is why banks will also consider rating agencies specialized in a specific sector or a given country[48]. Of course, rating agencies must be first registered and certified by the national supervisor in order to be used by the banks. The validation process consists of two steps, which are the assessment of the six required criteria (objectivity, independence, transparency, disclosure, resources and credibility) and the mapping process between the ratings and the Basel matrix of risk weights.

Table 3.18 shows the rating systems of S&P, Moody's and Fitch, which are very similar. Examples of S&P's rating are given in Tables 3.19, 3.20 and 3.21. We note that the rating of many sovereign counterparties has been downgraded by at least one notch, except China which has now a better rating than before the 2008 GFC. For some countries, the rating in local currency is different from the rating in foreign currency, for instance Argentina, Brazil, Russia and Ukraine[49]. We observe the same evolution for banks and it is now rare to find a bank with a AAA rating. This is not the case of corporate counterparties, which present more stable ratings across time.

Remark 29 *Credit conversion factors for off-balance sheet items are similar to those defined in the original Basel Accord. For instance, any commitment that is unconditionally cancelable receives a* 0% *CCF. A CCF of* 20% *(resp.* 50%*) is applied to commitments with*

[48]For instance, banks may use Japan Credit Rating Agency Ltd for Japanese public and corporate entities, DBRS Ratings Limited for bond issuers, Cerved Rating Agency for Italian small and medium-sized enterprises, etc.

[49]An SD rating is assigned in case of selective default of the obligor.

TABLE 3.18: Credit rating system of S&P, Moody's and Fitch

	Prime Maximum Safety	High Grade High Quality			Upper Medium Grade		
S&P/Fitch	AAA	AA+	AA	AA−	A+	A	A−
Moody's	Aaa	Aa1	Aa2	Aa3	A1	A2	A3

	Lower Medium Grade			Non Investment Grade Speculative		
S&P/Fitch	BBB+	BBB	BBB−	BB+	BB	BB−
Moody's	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3

	Highly Speculative			Substantial Risk	In Poor Standing		Extremely Speculative
S&P/Fitch	B+	B	B−	CCC+	CCC	CCC−	CC
Moody's	B1	B2	B3	Caa1	Caa2	Caa3	Ca

TABLE 3.19: Examples of country's S&P rating

Country	Local currency		Foreign currency	
	Jun. 2009	Oct. 2015	Jun. 2009	Oct. 2015
Argentina	B-	CCC+	B-	SD
Brazil	BBB+	BBB-	BBB-	BB+
China	A+	AA-	A+	AA-
France	AAA	AA	AAA	AA
Italy	A+	BBB-	A+	BBB-
Japan	AA	A+	AA	A+
Russia	BBB+	BBB-	BBB	BB+
Spain	AA+	BBB+	AA+	BBB+
Ukraine	B-	CCC+	CCC+	SD
US	AAA	AA+	AA+	AA+

Source: Standard & Poor's, www.standardandpoors.com.

TABLE 3.20: Examples of bank's S&P rating

Bank	Oct. 2001	Jun. 2009	Oct. 2015
Barclays Bank PLC	AA	AA-	A-
Credit Agricole S.A.	AA	AA-	A
Deutsche Bank AG	AA	A+	BBB+
International Industrial Bank	CCC+	BB-	
JPMorgan Chase & Co.	AA-	A+	A
UBS AG	AA+	A+	A

Source: Standard & Poor's, www.standardandpoors.com.

TABLE 3.21: Examples of corporate's S&P rating

Corporate	Jul. 2009	Oct. 2015
Danone	A-	A-
Exxon Mobil Corp.	AAA	AAA
Ford Motor Co.	CCC+	BBB-
General Motors Corp.	D	BBB-
L'Oreal S.A.	NR	NR
Microsoft Corp.	AAA	AAA
Nestle S.A.	AA	AA
The Coca-Cola Co.	A+	AA
Unilever PLC	A+	A+

Source: Standard & Poor's, www.standardandpoors.com.

an original maturity up to one year (resp. greater than one year). For revolving underwriting facilities, the CCF is equal to 50% *whereas it is equal to* 100% *for other off-balance sheet items (e.g. direct credit substitutes, guarantees, sale and repurchase agreements, forward asset purchases).*

3.2.2.2 Credit risk mitigation

Credit risk mitigation (CRM) refers to the various techniques used by banks for reducing the credit risk. These methods allow to decrease the credit exposure or to increase the recovery in case of default. The most common approaches are collateralized transactions, guarantees, credit derivatives and netting agreements.

Collateralized transactions In such operations, the credit exposure of the bank is partially hedged by collateral posted by the counterparty. BCBS (2006) defines then the following eligible instruments:

1. Cash and comparable instruments;
2. Gold;
3. Debt securities which are rated AAA to BB- when issued by sovereigns or AAA to BBB- when issued by other entities or at least A-3/P-3 for short-term debt instruments;
4. Debt securities which are not rated but fulfill certain criteria (senior debt issued by banks, listed on a recognisee exchange and sufficiently liquid);
5. Equities that are included in a main index;
6. UCITS and mutual funds, whose assets are eligible instruments and which offer a daily liquidity;
7. Equities which are listed on a recognized exchange and UCITS/mutual funds which include such equities.

The bank has the choice between two approaches to take into account collateralized transactions. In the simple approach[50], the risk weight of the collateral (with a floor of

[50] Collateral instruments (7) are not eligible for this approach.

20%) is applied to the market value of the collateral C whereas the non-hedged exposure $(\text{EAD} - C)$ receives the risk weight of the counterparty:

$$\text{RWA} = (\text{EAD} - C) \cdot \text{RW} + C \cdot \max(\text{RW}_C, 20\%) \tag{3.16}$$

where EAD is the exposure at default, C is the market value of the collateral, RW is the risk weight appropriate to the exposure and RW_C is the risk weight of the collateral. The second method, called the comprehensive approach, is based on haircuts. The risk-weighted asset amount after risk mitigation is $\text{RWA} = \text{RW} \cdot \text{EAD}^\star$ whereas $\text{EAD}^\star$ is the modified exposure at default defined as follows:

$$\text{EAD}^\star = \max(0, (1 + H_E) \cdot \text{EAD} - (1 - H_C - H_{FX}) \cdot C) \tag{3.17}$$

where H_E is the haircut applied to the exposure, H_C is the haircut applied to the collateral and H_{FX} is the haircut for currency risk. Table 3.22 gives the standard supervisory values of haircuts. If the bank uses an internal model to calculate haircuts, they must be based on the value-at-risk with a 99% confidence level and an holding period which depends on the collateral type and the frequency of remargining. The standard supervisory haircuts have been calibrated by assuming daily mark-to-market, daily remargining and a 10-business day holding period.

TABLE 3.22: Standardized supervisory haircuts for collateralized transactions

Rating	Residual Maturity	Sovereigns	Others
AAA to AA−	0−1Y	0.5%	1%
	1−5Y	2%	4%
	5Y+	4%	8%
A+ to BBB−	0−1Y	1%	2%
	1−5Y	3%	6%
	5Y+	6%	12%
BB+ to BB−		15%	
Cash		0%	
Gold		15%	
Main index equities		15%	
Equities listed on a recognized exchange		25%	
FX risk		8%	

Example 27 *We consider a 10-year credit of* \$100 *mn to a corporate firm rated* A. *The credit is guaranteed by five collateral instruments: a cash deposit (*\$2 *mn), a gold deposit (*\$5 *mn), a sovereign bond rated* AA *with a 2-year residual maturity (*\$15 *mn) and repurchase transactions on Microsoft stocks (*\$20 *mn) and Wirecard*[51] *stocks (*\$20 *mn).*

Before credit risk mitigation, the risk-weighted asset amount is equal to:

$$\text{RWA} = 100 \times 50\% = \$50 \text{ mn}$$

If we consider the simple approach, the repurchase transaction on Wirecard stocks is not eligible, because it does not fall within categories (1)-(6). The risk-weighted asset amount

[51] Wirecard is a German financial company specialized in payment processing and issuing services. The stock belongs to the MSCI Small Cap Europe index.

becomes[52]:

$$\begin{aligned} \text{RWA} &= (100 - 2 - 5 - 15 - 20) \times 50\% + (2 + 5 + 15 + 20) \times 20\% \\ &= \$37.40 \text{ mn} \end{aligned}$$

The repurchase transaction on Wirecard stocks is eligible in the comprehensive approach, because these equity stocks are traded in Börse Frankfurt. The haircuts are 15% for gold, 2% for the sovereign bond and 15% for Microsoft stocks[53]. For Wirecard stocks, a first haircut of 25% is applied because this instrument belongs to the seventh category and a second haircut of 8% is applied because there is a foreign exchange risk. The adjusted exposure at default is then equal to:

$$\begin{aligned} \text{EAD}^{\star} &= (1 + 8\%) \times 100 - 2 - (1 - 15\%) \times 5 - (1 - 2\%) \times 15 - \\ &\quad (1 - 15\%) \times 20 - (1 - 25\% - 8\%) \times 20 \\ &= \$73.65 \text{ mn} \end{aligned}$$

It follows that:

$$\text{RWA} = 73.65 \times 50\% = \$36.82 \text{ mn}$$

Guarantees and credit derivatives Banks can use these credit protection instruments if they are direct, explicit, irrevocable and unconditional. In this case, banks use the simple approach given by Equation (3.16). The case of credit default tranches is covered by rules described in the securitization framework.

Maturity mismatches A maturity mismatch occurs when the residual maturity of the hedge is less than that of the underlying asset. In this case, the bank uses the following adjustment:

$$C_A = C \cdot \frac{\min (T_G, T, 5) - 0.25}{\min (T, 5) - 0.25} \tag{3.18}$$

where T is the residual maturity of the exposure and T_G is the residual maturity of the collateral (or guarantee).

Example 28 *The bank A has granted a credit of \$30 mn to a corporate firm B, which is rated* BB. *In order to hedge the default risk, the bank A buys \$20 mn of a 3-year CDS protection on B to the bank C, which is rated* A+.

If the residual maturity of the credit is lower than 3 years, we obtain:

$$\text{RWA} = (30 - 20) \times 100\% + 20 \times 50\% = \$20 \text{ mn}$$

If the residual maturity is greater than 3 years, we first have to calculate the adjusted value of the guarantee. Assuming that the residual maturity is 4 years, we have:

$$G_A = 20 \times \frac{\min (3, 4, 5) - 0.25}{\min (4, 5) - 0.25} = \$14.67 \text{ mn}$$

It follows that:

$$\text{RWA} = (30 - 14.67) \times 100\% + 14.67 \times 50\% = \$22.67 \text{ mn}$$

[52]The floor of 20% is applied to the cash, gold and sovereign bond collateral instruments. The risk weight for Microsoft stocks is 20% because the rating of Microsoft is AAA.

[53]Because Microsoft belongs to the S&P 500 index, which is a main equity index.

3.2.3 The Basel II internal ratings-based approach

The completion of the internal ratings-based (IRB) approach was a complex task, because it required many negotiations between regulators, banks and politics. Tarullo (2008) points out that the publication of the first consultative paper (CP1) in June 1999 was both "*anticlimactic and contentious*". The paper is curiously vague without a precise direction. The only tangible proposal is the use of external ratings. The second consultative paper is released in January 2001 and includes in particular the IRB approach, which has been essentially developed by US members of the Basel Committee with the support of large international banks. The press release dated 16 January 2001 indicated that the Basel Committee would finalize the New Accord by the end of 2001, for an implementation in 2004. However, it has taken much longer than originally anticipated and the final version of the New Accord was published in June 2004 and implemented from December 2006[54]. The main reason is the difficulty of calibrating the IRB approach in order to satisfy a large part of international banks. The IRB formulas of June 2004 are significantly different from the original ones and reflect compromises between the different participants without really being satisfactory.

3.2.3.1 The general framework

Contrary to the standardized approach, the IRB approach is based on internal rating systems. With such a method, the objectives of the Basel Committee were to propose a more sensitive credit risk measure and define a common basis between internal credit risk models. The IRB approach may be seen as an external credit risk model, whose parameters are provided by the bank. Therefore, it is not an internal model, but a first step to harmonize the internal risk management practices by focusing on the main risk components, which are:

- the exposure at default (EAD);
- the probability of default (PD);
- the loss given default (LGD);
- the effective maturity (M).

The exposure at default is defined as the outstanding debt at the time of default. For instance, it is equal to the principal amount for a loan. The loss given default is the expected percentage of exposure at default that is lost if the debtor defaults. At first approximation, one can consider that $\mathrm{LGD} \simeq 1 - \boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{R}}$ is the recovery rate. While EAD is expressed in \$, LGD is measured in %. For example, if EAD is equal to \$10 mn and LGD is set to 70%, the expected loss due to the default is equal to \$7 mn. The probability of default measures the default risk of the debtor. In Basel II, the time horizon of PD is set to one year. When the duration of the credit is not equal to one year, one has to specify its effective maturity M. This is the combination of the one-year default probability PD and the effective maturity M that measures the default risk of the debtor until the duration of the credit.

In this approach, the credit risk measure is the sum of individual risk contributions:

$$\mathcal{R}(w) = \sum_{i=1}^{n} \mathcal{RC}_i$$

[54] See Chapter 4 entitled "*Negotiating Basel II*" of Tarullo (2008) for a comprehensive story of the Basel II Accord.

where $\mathcal{RC}_i$ is a function of the four risk components:

$$\mathcal{RC}_i = f_{\text{IRB}}\left(\text{EAD}_i, \text{LGD}_i, \text{PD}_i, \text{M}_i\right)$$

and f_{IRB} is the IRB fomula. In fact, there are two IRB methodologies. In the foundation IRB approach (FIRB), banks use their internal estimates of PD whereas the values of the other components (EAD, LGD and M) are set by regulators. Banks that adopt the advanced IRB approach (AIRB) may calculate all the four parameters (PD, EAD, LGD and M) using their own internal models and not only the probability of default. The mechanism of the IRB approach is then the following:

- a classification of exposures (sovereigns, banks, corporates, retail portfolios, etc.);
- for each credit i, the bank estimates the probability of default PD_i;
- it uses the standard regulatory values of the other risk components (EAD_i, LGD_i and M_i) or estimates them in the case of AIRB;
- the bank calculate then the risk-weighted assets RWA_i of the credit by applying the right IRB formula f_{IRB} to the risk components.

Internal ratings are central to the IRB approach. Table 3.23 gives an example of an internal rating system, where risk increases with the number grade (1, 2, 3, etc.). Another approach is to consider alphabetical letter grades[55]. A third approach is to use an internal rating scale similar to that of S&P[56].

3.2.3.2 The credit risk model of Basel II

Decomposing the value-at-risk into risk contributions BCBS (2004a) used the Merton-Vasicek model (Merton, 1974; Vasicek, 2002) to derive the IRB formula. In this framework, the portfolio loss is equal to:

$$L = \sum_{i=1}^{n} w_i \cdot \text{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\} \tag{3.19}$$

where w_i and T_i are the exposure at default and the residual maturity of the i^{th} credit. We assume that the loss given default LGD_i is a random variable and the default time $\boldsymbol{\tau}_i$ depends on a set of risk factors X, whose probability distribution is denoted by $\mathbf{H}$. Let $p_i(X)$ be the conditional default probability. It follows that the (unconditional or long-term) default probability is:

$$\begin{aligned} p_i &= \mathbb{E}_X\left[\mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}\right] \\ &= \mathbb{E}_X\left[p_i(X)\right] \end{aligned}$$

We also introduce the notation $D_i = \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$, which is the default indicator function. Conditionally to the risk factors X, D_i is a Bernoulli random variable with probability $p_i(X)$. If we consider the standard assumption that the loss given default is independent

[55] For instance, the rating system of Crédit Agricole is: A+, A, B+, B, C+, C, C-, D+, D, D-, E+, E and E- (source: Credit Agricole, Annual Financial Report 2014, page 201).

[56] This is the case of JPMorgan Chase & Co. (source: JPMorgan Chase & Co., Annual Report 2014, page 104).

TABLE 3.23: An example of internal rating system

Rating		Degree of risk	Definition	Borrower category by self-assessment
1		No essential risk	Extremely high degree of certainty of repayment	Normal
2		Negligible risk	High degree of certainty of repayment	
3		Some risk	Sufficient certainty of repayment	
4	A B C	Better than average	There is certainty of repayment but substantial changes in the environment in the future may have some impact on this uncertainty	
5	A B C	Average	There are no problems foreseeable in the future, but a strong likelihood of impact from changes in the environment	
6	A B C	Tolerable	There are no problems foreseeable in the future, but the future cannot be considered entirely safe	
7		Lower than average	There are no problems at the current time but the financial position of the borrower is relatively weak	
8	A B	Needs preventive management	There are problems with lending terms or fulfilment, or the borrower's business conditions are poor or unstable, or there are other factors requiring careful management	Needs attention
9		Needs serious management	There is a high likelihood of bankruptcy in the future	In danger of bankruptcy
10	I		The borrower is in serious financial straits and "effectively bankrupt"	Effectively bankruptcy
	II		The borrower is bankrupt	Bankrupt

Source: Ieda *et al.* (2000).

from the default time and we also assume that the default times are conditionally independent[57], we obtain:

$$\begin{aligned}\mathbb{E}\left[L \mid X\right] &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \mathbb{E}\left[D_i \mid X\right] \\ &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i\left(X\right) \end{aligned} \tag{3.20}$$

and[58]:

$$\begin{aligned}\sigma^2\left(L \mid X\right) &= \mathbb{E}\left[L^2 \mid X\right] - \mathbb{E}^2\left[L \mid X\right] \\ &= \sum_{i=1}^{n} w_i^2 \cdot \left(\mathbb{E}\left[\mathrm{LGD}_i^2\right] \cdot \mathbb{E}\left[D_i^2 \mid X\right] - \mathbb{E}^2\left[\mathrm{LGD}_i\right] \cdot p_i^2\left(X\right)\right)\end{aligned}$$

[57] The default times are not independent, because they depend on the common risk factors X. However, conditionally to these factors, they become independent because idiosyncratic risk factors are not correlated.

[58] Because the conditional covariance between D_i and D_j is equal to zero. The derivation of this formula is given in Exercise 3.4.8 on page 255.

We have $\mathbb{E}\left[D_i^2 \mid X\right] = p_i(X)$ and $\mathbb{E}\left[\mathrm{LGD}_i^2\right] = \sigma^2(\mathrm{LGD}_i) + \mathbb{E}^2[\mathrm{LGD}_i]$. We deduce that:

$$\sigma^2(L \mid X) = \sum_{i=1}^n w_i^2 \cdot A_i \tag{3.21}$$

where:

$$A_i = \mathbb{E}^2[\mathrm{LGD}_i] \cdot p_i(X) \cdot (1 - p_i(X)) + \sigma^2(\mathrm{LGD}_i) \cdot p_i(X)$$

BCBS (2004a) assumes that the portfolio is infinitely fine-grained, which means that there is no concentration risk:

$$\lim_{n \to \infty} \max \frac{w_i}{\sum_{j=1}^n w_j} = 0 \tag{3.22}$$

In this case, Gordy (2003) shows that the conditional distribution of L degenerates to its conditional expectation $\mathbb{E}[L \mid X]$. The intuition of this result is given by Wilde (2001a). He considers a fine-grained portfolio equivalent to the original portfolio by replacing the original credit i by m credits with the same default probability p_i, the same loss given default LGD_i but an exposure at default divided by m. Let L_m be the loss of the equivalent fine-grained portfolio. We have:

$$\begin{aligned}
\mathbb{E}[L_m \mid X] &= \sum_{i=1}^n \left(\sum_{j=1}^m \frac{w_i}{m}\right) \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \mathbb{E}[D_i \mid X] \\
&= \sum_{i=1}^n w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(X) \\
&= \mathbb{E}[L \mid X]
\end{aligned}$$

and:

$$\begin{aligned}
\sigma^2(L_m \mid X) &= \sum_{i=1}^n \left(\sum_{j=1}^m \frac{w_i^2}{m^2}\right) \cdot A_i \\
&= \frac{1}{m} \sum_{i=1}^n w_i^2 \cdot A_i \\
&= \frac{1}{m} \sigma^2(L_m \mid X)
\end{aligned}$$

When m tends to ∞, we obtain the infinitely fine-grained portfolio. We note that $\mathbb{E}[L_\infty \mid X] = \mathbb{E}[L \mid X]$ and $\sigma^2(L_\infty \mid X) = 0$. Conditionally to the risk factors X, the portfolio loss L_∞ is equal to the conditional mean $\mathbb{E}[L \mid X]$. The associated probability distribution $\mathbf{F}$ is then:

$$\begin{aligned}
\mathbf{F}(\ell) &= \Pr\{L_\infty \leq \ell\} \\
&= \Pr\{\mathbb{E}[L \mid X] \leq \ell\} \\
&= \Pr\left\{\sum_{i=1}^n w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(X) \leq \ell\right\}
\end{aligned}$$

Let $g(x)$ be the function $\sum_{i=1}^n w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(x)$. We have:

$$\mathbf{F}(\ell) = \int \cdots \int \mathbb{1}\{g(x) \leq \ell\} \, \mathrm{d}\mathbf{H}(x)$$

However, it is not possible to obtain a closed-form formula for the value-at-risk $\mathbf{F}^{-1}(\alpha)$ defined as follows:

$$\mathbf{F}^{-1}(\alpha) = \{\ell : \Pr\{g(X) \leq \ell\} = \alpha\}$$

If we consider a single risk factor and assume that $g(x)$ is an increasing function, we obtain:

$$\begin{aligned}\Pr\{g(X) \leq \ell\} = \alpha & \Leftrightarrow \Pr\left\{X \leq g^{-1}(\ell)\right\} = \alpha \\ & \Leftrightarrow \mathbf{H}\left(g^{-1}(\ell)\right) = \alpha \\ & \Leftrightarrow \ell = g\left(\mathbf{H}^{-1}(\alpha)\right)\end{aligned}$$

We finally deduce that the value-at-risk has the following expression:

$$\begin{aligned}\mathbf{F}^{-1}(\alpha) & = g\left(\mathbf{H}^{-1}(\alpha)\right) \\ & = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(\alpha)\right)\end{aligned} \tag{3.23}$$

Equation (3.23) is appealing because the value-at-risk satisfies the Euler allocation principle. Indeed, we have:

$$\begin{aligned}\mathcal{RC}_i & = w_i \cdot \frac{\partial\, \mathbf{F}^{-1}(\alpha)}{\partial\, w_i} \\ & = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(\alpha)\right)\end{aligned} \tag{3.24}$$

and:

$$\sum_{i=1}^{n} \mathcal{RC}_i = \mathbf{F}^{-1}(\alpha)$$

Remark 30 *If $g(x)$ is a decreasing function, we obtain $\Pr\left\{X \geq g^{-1}(\ell)\right\} = \alpha$ and:*

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(1-\alpha)\right)$$

The risk contribution becomes:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i\left(\mathbf{H}^{-1}(1-\alpha)\right) \tag{3.25}$$

We reiterate that Equation (3.24) has been obtained under the following assumptions:

- $\mathcal{H}_1$ the loss given default LGD_i is independent from the default time $\boldsymbol{\tau}_i$;
- $\mathcal{H}_2$ the default times $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ depend on a single risk factor X and are conditionally independent with respect to X;
- $\mathcal{H}_3$ the portfolio is infinitely fine-grained, meaning that there is no exposure concentration.

Equation (3.24) is a very important result for two main reasons. First, it implies that, under the previous assumptions, the value-at-risk of an infinitely fine-grained portfolio can be decomposed as a sum of independent risk contributions. Indeed, $\mathcal{RC}_i$ depends solely on the characteristics of the i^{th} credit (exposure at default, loss given default and probability of default). This facilitates the calculation of the value-at-risk for large portfolios. Second, the risk contribution $\mathcal{RC}_i$ is related to the expected value of the loss given default. We don't need to model the probability distribution of LGD_i, only the mean $\mathbb{E}[\mathrm{LGD}_i]$ is taken into account.

Closed-form formula of the value-at-risk In order to obtain a closed-form formula, we need a model of default times. BCBS (2004a) has selected the one-factor model of Merton (1974), which has been formalized by Vasicek (1991). Let Z_i be the normalized asset value of the entity i. In the Merton model, the default occurs when Z_i is below a given barrier B_i:

$$D_i = 1 \Leftrightarrow Z_i < B_i$$

By assuming that Z_i is Gaussian, we deduce that:

$$\begin{aligned} p_i &= \Pr\{D_i = 1\} \\ &= \Pr\{Z_i < B_i\} \\ &= \Phi(B_i) \end{aligned}$$

The value of the barrier B_i is then equal to $\Phi^{-1}(p_i)$. We assume that the asset value Z_i depends on the common risk factor X and an idiosyncratic risk factor ε_i as follows:

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$$

X and ε_i are two independent standard normal random variables. We note that[59]:

$$\begin{aligned} \mathbb{E}[Z_i Z_j] &= \mathbb{E}\left[\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i\right)\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_j\right)\right] \\ &= \mathbb{E}\left[\rho X^2 + (1-\rho)\varepsilon_i\varepsilon_j + X\sqrt{\rho(1-\rho)}(\varepsilon_i + \varepsilon_j)\right] \\ &= \rho \end{aligned}$$

where ρ is the constant asset correlation. We now calculate the conditional default probability:

$$\begin{aligned} p_i(X) &= \Pr\{D_i = 1 \mid X\} \\ &= \Pr\{Z_i < B_i \mid X\} \\ &= \Pr\left\{\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i < B_i\right\} \\ &= \Pr\left\{\varepsilon_i < \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right\} \\ &= \Phi\left(\frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \end{aligned}$$

Using the framework of the previous paragraph, we obtain:

$$\begin{aligned} g(x) &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(x) \\ &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \end{aligned}$$

We note that $g(x)$ is a decreasing function if $w_i \geq 0$. Using Equation (3.25) and the relationship $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$, it follows that:

$$\mathcal{RC}_i = w_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{3.26}$$

[59] We have $\mathbb{E}[\varepsilon_i\varepsilon_j] = 0$ because ε_i and ε_j are two specific risk factors.

Remark 31 *We verify that p_i is the unconditional default probability. Indeed, we have:*

$$\begin{aligned}\mathbb{E}_X\left[p_i(X)\right] &= \mathbb{E}_X\left[\Phi\left(\frac{\Phi^{-1}(p_i)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right)\right] \\ &= \int_{-\infty}^{\infty}\Phi\left(\frac{\Phi^{-1}(p_i)-\sqrt{\rho}x}{\sqrt{1-\rho}}\right)\phi(x)\,\mathrm{d}x\end{aligned}$$

We recognize the integral function analyzed in Appendix A.2.2.5 on page 1063. We deduce that:

$$\begin{aligned}\mathbb{E}_X\left[p_i(X)\right] &= \Phi_2\left(\infty,\frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}}\cdot\left(\frac{1}{1-\rho}\right)^{-1/2};\frac{\sqrt{\rho}}{\sqrt{1-\rho}}\left(\frac{1}{1-\rho}\right)^{-1/2}\right) \\ &= \Phi_2\left(\infty,\Phi^{-1}(p_i);\sqrt{\rho}\right) \\ &= \Phi\left(\Phi^{-1}(p_i)\right) \\ &= p_i\end{aligned}$$

Example 29 *We consider a homogeneous portfolio with 100 credits. For each credit, the exposure at default, the expected LGD and the probability of default are set to \$1 mn, 50% and 5%.*

Let us assume that the asset correlation ρ is equal to 10%. We have reported the numerical values of $\mathbf{F}^{-1}(\alpha)$ for different values of α in Table 3.24. If we are interested in the cumulative distribution function, $\mathbf{F}(\ell)$ is equal to the numerical solution α of the equation $\mathbf{F}^{-1}(\alpha)=\ell$. Using a bisection algorithm, we find the probabilities given in Table 3.24. For instance, the probability to have a loss less than or equal to \$3 mn is equal to 70.44%. Finally, to calculate the probability density function of the portfolio loss, we use the following relationship[60]:

$$f(x)=\frac{1}{\partial_\alpha\,\mathbf{F}^{-1}(\mathbf{F}(x))}$$

where:

$$\begin{aligned}\partial_\alpha\,\mathbf{F}^{-1}(\alpha) = &\sum_{i=1}^{n} w_i\cdot\mathbb{E}\left[\mathrm{LGD}_i\right]\cdot\sqrt{\frac{\rho}{1-\rho}}\cdot\frac{1}{\phi\left(\Phi^{-1}(\alpha)\right)}\cdot \\ &\phi\left(\frac{\Phi^{-1}(p_i)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\end{aligned}$$

In Figure 3.21, we compare the probability functions for two different values of the asset correlation ρ. We note that the level of ρ has a big impact on the quantile function and the shape of the density function.

TABLE 3.24: Numerical values of $f(\ell)$, $\mathbf{F}(\ell)$ and $\mathbf{F}^{-1}(\alpha)$ when ρ is equal to 10%

ℓ	(in \$ mn)	0.10	1.00	2.00	3.00	4.00	5.00
$\mathbf{F}(\ell)$	(in %)	0.03	16.86	47.98	70.44	83.80	91.26
$f(\ell)$	(in %)	1.04	31.19	27.74	17.39	9.90	5.43
α	(in %)	10.00	25.00	50.00	75.00	90.00	95.00
$\mathbf{F}^{-1}(\alpha)$	(in \$ mn)	0.77	1.25	2.07	3.28	4.78	5.90

[60] See Appendix A.2.2.3 on page 1062.

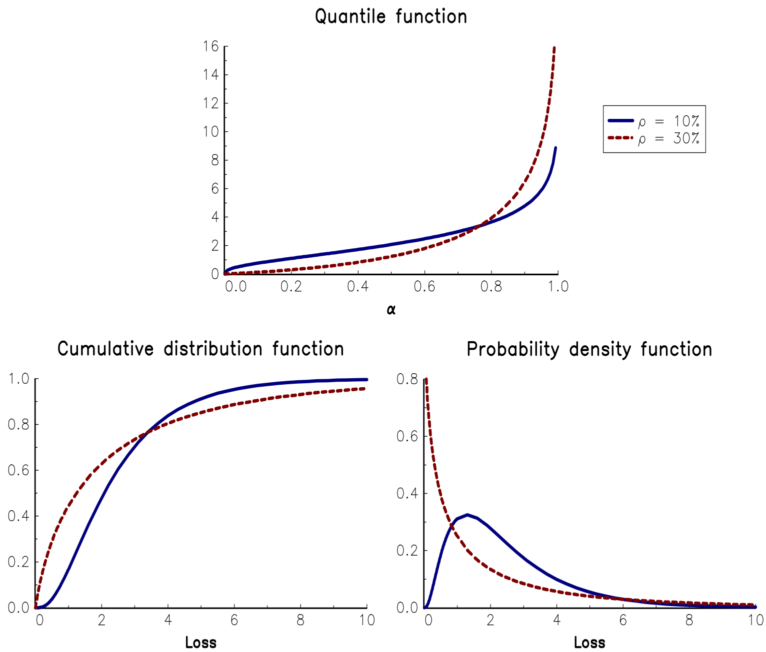

FIGURE 3.21: Probability functions of the credit portfolio loss

The risk contribution $\mathcal{RC}_i$ depends on three credit parameters (the exposure at default w_i, the expected loss given default $\mathbb{E}[\mathrm{LGD}_i]$ and the probability of default p_i) and two model parameters (the asset correlation ρ and the confidence level α of the value-at-risk). It is obvious that $\mathcal{RC}_i$ is an increasing function of the different parameters with the exception of the correlation. We obtain:

$$\operatorname{sign} \frac{\partial\, \mathcal{RC}_i}{\partial\, \rho} = \operatorname{sign} \frac{1}{2\left(1-\rho\right)^{3/2}} \left(\Phi^{-1}\left(p_i\right) + \frac{\Phi^{-1}\left(\alpha\right)}{\sqrt{\rho}} \right)$$

We deduce that the risk contribution is not a monotone function with respect to ρ. It increases if the term $\sqrt{\rho}\Phi^{-1}\left(p_i\right)+\Phi^{-1}\left(\alpha\right)$ is positive. This implies that the risk contribution may decrease if the probability of default is very low and the confidence level is larger than 50%. The two limiting cases are $\rho = 0$ and $\rho = 1$. In the first case, the risk contribution is equal to the expected loss:

$$\mathcal{RC}_i = \mathbb{E}\left[L_i\right] = w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot p_i$$

In the second case, the risk contribution depends on the value of the probability of default:

$$\lim_{\rho \to 1} \mathcal{RC}_i = \begin{cases} 0 & \text{if } p_i < 1-\alpha \\ 0.5 \cdot w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] & \text{if } p_i = 1-\alpha \\ w_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] & \text{if } p_i > 1-\alpha \end{cases}$$

The behavior of the risk contribution is illustrated in Figure 3.22 with the following base parameter values: $w_i = 100$, $\mathbb{E}\left[\mathrm{LGD}_i\right] = 70\%$, $\rho = 20\%$ and $\alpha = 90\%$. We verify that the risk contribution is an increasing function of $\mathbb{E}\left[\mathrm{LGD}_i\right]$ (top/left panel) and α (top/right panel). When p_i and α are set to 10% and 90%, the risk contribution increases with ρ and reaches the value 35, which corresponds to half of the nominal loss given default. When p_i and α are set to 5% and 90%, the risk contribution increases in a first time and then decreases (bottom/left panel). The maximum is reached for the value[61] $\rho^\star = 60.70\%$. When α is equal to 99%, this behavior vanishes (bottom/right panel).

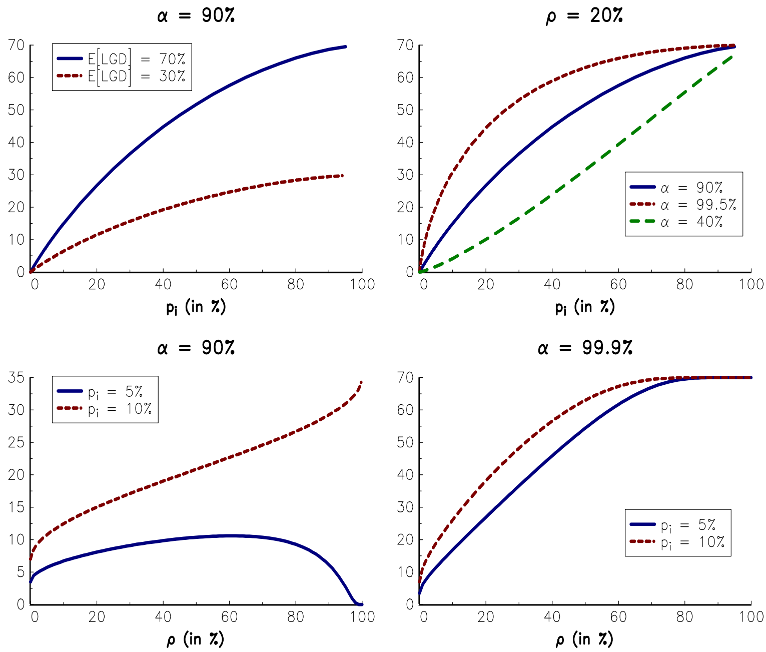

FIGURE 3.22: Relationship between the risk contribution $\mathcal{RC}_i$ and model parameters

In this model, the maturity T_i is taken into account through the probability of default. Indeed, we have $p_i = \Pr\{\boldsymbol{\tau}_i \leq T_i\}$. Let us denote PD_i the annual default probability of the obligor. If we assume that the default time is Markovian, we have the following relationship:

$$\begin{aligned} p_i &= 1 - \Pr\{\boldsymbol{\tau}_i > T_i\} \\ &= 1 - (1 - \mathrm{PD}_i)^{T_i} \end{aligned}$$

We can then rewrite Equation (3.26) such that the risk contribution depends on the exposure at default, the expected loss given default, the annualized probability of default and the maturity, which are the 4 parameters of the IRB approach.

3.2.3.3 The IRB formulas

A long process to obtain the finalized formulas The IRB formula of the second consultative portfolio was calibrated with $\alpha = 99.5\%$, $\rho = 20\%$ and a standard maturity of three years. To measure the impact of this approach, the Basel Committee conducted a quantitative impact study (QIS) in April 2001. A QIS is an Excel workbook to be filled by the bank. It allows the Basel Committee to gauge the impact of the different proposals for capital requirements. The answers are then gathered and analyzed at the industry level. Results were published in November 2001. Overall, 138 banks from 25 countries participated in the QIS. Not all participating banks managed to calculate the capital requirements under the

[61] We have:

$$\rho^\star = \max{}^2\left(0, -\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(p_i)}\right) = \left(\frac{1.282}{1.645}\right)^2 = 60.70\%$$

three methods (SA, FIRB and AIRB). However, 127 banks provided complete information on the SA approach and 55 banks on the FIRB approach. Only 22 banks were able to calculate the AIRB approach for all portfolios.

TABLE 3.25: Percentage change in capital requirements under CP2 proposals

		SA	FIRB	AIRB
G10	Group 1	6%	14%	−5%
	Group 2	1%		
EU	Group 1	6%	10%	−1%
	Group 2	−1%		
Others		5%		

Source: BCBS (2001b).

In Table 3.25, we report the difference in capital requirements between CP2 proposals and Basel I. Group 1 corresponds to diversified, internationally active banks with tier 1 capital of at least €3 bn whereas Group 2 consists of smaller or more specialized banks. BCBS (2001b) concluded that "*on average, the QIS2 results indicate that the CP2 proposals for credit risk would deliver an increase in capital requirements for all groups under both the SA and FIRB approaches*". It was obvious that these figures were not satisfactory. The Basel Committee considered then several modifications in order to (1) maintain equivalence on average between current required capital and the revised SA approach and (2) provide incentives under the FIRB approach. A third motivation has emerged rapidly. According to many studies[62], Basel II may considerably increase the procyclicality of capital requirements. Indeed, capital requirements may increase in an economic meltdown, because LGD increases in bad times and credits receive lower ratings. In this case, capital requirements may move in an opposite direction than the macroeconomic cycle, leading banks to reduce their supply of credit during a crisis. In this scenario, Basel II proposals may amplify credit crises and economic downturns. All these reasons explain the long period to finalize the Basel II Accord. After two new QIS (QIS 2.5 in July 2002 and QIS 3 in May 2003) and a troubled period at the end of 2003, the new Capital Accord is finally published in June 2004. However, there was a shared feeling that it was more a compromise than a terminated task. Thus, several issues remained unresolved and two new QIS will be conducted in 2004 and 2005 before the implementation in order to confirm the calibration.

The supervisory formula If we use the notations of the Basel Committee, the risk contribution has the following expression:

$$\mathcal{RC} = \mathrm{EAD} \cdot \mathrm{LGD} \cdot \Phi\left(\frac{\Phi^{-1}\left(1-(1-\mathrm{PD})^{\mathrm{M}}\right)+\sqrt{\rho}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

where EAD is the exposure at default, LGD is the (expected) loss given default, PD is the (one-year) probability of default and M is the effective maturity. Because $\mathcal{RC}$ is directly the capital requirement ($\mathcal{RC} = 8\% \times \mathrm{RWA}$), we deduce that the risk-weighted asset amount is equal to:

$$\mathrm{RWA} = 12.50 \cdot \mathrm{EAD} \cdot \boldsymbol{\mathcal{K}}^{\star} \tag{3.27}$$

[62]See for instance Goodhart *et al.* (2004) or Kashyap and Stein (2004).

where $\mathcal{K}^{\star}$ is the normalized required capital for a unit exposure:

$$\mathcal{K}^{\star} = \mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1} \left(1 - (1 - \mathrm{PD})^{\mathrm{M}} \right) + \sqrt{\rho} \Phi^{-1} (\alpha)}{\sqrt{1 - \rho}} \right) \tag{3.28}$$

In order to obtain the finalized formulas, the Basel Committee has introduced the following modifications:

- a maturity adjustment $\varphi(\mathrm{M})$ has been added in order to separate the impact of the one-year probability of default and the effect of the maturity; the function $\varphi(\mathrm{M})$ has then been calibrated such that Expression (3.28) becomes:

$$\mathcal{K}^{\star} \approx \mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1} (\mathrm{PD}) + \sqrt{\rho} \Phi^{-1} (\alpha)}{\sqrt{1 - \rho}} \right) \cdot \varphi(\mathrm{M}) \tag{3.29}$$

- it has used a confidence level of 99.9% instead of the 99.5% value;
- it has defined a parametric function $\rho(\mathrm{PD})$ for the default correlation in order that low ratings are not too penalizing for capital requirements;
- it has considered the unexpected loss as the credit risk measure:

$$\mathrm{UL}_{\alpha} = \mathrm{VaR}_{\alpha} - \mathbb{E}[L]$$

In summary, the risk-weighted asset amount in the IRB approach is calculated using Equation (3.27) and the following normalized required capital:

$$\mathcal{K}^{\star} = \left(\mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1} (\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})} \Phi^{-1} (99.9\%)}{\sqrt{1 - \rho(\mathrm{PD})}} \right) - \mathrm{LGD} \cdot \mathrm{PD} \right) \cdot \varphi(\mathrm{M}) \tag{3.30}$$

Risk-weighted assets for corporate, sovereign, and bank exposures The three asset classes use the same formula:

$$\begin{aligned} \mathcal{K}^{\star} \quad = \quad & \left(\mathrm{LGD} \cdot \Phi \left(\frac{\Phi^{-1} (\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})} \Phi^{-1} (99.9\%)}{\sqrt{1 - \rho(\mathrm{PD})}} \right) - \mathrm{LGD} \cdot \mathrm{PD} \right) \cdot \\ & \left(\frac{1 + (\mathrm{M} - 2.5) \cdot b(\mathrm{PD})}{1 - 1.5 \cdot b(\mathrm{PD})} \right) \end{aligned} \tag{3.31}$$

with $b(\mathrm{PD}) = (0.11852 - 0.05478 \cdot \ln(\mathrm{PD}))^2$ and:

$$\rho(\mathrm{PD}) = 12\% \times \left(\frac{1 - e^{-50 \times \mathrm{PD}}}{1 - e^{-50}} \right) + 24\% \times \left(1 - \frac{1 - e^{-50 \times \mathrm{PD}}}{1 - e^{-50}} \right) \tag{3.32}$$

We note that the maturity adjustment $\varphi(\mathrm{M})$ vanishes when the effective maturity is one year. For a defaulted exposure, we have:

$$\mathcal{K}^{\star} = \max(0, \mathrm{LGD} - \mathrm{EL})$$

where EL is the bank's best estimate of the expected loss[63].

[63] We can assimilate it to specific provisions.

For small and medium-sized enterprises[64], a firm-size adjustment is introduced by defining a new parametric function for the default correlation:

$$\rho^{\mathrm{SME}}(\mathrm{PD}) = \rho(\mathrm{PD}) - 0.04 \cdot \left(1 - \frac{(\max(S,5)-5)}{45}\right)$$

where S is the reported sales expressed in € mn. This adjustment has the effect to reduce the default correlation and then the risk-weighted assets. Similarly, the Basel Committee proposes specific arrangements for specialized lending and high-volatility commercial real estate (HVCRE).

In the foundation IRB approach, the bank estimates the probability of default, but uses standard values for the other parameters. In the advanced IRB approach, the bank always estimates the parameters PD and M, and may use its own estimates for the parameters EAD and LGD subject to certain minimum requirements. The risk components are defined as follows:

1. The exposure at default is the amount of the claim, without taking into account specific provisions or partial write-offs. For off-balance sheet positions, the bank uses similar credit conversion factors for the FIRB approach as for the SA approach. In the AIRB approach, the bank may use its own internal measures of CCF.

2. In the FIRB approach, the loss given default is set to 45% for senior claims and 75% for subordinated claims. In the AIRB approach, the bank may use its own estimates of LGD. However, they must be conservative and take into account adverse economic conditions. Moreover, they must include all recovery costs (litigation cost, administrative cost, etc.).

3. PD is the one-year probability of default calculated with the internal rating system. For corporate and bank exposures, a floor of 3 bps is applied.

4. The maturity is set to 2.5 years in the FIRB approach. In the advanced approach, M is the weighted average time of the cash flows, with a one-year floor and a five-year cap.

Example 30 *We consider a senior debt of* \$3 *mn on a corporate firm. The residual maturity of the debt is equal to* 2 *years. We estimate the one-year probability of default at* 5%*.*

To determine the capital charge, we first calculate the default correlation:

$$\begin{aligned}
\rho(\mathrm{PD}) &= 12\% \times \left(\frac{1-e^{-50\times 0.05}}{1-e^{-50}}\right) + 24\% \times \left(1 - \frac{1-e^{-50\times 0.05}}{1-e^{-50}}\right) \\
&= 12.985\%
\end{aligned}$$

We have:

$$\begin{aligned}
b(\mathrm{PD}) &= (0.11852 - 0.05478 \times \ln(0.05))^2 \\
&= 0.0799
\end{aligned}$$

It follows that the maturity adjustment is equal to:

$$\begin{aligned}
\varphi(\mathrm{M}) &= \frac{1 + (2-2.5)\times 0.0799}{1 - 1.5 \times 0.0799} \\
&= 1.0908
\end{aligned}$$

[64] They are defined as corporate entities where the reported sales for the consolidated group of which the firm is a part is less than € 50 mn.

The normalized capital charge with a one-year maturity is:

$$\begin{aligned}\mathcal{K}^{\star} &= 45\% \times \Phi\left(\frac{\Phi^{-1}(5\%)+\sqrt{12.985\%}\Phi^{-1}(99.9\%)}{\sqrt{1-12.985\%}}\right) - 45\% \times 5\% \\ &= 0.1055\end{aligned}$$

When the maturity is two years, we obtain:

$$\begin{aligned}\mathcal{K}^{\star} &= 0.1055 \times 1.0908 \\ &= 0.1151\end{aligned}$$

We deduce the value taken by the risk weight:

$$\begin{aligned}\mathrm{RW} &= 12.5 \times 0.1151 \\ &= 143.87\%\end{aligned}$$

It follows that the risk-weighted asset amount is equal to \$4.316 mn whereas the capital charge is \$345 287. Using the same process, we have calculated the risk weight for different values of PD, LGD and M in Table 3.26. The last two columns are for a SME claim by considering that sales are equal to €5 mn.

TABLE 3.26: IRB risk weights (in %) for corporate exposures

Maturity		M = 1		M = 2.5		M = 2.5 (SME)	
LGD		45%	75%	45%	75%	45%	75%
PD (in %)	0.10	18.7	31.1	29.7	49.4	23.3	38.8
	0.50	52.2	86.9	69.6	116.0	54.9	91.5
	1.00	73.3	122.1	92.3	153.9	72.4	120.7
	2.00	95.8	159.6	114.9	191.4	88.5	147.6
	5.00	131.9	219.8	149.9	249.8	112.3	187.1
	10.00	175.8	292.9	193.1	321.8	146.5	244.2
	20.00	223.0	371.6	238.2	397.1	188.4	314.0

Risk-weighted assets for retail exposures Claims can be included in the regulatory retail portfolio if they meet certain criteria: in particular, the exposure must be to an individual person or to a small business; it satisfies the granularity criterion, meaning that no aggregate exposure to one counterpart can exceed 0.2% of the overall regulatory retail portfolio; the aggregated exposure to one counterparty cannot exceed €1 mn. In these cases, the bank uses the following IRB formula:

$$\mathcal{K}^{\star} = \mathrm{LGD} \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD})+\sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \cdot \mathrm{PD} \tag{3.33}$$

We note that this IRB formula correspond to a one-year fixed maturity. The value of the default correlation depends on the categories. For residential mortgage exposures, we have $\rho(\mathrm{PD}) = 15\%$ whereas the default correlation $\rho(\mathrm{PD})$ is equal to 4% for qualifying revolving retail exposures. For other retail exposures, it is defined as follows:

$$\rho(\mathrm{PD}) = 3\% \times \left(\frac{1-e^{-35\times\mathrm{PD}}}{1-e^{-35}}\right) + 16\% \times \left(1-\frac{1-e^{-35\times\mathrm{PD}}}{1-e^{-35}}\right) \tag{3.34}$$

In Table 3.27, we report the corresponding risk weights for the three categories and for two different values of LGD.

TABLE 3.27: IRB risk weights (in %) for retail exposures

		Mortgage		Revolving		Other retail	
LGD		45%	25%	45%	85%	45%	85%
PD (in %)	0.10	10.7	5.9	2.7	5.1	11.2	21.1
	0.50	35.1	19.5	10.0	19.0	32.4	61.1
	1.00	56.4	31.3	17.2	32.5	45.8	86.5
	2.00	87.9	48.9	28.9	54.6	58.0	109.5
	5.00	148.2	82.3	54.7	103.4	66.4	125.5
	10.00	204.4	113.6	83.9	158.5	75.5	142.7
	20.00	253.1	140.6	118.0	222.9	100.3	189.4

The other two pillars The first pillar of Basel II, which concerns minimum capital requirements, is completed by two new pillars. The second pillar is the supervisory review process (SRP) and is composed of two main processes: the supervisory review and evaluation process (SREP) and the internal capital adequacy assessment process (ICAAP). The SREP defines the regulatory response to the first pillar, in particular the validation processes of internal models. Nevertheless, the SREP is not limited to capital requirements. More generally, the SREP evaluates the global strategy and resilience of the bank. ICAAP addresses risks that are not captured in Pillar 1 like concentration risk or non-granular portfolios in the case of credit risk[65]. For instance, stress tests are part of Pillar 2. The goal of the second pillar is then to encourage banks to continuously improve their internal models and processes for assessing the adequacy of their capital and to ensure that supervisors have the adequate tools to control them. The third pillar, which is also called market discipline, requires banks to publish comprehensive information about their risk management process. This is particularly true since the publication in January 2015 of the revised Pillar 3 disclosure requirements. Indeed, BCBS (2015a) imposes the use of templates for quantitative disclosure with a fixed format in order to facilitate the comparison between banks.

3.2.4 The Basel III revision

For credit risk capital requirements, Basel III is close to the Basel II framework with some adjustments, which mainly concern the parameters[66]. Indeed, the SA and IRB methods continue to be the two approaches for computing the capital charge for credit risk.

3.2.4.1 The standardized approach

Risk-weighted exposures External credit ratings continue to be the backbone of the standardized approach in Basel III. Nevertheless, they are not the only tool for measuring the absolute riskiness of debtors and loans. First, the Basel Committee recognizes that external credit ratings are prohibited in some jurisdictions for computing regulatory capital. For example, this is the case of the United States, which had abandoned in 2010 the use of commercial credit ratings after the Dodd-Frank reform. Second, the Basel Committee links risk weights to the loan-to-value ratio (LTV) for some categories.

When external ratings are allowed[67], the Basel Committee defines a new table of risk weights, which is close to the Basel II table. In Table 3.28, we indicate the main categories and the risk weights associated to credit ratings. We notice that the risk weights for

[65] Since Basel III, ICAAP is completed by the internal liquidity adequacy assessment process (ILAAP).
[66] The Basel III framework for credit risk is described in BCBS (2017c).
[67] This method is called the external credit risk assessment approach (ECRA).

TABLE 3.28: Risk weights of the SA approach (ECRA, Basel III)

Rating		AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to B−	CCC+ to C	NR
Sovereigns		0%	20%	50%	100%	150%	100%
PSE	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
MDB		20%	30%	50%	100%	150%	50%
Banks	2	20%	30%	50%	100%	150%	SCRA
	2 ST	20%	20%	20%	50%	150%	SCRA
	Covered	10%	20%	20%	50%	100%	(∗)
Corporates		20%	50%	75%	100%	150%	100%
Retail		75%					

(∗) For unrated covered bonds, the risk weight is generally half of the risk weight of the issuing bank.

sovereign exposures and non-central government public sector entities (PSE) are unchanged. The risk weights for multilateral development banks (MDB) continue to be related to the risk weights for banks. However, we notice that the first option is removed and we observe some differences for exposures to banks. First, the risk weight for the category A+/A− is reduced from 50% to 30%. Second, for unrated exposures, the standard figure of 50% is replaced by the standardized credit risk approach (SCRA). Third, the Basel Committee considers the special category of covered bonds, whose development has emerged after the 2008 Global Financial Crisis and the introduction of capital requirements for systemic risks[68]. For exposures to corporates, the Basel Committee uses the same scale than for other categories contrary to Basel II (see Table 3.16 on page 163). Finally, the risk weight for retail exposures remains unchanged.

The standardized credit risk approach (SCRA) must be used for all exposures to banks in two situations: (1) when the exposure is unrated; (2) when external credit ratings are prohibited. In this case, the bank must conduct a due diligence analysis in order to classify the exposures into three grades: A, B, and C. Grade A refers to the most solid banks, whose capital exceeds the minimum regulatory capital requirements, whereas Grade C refers to the most vulnerable banks. The risk weight is respectively equal to 40%, 75% and 150% (20%, 50% and 150% for short-term exposures).

When external credit ratings are prohibited, the risk weight of exposures to corporates is equal to 100% with two exceptions. A 65% risk weight is assigned to corporates, which can be considered investment grade (IG). For exposures to small and medium-sized enterprises, a 75% risk weight can be applied if the exposure can be classified in the retail category and 85% for the others.

The case of retail is particular because we have to distinguish real estate exposures and other retail exposures. By default, the risk weight is equal to 75% for this last category, which includes revolving credits, credit cards, consumer credit loans, auto loans, student loans, etc. For real estate exposures, the risk weights depend on the loan-to-value ratio (LTV). Suppose that someone borrows \$100 000 to purchase a house of \$150 000, the LTV ratio is 100 000/150 000 or 66.67%. This ratio is extensively used in English-speaking

[68]See Chapter 8 on page 453.

countries (e.g. the United States) to measure the risk of the loan. The idea is that the lender's haircut (\$100 000 in our example) represents the lender risk. If the borrower defaults, the lender recovers the property, that will be sold. The risk is then to sell the property below the lender's haircut. The higher the LTV ratio, the riskier the loan is for the lender. In continental Europe, the risk of home property loans is measured by the ability of the borrower to repay the capital and service his debt. In this case, the risk of the loan is generally related to the income of the borrower. It is obvious that these two methods for assessing the credit risk are completely different and this explains the stress in Europe to adopt the LTV approach. In Table 3.29, we have reported the value of risk weights with respect to the LTV (expressed in %) in the case of residential real estate exposures. The Basel Committee considers two categories depending if the repayment depends on the cash flows generated by property (D) or not (ND). The risk weight ranges from 20% to 105% in Basel III, whereas it was equal to 35% in Basel II.

TABLE 3.29: Risk weights of the SA approach (ECRA, Basel III)

Residential real estate			Commercial real estate		
Cash flows	ND	D	Cash flows	ND	D
$\text{LTV} \leq 50$	20%	30%	$\text{LTV} \leq 60$	$\min(60\%, \text{RW}_C)$	70%
$50 < \text{LTV} \leq 60$	25%	35%			
$60 < \text{LTV} \leq 80$	30%	45%	$60 < \text{LTV} \leq 80$	RW_C	90%
$80 < \text{LTV} \leq 90$	40%	60%	$\text{LTV} > 80$	RW_C	110%
$90 < \text{LTV} \leq 100$	50%	75%			
$\text{LTV} > 100$	70%	105%			

The LTV ratio is also used to determine the risk weight of commercial real estate, land acquisition, development and construction exposures. Table 3.29 gives the risk weight for commercial real estate exposures. If the repayment does not depend on the cash flows generated by property (ND), we use the risk weight of the counterparty with a cap of 60%. If the repayment depends on the cash flows generated by the property (D), the risk weight ranges from 70% to 110%, whereas it was equal to 100% in Basel II. Commercial real estate exposures that do not meet specific qualitative requirements will be risk-weighted at 150%, which is also the default figure for land acquisition, development and construction exposures.

For off-balance sheet items, credit conversion factors (CCF) have been revised. They can take the values 10%, 20%, 40%, 50% and 100%. This is a more granular scale without the possibility to set the CCF to 0%. Generally speaking, the CCF values in Basel III are more conservative than in Basel II.

Credit risk mitigation The regulatory framework for credit risk mitigation techniques changes very little from Basel II to Basel III: the two methods remain the simple and comprehensive approaches; the treatment of maturity mismatches is the same; the formulas for computing the risk weighted assets are identical, etc. Minor differences concern the description of eligible financial collateral and the haircut parameters, which are given in Table 3.30. For instance, we see that the Basel Committee makes the distinction of issuers for debt securities between sovereigns, other issuers and securitization exposures. While the haircuts do not change for sovereign debt securities with respect to Basel II, the scale is more granular for the two other categories. Haircuts are also increased by 5% for gold and equity collateral instruments.

The major difference concerns the treatment of securities financing transactions (SFT) such as repo-style transactions, since the Basel Committee has developed a specific approach

TABLE 3.30: Standardized supervisory haircuts for collateralized transactions (Basel III)

Rating	Residual Maturity	Sovereigns	Others	Securitization exposures
AAA to AA−	0−1Y	0.5%	1%	2%
	1−3Y	2%	3%	8%
	3−5Y	2%	4%	8%
	5Y−10Y	4%	6%	16%
	10Y+	4%	12%	16%
A+ to BBB−	0−1Y	1%	2%	4%
	1−3Y	3%	4%	12%
	3−5Y	3%	6%	12%
	5Y−10Y	6%	12%	24%
	10Y+	6%	20%	24%
BB+ to BB−		15%		
Cash			0%	
Gold			20%	
Main index equities			20%	
Equities listed on a recognized exchange			30%	
FX risk			8%	

for calculating the modified exposure $\mathrm{EAD}^{\star}$ of these instruments in the comprehensive approach (BCBS, 2017c, pages 43-47).

3.2.4.2 The internal ratings-based approach

The methodology of the IRB approach does not change with respect to Basel II, since the formulas are the same[69]. The only exception is the correlation parameter for bank exposures[70], which becomes:

$$\begin{aligned}
\rho(\mathrm{PD}) &= 1.25 \times \left(12\% \times \left(\frac{1-e^{-50\times\mathrm{PD}}}{1-e^{-50}}\right) + 24\% \times \frac{1-\left(1-e^{-50\times\mathrm{PD}}\right)}{1-e^{-50}}\right) \\
&= 15\% \times \left(\frac{1-e^{-50\times\mathrm{PD}}}{1-e^{-50}}\right) + 30\% \times \left(\frac{1-\left(1-e^{-50\times\mathrm{PD}}\right)}{1-e^{-50}}\right) \qquad (3.35)
\end{aligned}$$

Therefore, the correlation range for the bank category increases from 12% − 24% to 15% − 30%. In fact, the main differences concern the computation of the LGD parameter, and the validation of the IRB approach, which is much more restrictive. For instance, the IRB approaches are not permitted for exposures to equities, and we cannot develop an AIRB approach for exposures to banks and exposures to corporates with annual revenues greater than €500 mn. For banks and large corporates, only the FIRB approach is available.

The Basel Committee still considers five asset classes: corporates, sovereigns, banks, retail and equities. In the FIRB approach, the bank estimates the PD parameter, while

[69] This concerns Equation (3.27) for risk-weighted assets, Equations (3.31) and (3.32) for corporate, sovereign, and bank exposures, Equations (3.33) and (3.34) for retail exposures, the maturity adjustment $b(\mathrm{PD})$, the correlation formula $\rho^{\mathrm{SME}}(\mathrm{PD})$ for SME exposures, the correlation parameters for retail exposures, etc.

[70] The multiplier of 1.25 is applied for regulated financial institutions with a total asset larger than \$100 bn and all unregulated financial institutions.

it uses the regulatory estimates of EAD, LGD and M[71]. In the AIRB approach, the bank estimates all the parameters, but they are subject to some input floors. For example, the minimum PD is set to 5 bps for corporate and bank exposures.

Certainly, LGD is the most challenging parameter in Basel III. In the FIRB approach, the default values are 75% for subordinated claims, 45% for senior claims on financial institutions and 40% for senior claims on corporates. When considering a collateral, the LGD parameter becomes:

$$\mathrm{LGD}_\star = \omega \cdot \mathrm{LGD} + (1-\omega) \cdot \mathrm{LGD}_C$$

where LGD and LGD_C apply to the unsecured exposure and the collateralized part, and ω is the relative weight between LGD and LGD_C:

$$\omega = 1 - \frac{(1-H_C) \cdot C}{(1+H_E) \cdot \mathrm{EAD}}$$

Here, H_E is the SA haircut for the exposure, C is the value of the collateral, and H_C is the specific haircut for the collateral. LGD_C is equal to 0% for financial collateral, 20% for receivables and real estate and 25% for other physical collateral, whereas H_C can be from 0% to 100%. In the AIRB approach, the LGD parameter may be estimated by the bank, under the constraint that it is greater than the input floor $\mathrm{LGD}^{\mathrm{Floor}}$. For unsecured exposures, we have $\mathrm{LGD} \geq \mathrm{LGD}^{\mathrm{Floor}}$ where $\mathrm{LGD}^{\mathrm{Floor}} = 25\%$. For secured exposures, we have $\mathrm{LGD}_\star \geq \mathrm{LGD}_\star^{\mathrm{Floor}}$ where:

$$\mathrm{LGD}_\star^{\mathrm{Floor}} = \omega \cdot \mathrm{LGD}^{\mathrm{Floor}} + (1-\omega) \cdot \mathrm{LGD}_C^{\mathrm{Floor}}$$

$\mathrm{LGD}^{\mathrm{Floor}} = 25\%$ and $\mathrm{LGD}_C^{\mathrm{Floor}}$ depends on the collateral type: 0% for financial collateral, 10% for receivables and real estate and 15% for other physical collateral.

Remark 32 *Since the capital requirement is based on the unexpected loss, the Basel Committee imposes that the expected loss is deduced from regulatory capital.*

3.2.5 The securitization framework

Capital calculations for securitization require developing a more complex approach than the IRB approach, because the bank is not directly exposed to the loss of the credit portfolio, but to the conditional loss of the credit portfolio. This is particularly true if we consider a CDO tranche since we cannot measure the risk of equity, mezzanine and senior tranches in the same way. In what follows, we do not study the Basel II framework, which was very complex, but presented many weaknesses during the 2008 Global Financial Crisis. We prefer to focus on the Basel III framework (BCBS, 2016e), which is implemented since January 2018.

3.2.5.1 Overview of the approaches

The securitization framework consists of three approaches:

1. Securitization internal ratings-based approach (SEC-IRBA)
2. Securitization external ratings-based approach (SEC-ERBA)
3. Securitization standardized approach (SEC-SA)

[71]We recall that M is set to 2.5 years for all exposures, except for repo-style and retail exposures where the maturity is set to 6 and 12 months.

Contrary to credit risk, the hierarchy is reversed. The SEC-IRBA must be first used and is based on the capital charge $\mathcal{K}_{\mathrm{IRB}}$ of the underlying exposures. If the bank cannot calculate $\mathcal{K}_{\mathrm{IRB}}$ for a given securitization exposure, because it has not access to the collateral pool of the debt[72], it has to use the SEC-ERBA. If the tranche is unrated or if external ratings are not allowed, the bank must finally use the SEC-SA. When it is not possible to use one of the three approaches, the risk weight of the securitization exposure is set to 1 250%.

This framework has been developed for three types of exposures: STC securitization, non-STC securitization and resecuritization. STC stands for simple, transparent and comparable securitizations. In July 2015, the BCBS and the Board of IOSCO have published a set of 14 criteria for identifying STC exposures. These criteria are related to the collateral pool (asset risk), the transparency (structural risk) and the governance (fiduciary and servicer risk) of the SPV. Examples of criteria are the nature of the assets, the payment status, alignment of interests, transparency to investors, etc. Resecuritization implies that some underlying assets are themselves securitization exposures. For example, a CDO-squared is a resecuritization, because the asset pool is a basket of CDO tranches.

3.2.5.2 Internal ratings-based approach (SEC-IRBA)

In order to implement SEC-IRBA, the bank must conduct a strict due diligence of the pay-through securitization exposure in order to have a comprehensive information of the underlying exposures. For each asset that composes the collateral pool, it calculates the capital charge. Then, the bank determines $\mathcal{K}_{\mathrm{IRB}}$ as the ratio between the sum of individual capital charges and the exposure amount of the collateral pool. If the bank has not all the information, it can use the following formula:

$$\mathcal{K}_{\mathrm{IRB}} = \omega \cdot \mathcal{K}_{\mathrm{IRB}}^{\star} + (1 - \omega) \cdot \mathcal{K}_{\mathrm{SA}}$$

where $\mathcal{K}_{\mathrm{IRB}}^{\star}$ is the IRB capital requirement for the IRB pool[73], $\mathcal{K}_{\mathrm{SA}}$ is the SA capital requirement for the underlying exposures and ω is the percentage of the IRB pool. However, this formula is only valid if $\omega \geq 95\%$. Otherwise, the bank must use the SEC-SA.

We consider a tranche, where A is the attachment point and D is the detachment point. If $\mathcal{K}_{\mathrm{IRB}} \geq D$, the Basel Committee considers that the risk is very high and RW is set to 1 250%. Otherwise, we have:

$$\begin{aligned} \mathrm{RW} &= 12.5 \cdot \left(\frac{\max\left(\mathcal{K}_{\mathrm{IRB}}, A\right) - A}{D - A} \right) + \\ & \quad 12.5 \cdot \left(\frac{D - \max\left(\mathcal{K}_{\mathrm{IRB}}, A\right)}{D - A} \right) \cdot \mathcal{K}_{\mathrm{SSFA}}\left(\mathcal{K}_{\mathrm{IRB}}\right) \end{aligned} \tag{3.36}$$

where $\mathcal{K}_{\mathrm{SSFA}}\left(\mathcal{K}_{\mathrm{IRB}}\right)$ is the capital charge for one unit of securitization exposure[74]. Therefore, we obtain two cases. If $A < \mathcal{K}_{\mathrm{IRB}} < D$, we replace $\max\left(\mathcal{K}_{\mathrm{IRB}}, A\right)$ by $\mathcal{K}_{\mathrm{IRB}}$ in the previous formula. It follows that the capital charge between the attachment point A and $\mathcal{K}_{\mathrm{IRB}}$ is risk-weighted by 1 250% and the remaining part between $\mathcal{K}_{\mathrm{IRB}}$ and the detachment point D is risk-weighted by $12.5 \cdot \mathcal{K}_{\mathrm{SSFA}}\left(\mathcal{K}_{\mathrm{IRB}}\right)$. This is equivalent to consider that the sub-tranche $\mathcal{K}_{\mathrm{IRB}} - A$ has already defaulted, while the credit risk is on the sub-tranche $D - \mathcal{K}_{\mathrm{IRB}}$. In the second case $\mathcal{K}_{\mathrm{IRB}} < A < D$, the first term of the formula vanishes, and we retrieve the RWA formula (3.27) on page 177.

[72] The structure of pay-through securitization is shown in Figure 3.12 on page 139.

[73] It corresponds to the part of the collateral pool, for which the bank has the information on the individual underlying exposures.

[74] It corresponds to the variable $\mathcal{K}^{\star}$ in the IRB formula on page 177.

The capital charge for one unit of securitization exposure is equal to[75]:

$$\mathcal{K}_{\text{SSFA}}\left(\mathcal{K}_{\text{IRB}}\right) = \frac{\exp\left(cu\right) - \exp\left(cl\right)}{c\left(u-l\right)}$$

where $c = -\left(p\mathcal{K}_{\text{IRB}}\right)^{-1}$, $u = D - \mathcal{K}_{\text{IRB}}$, $l = \left(A - \mathcal{K}_{\text{IRB}}\right)^{+}$ and:

$$p = \max\left(0.3; m_{\text{STC}}\left(\alpha + \frac{\beta}{N} + \gamma \cdot \mathcal{K}_{\text{IRB}} + \delta \cdot \text{LGD} + \epsilon \cdot M_{[A;D]}\right)\right)$$

The parameter p is called the supervisory parameter and is a function of the effective number[76] of loans N, the average LGD and the effective maturity[77] $M_{[A;D]}$ of the tranche. The coefficient m_{STC} is equal to 1 for non-STC securitizations and 0.5 for STC securitizations, while the other parameters α, β, γ, δ and ϵ are given in Table 3.31. We notice that the values depend on the underlying portfolio (wholesale or retail), the granularity ($N < 25$ or $N \geq 25$) and the seniority.

TABLE 3.31: Value of the parameters α, β, γ, δ and ϵ (SEC-IRBA)

Category	Senior	Granularity	α	β	γ	δ	ϵ
Wholesale	✓	$N \geq 25$	0.00	3.56	−1.85	0.55	0.07
	✓	$N < 25$	0.11	2.61	−2.91	0.68	0.07
		$N \geq 25$	0.16	2.87	−1.03	0.21	0.07
		$N < 25$	0.22	2.35	−2.46	0.48	0.07
Retail	✓		0.00	0.00	−7.48	0.71	0.24
			0.00	0.00	−5.78	0.55	0.27

Remark 33 *The derivation of these formulas is based on the model of Gordy and Jones (2003).*

Example 31 *We consider a non-STC CDO based on wholesale assets with three tranches: equity (0%−5%), mezzanine (5%−30%) and senior (30%−100%). The remaining maturity is equal to 10 years. The analysis of the underlying portfolio shows that the effective number of loans N is equal to 30 and the average LGD is equal to 30%. We also assume that $\mathcal{K}^{\star}_{\text{IRB}} = 18\%$, $\mathcal{K}_{\text{SA}} = 20\%$ and $\omega = 95\%$.*

We have $\mathcal{K}_{\text{IRB}} = 0.95 \times 18\% + 0.05 \times 20\% = 18.1\%$. Since $\mathcal{K}_{\text{IRB}} > D_{\text{equity}}$, we deduce that $\text{RW}_{\text{equity}} = 1\,250\%$. For the mezzanine tranche, we have $1 + 0.8 \times (M-1) = 8.2$ years, meaning that the 5-year cap is applied. Using Table 3.31 (fourth row), we deduce that $\alpha = 0.16$, $\beta = 2.87$, $\gamma = -1.03$, $\delta = 0.21$ and $\epsilon = 0.07$. It follows that:

$$\begin{aligned} p &= \max\left(0.30; 0.16 + \frac{2.87}{30} - 1.03 \times 18.1\% + 0.21 \times 30\% + 0.07 \times 5\right) \\ &= 48.22\% \end{aligned}$$

[75] SSFA means simplified supervisory formula approach.

[76] The effective number is equal to the inverse of the Herfindahl index H where $H = \sum_{i=1}^{n} w_i^2$ and w_i is the weight of the i^{th} asset. In our case, we have $w_i = \text{EAD}_i \,/\, \sum_{j=1}^{n} \text{EAD}_j$, implying that:

$$N = \frac{\left(\sum_{i=1}^{n} \text{EAD}_i\right)^2}{\sum_{i=1}^{n} \text{EAD}_i^2}$$

[77] Like for the IRB approach, $M_{[A;D]}$ is the effective maturity with a one-year floor and five-year cap. The effective maturity can be calculated as the weighted-average maturity of the cash-flows of the tranche or $1 + 0.8 \cdot (M-1)$ where M is the legal maturity of the tranche.

Since we have $c = -11.46$, $u = 11.90\%$ and $l = 0\%$, we obtain $\mathcal{K}_{\mathrm{SSFA}}(\mathcal{K}_{\mathrm{IRB}}) = 54.59\%$. Finally, Equation (3.36) gives $\mathrm{RW}_{\mathrm{mezzanine}} = 979.79\%$. If we perform the same analysis for the senior tranche[78], we obtain $\mathrm{RW}_{\mathrm{senior}} = 10.84\%$.

3.2.5.3 External ratings-based approach (SEC-ERBA)

Under the ERBA, we have:

$$\mathrm{RWA} = \mathrm{EAD} \cdot \mathrm{RW}$$

where EAD is the securitization exposure amount and RW is the risk weight that depends on the external rating[79] and four other parameters: the STC criterion, the seniority of the tranche, the maturity and the thickness of the tranche. In the case of short-term ratings, the risk weights are given below:

Rating	A-1/P-1	A-2/P-2	A-3/P-3	Other
STC	10%	30%	60%	1 250%
non-STC	15%	50%	100%	1 250%

For long term ratings, the risk weight goes from 15% for AAA-grade to 1 250% (Table 2, BCBS 2016e, page 27). An example of risk weights for non-STC securitizations is given below:

Rating	Senior		Non-senior	
	1Y	5Y	1Y	5Y
AAA	15%	20%	15%	70%
AA	25%	40%	30%	120%
A	50%	65%	80%	180%
BBB	90%	105%	220%	310%
BB	160%	180%	620%	760%
B	310%	340%	1 050%	1 050%
CCC	460%	505%	1 250%	1 250%
Below CCC-	1 250%	1 250%	1 250%	1 250%

These risk weights are then adjusted for taking into account the effective maturity $M_{[A;D]}$ and the thickness $D - A$ of the tranche. The maturity adjustment corresponds to a linear interpolation between one and five years. The thickness adjustment must be done for non-senior tranches by multiplying the risk weight by the factor $1 - \min(D - A; 0.5)$.

Example 32 *We consider Example 31 and we assume that the mezzanine and senior tranches are rated BB and AAA.*

Using the table above, we deduce that the non-adjusted risk weights are equal to 1 250% for the equity tranche, 760% for the mezzanine tranche and 20% for the senior tranche. There is no maturity adjustment because $M_{[A;D]}$ is equal to five years. Finally, we obtain $\mathrm{RW}_{\mathrm{equity}} = 1\,250\% \times (1 - \min(5\%, 50\%)) = 1187.5\%$, $\mathrm{RW}_{\mathrm{mezzanine}} = 760\% \times (1 - \min(25\%, 50\%)) = 570\%$ and $\mathrm{RW}_{\mathrm{senior}} = 20\%$.

3.2.5.4 Standardized approach (SEC-SA)

The SA is very close to the IRBA since it uses Equation (3.36) by replacing $\mathcal{K}_{\mathrm{IRB}}$ by $\mathcal{K}_{\mathrm{A}}$ and the supervisory parameter p by the default values 0.5 and 1 for STC and non-STC securitizations. To calculate $\mathcal{K}_{\mathrm{A}}$, we first determine $\mathcal{K}_{\mathrm{SA}}$ which is the ratio between the

[78] In this case, the parameters are $\alpha = 0$, $\beta = 3.56$, $\gamma = -1.85$, $\delta = 0.55$ and $\epsilon = 0.07$ (second row in Table 3.31). We have $p = \max(30\%; 29.88\%) = 30\%$, $c = -18.42$, $u = 81.90\%$, $l = 11.90\%$, and $\mathcal{K}_{\mathrm{SSFA}}(\mathcal{K}_{\mathrm{IRB}}) = 0.87\%$.

[79] By definition, this approach is only available for tranches that are rated.

weighted average capital charge of the underlying portfolio computed with the SA approach and the exposure amount of the underlying portfolio. Then, we have:

$$\mathcal{K}_{\mathrm{A}} = (1 - \varpi) \cdot \mathcal{K}_{\mathrm{SA}} + \varpi \cdot 50\%$$

where ϖ is the percentage of underlying exposures that are 90 days or more past due.

Remark 34 *The SEC-SA is the only approach allowed for calculating the capital requirement of resecuritization exposures. In this case, ϖ is set to zero and the supervisory parameter p is equal to* 1.5*.*

If we consider Example 31 on page 187 and assume that $\varpi = 0$, we obtain $\mathrm{RW}_{\mathrm{equity}} = 1\,250\%$, $\mathrm{RW}_{\mathrm{mezzanine}} = 1\,143\%$ and $\mathrm{RW}_{\mathrm{senior}} = 210.08\%$.

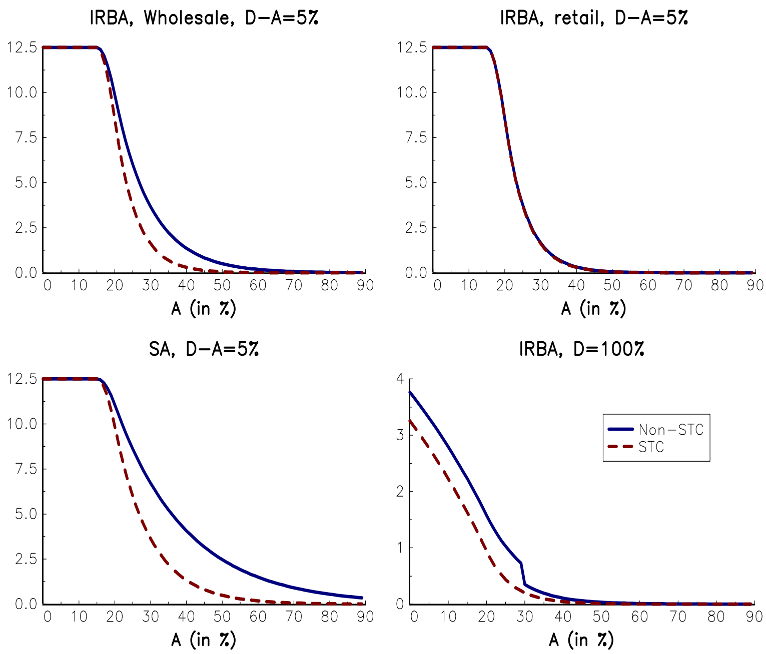

FIGURE 3.23: Risk weight of securitization exposures

Example 33 *We consider a CDO tranche, whose attachment and detachment points are A and D. We assume that $\mathcal{K}_{\mathrm{IRB}} = \mathcal{K}_{\mathrm{A}} = 20\%$, $N = 30$, $\mathrm{LGD} = 50\%$ and $\varpi = 0$.*

In Figure 3.23, we have represented the evolution of the risk weight RW of the tranche $[A, D]$ for different values of A and D. For the first third panels, the thickness of the tranche is equal to 5%, while the detachment point is set to 100% for the fourth panel. In each panel, we consider two cases: non-STC and STC. If we compare the first and second panels, we notice the impact of the asset category (wholesale vs retail) on the risk weight. The third panel shows that the SA approach penalizes more non-STC securitization exposures. Since the detachment point is equal to 100%, the fourth panel corresponds to a senior tranche for high values of the attachment point A and a non-senior tranche when the attachment point A is low. In this example, we assume that the tranche becomes non-senior when $A < 30\%$. We observe a small cliff effect for non-STC securitization exposures.

3.3 Credit risk modeling

We now address the problem of parameter specification. This mainly concerns the exposure at default, the loss given default and the probability of default because the effective maturity is well defined. This section also analyzes default correlations and non granular portfolios when the bank develops its own credit model for calculating economic capital and satisfying Pillar 2 requirements.

3.3.1 Exposure at default

According to BCBS (2017c), the exposure at default "*for an on-balance sheet or off-balance sheet item is defined as the expected gross exposure of the facility upon default of the obligor*". Generally, the computation of EAD for on-balance sheet assets is not an issue. For example, EAD corresponds to the gross notional in the case of a loan or a credit. In fact, the big issue concerns off-balance sheet items, such as revolving lines of credit, credit cards or home equity lines of credit (HELOC). At the default time $\boldsymbol{\tau}$, we have (Taplin *et al.*, 2007):

$$\mathrm{EAD}\left(\boldsymbol{\tau} \mid t\right) = B\left(t\right) + \mathrm{CCF} \cdot \left(L\left(t\right) - B\left(t\right)\right) \tag{3.37}$$

where $B(t)$ is the outstanding balance (or current drawn) at time t, $L(t)$ is the current undrawn limit of the credit facility[80] and CCF is the credit conversion factor. This means that the exposure at default for off-balance sheet items has two components: the current drawn, which is a non-random component and the future drawn, which is a random component.

From Equation (3.37), we deduce that:

$$\mathrm{CCF} = \frac{\mathrm{EAD}\left(\boldsymbol{\tau} \mid t\right) - B\left(t\right)}{L\left(t\right) - B\left(t\right)} \tag{3.38}$$

At first sight, it looks easy to estimate the credit conversion factor. Let us consider the off-balance sheet item i that has defaulted. We have:

$$\mathrm{CCF}_i\left(\boldsymbol{\tau}_i - t\right) = \frac{B_i\left(\boldsymbol{\tau}_i\right) - B_i\left(t\right)}{L_i\left(t\right) - B_i\left(t\right)}$$

At time $\boldsymbol{\tau}_i$, we observe the default of Asset i and the corresponding exposure at default, which is equal to the outstanding balance $B_i(\boldsymbol{\tau}_i)$. Then, we have to choose a date $t < \boldsymbol{\tau}_i$ to observe $B_i(t)$ and $L_i(t)$ in order to calculate the CCF. We notice that it is sensitive to the time period $\boldsymbol{\tau}_i - t$, but banks generally use a one-year time period. Therefore, we can calculate the mean or the quantile α of a sample $\{\mathrm{CCF}_1, \ldots, \mathrm{CCF}_n\}$ for a given homogenous category of off-balance sheet items. Like the supervisory CCF values, the estimated CCF is a figure between 0% and 100%.

In practice, it is difficult to estimate CCF values for five reasons:

1. As explained by Qi (2009), there is a '*race to default*' between borrowers and lenders. Indeed, "*as borrowers approach default, their financial conditions deteriorate and they may use the current undrawn as a source of funding, whereas lenders may cut back credit lines to reduce potential losses*" (Qi, 2009, page 4).

2. $L_i(t)$ depends on the current time t, meaning that it could evolve over time.

[80] The current undrawn $L(t) - B(t)$ is the amount that the debtor is able to draw upon in addition to the current drawn $B(t)$.

3. The computation of the CCF is sensitive to the denominator $L_i(t) - B_i(t)$, which can be small. When $L_i(t) \approx B_i(t)$, the CCF ratio is unstable.

4. We have made the assumption that $\text{CCF}_i(\boldsymbol{\tau}_i - t) \in [0,1]$, implying that $B_i(\boldsymbol{\tau}_i) \geq B_i(t)$ and $B_i(\boldsymbol{\tau}_i) \leq L_i(t)$. This is not always true. We can imagine that the outstanding balance decreases between the current time and the default time ($\text{CCF}_i(\boldsymbol{\tau}_i - t) < 0$) or the outstanding balance at the default time is greater than the limit $L_i(t)$. Jacobs, Jr. (2010) reports extreme variation larger than $\pm 3\,000\%$ when computing raw CCF values!

5. The credit conversion factor is generally an increasing function of the default probability of the borrower.

Because of the previous issues, the observed CCF is floored at 0% and capped at 100%. Tong *et al.* (2016) report the distribution of the credit conversion factor of credit cards from a UK bank[81], and notice that the observations are mainly concentred on the two extreme points 0% and 100% after truncation. Another measure for modeling the exposure at default is to consider the facility utilization change factor (Yang and Tkachenko, 2012):

$$\text{UCF} = \frac{B_i(\boldsymbol{\tau}_i) - B_i(t)}{L_i(t)}$$

It corresponds to the credit conversion factor, where the current undrawn amount $L_i(t) - B_i(t)$ is replaced by the current authorized limit $L_i(t)$. It has the advantage to be more stable, in particular around the singularity $L_i(t) = B_i(t)$.

The econometrics of CCF is fairly basic. As said previously, it consists in estimating the mean or the quantile α of a sample $\{\text{CCF}_1, \ldots, \text{CCF}_n\}$. For that, we can use the cohort method or the time horizon approach (Witzany, 2011). In the cohort method, we divide the study period into fixed intervals (6 or 12 months). For each asset, we identify if it has defaulted during the interval, and then we set t to the starting date of the interval. In the time horizon approach, t is equal to the default time $\boldsymbol{\tau}_i$ minus a fixed horizon (e.g. one, three or 12 months). Sometimes, it can be useful to include some explanatory variables. In this case, the standard model is the Tobit linear regression, which is presented on page 708, because data are censored and the predicted value of CCF must lie in the interval $[0,1]$.

3.3.2 Loss given default

3.3.2.1 Definition

The recovery rate $\boldsymbol{\mathcal{R}}$ is the percentage of the notional on the defaulted debt that can be recovered. In the Basel framework, the recovery rate is not explicitly used, and the concept of loss given default is preferred for measuring the credit portfolio loss. The two metrics are expressed as a percentage of the face value, and we have:

$$\text{LGD} \geq 1 - \boldsymbol{\mathcal{R}}$$

Let us consider a bank that is lending \$100 mn to a corporate firm. We assume that the firm defaults at one time and the bank recovers \$60 mn. We deduce that the recovery rate is equal to:

$$\boldsymbol{\mathcal{R}} = \frac{60}{100} = 60\%$$

[81] See Figure 1 on page 912 in Tong *et al.* (2016).

In order to recover \$60 mn, the bank has incurred some operational and litigation costs, whose amount is \$5 mn. In this case, the bank has lost \$40 mn plus \$5 mn, implying that the loss given default is equal to:

$$\mathrm{LGD} = \frac{40+5}{100} = 45\%$$

In fact, this example shows that $\boldsymbol{\mathcal{R}}$ and LGD are related in the following way:

$$\mathrm{LGD} = 1 - \boldsymbol{\mathcal{R}} + c$$

where c is the litigation cost. We now understand why the loss given default is the right measure when computing the portfolio loss.

Schuermann (2004) identifies three approaches for calculating the loss given default:

1. Market LGD
2. Implied LGD
3. Workout LGD

The market LGD is deduced from the bond price just after the default[82]. It is easy to calculate and available for large corporates and banks. The implied LGD is calculated from a theoretical pricing model of bonds or CDS. The underlying idea is to estimate the implied loss given default, which is priced by the market. As for the first method, this metric is easy to calculate, but it depends on the model assumptions. The last approach is the workout or ultimate LGD. Indeed, the loss given default has three components: the direct loss of principal, the loss of carrying non-performing loans and the workout operational and legal costs. The workout LGD is the right measure when considering the IRB approach. Nevertheless, Schuermann (2004) notices that between two and three years are needed on average to obtain the recovery.

In what follows, we present two approaches for modeling LGD. The first approach considers that LGD is a random variable, whose probability distribution has to be estimated:

$$\mathrm{LGD} \sim \mathbf{F}(x) \tag{3.39}$$

However, we recall that the loss given default in the Basel IRB formulas does not correspond to the random variable, but to its expectation $\mathbb{E}[\mathrm{LGD}]$. Therefore, the second approach consists in estimating the conditional expectation:

$$\begin{aligned}\mathbb{E}[\mathrm{LGD}] &= \mathbb{E}[\mathrm{LGD} \mid X_1 = x_1, \ldots, X_m = x_m] \\ &= g(x_1, \ldots, x_m)\end{aligned} \tag{3.40}$$

where $(X_1, \ldots, X_m)$ are the risk factors that determine the loss given default.

Remark 35 *We notice that* $\boldsymbol{\mathcal{R}} \in [0,1]$*, but* $\mathrm{LGD} \geq 0$*. Indeed, we can imagine that the litigation cost can be high compared to the recovery part of the debt. In this case, we can have* $c > \boldsymbol{\mathcal{R}}$*, implying that* $\mathrm{LGD} > 100\%$*. For instance, if* $\boldsymbol{\mathcal{R}} = 20\%$ *and* $c = 30\%$*, we obtain* $\mathrm{LGD} = 110\%$*. This situation is not fanciful, because* $\boldsymbol{\mathcal{R}}$ *and* c *are not known at the default time. The bank will then begin to engage costs without knowing the recovery amount. For example, one typical situation is* $\boldsymbol{\mathcal{R}} = 0\%$ *and* $c > 0$*, when the bank discovers that there is no possible recovery, but has already incurs some litigation costs. Even if* LGD *can be larger than* 100%*, we assume that* $\mathrm{LGD} \in [0,1]$ *because these situations are unusual.*

[82] This measure is also called '*trading price recovery*'.

3.3.2.2 Stochastic modeling

Using a parametric distribution In this case, we generally use the beta distribution $\mathcal{B}(\alpha, \beta)$, which is described on page 1053. Its density function is given by:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathfrak{B}(\alpha, \beta)}$$

where $\mathfrak{B}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt$. The mean and the variance are:

$$\mu(X) = \mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$$

and:

$$\sigma^2(X) = \operatorname{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

When α and β are greater than 1, the distribution has one mode $x_{\text{mode}} = (\alpha-1)/(\alpha+\beta-2)$. This probability distribution is very flexible and allows to obtain various shapes that are given in Figure 3.24:

- if $\alpha = 1$ and $\beta = 1$, we obtain the uniform distribution; if $\alpha \to \infty$ and $\beta \to \infty$, we obtain the Dirac distribution at the point $x = 0.5$; if one parameter goes to zero, we obtain a Bernoulli distribution;
- if $\alpha = \beta$, the distribution is symmetric around $x = 0.5$; we have a bell curve when the two parameters α and β are higher than 1, and a **U**-shape curve when the two parameters α and β are lower than 1;
- if $\alpha > \beta$, the skewness is negative and the distribution is left-skewed, if $\alpha < \beta$, the skewness is positive and the distribution is right-skewed.

Given the estimated mean $\hat{\mu}_{\text{LGD}}$ and standard deviation $\hat{\sigma}_{\text{LGD}}$ of a sample of losses given default, we can calibrate the parameters α and β using the method of moments[83]:

$$\hat{\alpha} = \frac{\hat{\mu}_{\text{LGD}}^2(1-\hat{\mu}_{\text{LGD}})}{\hat{\sigma}_{\text{LGD}}^2} - \hat{\mu}_{\text{LGD}} \tag{3.41}$$

and:

$$\hat{\beta} = \frac{\hat{\mu}_{\text{LGD}}(1-\hat{\mu}_{\text{LGD}})^2}{\hat{\sigma}_{\text{LGD}}^2} - (1-\hat{\mu}_{\text{LGD}}) \tag{3.42}$$

The other approach is to use the method of maximum likelihood, which is described in Section 10.1.2 on page 614.

Example 34 *We consider the following sample of losses given default:* 68%, 90%, 22%, 45%, 17%, 25%, 89%, 65%, 75%, 56%, 87%, 92% *and* 46%.

We obtain $\hat{\mu}_{\text{LGD}} = 59.77\%$ and $\hat{\sigma}_{\text{LGD}} = 27.02\%$. Using the method of moments, the estimated parameters are $\hat{\alpha}_{\text{MM}} = 1.37$ and $\hat{\beta}_{\text{MM}} = 0.92$, whereas we have $\hat{\alpha}_{\text{ML}} = 1.84$ and $\hat{\beta}_{\text{ML}} = 1.25$ for the method of maximum likelihood. We notice that the two calibrated probability distributions have different shapes (see Figure 3.25).

[83] See Section 10.1.3.1 on page 628.

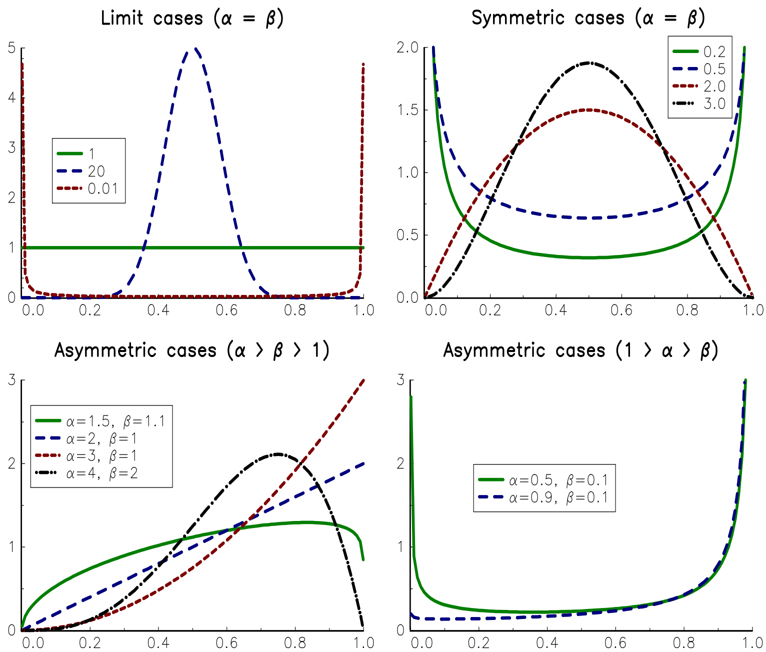

FIGURE 3.24: Probability density function of the beta distribution $\mathcal{B}(\alpha, \beta)$

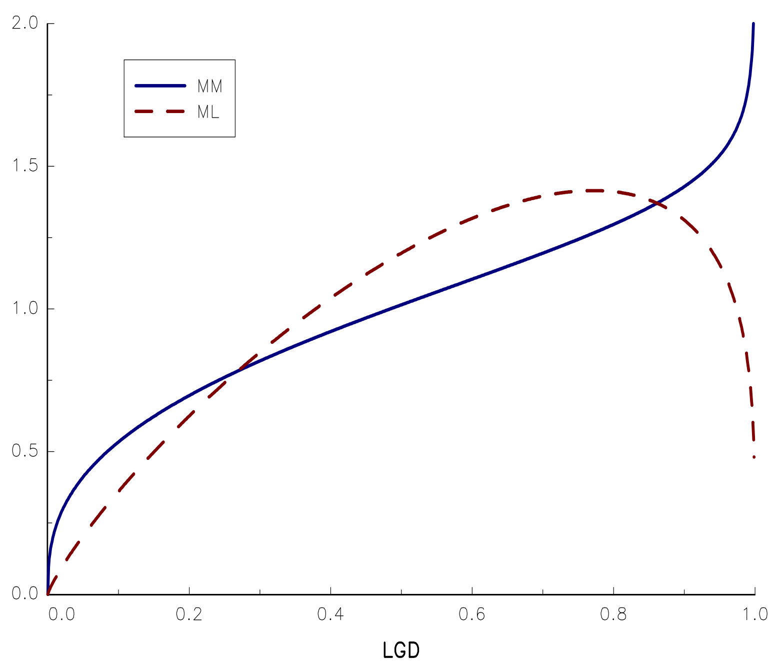

FIGURE 3.25: Calibration of the beta distribution

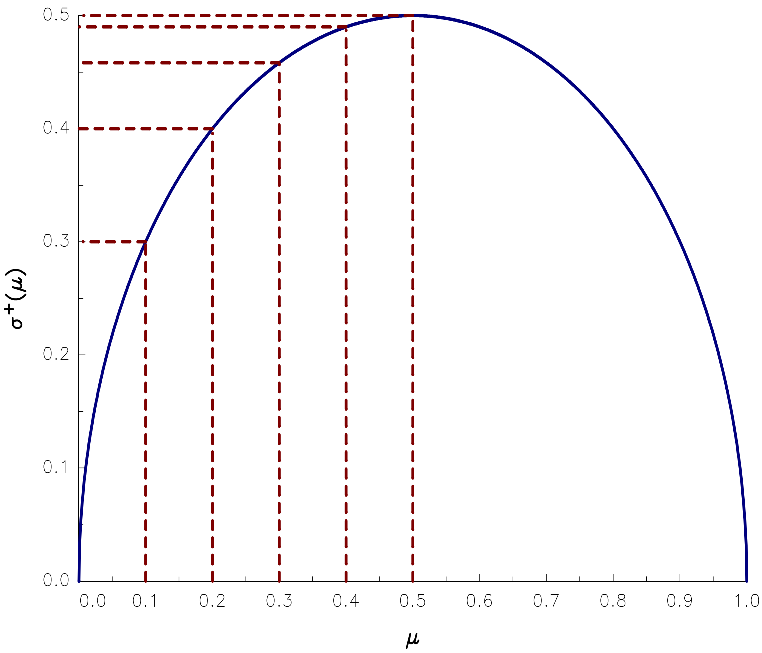

FIGURE 3.26: Maximum standard deviation $\sigma^+(\mu)$

Remark 36 *We can calibrate the beta distribution as long as we respect some constraints on $\hat{\mu}_{\mathrm{LGD}}$ and $\hat{\sigma}_{\mathrm{LGD}}$. Using Equations (3.41) and (3.42), we deduce that:*

$$\hat{\sigma}_{\mathrm{LGD}} < \sqrt{\hat{\mu}_{\mathrm{LGD}}\left(1 - \hat{\mu}_{\mathrm{LGD}}\right)}$$

because $\hat{\alpha}$ and $\hat{\beta}$ must be positive. This condition is not well restrictive. Indeed, if we consider a general random variable X on $[0,1]$, we have $\mathbb{E}\left[X^2\right] \leq \mathbb{E}[X]$, implying that:

$$\sigma(X) \leq \sigma^+(\mu) = \sqrt{\mu(1-\mu)}$$

where $\mu = \mathbb{E}[X]$. Therefore, only the limit case cannot be reached by the beta distribution[84]. However, we notice that the standard deviation cannot be arbitrary fixed to a high level. For example, Figure 3.26 shows that there is no random variable on $[0,1]$ such that $\mu = 10\%$ and $\sigma > 30\%$, $\mu = 20\%$ and $\sigma > 40\%$, $\mu = 50\%$ and $\sigma > 50\%$, etc.

In Figure 3.27, we have reported the calibrated beta distribution using the method of moments for several values of μ_{LGD} and $\sigma_{\mathrm{LGD}} = 30\%$. We obtain **U**-shaped probability distributions. In order to obtain a concave (or bell-shaped) distribution, the standard deviation σ_{LGD} must be lower (see Figure 3.28).

Remark 37 *The previous figures may leave us believing that the standard deviation must be very low in order to obtain a concave beta probability density function. In fact, this is not a restriction due to the beta distribution, since it is due to the support $[0,1]$ of the random variable. Indeed, we can show that the standard deviation is bounded[85] by $\sqrt{1/12} \simeq 28.86\%$ when the probability distribution has one mode on $[0,1]$.*

[84] The limit case corresponds to the Bernoulli distribution $\mathcal{B}(p)$ where $p = \mu$.
[85] The bound is the standard deviation of the uniform distribution $\mathcal{U}_{[0,1]}$.

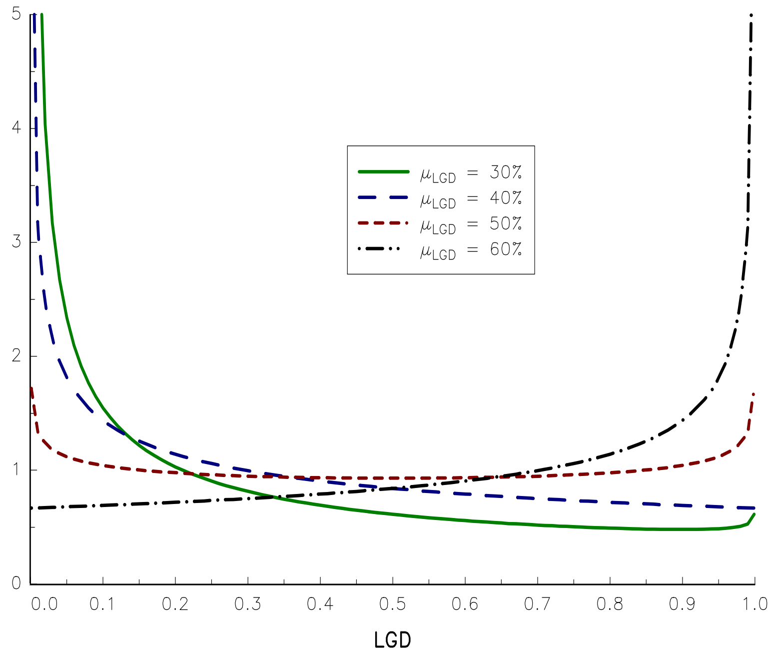

FIGURE 3.27: Calibration of the beta distribution when $\sigma_{\mathrm{LGD}} = 30\%$

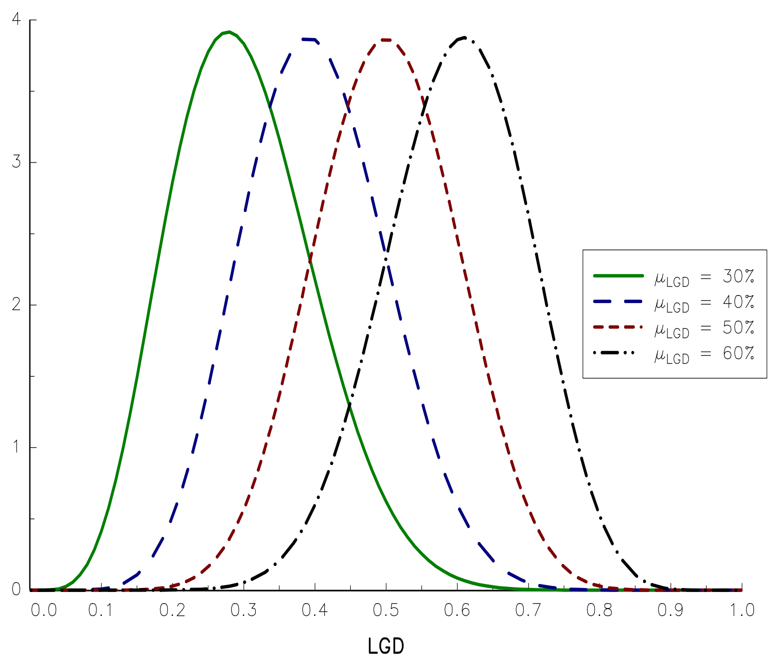

FIGURE 3.28: Calibration of the beta distribution when $\sigma_{\mathrm{LGD}} = 10\%$

As noted by Altman and Kalotay (2014), the beta distribution is not always appropriate for modeling loss given default even if it is widespread used by the industry. Indeed, we observe that losses given default tend to be bimodal, meaning that the recovery rate is quite high or quite low (Loterman *et al.*, 2012). This is why Altman and Kalotay (2014) propose to model the loss given default as a Gaussian mixture model. They first apply the transformation $y_i = \Phi^{-1}(\mathrm{LGD}_i)$ to the sample, then calibrate[86] the 4-component mixture model on the transformed data $(y_1, \ldots, y_n)$ and finally perform the inverse transform for estimating the parametric distribution. They show that the estimated distribution fits relatively well the non-parametric distribution estimated with the kernel method.

Using a non-parametric distribution The beta distribution is either bell-shaped or **U**-shaped. In this last case, the limit is the Bernoulli distribution:

LGD	0%	100%
Probability	$(1 - \mu_{\mathrm{LGD}})$	μ_{LGD}

This model is not necessarily absurd, since it means that the recovery can be very high or very low. Figure 2 in Bellotti and Crook (2012) represents the histogram of recovery rates of 55 000 defaulted credit card accounts from 1999 to 2005 in the UK. The two extreme cases ($\boldsymbol{\mathcal{R}} = 0\%$ and $\boldsymbol{\mathcal{R}} = 100\%$) are the most frequent cases. Therefore, it is interesting to consider the empirical distribution instead of an estimated distribution. In this case, we generally consider risk classes, e.g. $0\% - 5\%, 5\% - 10\%, 10\% - 20\%, \ldots, 80\% - 90\%, 90\% - 100\%$.

Example 35 *We consider the following empirical distribution of* LGD*:*

LGD *(in %)*	0	10	20	25	30	40	50	60	70	75	80	90	100
$\hat{p}$ *(in %)*	1	2	10	25	10	2	0	2	10	25	10	2	1

This example illustrates the shortcoming of the beta modeling when we have a bimodal LGD distribution. In Figure 3.29, we have reported the empirical distribution, and the corresponding (rescaled) calibrated beta distribution. We notice that it is very far from the empirical distribution.

Remark 38 *Instead of using the empirical distribution by risk classes, we can also consider the kernel approach, which is described on page 637.*

Example 36 *We consider a credit portfolio of* 10 *loans, whose loss is equal to:*

$$L = \sum_{i=1}^{10} \mathrm{EaD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\{\boldsymbol{\tau}_i \leq T_i\}$$

where the maturity T_i is equal to 5 years, the exposure at default EaD_i *is equal to* \$1 000 *and the default time $\boldsymbol{\tau}_i$ is exponential with the following intensity parameter λ_i:*

i	1	2	3	4	5	6	7	8	9	10
λ_i *(in bps)*	10	10	25	25	50	100	250	500	500	1 000

The loss given default LGD_i *is given by the empirical distribution, which is described in Example 35.*

[86] The estimation of Gaussian mixture models is presented on page 624.

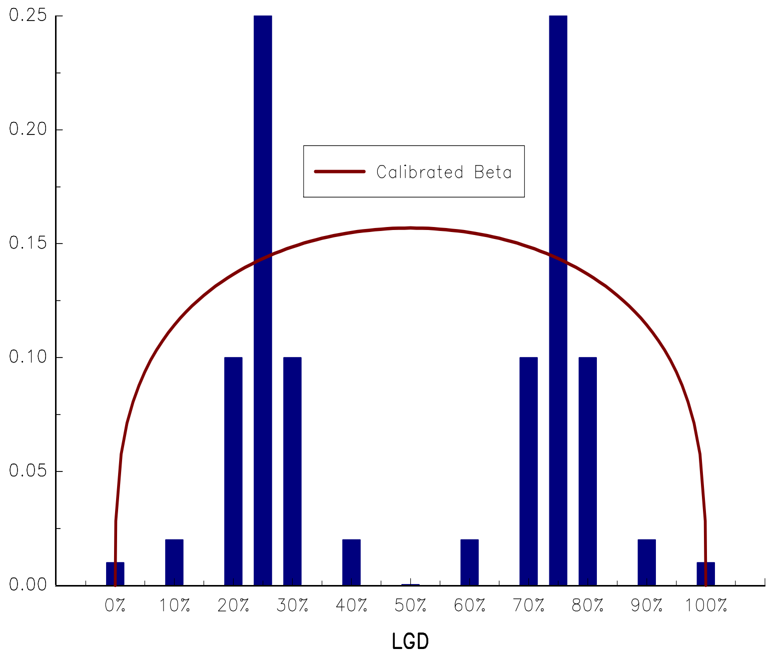

FIGURE 3.29: Calibration of a bimodal LGD distribution

In Figure 3.30, we have calculated the distribution of the portfolio loss with the Monte Carlo method. We compare the loss distribution when we consider the empirical distribution and the calibrated beta distribution for the loss given default. We also report the loss distribution when we replace the random variable LGD_i by its expected value $\mathbb{E}[\mathrm{LGD}_i] = 50\%$. We observe that the shape of L highly depends on the LGD model. For example, we observe a more pronounced fat tail with the calibrated beta distribution. This implies that the LGD model has a big impact for calculating the value-at-risk. For instance, we have reported the loss distribution using the beta model for different values of $(\mu_{\mathrm{LGD}}, \sigma_{\mathrm{LGD}})$ in Figure 3.31. We conclude that the modeling of LGD must not be overlooked. In many cases, the model errors have more impact when they concern the loss given default than the probability of default.

Remark 39 *The expression of the portfolio loss is:*

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\{\boldsymbol{\tau}_i \leq T_i\}$$

If the portfolio is fined grained, we have:

$$\mathbb{E}[L \mid X] = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(X)$$

We deduce that the distribution of the portfolio loss is equal to:

$$\Pr\{L \leq \ell\} = \int \cdots \int \mathbb{1}\left\{\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot p_i(x) \leq \ell\right\} \mathrm{d}\mathbf{H}(x)$$

This loss distribution does not depend on the random variables LGD_i*, but on their expected values* $\mathbb{E}[\mathrm{LGD}_i]$*. This implies that it is not necessary to model the loss given default, but*

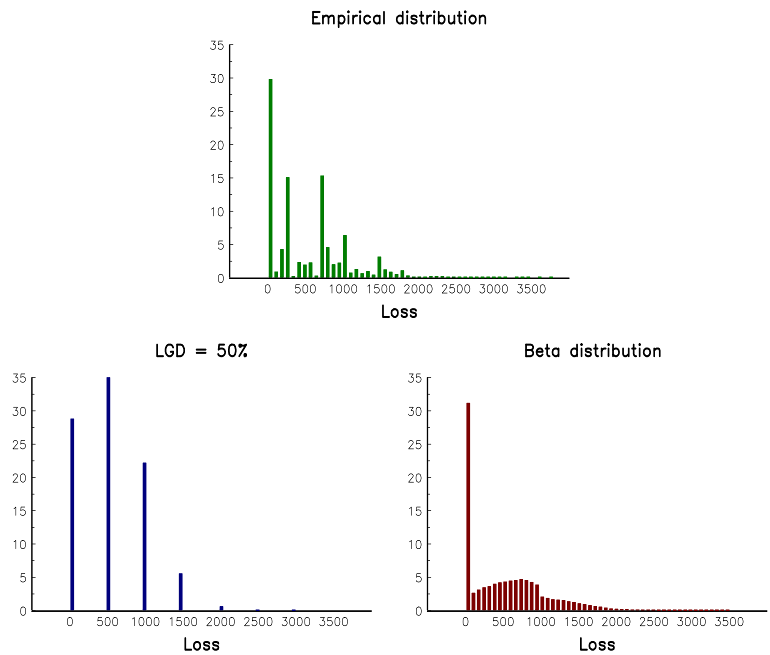

FIGURE 3.30: Loss frequency in % of the three LGD models

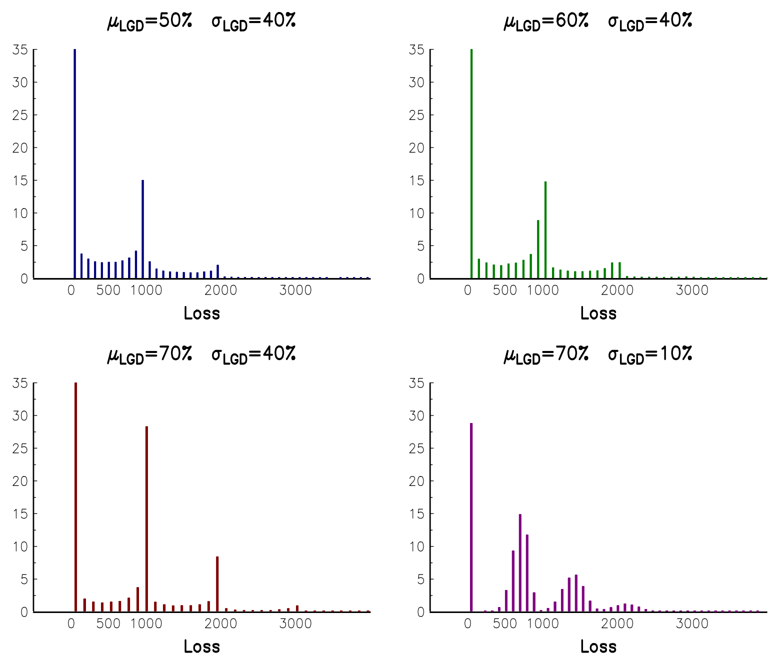

FIGURE 3.31: Loss frequency in % for different values of μ_{LGD} and σ_{LGD}

only the mean. Therefore, we can replace the previous expression of the portfolio loss by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$$

3.3.2.3 Economic modeling

There are many factors that influence the recovery process. In the case of corporate debt, we distinguish between specific and economic factors. For instance, specific factors are the relative seniority of the debt or the guarantees. Senior debt must be repaid before subordinated or junior debt is repaid. If the debt is collateralized, this affects the loss given default. Economic factors are essentially the business cycle and the industry. In the third version of Moody's LossCalc, Dwyer and Korablev (2009) consider seven factors that are grouped in three major categories:

1. factors external to the issuer: geography, industry, credit cyle stage;
2. factors specific to the issuer: distance-to-default, probability of default (or leverage for private firms);
3. factors specific to the debt issuance: debt type, relative standing in capital structure, collateral.

Curiously, Dwyer and Korablev (2009) explain that "*some regions have been characterized as creditor-friendly, while others are considered more creditor-unfriendly*". For instance, recovery rates are lower in the UK and Europe than in the rest of the world. However, the most important factors are the seniority followed by the industry, as it is illustrated by the Moody's statistics on ultimate recoveries. From 1987 to 2017, the average corporate debt recovery rate is equal to 80.4% for loans, 62.3% for senior secured bonds, 47.9% for senior unsecured bonds and 28.0% for subordinated bonds (Moody's, 2018). It is interesting to notice that the recovery rate and the probability of default are negatively correlated. Indeed, Dwyer and Korablev (2009) take the example of two corporate firms A and B, and they assume that $\mathrm{PD}_B \gg \mathrm{PD}_A$. In this case, we may think that the assets of A relative to its liabilities is larger than the ratio of B. Therefore, we must observe a positive relationship between the loss given default and the probability of default.

Remark 40 *The factors depend of the asset class. For instance, we will consider more microeconomic variables when modeling the loss given default for mortgage loans (Tong et al., 2013).*

Once the factors are identified, we must estimate the LGD model:

$$\mathrm{LGD} = f\left(X_1, \ldots, X_m\right)$$

where $X_1, \ldots, X_m$ are the m factors, and f is a non-linear function. Generally, we consider a transformation of LGD in order to obtain a more tractable variable. We can apply a logit transform $Y = \ln\left(\mathrm{LGD}\right) - \ln\left(1 - \mathrm{LGD}\right)$, a probit transform $Y = \Phi^{-1}\left(\mathrm{LGD}\right)$ or a beta transformation (Bellotti and Crook, 2012). In this case, we can use the different statistical tools given in Chapters 10 and 15 to model the random variable Y. The most popular models are the logistic regression, regression trees and neural networks (Bastos, 2010). However, according to EBA (2017), multivariate regression remains the most widely used methods, despite the strong development of machine learning techniques, that are presented on page 943.

Remark 41 *We do not develop here the econometric approach, because it is extensively presented in Chapter 15 dedicated to the credit scoring. Indeed, statistical models of* LGD *use the same methods than statistical models of* PD. *We also refer to Chapter 14 dedicated to stress testing methods when we would like to calculate stressed LGD parameters.*

3.3.3 Probability of default

3.3.3.1 Survival function

The survival function is the main tool to characterize the probability of default. It is also known as reduced-form modeling.

Definition and main properties Let $\boldsymbol{\tau}$ be a default (or survival) time. The survival function[87] is defined as follows:

$$\begin{aligned} \mathbf{S}(t) &= \Pr\{\boldsymbol{\tau} > t\} \\ &= 1 - \mathbf{F}(t) \end{aligned}$$

where $\mathbf{F}$ is the cumulative distribution function. We deduce that the probability density function is related to the survival function in the following manner:

$$f(t) = -\frac{\partial\, \mathbf{S}(t)}{\partial\, t} \tag{3.43}$$

In survival analysis, the key concept is the hazard function $\lambda(t)$, which is the instantaneous default rate given that the default has not occurred before t:

$$\lambda(t) = \lim_{\mathrm{d}t \rightarrow 0^+} \frac{\Pr\{t \leq \boldsymbol{\tau} \leq t + \mathrm{d}t \mid \boldsymbol{\tau} \geq t\}}{\mathrm{d}t}$$

We deduce that:

$$\begin{aligned} \lambda(t) &= \lim_{\mathrm{d}t \rightarrow 0^+} \frac{\Pr\{t \leq \boldsymbol{\tau} \leq t + \mathrm{d}t\}}{\mathrm{d}t} \cdot \frac{1}{\Pr\{\boldsymbol{\tau} \geq t\}} \\ &= \frac{f(t)}{\mathbf{S}(t)} \end{aligned}$$

Using Equation (3.43), another expression of the hazard function is:

$$\begin{aligned} \lambda(t) &= -\frac{\partial_t\, \mathbf{S}(t)}{\mathbf{S}(t)} \\ &= -\frac{\partial\, \ln \mathbf{S}(t)}{\partial\, t} \end{aligned}$$

The survival function can then be rewritten with respect to the hazard function and we have:

$$\mathbf{S}(t) = e^{-\int_0^t \lambda(s)\,\mathrm{d}s} \tag{3.44}$$

In Table 3.32, we have reported the most common hazard and survival functions. They can be extended by adding explanatory variables in order to obtain proportional hazard models (Cox, 1972). In this case, the expression of the hazard function is $\lambda(t) = \lambda_0(t)\exp\left(\beta^\top x\right)$ where $\lambda_0(t)$ is the baseline hazard rate and x is the vector of explanatory variables, which are not dependent on time.

[87] Previously, we have noted the survival function as $\mathbf{S}_{t_0}(t)$. Here, we assume that the current time t_0 is 0.

TABLE 3.32: Common survival functions

Model	$\mathbf{S}(t)$	$\lambda(t)$
Exponential	$\exp(-\lambda t)$	λ
Weibull	$\exp(-\lambda t^{\gamma})$	$\lambda\gamma t^{\gamma-1}$
Log-normal	$1-\Phi(\gamma\ln(\lambda t))$	$\gamma t^{-1}\phi(\gamma\ln(\lambda t))/(1-\Phi(\gamma\ln(\lambda t)))$
Log-logistic	$1/\left(1+\lambda t^{\frac{1}{\gamma}}\right)$	$\lambda\gamma^{-1}t^{\frac{1}{\gamma}}/\left(t+\lambda t^{1+\frac{1}{\gamma}}\right)$
Gompertz	$\exp(\lambda(1-e^{\gamma t}))$	$\lambda\gamma\exp(\gamma t)$

The exponential model holds a special place in default time models. It can be justified by the following problem in physics:

> "*Assume that a system consists of n identical components which are connected in series. This means that the system fails as soon as one of the components fails. One can assume that the components function independently. Assume further that the random time interval until the failure of the system is one n^{th} of the time interval of component failure*" (Galambos, 1982).

We have $\Pr\{\min(\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_n)\leq t\}=\Pr\{\boldsymbol{\tau}_i\leq n\cdot t\}$. The problem is then equivalent to solve the functional equation $\mathbf{S}(t)=\mathbf{S}^n(t/n)$ with $\mathbf{S}(t)=\Pr\{\boldsymbol{\tau}_1>t\}$. We can show that the unique solution for $n\geq 1$ is the exponential distribution. Following Galambos and Kotz (1978), its other main properties are:

1. the mean residual life $\mathbb{E}[\boldsymbol{\tau}\mid\boldsymbol{\tau}\geq t]$ is constant;
2. it satisfies the famous lack of memory property:
$$\Pr\{\boldsymbol{\tau}\geq t+u\mid\boldsymbol{\tau}\geq t\}=\Pr\{\boldsymbol{\tau}\geq u\}$$
or equivalently $\mathbf{S}(t+u)=\mathbf{S}(t)\,\mathbf{S}(u)$;
3. the probability distribution of $n\cdot\boldsymbol{\tau}_{1:n}$ is the same as probability distribution of $\boldsymbol{\tau}_i$.

Piecewise exponential model In credit risk models, the standard probability distribution to define default times is a generalization of the exponential model by considering piecewise constant hazard rates:

$$\begin{aligned}\lambda(t) &= \sum_{m=1}^{M}\lambda_m\cdot\mathbb{1}\left\{t^\star_{m-1}<t\leq t^\star_m\right\}\\ &= \lambda_m \qquad \text{if } t\in\left]t^\star_{m-1},t^\star_m\right]\end{aligned}$$

where $t^\star_m$ are the knots of the function[88]. For $t\in\left]t^\star_{m-1},t^\star_m\right]$, the expression of the survival function becomes:

$$\begin{aligned}\mathbf{S}(t) &= \exp\left(-\sum_{k=1}^{m-1}\lambda_k\left(t^\star_k-t^\star_{k-1}\right)-\lambda_m\left(t-t^\star_{m-1}\right)\right)\\ &= \mathbf{S}\left(t^\star_{m-1}\right)e^{-\lambda_m\left(t-t^\star_{m-1}\right)}\end{aligned}$$

[88] We have $t^\star_0=0$ and $t^\star_{M+1}=\infty$.

It follows that the density function is equal to[89]:

$$f(t) = \lambda_m \exp\left(-\sum_{k=1}^{m-1} \lambda_k \left(t_k^\star - t_{k-1}^\star\right) - \lambda_m \left(t - t_{m-1}^\star\right)\right)$$

In Figure 3.32, we have reported the hazard, survival and density functions for three set of parameters $\{(t_m^\star, \lambda_m), m = 1, \ldots, M\}$:

$$\begin{array}{rl} \{(1, 1\%), (2, 1.5\%), (3, 2\%), (4, 2.5\%), (\infty, 3\%)\} & \text{for } \lambda_1(t) \\ \{(1, 10\%), (2, 7\%), (5, 5\%), (7, 4.5\%), (\infty, 6\%)\} & \text{for } \lambda_2(t) \end{array}$$

and $\lambda_3(t) = 4\%$. We note the special shape of the density function, which is not smooth at the knots.

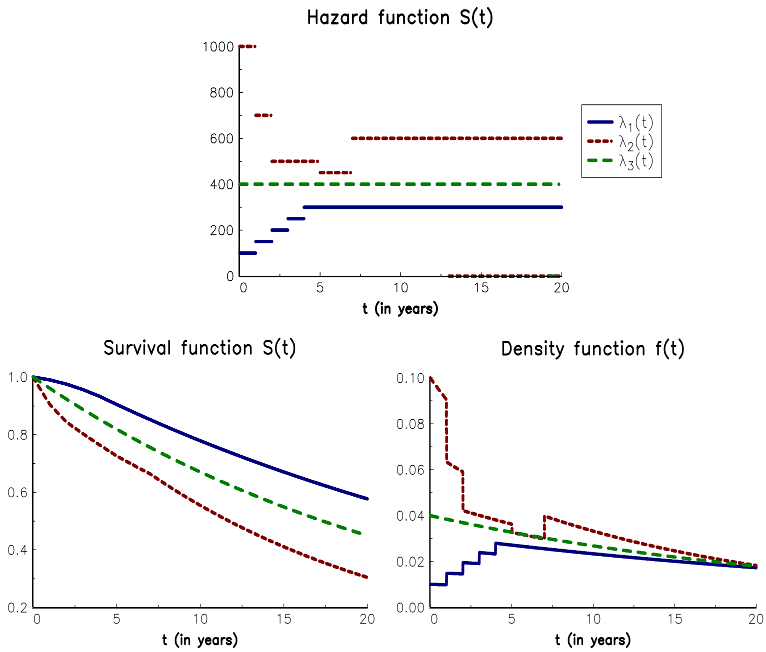

FIGURE 3.32: Example of the piecewise exponential model

Estimation To estimate the parameters of the survival function, we can use the cohort approach. Under this method, we estimate the empirical survival function by counting the number of entities for a given population that do not default over the period Δt:

$$\hat{\mathbf{S}}(\Delta t) = 1 - \frac{\sum_{i=1}^{n} \mathbb{1}\{t < \boldsymbol{\tau}_i \leq t + \Delta t\}}{n}$$

where n is the number of entities that compose the population. We can then fit the survival function by using for instance the least squares method.

[89]We verify that:

$$\frac{f(t)}{\mathbf{S}(t)} = \lambda_m \qquad \text{if } t \in \left]t_{m-1}^\star, t_m^\star\right]$$

Example 37 *We consider a population of* 1 000 *corporate firms. The number of defaults* $n_D(\Delta t)$ *over the period* Δt *is given in the table below:*

Δt *(in months)*	3	6	9	12	15	18	21	22
$n_D(\Delta t)$	2	5	9	12	16	20	25	29

We obtain $\hat{\mathbf{S}}(0.25) = 0.998$, $\hat{\mathbf{S}}(0.50) = 0.995$, $\hat{\mathbf{S}}(0.75) = 0.991$, $\hat{\mathbf{S}}(1.00) = 0.988$, $\hat{\mathbf{S}}(1.25) = 0.984$, $\hat{\mathbf{S}}(1.50) = 0.980$, $\hat{\mathbf{S}}(1.75) = 0.975$ and $\hat{\mathbf{S}}(2.00) = 0.971$. For the exponential model, the least squares estimator $\hat{\lambda}$ is equal to 1.375%. In the case of the Gompertz survival function, we obtain $\hat{\lambda} = 2.718\%$ and $\hat{\gamma} = 0.370$. If we consider the piecewise exponential model, whose knots correspond to the different periods Δt, we have $\hat{\lambda}_1 = 0.796\%$, $\hat{\lambda}_2 = 1.206\%$, $\hat{\lambda}_3 = 1.611\%$, $\hat{\lambda}_4 = 1.216\%$, $\hat{\lambda}_5 = 1.617\%$, $\hat{\lambda}_6 = 1.640\%$, $\hat{\lambda}_7 = 2.044\%$ and $\hat{\lambda}_8 = 1.642\%$. To compare these three calibrations, we report the corresponding hazard functions in Figure 3.33. We deduce that the one-year default probability[90] is respectively equal to 1.366%, 1.211% and 1.200%.

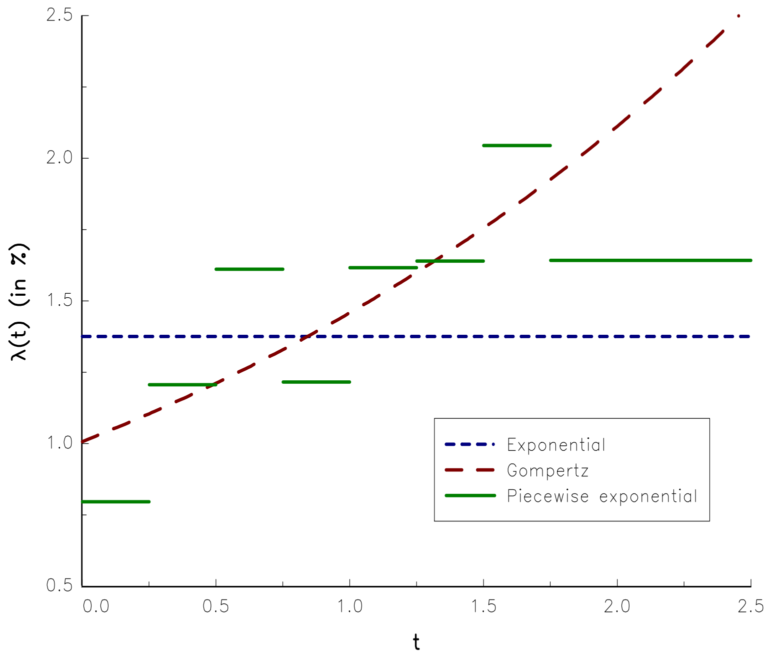

FIGURE 3.33: Estimated hazard function

In the piecewise exponential model, we can specify an arbitrary number of knots. In the previous example, we use the same number of knots than the number of observations to calibrate. In such case, we can calibrate the parameters using the following iterative process:

1. We first estimate the parameter λ_1 for the earliest maturity Δt_1.
2. Assuming that $\left(\hat{\lambda}_1, \ldots, \hat{\lambda}_{i-1}\right)$ have been estimated, we calculate $\hat{\lambda}_i$ for the next maturity Δt_i.
3. We iterate step 2 until the last maturity Δt_m.

[90] We have $\text{PD} = 1 - \mathbf{S}(1)$.

This algorithm works well if the knots $t_m^\star$ exactly match the maturities. It is known as the bootstrap method and is very popular to estimate the survival function from market prices. Let $\{\mathcal{s}(T_1), \ldots, \mathcal{s}(T_M)\}$ be a set of CDS spreads for a given name. Assuming that $T_1 < T_2 < \ldots < T_M$, we consider the piecewise exponential model with $t_m^\star = T_m$. We first estimate $\hat{\lambda}_1$ such that the theoretical spread is equal to $\mathcal{s}(T_1)$. We then calibrate the hazard function in order to retrieve the spread $\mathcal{s}(T_2)$ of the second maturity. This means to consider that $\lambda(t)$ is known and equal to $\hat{\lambda}_1$ until time T_1 whereas $\lambda(t)$ is unknown from T_1 to T_2:

$$\lambda(t) = \begin{cases} \hat{\lambda}_1 & \text{if } t \in \left]0, T_1\right] \\ \lambda_2 & \text{if } t \in \left]T_1, T_2\right] \end{cases}$$

Estimating $\hat{\lambda}_2$ is therefore straightforward because it is equivalent to solve one equation with one variable. We proceed in a similar way for the other maturities.

Example 38 *We assume that the term structure of interest rates is generated by the Nelson-Siegel model with $\theta_1 = 5\%$, $\theta_2 = -5\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. We consider three credit curves, whose CDS spreads expressed in bps are given in the following table:*

Maturity (in years)	#1	#2	#3
1	50	50	350
3	60	60	370
5	70	90	390
7	80	115	385
10	90	125	370

The recovery rate $\mathcal{R}$ is set to 40%.

TABLE 3.33: Calibrated piecewise exponential model from CDS prices

Maturity (in years)	#1	#2	#3
1	83.3	83.3	582.9
3	110.1	110.1	637.5
5	140.3	235.0	702.0
7	182.1	289.6	589.4
10	194.1	241.9	498.5

Using the bootstrap method, we obtain results in Table 3.33. We notice that the piecewise exponential model coincide for the credit curves #1 and #2 for $t < 3$ years. This is normal because the CDS spreads of the two credit curves are equal when the maturity is less or equal than 3 years. The third credit curve illustrates that the bootstrap method is highly sensitive to small differences. Indeed, the calibrated intensity parameter varies from 499 to 702 bps while the CDS spreads varies from 350 to 390 bps. Finally, the survival function associated to these 3 bootstrap calibrations are shown in Figure 3.34.

Remark 42 *Other methods for estimating the probability of default are presented in Chapter 19 dedicated to credit scoring models.*

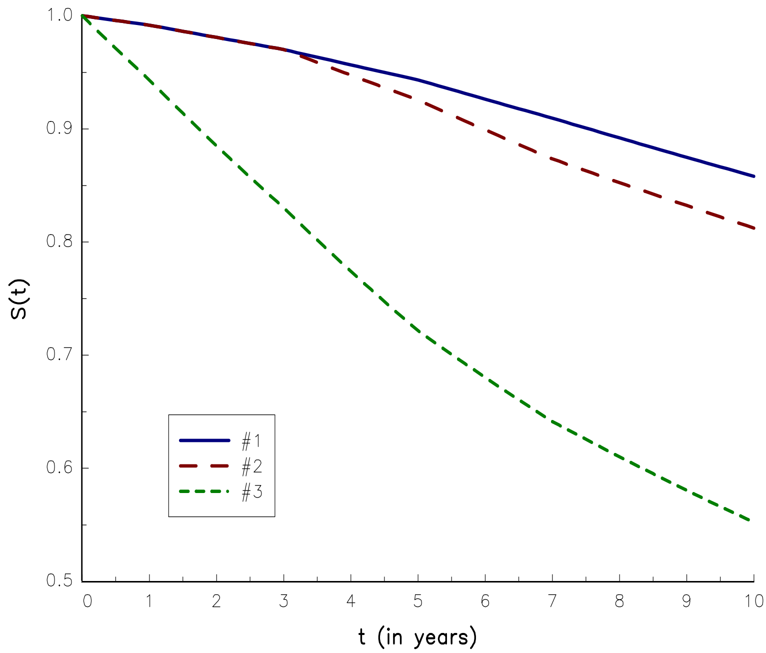

FIGURE 3.34: Calibrated survival function from CDS prices

3.3.3.2 Transition probability matrix

When dealing with risk classes, it is convenient to model a transition probability matrix. For instance, this approach is used for modeling credit rating migration.

Discrete-time modeling We consider a time-homogeneous Markov chain $\mathfrak{R}$, whose transition matrix is $P = (p_{i,j})$. We note $\mathcal{S} = \{1, 2, \ldots, K\}$ the state space of the chain and $p_{i,j}$ is the probability that the entity migrates from rating i to rating j. The matrix P satisfies the following properties:

- $\forall i, j \in \mathcal{S}$, $p_{i,j} \geq 0$;
- $\forall i \in \mathcal{S}$, $\sum_{j=1}^{K} p_{i,j} = 1$.

In credit risk, we generally assume that K is the absorbing state (or the default state), implying that any entity which has reached this state remains in this state. In this case, we have $p_{K,K} = 1$. Let $\mathfrak{R}(t)$ be the value of the state at time t. We define $p(s, i; t, j)$ as the probability that the entity reaches the state j at time t given that it has reached the state i at time s. We have:

$$\begin{aligned} p(s, i; t, j) &= \Pr\{\mathfrak{R}(t) = j \mid \mathfrak{R}(s) = i\} \\ &= p_{i,j}^{(t-s)} \end{aligned}$$

This probability only depends on the duration between s and t because of the Markov property. Therefore, we can restrict the analysis by calculating the n-step transition probability:

$$p_{i,j}^{(n)} = \Pr\{\mathfrak{R}(t+n) = j \mid \mathfrak{R}(t) = i\}$$

and the associated n-step transition matrix $P^{(n)} = \left(p_{i,j}^{(n)}\right)$. For $n = 2$, we obtain:

$$\begin{aligned}
p_{i,j}^{(2)} &= \Pr\left\{\mathfrak{R}(t+2) = j \mid \mathfrak{R}(t) = i\right\} \\
&= \sum_{k=1}^{K} \Pr\left\{\mathfrak{R}(t+2) = j, \mathfrak{R}(t+1) = k \mid \mathfrak{R}(t) = i\right\} \\
&= \sum_{k=1}^{K} \Pr\left\{\mathfrak{R}(t+2) = j \mid \mathfrak{R}(t+1) = k\right\} \cdot \Pr\left\{\mathfrak{R}(t+1) = k \mid \mathfrak{R}(t) = i\right\} \\
&= \sum_{k=1}^{K} p_{i,k} \cdot p_{k,j}
\end{aligned}$$

In a similar way, we obtain:

$$p_{i,j}^{(n+m)} = \sum_{k=1}^{K} p_{i,k}^{(n)} \cdot p_{k,j}^{(m)} \qquad \forall n, m > 0 \tag{3.45}$$

This equation is called the Chapman-Kolmogorov equation. In matrix form, we have:

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

with the convention $P^{(0)} = I$. In particular, we have:

$$\begin{aligned}
P^{(n)} &= P^{(n-1)} \cdot P^{(1)} \\
&= P^{(n-2)} \cdot P^{(1)} \cdot P^{(1)} \\
&= \prod_{t=1}^{n} P^{(1)} \\
&= P^{n}
\end{aligned}$$

We deduce that:

$$p(t, i; t+n, j) = p_{i,j}^{(n)} = \mathbf{e}_i^\top P^n \mathbf{e}_j \tag{3.46}$$

When we apply this framework to credit risk, $\mathfrak{R}(t)$ denotes the rating (or the risk class) of the firm at time t, $p_{i,j}$ is the one-period transition probability from rating i to rating j and $p_{i,K}$ is the one-period default probability of rating i. In Table 3.34, we report the S&P one-year transition probability matrix for corporate bonds estimated by Kavvathas (2001). We read the figures as follows[91]: a firm rated AAA has a one-year probability of 92.82% to remain AAA; its probability to become AA is 6.50%; a firm rated CCC defaults one year later with a probability equal to 23.50%; etc. In Tables 3.35 and 3.36, we have reported the two-year and five-year transition probability matrices. We detail below the calculation of $p_{\mathsf{AAA},\mathsf{AAA}}^{(2)}$:

$$\begin{aligned}
p_{\mathsf{AAA},\mathsf{AAA}}^{(2)} &= p_{\mathsf{AAA},\mathsf{AAA}} \times p_{\mathsf{AAA},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{AA}} \times p_{\mathsf{AA},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{A}} \times p_{\mathsf{A},\mathsf{AAA}} + \\
&\quad p_{\mathsf{AAA},\mathsf{BBB}} \times p_{\mathsf{BBB},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{BB}} \times p_{\mathsf{BB},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{B}} \times p_{\mathsf{B},\mathsf{AAA}} + \\
&\quad p_{\mathsf{AAA},\mathsf{CCC}} \times p_{\mathsf{CCC},\mathsf{AAA}} \\
&= 0.9283^2 + 0.0650 \times 0.0063 + 0.0056 \times 0.0008 + \\
&\quad 0.0006 \times 0.0005 + 0.0006 \times 0.0004 \\
&= 86.1970\%
\end{aligned}$$

TABLE 3.34: Example of credit migration matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.82	6.50	0.56	0.06	0.06	0.00	0.00	0.00
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.00
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.04
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.22
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.06
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Source: Kavvathas (2001).

We note $\pi_i^{(n)}$ the probability of the state i at time n:

$$\pi_i^{(n)} = \Pr\left\{\mathfrak{R}(n) = i\right\}$$

and $\pi^{(n)} = \left(\pi_1^{(n)}, \ldots, \pi_K^{(n)}\right)$ the probability distribution. By construction, we have:

$$\pi^{(n+1)} = P^\top \pi^{(n)}$$

The Markov chain $\mathfrak{R}$ admits a stationary distribution $\pi^\star$ if[92]:

$$\pi^\star = P^\top \pi^\star$$

In this case, $\pi_i^\star$ is the limiting probability of state i:

$$\lim_{n\to\infty} p_{k,i}^{(n)} = \pi_i^\star$$

We can interpret $\pi_i^\star$ as the average duration spent by the chain $\mathfrak{R}$ in the state i. Let $\mathcal{T}_i$ be the return period[93] of state i:

$$\mathcal{T}_i = \inf\left\{n : \mathfrak{R}(n) = i \mid \mathfrak{R}(0) = i\right\}$$

The average return period is then equal to:

$$\mathbb{E}\left[\mathcal{T}_i\right] = \frac{1}{\pi_i^\star}$$

For credit migration matrices, there is no stationary distribution because the long-term rating $\mathfrak{R}(\infty)$ is the absorbing state as noted by Jafry and Schuermann:

> "*Given sufficient time, all firms will eventually sink to the default state. This behavior is clearly a mathematical artifact, stemming from the idealized linear, time invariant assumptions inherent in the simple Markov model. In reality the economy (and hence the migration matrix) will change on time-scales far shorter than required to reach the idealized default steady-state proscribed by an assumed constant migration matrix*" (Jafry and Schuermann, 2004, page 2609).

[91] The rows represent the initial rating whereas the columns indicate the final rating.
[92] Not all Markov chains behave in this way, meaning that $\pi^\star$ does not necessarily exist.
[93] This concept plays an important role when designing stress scenarios (see Chapter 18).

TABLE 3.35: Two-year transition probability matrix P^2 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	86.20	12.02	1.47	0.18	0.11	0.01	0.00	0.00
AA	1.17	84.59	12.23	1.51	0.18	0.22	0.07	0.02
A	0.16	4.17	84.47	9.23	1.31	0.51	0.04	0.11
BBB	0.10	0.63	10.53	77.66	8.11	2.10	0.32	0.56
BB	0.08	0.24	1.60	13.33	66.79	13.77	1.59	2.60
B	0.01	0.21	0.61	1.29	11.20	70.03	5.61	11.03
CCC	0.29	0.04	0.68	1.37	4.31	17.51	37.34	38.45
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

TABLE 3.36: Five-year transition probability matrix P^5 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	69.23	23.85	5.49	0.96	0.31	0.12	0.02	0.03
AA	2.35	66.96	24.14	4.76	0.86	0.62	0.13	0.19
A	0.43	8.26	68.17	17.34	3.53	1.55	0.18	0.55
BBB	0.24	1.96	19.69	56.62	13.19	5.32	0.75	2.22
BB	0.17	0.73	5.17	21.23	40.72	20.53	2.71	8.74
B	0.07	0.47	1.73	4.67	16.53	44.95	5.91	25.68
CCC	0.38	0.24	1.37	2.92	7.13	18.51	9.92	59.53
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

We note that the survival function $\mathbf{S}_i(t)$ of a firm whose initial rating is the state i is given by:

$$\begin{aligned} \mathbf{S}_i(t) &= 1 - \Pr\{\mathfrak{R}(t) = K \mid \mathfrak{R}(0) = i\} \\ &= 1 - \mathbf{e}_i^\top P^t \mathbf{e}_K \end{aligned} \tag{3.47}$$

In the piecewise exponential model, we recall that the survival function has the following expression:

$$\mathbf{S}(t) = \mathbf{S}\left(t_{m-1}^\star\right) e^{-\lambda_m\left(t - t_{m-1}^\star\right)}$$

for $t \in \left]t_{m-1}^\star, t_m^\star\right]$. We deduce that $\mathbf{S}(t_m^\star) = \mathbf{S}\left(t_{m-1}^\star\right) e^{-\lambda_m\left(t_m^\star - t_{m-1}^\star\right)}$, implying that:

$$\ln \mathbf{S}(t_m^\star) = \ln \mathbf{S}\left(t_{m-1}^\star\right) - \lambda_m\left(t_m^\star - t_{m-1}^\star\right)$$

and:

$$\lambda_m = \frac{\ln \mathbf{S}\left(t_{m-1}^\star\right) - \ln \mathbf{S}(t_m^\star)}{t_m^\star - t_{m-1}^\star}$$

It is then straightforward to estimate the piecewise hazard function:

- the knots of the piecewise function are the years $m \in \mathbb{N}^*$;
- for each initial rating i, the hazard function $\lambda_i(t)$ is defined as:

  $$\lambda_i(t) = \lambda_{i,m} \qquad \text{if } t \in \left]m-1, m\right]$$

  where:

  $$\begin{aligned} \lambda_{i,m} &= \frac{\ln \mathbf{S}_i(m-1) - \ln \mathbf{S}_i(m)}{m - (m-1)} \\ &= \ln\left(\frac{1 - \mathbf{e}_i^\top P^{m-1}\mathbf{e}_K}{1 - \mathbf{e}_i^\top P^m \mathbf{e}_K}\right) \end{aligned}$$

  and $P^0 = I$.

If we consider the credit migration matrix given in Table 3.34 and estimate the piecewise exponential model, we obtain the hazard function[94] $\lambda_i(t)$ shown in Figure 3.35. For good initial ratings, hazard rates are low for short maturities and increase with time. For bad initial ratings, we obtain the opposite effect, because the firm can only improve its rating if it did not default. We observe that the hazard function of all the ratings converges to the same level, which is equal to 102.63 bps. This indicates the long-term hazard rate of the Markov chain, meaning that 1.02% of firms default every year on average.

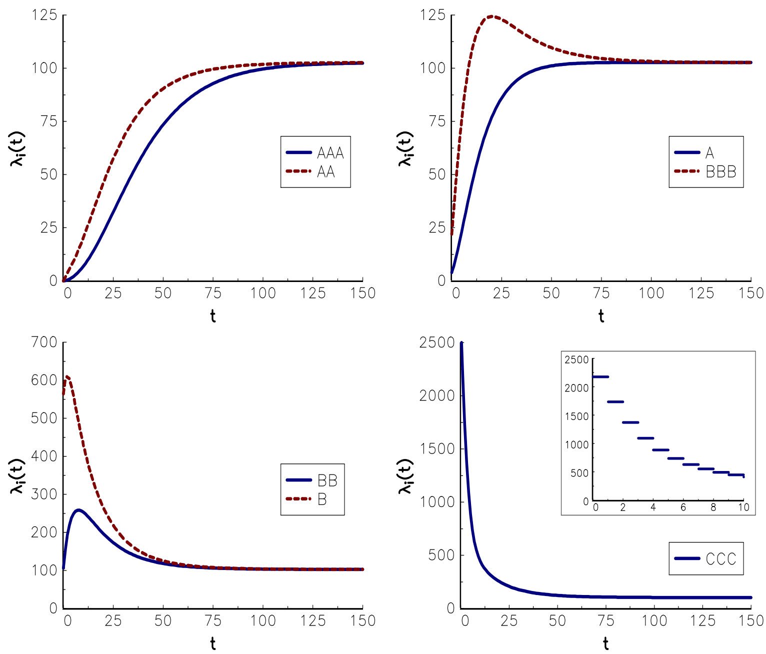

FIGURE 3.35: Estimated hazard function $\lambda_i(t)$ from the credit migration matrix

Continuous-time modeling We now consider the case $t \in \mathbb{R}_+$. We note $P(s;t)$ the transition matrix defined as follows:

$$\begin{aligned} P_{i,j}(s;t) &= p(s,i;t,j) \\ &= \Pr\{\mathfrak{R}(t) = j \mid \mathfrak{R}(s) = i\} \end{aligned}$$

Assuming that the Markov chain is time-homogenous, we have $P(t) = P(0;t)$. Jarrow *et al.* (1997) introduce the generator matrix $\Lambda = (\lambda_{i,j})$ where $\lambda_{i,j} \geq 0$ for all $i \neq j$ and:

$$\lambda_{i,i} = -\sum_{j \neq i}^{K} \lambda_{i,j}$$

In this case, the transition matrix satisfies the following relationship:

$$P(t) = \exp(t\Lambda) \tag{3.48}$$

[94] Contrary to what the graph suggests, $\lambda_i(t)$ is a piecewise constant function (see details of the curve in the fifth panel for very short maturities).

where $\exp(A)$ is the matrix exponential of A. Let us give a probabilistic interpretation of Λ. If we assume that the probability of jumping from rating i to rating j in a short time period Δt is proportional to Δt, we have:

$$p(t, i; t+\Delta t, j) = \lambda_{i,j} \Delta t$$

The matrix form of this equation is $P(t; t+\Delta t) = \Lambda \, \Delta t$. We deduce that:

$$\begin{aligned} P(t+\Delta t) &= P(t) \, P(t; t+\Delta t) \\ &= P(t) \, \Lambda \, \Delta t \end{aligned}$$

and:

$$\mathrm{d}P(t) = P(t) \, \Lambda \, \mathrm{d}t$$

Because we have $\exp(\mathbf{0}) = I$, we obtain the solution $P(t) = \exp(t\Lambda)$. We then interpret $\lambda_{i,j}$ as the instantaneous transition rate of jumping from rating i to rating j.

Remark 43 *In Appendix A.1.1.3, we present the matrix exponential function and its mathematical properties. In particular, we have $e^{A+B} = e^A e^B$ and $e^{A(s+t)} = e^{As} e^{At}$ where A and B are two square matrices such that $AB = BA$ and s and t are two real numbers.*

Example 39 *We consider a rating system with three states: A (good rating), B (bad rating) and D (default). The Markov generator is equal to:*

$$\Lambda = \begin{pmatrix} -0.30 & 0.20 & 0.10 \\ 0.15 & -0.40 & 0.25 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}$$

The one-year transition probability matrix is equal to:

$$P(1) = e^{\Lambda} = \begin{pmatrix} 75.16\% & 14.17\% & 10.67\% \\ 10.63\% & 68.07\% & 21.30\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

For the two-year maturity, we get:

$$P(2) = e^{2\Lambda} = \begin{pmatrix} 58.00\% & 20.30\% & 21.71\% \\ 15.22\% & 47.85\% & 36.93\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

We verify that $P(2) = P(1)^2$. This derives from the property of the matrix exponential:

$$P(t) = e^{t\Lambda} = \left(e^{\Lambda}\right)^t = P(1)^t$$

The continuous-time framework allows to calculate transition matrices for non-integer maturities, which do not correspond to full years. For instance, the one-month transition probability matrix of the previous example is equal to:

$$P(2) = e^{\frac{1}{12}\Lambda} = \begin{pmatrix} 97.54\% & 1.62\% & 0.84\% \\ 1.21\% & 96.73\% & 2.05\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

One of the issues with the continuous-time framework is to estimate the Markov generator Λ. One solution consists in using the empirical transition matrix $\hat{P}(t)$, which have been calculated for a given time horizon t. In this case, the estimate $\hat{\Lambda}$ must satisfy the relationship $\hat{P}(t) = \exp\left(t\hat{\Lambda}\right)$. We deduce that:

$$\hat{\Lambda} = \frac{1}{t} \ln\left(\hat{P}(t)\right)$$

where $\ln A$ is the matrix logarithm of A. However, the matrix $\hat{\Lambda}$ cannot verify the Markov conditions $\hat{\lambda}_{i,j} \geq 0$ for all $i \neq j$ and $\sum_{j=1}^{K} \lambda_{i,j} = 0$. For instance, if we consider the previous S&P transition matrix, we obtain the generator $\hat{\Lambda}$ given in Table 3.37. We notice that six off-diagonal elements of the matrix are negative[95]. This implies that we can obtain transition probabilities which are negative for short maturities. In this case, Israel *et al.* (2001) propose two estimators to obtain a valid generator:

1. the first approach consists in adding the negative values back into the diagonal values:
$$\begin{cases} \bar{\lambda}_{i,j} = \max\left(\hat{\lambda}_{i,j}, 0\right) & i \neq j \\ \bar{\lambda}_{i,i} = \hat{\lambda}_{i,i} + \sum_{j \neq i} \min\left(\hat{\lambda}_{i,j}, 0\right) & \end{cases}$$

2. in the second method, we carry forward the negative values on the matrix entries which have the correct sign:
$$\begin{cases} G_i = \left|\hat{\lambda}_{i,i}\right| + \sum_{j \neq i} \max\left(\hat{\lambda}_{i,j}, 0\right) \\ B_i = \sum_{j \neq i} \max\left(-\hat{\lambda}_{i,j}, 0\right) \\ \tilde{\lambda}_{i,j} = \begin{cases} 0 & \text{if } i \neq j \text{ and } \hat{\lambda}_{i,j} < 0 \\ \hat{\lambda}_{i,j} - B_i \left|\hat{\lambda}_{i,j}\right| / G_i & \text{if } G_i > 0 \\ \hat{\lambda}_{i,j} & \text{if } G_i = 0 \end{cases} \end{cases}$$

Using the estimator $\hat{\Lambda}$ and the two previous algorithms, we obtain the valid generators given in Tables 3.39 and 3.40. We find that $\left\|\hat{P} - \exp\left(\bar{\Lambda}\right)\right\|_1 = 11.02 \times 10^{-4}$ and $\left\|\hat{P} - \exp\left(\tilde{\Lambda}\right)\right\|_1 = 10.95 \times 10^{-4}$, meaning that the Markov generator $\tilde{\Lambda}$ is the estimator that minimizes the distance to $\hat{P}$. We can then calculate the transition probability matrix for all maturities, and not only for calendar years. For instance, we report the 207-day transition probability matrix $P\left(\frac{207}{365}\right) = \exp\left(\frac{207}{365}\tilde{\Lambda}\right)$ in Table 3.41.

Remark 44 *The continuous-time framework is more flexible when modeling credit risk. For instance, the expression of the survival function becomes:*

$$\mathbf{S}_i(t) = \Pr\left\{\mathfrak{R}(t) = K \mid \mathfrak{R}(0) = i\right\} = 1 - \mathbf{e}_i^\top \exp(t\Lambda)\, \mathbf{e}_K$$

We can therefore calculate the probability density function in an easier way:

$$f_i(t) = -\partial_t\, \mathbf{S}_i(t) = \mathbf{e}_i^\top \Lambda \exp(t\Lambda)\, \mathbf{e}_K$$

For illustration purposes, we represent the probability density function of S&P ratings estimated with the valid generator $\tilde{\Lambda}$ in Figure 3.36.

[95] We have also calculated the estimator described in Israel *et al.* (2001):

$$\breve{\Lambda} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\hat{P} - I\right)^n}{n}$$

We do not obtain the same matrix as for the estimator $\hat{\Lambda}$, but there are also six negative off-diagonal elements (see Table 3.38).

TABLE 3.37: Markov generator $\hat{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.49	703.67	35.21	3.04	6.56	−0.79	−0.22	0.02
AA	67.94	−859.31	722.46	51.60	2.57	10.95	4.92	−1.13
A	7.69	245.59	−898.16	567.70	53.96	20.65	−0.22	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	−0.84	11.83	30.11	8.71	818.31	−1936.82	539.18	529.52
CCC	25.11	−2.89	44.11	84.87	272.05	1678.69	−5043.00	2941.06
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.38: Markov generator $\breve{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−745.85	699.11	38.57	2.80	6.27	−0.70	−0.16	−0.05
AA	67.54	−855.70	716.56	54.37	2.81	10.81	4.62	−1.01
A	7.77	243.62	−891.46	560.45	56.33	20.70	0.07	2.53
BBB	5.06	22.68	641.55	−1335.03	542.46	91.05	16.09	16.15
BB	4.18	10.12	48.00	903.40	−2111.65	965.71	98.28	81.96
B	−0.56	11.61	29.31	19.39	789.99	−1887.69	491.46	546.49
CCC	23.33	−1.94	42.22	81.25	272.44	1530.66	−4725.22	2777.25
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.39: Markov generator $\bar{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−748.50	703.67	35.21	3.04	6.56	0.00	0.00	0.02
AA	67.94	−860.44	722.46	51.60	2.57	10.95	4.92	0.00
A	7.69	245.59	−898.38	567.70	53.96	20.65	0.00	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	0.00	11.83	30.11	8.71	818.31	−1937.66	539.18	529.52
CCC	25.11	0.00	44.11	84.87	272.05	1678.69	−5045.89	2941.06
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.40: Markov generator $\tilde{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.99	703.19	35.19	3.04	6.55	0.00	0.00	0.02
AA	67.90	−859.88	721.98	51.57	2.57	10.94	4.92	0.00
A	7.69	245.56	−898.27	567.63	53.95	20.65	0.00	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	0.00	11.83	30.10	8.71	818.14	−1937.24	539.06	529.40
CCC	25.10	0.00	44.10	84.84	271.97	1678.21	−5044.45	2940.22
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.41: 207-day transition probability matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.85	3.81	0.27	0.03	0.04	0.00	0.00	0.00
AA	0.37	95.28	3.90	0.34	0.03	0.06	0.02	0.00
A	0.04	1.33	95.12	3.03	0.33	0.12	0.00	0.02
BBB	0.03	0.14	3.47	92.75	2.88	0.53	0.09	0.11
BB	0.02	0.06	0.31	4.79	88.67	5.09	0.53	0.53
B	0.00	0.06	0.17	0.16	4.16	89.84	2.52	3.08
CCC	0.12	0.01	0.23	0.45	1.45	7.86	75.24	14.64
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

3.3.3.3 Structural models

The previous approaches are purely statistical and are called reduced-form models. We now consider economic models for modeling default times. These approaches are based on accounting and market data and are called structural models.

The Merton model The structural approach of credit risk has been formalized by Merton (1974). In this framework, the bond holders will liquidate the corporate firm if the asset value $A(t)$ goes below a threshold B related to the total amount of debt. The underlying idea is that bond holders monitor the asset value and compare $A(t)$ to the default barrier B.

Merton (1974) assumes that the dynamics of the assets $A(t)$ follows a geometric Brownian motion:

$$\mathrm{d}A(t) = \mu_A A(t)\,\mathrm{d}t + \sigma_A A(t)\,\mathrm{d}W(t)$$

where $A(0) = A_0$. The default occurs if the asset value $A(t)$ falls under the threshold B:

$$\boldsymbol{\tau} := \inf\{t : A(t) \leq B\}$$

In this case, the bond holders receive $A(T)$, and lose $B - A(T)$. The payoff of bond holders is then equal to:

$$D = B - \max(B - A(T), 0)$$

where D is the debt value of maturity T. The holding of a risky bond can be interpreted as a trading strategy where we have bought a zero-coupon and financed the cost by selling a put on $A(t)$ with an exercise price B and a maturity T. From the viewpoint of the equity holders, the payoff is equal to $\max(A(T) - D, 0)$. The holding of an equity share can be interpreted as a trading strategy where we have bought a call option with a strike equal to the debt value D. It follows that the current value E_0 of the equity is:

$$\begin{aligned} E_0 &= e^{-rT} \cdot \mathbb{E}[\max(A(T) - D, 0)] \\ &= A_0 \Phi(d_1) - e^{-rT} D \Phi(d_2) \end{aligned}$$

where:

$$d_1 = \frac{\ln A_0 - \ln D + rT}{\sigma_A \sqrt{T}} + \frac{1}{2}\sigma_A \sqrt{T}$$

and $d_2 = d_1 - \sigma_A \sqrt{T}$. We notice that the equity value depends on the current asset value A_0, the leverage ratio $L = D/A_0$, the asset volatility σ_A and the time of repayment T.

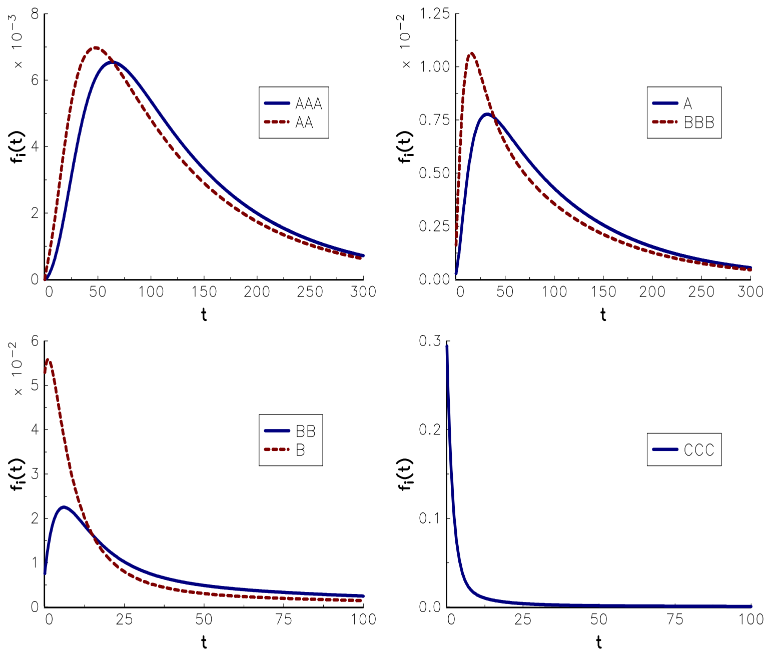

FIGURE 3.36: Probability density function $f_i(t)$ of S&P ratings

The KMV implementation In the nineties, the Merton model has been implemented by KMV[96] with a lot of success. The underlying idea of the KMV implementation is to estimate the default probability of a firm. One of the difficulties is to estimate the asset volatility σ_A. However, Jones *et al.* (1984) show that it is related to the equity volatility σ_E. Indeed, we have $E(t) = \mathcal{C}(t, A(t))$, implying that:

$$\begin{aligned} \mathrm{d}E(t) &= \partial_t \mathcal{C}(t, A(t))\,\mathrm{d}t + \mu_A A(t)\,\partial_A \mathcal{C}(t, A(t))\,\mathrm{d}t + \\ &\quad \frac{1}{2}\sigma_A^2 A^2(t)\,\partial_A^2 \mathcal{C}(t, A(t))\,\mathrm{d}t + \sigma_A A(t)\,\partial_A \mathcal{C}(t, A(t))\,\mathrm{d}W(t) \end{aligned}$$

Since the stochastic term is also equal to $\sigma_E E(t)\,\mathrm{d}W(t)$, we obtain the following equality at time $t = 0$:

$$\sigma_E E_0 = \sigma_A A_0 \Phi(d_1)$$

Therefore, Crosbie and Bohn (2002) deduce the following system of equations:

$$\left\{ \begin{array}{l} A_0 \Phi(d_1) - e^{-rT} D \Phi(d_2) - E_0 = 0 \\ \sigma_E E_0 - \sigma_A A_0 \Phi(d_1) = 0 \end{array} \right. \tag{3.49}$$

Once we have estimated A_0 and σ_A, we can calculate the survival function:

$$\begin{aligned} \mathbf{S}(t) &= \Pr\{A(t) \geq D \mid A(0) = A_0\} \\ &= \Phi\left(\frac{\ln A_0 - \ln D + \mu_A t}{\sigma_A \sqrt{t}} + \frac{1}{2}\sigma_A \sqrt{t}\right) \end{aligned}$$

and deduce the probability of default $\mathbf{F}(t) = 1 - \mathbf{S}(t)$ and the distance to default $\mathrm{DD}(t) = \Phi^{-1}(\mathbf{S}(t))$.

[96]KMV was a company dedicated to credit risk modeling, and was founded by Stephen Kealhofer, John McQuown and Oldrich Vasícek. In 2002, they sold KMV to Moody's.

Example 40 *Crosbie and Bohn (2002) assume that the market capitalization E_0 of the firm is \$3 bn, its debt liability D is \$10 bn, the corresponding equity volatility σ_E is equal to 40%, the maturity T is one year and the expected return μ_A is set to 7%.*

Using an interest rate $r = 5\%$ and solving Equation (3.49), we find that the asset value A_0 is equal to \$12.512 bn and the implied asset volatility σ_A is equal to 9.609%. Therefore, we can calculate the distance-to-default $\mathrm{DD}(1) = 3.012$ and the one-year probability $\mathrm{PD}(1) = 12.96$ bps. In Figure 3.37, we report the probability of default for different time horizons. We also show the impact of the equity volatility σ_E and the expected return μ_A, which can be interpreted as a return-on-equity ratio (ROE). We verify that the probability of default is an increasing function of the volatility risk and a decreasing function of the profitability.

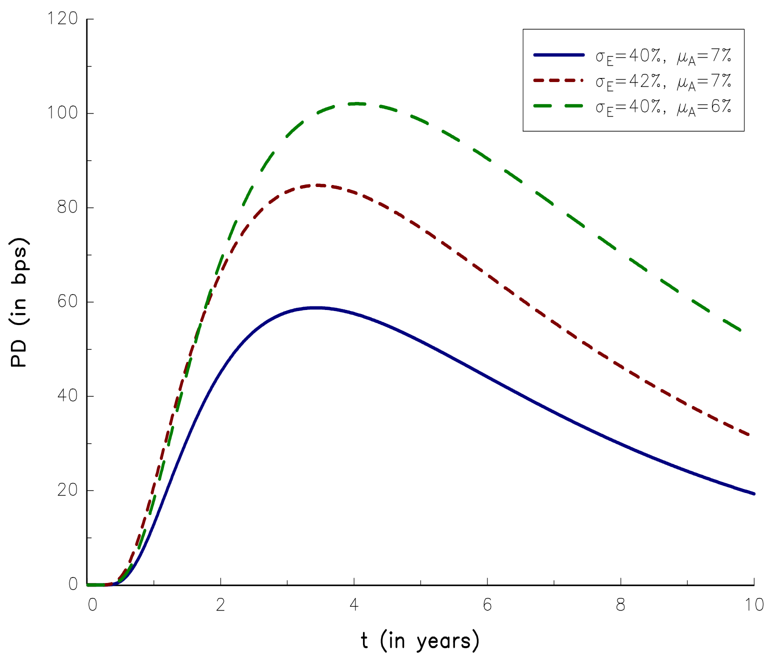

FIGURE 3.37: Probability of default in the KMV model

Remark 45 *The KMV model is more complex than the presentation above. In particular, the key variable is not the probability of default, but the distance-to-default (see Figure 3.38). Once this measure is calculated, it is converted into an expected default frequency (EDF) by considering an empirical distribution of* PD *conditionally to the distance-to-default. For instance,* $\mathrm{DD}(1) = 4$ *is equivalent to* $\mathrm{PD}(1) = 100$ *bps (Crosbie and Bohn, 2002).*

The CreditGrades implementation The CreditGrades approach is an extension of the Merton model, uses the framework of Black and Cox (1976) and has been developed by Finkelstein *et al.* (2002). They assume that the asset-per-share value $A(t)$ is a geometric Brownian motion without drift:

$$\mathrm{d}A(t) = \sigma_A A(t)\,\mathrm{d}W(t)$$

whereas the default barrier B is defined as the recovery value of bond holders. B is equal to the product $\boldsymbol{\mathcal{R}} \cdot D$, where $\boldsymbol{\mathcal{R}} \in [0, 1]$ is the recovery rate and D is the debt-per-share value.

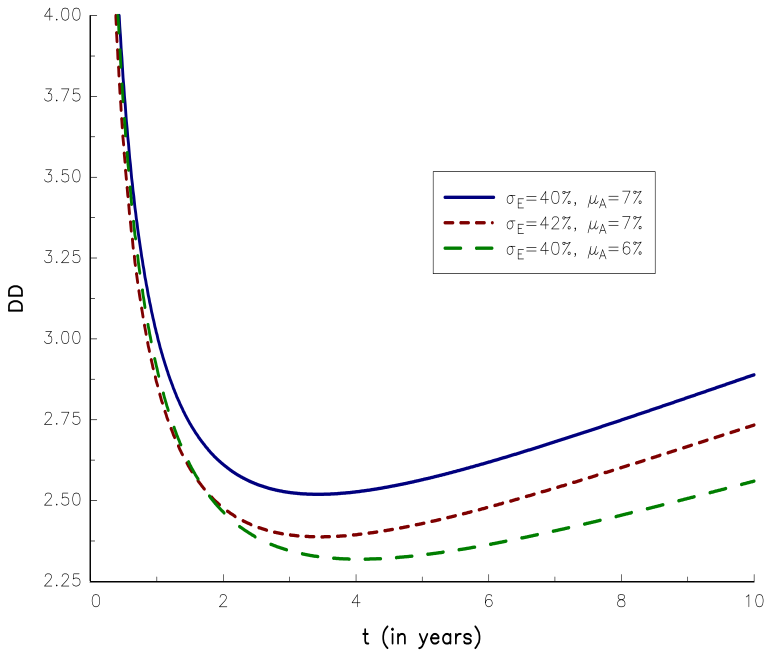

FIGURE 3.38: Distance-to-default in the KMV model

They also assume that $\mathcal{R}$ and $A(t)$ are independent and $\mathcal{R} \sim \mathcal{LN}(\mu_{\mathcal{R}}, \sigma_{\mathcal{R}})$. We recall that the default time is defined by:

$$\boldsymbol{\tau} := \inf \{t \geq 0 : t \in \mathcal{D}\}$$

where $\mathcal{D} = \{A(t) \leq B\}$. Since we have $A(t) = A_0 e^{\sigma_A W(t) - \sigma_A^2 t/2}$ and $B = D e^{\mu_{\mathcal{R}} + \sigma_{\mathcal{R}} \varepsilon}$ where $\varepsilon \sim \mathcal{N}(0,1)$, it follows that:

$$\mathcal{D} = \left\{ A_0 e^{\sigma_A W(t) - \sigma_A^2 t/2} \leq D e^{\mu_{\mathcal{R}} + \sigma_{\mathcal{R}} \varepsilon} \right\}$$

The authors introduce the average recovery rate $\bar{\mathcal{R}} = \mathbb{E}[\mathcal{R}] = e^{\mu_{\mathcal{R}} + \sigma_{\mathcal{R}}^2/2}$. We deduce that:

$$\begin{aligned} \mathcal{D} &= \left\{ A_0 e^{\sigma_A W(t) - \sigma_A^2 t/2} \leq \bar{\mathcal{R}} D e^{\sigma_{\mathcal{R}} \varepsilon - \sigma_{\mathcal{R}}^2/2} \right\} \\ &= \left\{ A_0 e^{\sigma_A W(t) - \sigma_A^2 t/2 - \sigma_{\mathcal{R}} \varepsilon + \sigma_{\mathcal{R}}^2/2} \leq \bar{\mathcal{R}} D \right\} \end{aligned} \tag{3.50}$$

Finkelstein *et al.* (2002) introduce the process $X(t)$ defined by:

$$X(t) = \sigma_A W(t) - \frac{1}{2}\sigma_A^2 t - \sigma_{\mathcal{R}} \varepsilon - \frac{1}{2}\sigma_{\mathcal{R}}^2$$

It follows that Inequality (3.50) becomes:

$$\mathcal{D} = \left\{ X(t) \leq \ln \left(\frac{\bar{\mathcal{R}} D}{A_0 e^{\sigma_{\mathcal{R}}^2}} \right) \right\}$$

By assuming that $X(t)$ can be approximated by a geometric Brownian motion with drift $-\sigma_A^2/2$ and diffusion rate σ_A, we can show that[97]:

$$\mathbf{S}(t) = \Phi\left(-\frac{\sigma(t)}{2} + \frac{\ln\varphi}{\sigma(t)}\right) - \varphi\Phi\left(-\frac{\sigma(t)}{2} - \frac{\ln\varphi}{\sigma(t)}\right)$$

where $\sigma(t) = \sqrt{\sigma_A^2 t + \sigma_{\boldsymbol{\mathcal{R}}}^2}$ and:

$$\varphi = \frac{A_0 e^{\sigma_{\boldsymbol{\mathcal{R}}}^2}}{\bar{\boldsymbol{\mathcal{R}}} D}$$

This survival function is then calibrated by assuming that $A_0 = S_0 + \bar{\boldsymbol{\mathcal{R}}} D$ and:

$$\sigma_A = \sigma_S \frac{S^\star}{S^\star + \bar{\boldsymbol{\mathcal{R}}} D}$$

where S_0 is the current stock price, $S^\star$ is the reference stock price and σ_S is the stock (implied or historical) volatility. All the parameters (S_0, $S^\star$, σ_S, $\bar{\boldsymbol{\mathcal{R}}}$, D) are easy to calibrate, except the volatility of the recovery rate $\sigma_{\boldsymbol{\mathcal{R}}}$. We have:

$$\sigma_{\boldsymbol{\mathcal{R}}}^2 = \operatorname{var}(\ln \boldsymbol{\mathcal{R}}) = \operatorname{var}(\ln B)$$

We deduce that $\sigma_{\boldsymbol{\mathcal{R}}}$ is the uncertainty of the default barrier B.

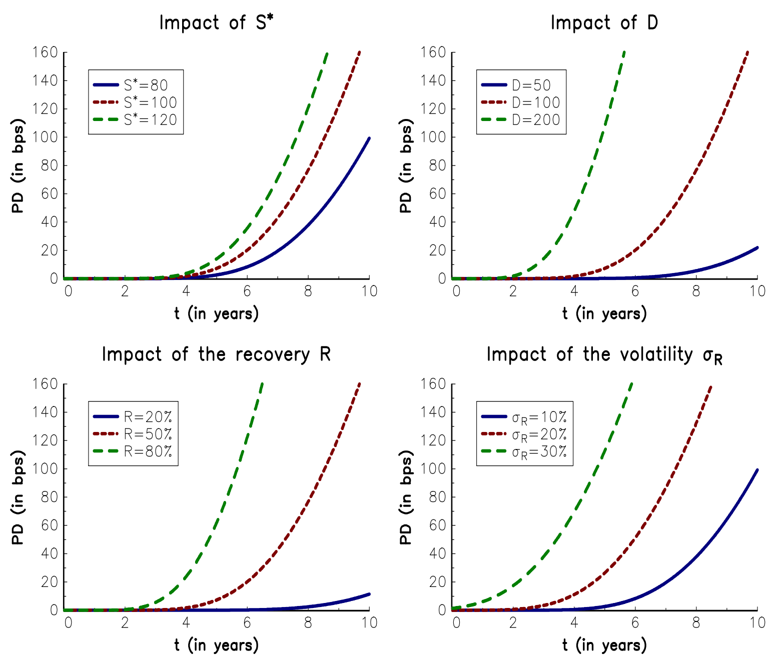

FIGURE 3.39: Probability of default in the CreditGrades model

[97]By considering the reflection principle and Equation (A.24) defined on page 1074, we deduce that:

$$\Pr\left\{\inf_{s\leq t}\mu s + \sigma W(s) > c\right\} = \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) - e^{2\mu c/\sigma^2}\Phi\left(\frac{\mu t + c}{\sigma\sqrt{t}}\right)$$

The expression of $\mathbf{S}(t)$ is obtained by setting $\mu = -\sigma_A^2/2$, $\sigma = \sigma_A$ and $c = \ln\left(\bar{\boldsymbol{\mathcal{R}}} D\right) - \ln\left(A_0 e^{\sigma_{\boldsymbol{\mathcal{R}}}^2}\right)$, and using the change of variable $u = t + \sigma_{\boldsymbol{\mathcal{R}}}^2/\sigma_A^2$.

In Figure 3.39, we illustrate the CreditGrades model by computing the probability of default when $S_0 = 100$, $S^\star = 100$, $\sigma_S = 20\%$, $\bar{\boldsymbol{\mathcal{R}}} = 50\%$, $\sigma_{\boldsymbol{\mathcal{R}}} = 10\%$ and $D = 100$. We notice that $\mathrm{PD}(t)$ is an increasing function of $S^\star$, σ_S, $\bar{\boldsymbol{\mathcal{R}}}$, and $\sigma_{\boldsymbol{\mathcal{R}}}$. The impact of the recovery rate may be curious, but bond holders may be encouraged to cause the default when the recovery rate is high.

Relationship with intensity (or reduced-form) models Let $\lambda(s)$ be a positive continuous process. We define the default time by $\boldsymbol{\tau} := \inf\left\{t \geq 0 : \int_0^t \lambda(s)\,\mathrm{d}s \geq \theta\right\}$ where θ is a standard exponential random variable. We have:

$$\begin{aligned}
\mathbf{S}(t) &= \Pr\{\boldsymbol{\tau} > t\} \\
&= \Pr\left\{\int_0^t \lambda(s)\,\mathrm{d}s \leq \theta\right\} \\
&= \mathbb{E}\left[\exp\left(-\int_0^t \lambda(s)\,\mathrm{d}s\right)\right]
\end{aligned}$$

Let $\Lambda(t) = \int_0^t \lambda(s)\,\mathrm{d}s$ be the integrated hazard function. If $\lambda(s)$ is deterministic, we obtain $\mathbf{S}(t) = \exp(-\Lambda(t))$. In particular, if $\lambda(s)$ is a piecewise constant function, we obtain the piecewise exponential model.

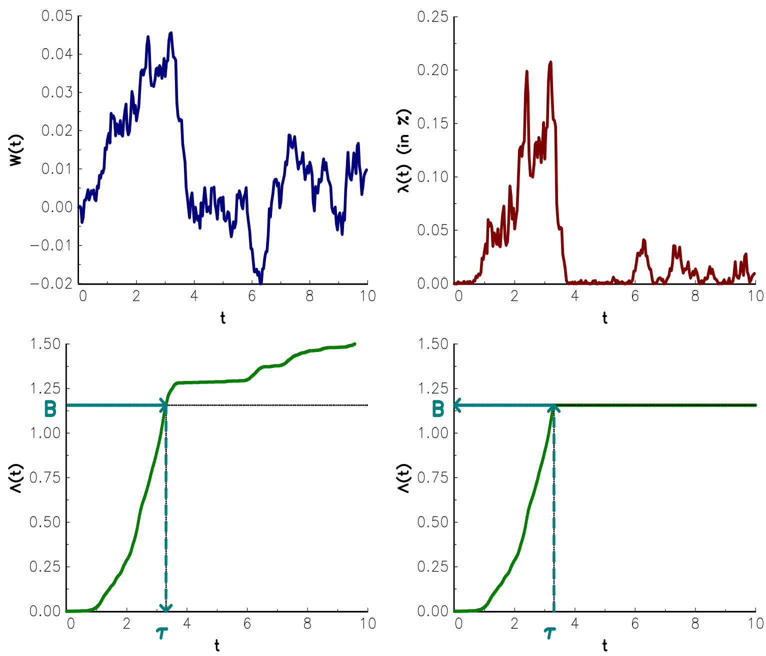

FIGURE 3.40: Intensity models and the default barrier issue

We now consider the stochastic case $\lambda(t) = \sigma W^2(t)$ where $W(t)$ is a Brownian motion. In Figure 3.40, we illustrate the simulation mechanism of defaults. First, we simulate the exponential variable B. In our example, it is equal to 1.157. Second, we simulate the Brownian motion $W(t)$ (top/left panel). Then, we calculate $\lambda(t)$ where $\sigma = 1.5\%$ (top/right panel), and the integrated hazard function $\Lambda(t)$ (bottom/left panel). Finally, we determine the default time when the integrated hazard function crosses the barrier B. In our example, $\boldsymbol{\tau}$ is equal to 3.30. In fact, the simulation mechanism may be confusing. Indeed, we have the impression that we know the barrier B, implying that the default is predictable. In intensity models, this is the contrary. We don't know the stochastic barrier B, but the occurrence of the default unveils the barrier B as illustrated in the bottom/right panel in Figure 3.40. In structural models, we assume that the barrier B is known and we can predict the default time because we observe the distance to the barrier. Intensity and structural models are then the two faces of the same coin. They use the same concept of default barrier, but its interpretation is completely different.

3.3.4 Default correlation

In this section, we consider the modeling of default correlations, which corresponds essentially to two approaches: the copula model and the factor model. Then, we see how to estimate default correlations. Finally, we show how to consider the dependence of default times in the pricing of basket derivatives.

3.3.4.1 The copula model

Copula functions are extensively studied in Chapter 11, and we invite the reader to examine this chapter before to go further. Let $\mathbf{F}$ be the joint distribution of the random vector $(X_1, \ldots, X_n)$, we show on page 719 that $\mathbf{F}$ admits a copula representation:

$$\mathbf{F}(x_1, \ldots, x_n) = \mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_n(x_n))$$

where $\mathbf{F}_i$ is the marginal distribution of X_i and $\mathbf{C}$ is the copula function associated to $\mathbf{F}$. Since there is a strong relationship between probability distributions and survival functions, we can also show that the survival function $\mathbf{S}$ of the random vector $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ has a copula representation:

$$\mathbf{S}(t_1, \ldots, t_n) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_n(t_n))$$

where $\mathbf{S}_i$ is the survival function of $\boldsymbol{\tau}_i$ and $\breve{\mathbf{C}}$ is the survival copula associated to $\mathbf{S}$. The copula $\breve{\mathbf{C}}$ is unique if the marginals are continuous. The copula functions $\mathbf{C}$ and $\breve{\mathbf{C}}$ are not necessarily the same, except when the copula $\mathbf{C}$ is radially symmetric (Nelsen, 2006). This is for example the case of the Normal (or Gaussian) copula and the Student's t copula. Since these two copula functions are the only ones that are really used by professionals[98], we assume that $\breve{\mathbf{C}} = \mathbf{C}$ in the sequel.

The Basel model We have seen that the Basel framework for modeling the credit risk is derived from the Merton model. Let $Z_i \sim \mathcal{N}(0,1)$ be the (normalized) asset value of the i^{th} firm. In the Merton model, the default occurs when Z_i is below a non-stochastic barrier B_i:

$$D_i = 1 \Leftrightarrow Z_i \leq B_i$$

The Basel Committee assumes that $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$ where $X \sim \mathcal{N}(0,1)$ is the systematic risk factor and $\varepsilon_i \sim \mathcal{N}(0,1)$ is the specific risk factor. We have shown that the default barrier B_i is equal to $\Phi^{-1}(p_i)$ where p_i is the unconditional default probability. We have also demonstrated that the conditional default probability is equal to:

$$p_i(X) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right)$$

[98] They can also use some Archimedean copulas that are not radially symmetric such as the Clayton copula, but it generally concerns credit portfolios with a small number of exposures.

Remark 46 *In the Basel framework, we assume a fixed maturity. If we introduce the time dimension, we obtain:*

$$\begin{aligned} p_i(t) &= \Pr\{\boldsymbol{\tau}_i \le t\} \\ &= 1 - S_i(t) \end{aligned}$$

and:

$$p_i(t, X) = \Phi\left(\frac{\Phi^{-1}(1 - \mathbf{S}_i(t)) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right)$$

where $S_i(t)$ is the survival function of the i^{th} firm.

The vector of assets $Z = (Z_1, \ldots, Z_n)$ is Gaussian with a constant covariance matrix $\mathbb{C} = \mathbb{C}_n(\rho)$:

$$\mathbb{C} = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

It follows that the joint default probability is:

$$\begin{aligned} p_{1,\ldots,n} &= \Pr\{D_1 = 1, \ldots, D_n = 1\} \\ &= \Pr\{Z_1 \le B_1, \ldots, Z_n \le B_n\} \\ &= \Phi(B_1, \ldots, B_n; \mathbb{C}) \end{aligned}$$

Since we have $B_i = \Phi^{-1}(p_i)$, we deduce that the Basel copula between the default indicator functions is a Normal copula, whose parameters are a constant correlation matrix $\mathbb{C}_n(\rho)$:

$$\begin{aligned} p_{1,\ldots,n} &= \Phi\left(\Phi^{-1}(p_1), \ldots, \Phi^{-1}(p_n); \mathbb{C}\right) \\ &= \mathbf{C}(p_1, \ldots, p_n; \mathbb{C}_n(\rho)) \end{aligned}$$

Let us now consider the dependence between the survival times:

$$\begin{aligned} \mathbf{S}(t_1, \ldots, t_n) &= \Pr\{\boldsymbol{\tau}_1 > t_1, \ldots, \boldsymbol{\tau}_n > t_n\} \\ &= \Pr\left\{Z_1 > \Phi^{-1}(p_1(t_1)), \ldots, Z_n > \Phi^{-1}(p_n(t_n))\right\} \\ &= \mathbf{C}(1 - p_1(t_1), \ldots, 1 - p_n(t_n); \mathbb{C}) \\ &= \mathbf{C}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_n(t_n); \mathbb{C}_n(\rho)) \end{aligned}$$

The dependence between the default times is again the Normal copula with the matrix of parameters $\mathbb{C}_n(\rho)$.

Extension to other copula models The Basel model assumes that the asset correlation is the same between the different firms. A first extension is to consider that the dependence between the default times remain a Normal copula, but with a general correlation matrix:

$$\mathbb{C} = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ & 1 & & \vdots \\ & & \ddots & \rho_{n-1,n} \\ & & & 1 \end{pmatrix} \tag{3.51}$$

This approach is explicitly proposed by Li (2000), but it was already implemented in CreditMetrics (Gupton *et al.*, 1997). The correlation matrix can be estimated using a structural model or approximated by the correlation of stock returns. However, this approach is only valid for publicly traded companies and is not always stable. This is why professionals prefer to use direct extensions of the one-factor model.

Let X_j be a Gaussian factor where $j = 1, \ldots, m$. We assume that the asset value Z_i depends on one of these common risk factors:

$$Z_i = \sum_{j=1}^{m} \beta_{i,j} X_j + \varepsilon_i \tag{3.52}$$

with $\sum_{j=1}^{m} \mathbb{1}\{\beta_{i,j} > 0\} = 1$. We assume that the common risk factors are correlated with each other, but they are independent of the specific risks $(\varepsilon_1, \ldots, \varepsilon_n)$, which are by definition not correlated. For instance, X_j can represent the systematic risk factor of the j^{th} sector or industry. Of course, we can extend this approach to a higher dimension such as sector $\times$ region. For example, if we consider three sectors (S_1, S_2 and S_3) and two geographical regions (R_1 and R_2), we obtain six common risk factors:

	S_1	S_2	S_3
R_1	X_1	X_2	X_3
R_2	X_4	X_5	X_6

These risk factors can then be seen as composite sectors. We note $\operatorname{map}(i)$ the mapping function, which indicates the composite sector j (or the risk factor j): $\operatorname{map}(i) = j$ if $i \in X_j$. We assume that the dependence between the default times $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ is a Normal copula function, whose correlation matrix $\mathbb{C}$ is equal to:

$$\mathbb{C} = \begin{pmatrix} 1 & \rho(\operatorname{map}(1), \operatorname{map}(2)) & \cdots & \rho(\operatorname{map}(1), \operatorname{map}(n)) \\ & 1 & & \vdots \\ & & \ddots & \rho(\operatorname{map}(n-1), \operatorname{map}(n)) \\ & & & 1 \end{pmatrix} \tag{3.53}$$

In practice, we have $m \ll n$ and many elements of the correlation matrix $\mathbb{C}$ are equal. In fact, there are only $m \times (m+1)/2$ different values, which correspond to inter-sector and intra-sector correlations.

Example 41 *Let us consider the case of four sectors:*

Factor	X_1	X_2	X_3	X_4
X_1	*30%*	***20%***	***10%***	***0%***
X_2		*40%*	***30%***	***20%***
X_3			*50%*	***10%***
X_4				*60%*

The inter-sector correlations are indicated in ***bold****, whereas the intra-sector correlations are underlined.*

If the portfolio is composed of seven loans of corporate firms that belong to the following sectors:

i	1	2	3	4	5	6	7
$j = \operatorname{map}(i)$	1	1	2	3	3	3	4

we obtain the following correlation matrix:

$$\mathbb{C} = \begin{pmatrix} 1.00 & 0.30 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & 1.00 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & & 1.00 & 0.30 & 0.30 & 0.30 & 0.20 \\ & & & 1.00 & 0.50 & 0.50 & 0.10 \\ & & & & 1.00 & 0.50 & 0.10 \\ & & & & & 1.00 & 0.10 \\ & & & & & & 1.00 \end{pmatrix}$$

Simulation of copula models With the exception of the Normal copula with a constant correlation matrix and an infinitely fine-grained portfolio, we cannot calculate analytically the value-at-risk or the expected shortfall of the portfolio loss. In this case, we consider Monte Carlo methods, and we use the method of transformation for simulating copula functions[99]. Since we have $\mathbf{S}_i(\boldsymbol{\tau}_i) \sim \mathcal{U}_{[0,1]}$, the simulation of correlated default times is obtained with the following algorithm:

1. we simulate the random vector $(u_1, \ldots, u_n)$ from the copula function $\mathbf{C}$;
2. we set $\boldsymbol{\tau}_i = \mathbf{S}_i^{-1}(u_i)$.

In many cases, we don't need to simulate the default time $\boldsymbol{\tau}_i$, but the indicator function $D_i(t_i) = \mathbb{1}\{\boldsymbol{\tau}_i \le t_i\}$. Indeed, D_i is a Bernoulli random variable with parameter $\mathbf{F}_i(t) = 1 - \mathbf{S}_i(t)$, implying that $D(t) = (D_1(t_1), \ldots, D_n(t_n))$ is a Bernoulli random vector with parameter $p(t) = (p_1(t_1), \ldots, p_n(t_n))$. Since the copula of $D(t)$ is the copula of the random vector $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$, we obtain the following algorithm to simulate correlated indicator functions:

1. we simulate the random vector $(u_1, \ldots, u_n)$ from the copula function $\mathbf{C}$;
2. we set $D_i(t_i) = 1$ if $u_i > \mathbf{S}_i(t_i)$.

In the case of the Normal copula, the simulation of $u = (u_1, \ldots, u_n)$ requires calculating the Cholesky decomposition of the correlation matrix $\mathbb{C}$. However, this approach is valid for a small size n of the credit portfolio, because we are rapidly limited by the memory storage capacity of the computer. In a 32-bit computer, the storage of a double requires 8 bytes, meaning that the storage of a $n \times n$ Cholesky matrix requires 78.125 KB if $n = 100$, 7.629 MB if $n = 1\,000$, 762.94 MB if $n = 10\,000$, etc. It follows that the traditional Cholesky algorithm is not adapted when considering a large credit portfolio. However, if we consider the Basel model, we can simulate the correlated default times using the following [BASEL] algorithm:

1. we simulate $n+1$ Gaussian independent random variables X and $(\varepsilon_1, \ldots, \varepsilon_n)$;
2. we simulate the Basel copula function:
$$(u_1, \ldots, u_n) = \left(\Phi\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_1\right), \ldots, \Phi\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_n\right)\right) \tag{3.54}$$
3. we set $\boldsymbol{\tau}_i = \mathbf{S}_i^{-1}(u_i)$.

[99] See Section 13.1.3.2 on page 802.

The [BASEL] algorithm is the efficient way to simulate the one-factor model and demonstrates that we don't always need to use the Cholesky decomposition for simulating the Normal (or the Student's t) copula function. Let us generalize the [BASEL] algorithm when we consider the Normal copula with the correlation matrix given by Equation (3.53). The eigendecomposition of $\mathbb{C}$ is equal to $V \Lambda V^\top$, where V is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. Let u be a vector of n independent Gaussian standardized random numbers. Then, $Z = V \Lambda^{1/2} u$ is a Gaussian vector with correlation $\mathbb{C}$. We note $\mathbb{C}^\star = \left(\rho^\star_{j_1,j_2}\right)$ the $m \times m$ correlation matrix based on intra- and inter-sector correlations[100] and we consider the corresponding eigendecomposition $\mathbb{C}^\star = V^\star \Lambda^\star V^{\star\top}$. Let $X^\star$ be a $m \times 1$ Gaussian standardized random vector. It follows that the random vector $Z = (Z_1, \ldots, Z_n)$ is a Gaussian random vector with correlation matrix $\mathbb{C} = \operatorname{map}(\mathbb{C}^\star)$ where[101]:

$$Z_i = \sum_{j=1}^m A^\star_{\operatorname{map}(i),j} X^\star_j + \sqrt{1 - \rho^\star_{\operatorname{map}(i),\operatorname{map}(i)}}\,\varepsilon_i$$

and $A^\star = V^\star (\Lambda^\star)^{1/2}$ and $V^\star$ are the L_2-normalized eigenvectors. The [EIG] algorithm proposed by Jouanin *et al.* (2004) consists then in replacing the second step of the [BASEL] algorithm:

1. we simulate $n + m$ Gaussian independent random variables $(X^\star_1, \ldots, X^\star_m)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$;
2. for the i^{th} credit, we calculate:
$$Z_i = \sum_{j=1}^m A^\star_{\operatorname{map}(i),j} X^\star_j + \sqrt{1 - \rho^\star_{\operatorname{map}(i),\operatorname{map}(i)}}\,\varepsilon_i \tag{3.55}$$
3. we simulate the copula function:
$$(u_1, \ldots, u_n) = (\Phi(Z_1), \ldots, \Phi(Z_n))$$
4. we set $\boldsymbol{\tau}_i = \mathbf{S}_i^{-1}(u_i)$.

Here is a comparison of the efficiency of the [EIG] algorithm with respect to the traditional [CHOL] algorithm:

Algorithm	Matrix dimension	Random numbers	Number of operations +	Number of operations ×
CHOL	$n \times n$	n	$n \times (n-1)$	$n \times n$
EIG	$m \times m$	$n + m$	$n \times (m+1)$	$n \times (m+1)$
		10 000 loans + 20 sectors		
CHOL	10^8	10 000	$\simeq 10^8$	10^8
EIG	400	10 020	2.1×10^5	2.1×10^5

These results explain why the [EIG] algorithm is faster than the [CHOL] algorithm[102]. We also notice that the [EIG] algorithm corresponds to the [BASEL] algorithm in the case $m = 1$ when there is only one common factor.

[100] The diagonal elements correspond to intra-sector correlations, whereas the off-diagonal elements correspond to inter-sector correlations.

[101] Jouanin *et al.* (2004) showed that if the eigenvalues of $\mathbb{C}^\star$ are positive, then $\mathbb{C} = \operatorname{map}(\mathbb{C}^\star)$ is a correlation matrix.

[102] On average, the computational time is divided by a factor of n/m.

Let us consider Example 41. We obtain:

$$A^{\star}=\begin{pmatrix} -0.2633 & 0.1302 & -0.3886 & 0.2504 \\ -0.5771 & -0.1980 & -0.1090 & 0.1258 \\ -0.5536 & 0.0943 & 0.3281 & 0.2774 \\ -0.4897 & 0.0568 & -0.0335 & -0.5965 \end{pmatrix}$$

We deduce that the second step of the [EIG] algorithm is:

- if the credit belongs to the first sector, we simulate Z_i as follows:

$$Z_i=-0.263\cdot X_1^{\star}-0.130\cdot X_2^{\star}+0.389\cdot X_3^{\star}+0.250\cdot X_4^{\star}+0.837\cdot\varepsilon_i$$

- if the credit belongs to the second sector, we simulate Z_i as follows:

$$Z_i=-0.577\cdot X_1^{\star}-0.198\cdot X_2^{\star}-0.109\cdot X_3^{\star}+0.126\cdot X_4^{\star}+0.775\cdot\varepsilon_i$$

- if the credit belongs to the third sector, we simulate Z_i as follows:

$$Z_i=-0.554\cdot X_1^{\star}+0.094\cdot X_2^{\star}+0.328\cdot X_3^{\star}+0.277\cdot X_4^{\star}+0.707\cdot\varepsilon_i$$

- if the credit belongs to the fourth sector, we simulate Z_i as follows:

$$Z_i=-0.490\cdot X_1^{\star}+0.057\cdot X_2^{\star}-0.034\cdot X_3^{\star}-0.597\cdot X_4^{\star}+0.632\cdot\varepsilon_i$$

Remark 47 *The extension to the Student's t copula is straightforward, because the multivariate Student's t distribution is related to the multivariate normal distribution*[103].

3.3.4.2 The factor model

In the previous paragraph, the multivariate survival function writes:

$$\mathbf{S}(t_1,\ldots,t_n)=\mathbf{C}(\mathbf{S}_1(t_1),\ldots,\mathbf{S}_n(t_n);\mathbb{C})$$

where $\mathbf{C}$ is the Normal copula and $\mathbb{C}$ is the matrix of default correlations. In the sector approach, we note $\mathbb{C}=\operatorname{map}(\mathbb{C}^{\star})$ where map is the mapping function and $\mathbb{C}^{\star}$ is the matrix of intra- and inter-correlations. In this model, we characterize the default time by the relationship $\boldsymbol{\tau}_i<t\Leftrightarrow Z_i<B_i(t)$ where $Z_i=\sum_{j=1}^{m}A^{\star}_{\operatorname{map}(i),j}X_j^{\star}+\sqrt{1-\rho^{\star}_{\operatorname{map}(i),\operatorname{map}(i)}}\varepsilon_i$ and $B_i(t)=\Phi^{-1}(\mathrm{PD}_i(t))=\Phi^{-1}(1-\mathbf{S}_i(t))$.

The risk factors $X_j^{\star}$ are not always easy to interpret. If $m=1$, we retrieve $Z_i=\sqrt{\rho}\cdot X+\sqrt{1-\rho}\cdot\varepsilon_i$ where ρ is the uniform correlation and X is the common factor. It generally corresponds to the economic cycle. Let us consider the case $m=2$:

$$\mathbb{C}^{\star}=\begin{pmatrix}\rho_1 & \rho \\ \rho & \rho_2\end{pmatrix}$$

where ρ_1 and ρ_2 are the intra-sector correlations and ρ is the inter-sector correlation. We have:

$$Z_i=A^{\star}_{\operatorname{map}(i),1}\cdot X_1^{\star}+A^{\star}_{\operatorname{map}(i),2}\cdot X_2^{\star}+\sqrt{1-\rho_{\operatorname{map}(i)}}\cdot\varepsilon_i$$

It is better to consider the following factor decomposition:

$$Z_i=\sqrt{\rho}\cdot X+\sqrt{\rho_{\operatorname{map}(i)}-\rho}\cdot X_{\operatorname{map}(i)}+\sqrt{1-\rho_{\operatorname{map}(i)}}\cdot\varepsilon_i \tag{3.56}$$

[103] See pages 737 and 1055.

In this case, we have three factors, and not two factors: X is the common factor, whereas X_1 and X_2 are the two specific sector factors. We can extend the previous approach to a factor model with $m+1$ factors:

$$Z_i = \sqrt{\rho} \cdot X + \sqrt{\rho_{\mathrm{map}(i)} - \rho} \cdot X_{\mathrm{map}(i)} + \sqrt{1 - \rho_{\mathrm{map}(i)}} \cdot \varepsilon_i \tag{3.57}$$

Equations (3.56) and (3.57) are exactly the same, except the number of factors. However, the copula function associated to the factor model described by Equation (3.57) is the copula of the sector model, when we assume that the inter-sector correlation is the same for all the sectors, meaning that the off-diagonal elements of $\mathbb{C}^{\star}$ are equal. In this case, we can use the previous decomposition for simulating the default times. This algorithm called [CISC] (constant inter-sector correlation) requires simulating one additional random number compared to the [EIG] algorithm. However, the number of operations is reduced[104].

Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be two default times, whose joint survival function is $\mathbf{S}(t_1, t_2) = \mathbf{C}(\mathbf{S}_1(t_1), \mathbf{S}_2(t_2))$. We have:

$$\begin{aligned} \mathbf{S}_1(t \mid \boldsymbol{\tau}_2 = t^{\star}) &= \Pr\{\boldsymbol{\tau}_1 > t \mid \boldsymbol{\tau}_2 = t^{\star}\} \\ &= \partial_2 \mathbf{C}(\mathbf{S}_1(t), \mathbf{S}_2(t^{\star})) \\ &= \mathbf{C}_{2|1}(\mathbf{S}_1(t), \mathbf{S}_2(t^{\star})) \end{aligned}$$

where $\mathbf{C}_{2|1}$ is the conditional copula function[105]. If $\mathbf{C} \neq \mathbf{C}^{\perp}$, the default probability of one firm changes when another firm defaults (Schmidt and Ward, 2002). This implies that the credit spread of the first firm jumps at the default time of the second firm. This phenomenon is called spread jump or jump-to-default (JTD). Sometimes it might be difficult to explain the movements of these spread jumps in terms of copula functions. The interpretation is easier when we consider a factor model. For example, we consider the Basel model. Figures 3.41 to 3.45 show the jumps of the hazard function of the S&P one-year transition matrix for corporate bonds given in Table 3.34 on page 208. We recall that the rating $\mathfrak{R}(t) = K$ corresponds to the default state and we note $\mathfrak{R}(t) = i$ the initial rating of the firm. We have seen that $\mathbf{S}_i(t) = 1 - \mathbf{e}_i^{\top} \exp(t\Lambda)\, \mathbf{e}_K$ where Λ is the Markov generator. The hazard function is equal to:

$$\lambda_i(t) = \frac{f_i(t)}{\mathbf{S}_i(t)} = \frac{\mathbf{e}_i^{\top} \Lambda \exp(t\Lambda)\, \mathbf{e}_K}{1 - \mathbf{e}_i^{\top} \exp(t\Lambda)\, \mathbf{e}_K}$$

We deduce that:

$$\lambda_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^{\star}) = \frac{f_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^{\star})}{\mathbf{S}_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^{\star})}$$

With the Basel copula, we have:

$$\mathbf{S}_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^{\star}) = \Phi\left(\frac{\Phi^{-1}(\mathbf{S}_{i_1}(t)) - \rho\Phi^{-1}(\mathbf{S}_{i_2}(t^{\star}))}{\sqrt{1-\rho^2}}\right)$$

and:

$$\begin{aligned} f_{i_1}(t \mid \boldsymbol{\tau}_{i_2} = t^{\star}) &= \phi\left(\frac{\Phi^{-1}(\mathbf{S}_{i_1}(t)) - \rho\Phi^{-1}(\mathbf{S}_{i_2}(t^{\star}))}{\sqrt{1-\rho^2}}\right) \cdot \\ &\quad \frac{f_{i_1}(t)}{\sqrt{1-\rho^2}\,\phi\left(\Phi^{-1}(\mathbf{S}_{i_1}(t))\right)} \end{aligned}$$

[104] For the [EIG] algorithm, we have $n \times (m+1)$ operations ($+$ and $\times$), while we have $3n$ elementary operations for the [CISC] algorithm.

[105] The mathematical analysis of conditional copulas is given on page 737.

The reference to the factor model allows an easier interpretation of the jumps of the hazard rate. For example, it is obvious that the default of a CCC-rated company in ten years implies a negative jump for the well rated companies (Figure 3.45). Indeed, this indicates that the high idiosyncratic risk of the CCC-rated firm has been compensated by a good economic cycle (the common risk factor X). If the default of the CCC-rated company has occurred at an early stage, the jumps were almost zero, because we can think that the default is due to the specific risk of the company. On the contrary, if a AAA-rated company defaults, the jump would be particularly high as the default is sudden, because it is more explained by the common risk factor than by the specific risk factor (Figure 3.42). We deduce that there is a relationship between jump-to-default and default correlation.

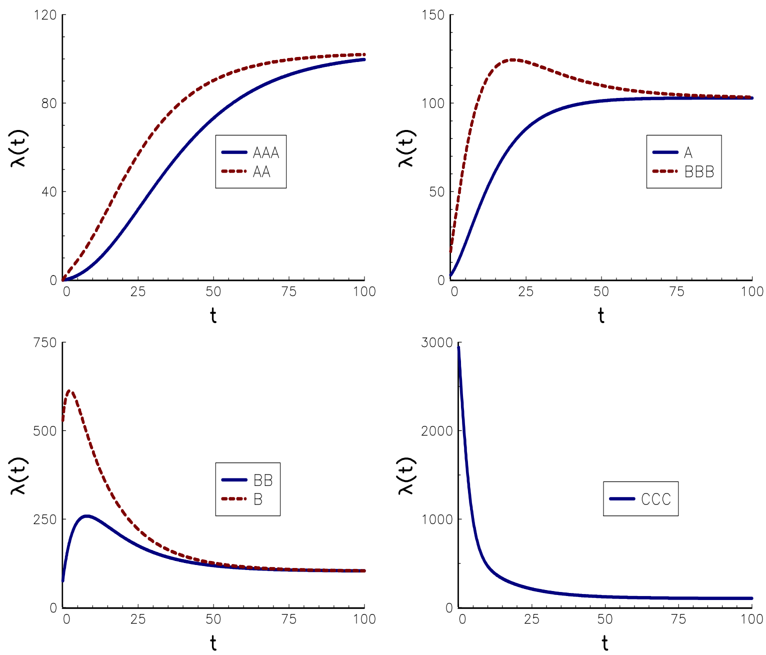

FIGURE 3.41: Hazard function $\lambda_i(t)$ (in bps)

3.3.4.3 Estimation methods

The Normal copula model with sector correlations requires the estimation of the matrix $\mathbb{C}^\star$, which is abusively called the default correlation matrix. In order to clarify this notion, we make the following distinctions:

- the '*canonical or copula correlations*' correspond to the parameter matrix of the copula function that models the dependence between the defaults;
- the '*default time correlations*' are the correlations between the default times $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$; they depend on the copula function, but also on the unidimensional survival functions;
- the '*discrete default correlations*' are the correlations between the indicator functions $(D_1(t), \ldots, D_n(t))$; they depend on the copula function, the unidimensional survival functions and the time horizon t; this is why we don't have a unique default correlation between two firms, but a term structure of default correlations;

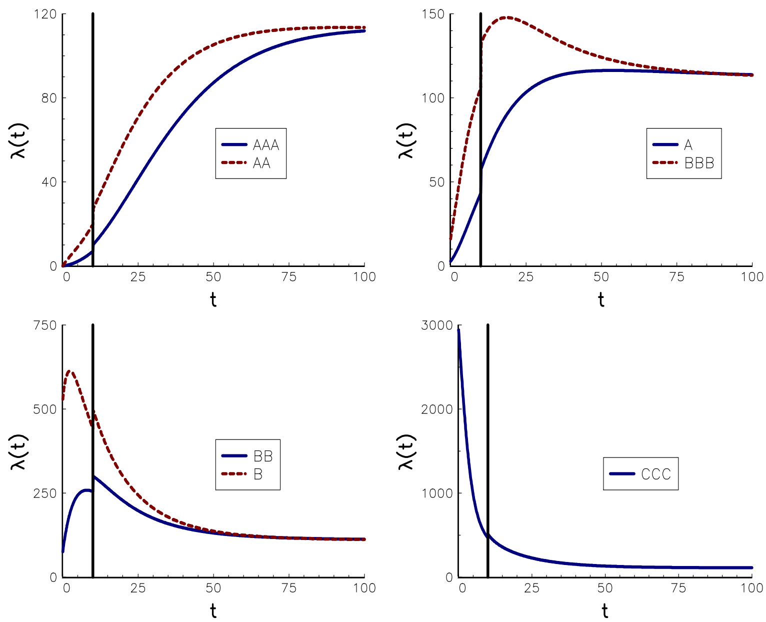

FIGURE 3.42: Hazard function $\lambda_i(t)$ (in bps) when a AAA-rated company defaults after 10 years ($\rho = 5\%$)

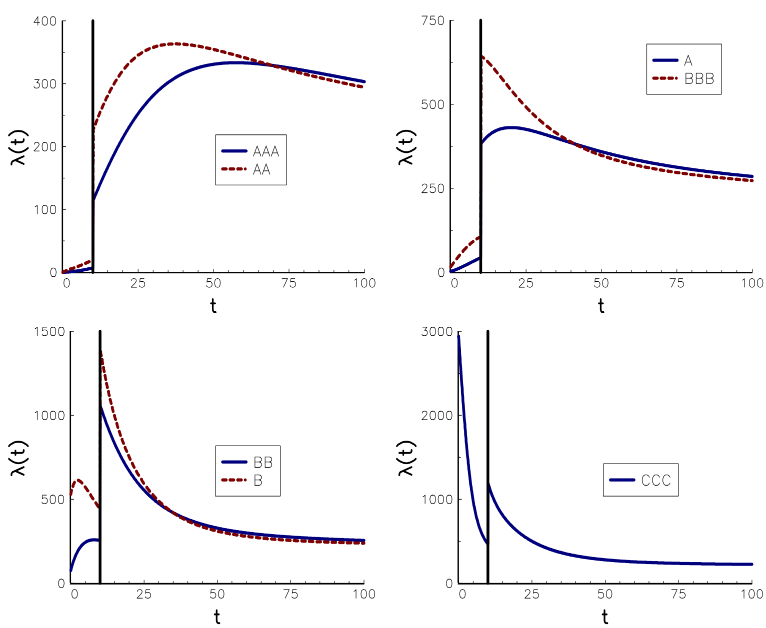

FIGURE 3.43: Hazard function $\lambda_i(t)$ (in bps) when a AAA-rated company defaults after 10 years ($\rho = 50\%$)

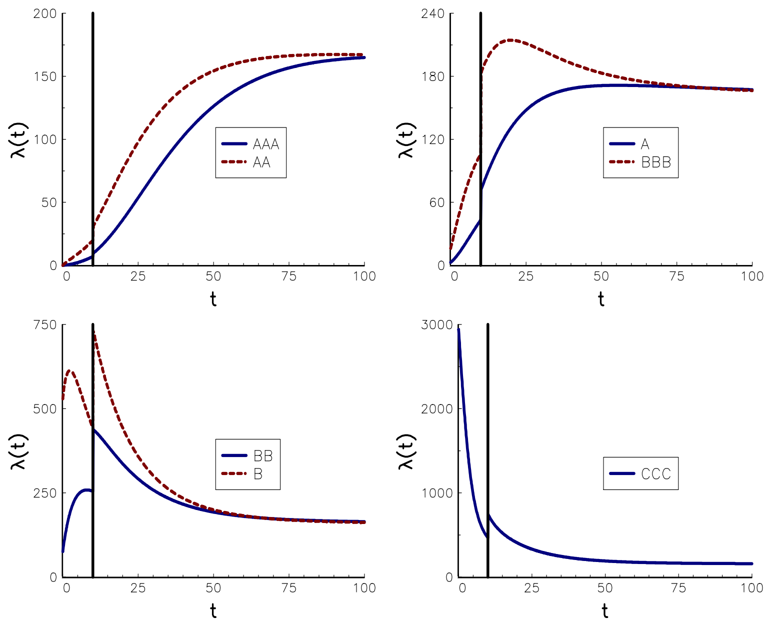

FIGURE 3.44: Hazard function $\lambda_i(t)$ (in bps) when a BB-rated company defaults after 10 years ($\rho = 50\%$)

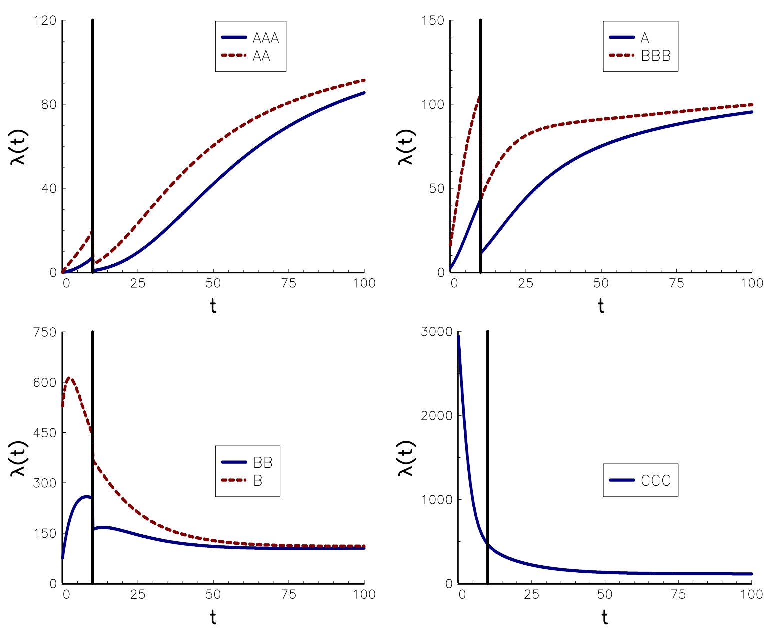

FIGURE 3.45: Hazard function $\lambda_i(t)$ (in bps) when a CCC-rated company defaults after 10 years ($\rho = 50\%$)

- the '*asset correlations*' are the correlations between the asset values in the Merton model;
- the '*equity correlations*' are the correlations between the stock returns; in a Merton-like model, they are assumed to be equal to the asset correlations.

In practice, the term '*default correlation*' is used as a generic term for these different measures.

Relationship between the different default correlations We consider two firms. Li (2000) introduces two measures of default correlation. The discrete default correlation is equal to:

$$\rho(t_1, t_2) = \frac{\mathbb{E}[D_1(t_1) D_2(t_2)] - \mathbb{E}[D_1(t_1)]\mathbb{E}[D_2(t_2)]}{\sigma(D_1(t_1))\,\sigma(D_2(t_2))}$$

where $D_i(t_i) = \mathbb{1}\{\boldsymbol{\tau}_i \leq t_i\}$, whereas the default (or survival) time correlation is equal to:

$$\rho(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \frac{\mathbb{E}[\boldsymbol{\tau}_1 \boldsymbol{\tau}_2] - \mathbb{E}[\boldsymbol{\tau}_1]\mathbb{E}[\boldsymbol{\tau}_2]}{\sigma(\boldsymbol{\tau}_1)\,\sigma(\boldsymbol{\tau}_2)}$$

These two measures give very different numerical results. Concerning the asset correlation, it is equal to:

$$\rho(Z_1, Z_2) = \frac{\mathbb{E}[Z_1 Z_2] - \mathbb{E}[Z_1]\mathbb{E}[Z_2]}{\sigma(Z_1)\,\sigma(Z_2)}$$

These three measures depend on the canonical correlation. Let us denote by ρ the copula parameter of the Normal copula between the two default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. We have:

$$\rho(t_1, t_2) = \frac{\mathbf{C}(\mathrm{PD}_1(t_1), \mathrm{PD}_2(t_2); \rho) - \mathrm{PD}_1(t_1) \cdot \mathrm{PD}_2(t_2)}{\sqrt{\mathrm{PD}_1(t_1)(1 - \mathrm{PD}_1(t_1))} \cdot \sqrt{\mathrm{PD}_2(t_2)(1 - \mathrm{PD}_2(t_2))}}$$

and:

$$\rho(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \frac{\operatorname{cov}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)}{\sqrt{\operatorname{var}(\boldsymbol{\tau}_1) \cdot \operatorname{var}(\boldsymbol{\tau}_1)}}$$

where $\operatorname{cov}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = \int_0^\infty \int_0^\infty (\mathbf{S}(t_1, t_2) - \mathbf{S}_1(t_1)\mathbf{S}_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$ and $\operatorname{var}(\boldsymbol{\tau}_i) = 2\int_0^\infty t\mathbf{S}_i(t)\,\mathrm{d}t - \left[\int_0^\infty \mathbf{S}_i(t)\,\mathrm{d}t\right]^2$. We verify that $\rho(t_1, t_2) \neq \rho$ and $\rho(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) \neq \rho$. We can also show that $\rho(t_1, t_2) < \rho$ and $\rho(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) < \rho$ for the Normal copula. In the Basel model, we have $\rho(Z_1, Z_2) = \rho$.

We consider two exponential default times $\boldsymbol{\tau}_1 \sim \mathcal{E}(\lambda_1)$ and $\boldsymbol{\tau}_2 \sim \mathcal{E}(\lambda_2)$. In Tables 3.42, 3.43 and 3.44, we report the discrete default correlations $\rho(t_1, t_2)$ for different time horizons. We notice that $\rho(t_1, t_2)$ is much lower than 20%,, which is the copula correlation. We have also calculated $\rho(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$, which is respectively equal to 17.0%, 21.5% and 18.0%. We notice that the correlations are higher for the Student's t copula than for the Normal copula[106].

Statistical inference of the default correlation In the case of a factor model, we have:

$$\tilde{Z}_{i,t} = \beta^\top \tilde{X}_{i,t} + \sqrt{1 - \|\beta\|_2^2} \cdot \tilde{\varepsilon}_{i,t}$$

where $\tilde{Z}_{i,t}$ is the standardized asset value of the i^{th} firm at time t and $\tilde{X}_{i,t}$ is the standardized vector of risk factors at time t for the i^{th} firm. We can then estimate the parameter β using OLS or GMM techniques. Let us consider the constant inter-sector correlation model:

$$Z_i = \sqrt{\rho} \cdot X + \sqrt{\rho_{\mathrm{map}(i)} - \rho} \cdot X_{\mathrm{map}(i)} + \sqrt{1 - \rho_{\mathrm{map}(i)}} \cdot \varepsilon_i$$

[106] This phenomenon is explained in the chapter dedicated to the copula functions.

TABLE 3.42: Discrete default correlation in % ($\lambda_1 = 100$ bps, $\lambda_2 = 50$ bps, Normal copula with $\rho = 20\%$)

t_1 / t_2	1	2	3	4	5	10	25	50
1	2.0	2.4	2.7	2.9	3.1	3.6	4.2	4.5
2	2.3	2.9	3.3	3.6	3.8	4.5	5.3	5.7
3	2.6	3.2	3.6	4.0	4.2	5.0	6.0	6.5
4	2.7	3.4	3.9	4.2	4.5	5.4	6.5	7.1
5	2.9	3.6	4.1	4.5	4.8	5.7	6.9	7.5
10	3.2	4.1	4.7	5.1	5.5	6.6	8.2	9.1
25	3.4	4.5	5.1	5.7	6.1	7.5	9.6	10.9
50	3.3	4.4	5.1	5.6	6.1	7.6	9.9	11.5

TABLE 3.43: Discrete default correlation in % ($\lambda_1 = 100$ bps, $\lambda_2 = 50$ bps, Student's t copula with $\rho = 20\%$ and $\nu = 4$)

t_1 / t_2	1	2	3	4	5	10	25	50
1	13.9	14.5	14.5	14.3	14.0	12.6	9.8	7.2
2	12.8	14.3	14.8	14.9	14.9	14.3	11.9	9.2
3	11.9	13.7	14.5	14.9	15.1	15.0	13.1	10.4
4	11.2	13.1	14.1	14.6	14.9	15.3	13.8	11.3
5	10.6	12.6	13.7	14.3	14.7	15.4	14.3	11.9
10	8.5	10.5	11.8	12.6	13.3	14.8	15.2	13.6
25	5.5	7.2	8.3	9.2	9.9	11.9	14.0	14.3
50	3.3	4.5	5.3	5.9	6.5	8.3	11.0	12.6

TABLE 3.44: Discrete default correlation in % ($\lambda_1 = 20\%$, $\lambda_2 = 10\%$, Normal copula with $\rho = 20\%$)

t_1 / t_2	1	2	3	4	5	10	25	50
1	8.8	10.2	10.7	11.0	11.1	10.4	6.6	2.4
2	9.4	11.0	11.8	12.1	12.3	11.9	7.9	3.1
3	9.3	11.0	11.9	12.4	12.7	12.5	8.6	3.4
4	9.0	10.8	11.7	12.2	12.6	12.6	8.9	3.7
5	8.6	10.4	11.3	11.9	12.3	12.4	9.0	3.8
10	6.3	7.8	8.7	9.3	9.7	10.3	8.1	3.7
25	1.9	2.4	2.8	3.1	3.3	3.8	3.5	1.9
50	0.2	0.3	0.3	0.3	0.4	0.5	0.5	0.3

The corresponding linear regression is:

$$\tilde{Z}_{i,t} = \beta_0 \cdot \tilde{X}_{0,t} + \beta^\top \tilde{X}_{i,t} + \sqrt{1 - \rho_{\mathrm{map}(i)}} \cdot \tilde{\varepsilon}_{i,t}$$

where $\tilde{X}_{i,t}$ is equal to $e_i \odot \tilde{X}_t$, $\tilde{X}_t$ is the set of the risk factors, which are specific to the sectors at time t and $\tilde{X}_{0,t}$ is the common risk factor. We deduce that the estimation of ρ and $\rho_1, \ldots, \rho_m$ are given by the following relationships: $\hat{\rho} = \hat{\beta}_0^2$ and $\hat{\rho}_j = \hat{\beta}_0^2 + \hat{\beta}_j^2$.

A second approach is to consider the correlation between the default rates of homogeneous cohorts[107]. This correlation converges asymptotically to the survival time correlation. Then, we have to inverse the relationship between the survival time correlation and the copula correlation for estimating the parameters of the copula function.

The third approach has been suggested by Gordy and Heitfield (2002). They consider the Basel model: $Z_i = \sqrt{\rho} \cdot X + \sqrt{1-\rho} \cdot \varepsilon_i$, where $X \sim \mathbf{H}$ and $\varepsilon_i \sim \mathcal{N}(0,1)$. The default probability conditionally to $X = x$ is equal to:

$$p_i(x; B_i, \rho) = \Phi\left(\frac{B_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)$$

We note d_t the number of defaulted firms and n_t the total number of firms at time t. If we have a historical sample of default rates, we can estimate ρ using the method of maximum likelihood. Let $\ell_t(\theta)$ be the log-likelihood of the observation t. If we assume that there is only one risk class $\mathcal{C}$ ($B_i = B$), the conditional number of defaults $\mathcal{D}$ is a binomial random variable:

$$\Pr\{\mathcal{D} = d_t \mid X = x\} = \binom{n_t}{d_t} p(x; B, \rho)^{d_t} (1 - p(x; B, \rho))^{n_t - d_t}$$

We deduce that:

$$\begin{aligned} \ell_t(\theta) &= \ln \int \Pr\{\mathcal{D} = d_t \mid X = x\} \, \mathrm{d}\mathbf{H}(x) \\ &= \ln \int \binom{n_t}{d_t} p(x; B, \rho)^{d_t} (1 - p(x; B, \rho))^{n_t - d_t} \, \mathrm{d}\mathbf{H}(x) \end{aligned}$$

Generally, we consider a one-year time horizon for calculating default rates. Moreover, if we assume that the common factor X is Gaussian, we deduce that $B = \Phi^{-1}(\mathrm{PD})$ where PD is the one-year default probability for the risk class $\mathcal{C}$. It follows that:

$$\ell_t(\theta) = \ln \int \binom{n_t}{d_t} p\left(x; \Phi^{-1}(\mathrm{PD}), \rho\right)^{d_t} \left(1 - p\left(x; \Phi^{-1}(\mathrm{PD}), \rho\right)\right)^{n_t - d_t} \, \mathrm{d}\Phi(x)$$

Therefore, we can estimate the parameter ρ. If there are several risk classes, we can assume that:

$$\ell_t(\theta) = \ln \int \binom{n_t}{d_t} p(x; B, \rho)^{d_t} (1 - p(x; B, \rho))^{n_t - d_t} \, \mathrm{d}\Phi(x)$$

In this case, we have two parameters to estimate: the copula correlation ρ and the implied default barrier B.

The underlying idea of this approach is that the distribution of the default rate depends on the default probability and the copula correlation. More specifically, the mean of the default rate of a risk class $\mathcal{C}$ is equal to the default probability of $\mathcal{C}$ whereas the volatility of the default rate is related to the default correlation. We introduce the notation:

$$f_t = \frac{d_t}{n_t}$$

[107] Each cohort corresponds to a risk class.

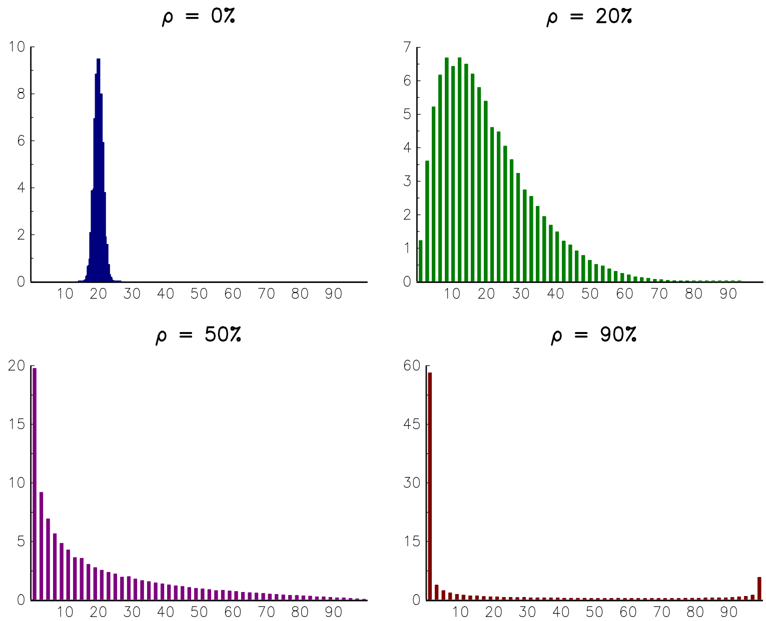

FIGURE 3.46: Distribution of the default rate (in %)

where f_t is the default rate at time t. We assume that the one-year default probability of $\mathcal{C}$ is equal to 20%. In Figure 3.46, we report the distribution of the one-year default rate for different values of ρ when the number of firms n_t is equal to 1 000. We also report some statistics (mean, standard deviation and quantile functions) in Table 3.45. By definition, the four probability distributions have the same mean, which is equal to 20%, but their standard deviations are different. If $\rho = 0\%$, $\sigma(f_t)$ is equal to 1.3% while $\sigma(f_t) = 33.2\%$ in the case $\rho = 90\%$.

TABLE 3.45: Statistics of the default rate (in %))

ρ	$\mu(f_t)$	$\sigma(f_t)$	$Q_\alpha(f_t)$ 1%	10%	25%	50%	75%	90%	99%
0%	20.0	1.3	17.1	18.4	19.1	20.0	20.8	21.6	23.0
20%	20.0	13.0	1.7	5.6	10.0	17.4	27.3	38.3	59.0
50%	20.0	21.7	0.0	0.6	3.1	11.7	30.3	53.8	87.3
90%	20.0	33.2	0.0	0.0	0.0	0.4	26.3	88.2	100.0

Example 42 *We consider a risk class $\mathcal{C}$, whose probability of default is equal to* 200 *bps. Over the last 20 years, we have observed the following annual number of defaults:* 3, 1, 14, 0, 33, 3, 53, 1, 4, 0, 1, 8, 7, 3, 5, 5, 0, 49, 0 *and* 7*. We assume that the number of firms is equal to* 500 *every year.*

If we estimate the Basel model with the method of maximum likelihood by assuming that $B = \Phi^{-1}(\mathrm{PD})$, we obtain $\hat{\rho} = 28.93\%$. If we estimate both the default correlation and the default barrier, we have $\hat{\rho} = 28.58\%$ and $\hat{B} = -2.063$, which is equivalent to a default probability of 195 bps. It is better to estimate the barrier if we don't trust the default probability of the risk class because the estimation can be biased. For instance, if we assume that PD = 100 bps, we obtain $\hat{\rho} = 21.82\%$, which is relatively lower than the previous estimate.

The previous estimation method has been generalized by Demey *et al.* (2004) to the CISC model with several intra-sector correlations, but a unique inter-sector correlation. In Table 3.46, we report their results for the period between 1981 and 2002. We notice that the default correlations are relatively low between 7% and 36%. The largest correlations are observed for the sectors of energy, finance, real estate, telecom and utilities. We also notice some significant differences between the Basel model and the CISC model.

TABLE 3.46: Estimation of canonical default correlations

Sector	CISC model	Basel model
Aerospace/Automobile	11.2%	11.6%
Consumer/Service sector	8.7%	7.5%
Energy/Natural resources	21.3%	11.5%
Financial institutions	15.7%	12.2%
Forest/Building products	6.8%	14.5%
Health	8.3%	9.2%
High technology	6.8%	4.7%
Insurance	12.2%	7.6%
Leisure time/Media	7.0%	7.0%
Real estate	35.9%	27.7%
Telecom	27.1%	34.3%
Transportation	6.8%	8.3%
Utilities	18.3%	21.2%
Inter-sector	6.8%	✓

Source: Demey *et al.* (2004).

Remark 48 *There are very few publications on the default correlations. Moreover, they generally concern the one-year discrete default correlations $\rho(1,1)$, not the copula correlation. For example, Nagpal and Bahar (2001) estimate $\rho(t_1,t_2)$ for US corporates and the period 1981-1999. They distinguish the different sectors, three time horizons (1Y, 5Y and 7Y) and IG/HY credit ratings. Even if the range goes from -5.35% to 39.35%, they obtain a very low correlation on average. However, these results should be taken with caution, because we know that the default correlation has increased since the 2008 Global Financial Crisis (Christoffersen* et al.*, 2017).*

3.3.4.4 Dependence and credit basket derivatives

Interpretation and pitfalls of the Basel copula The Basel copula is the basic model for pricing CDO tranches, just as the Black-Scholes model is for options. We define the implied correlation as the parameter ρ that gives the market spread of the CDO tranche. In some sense, the implied correlation for CDOs is the equivalent of the implied volatility for options. Since the implied correlation depends on attachment and detachment points of the CDO tranche, we don't have a single value, but a curve which is not flat. Therefore, we observe a correlation smile or skew, meaning that the correlation is not constant.

In order to understand this phenomenon, we come back to the economic interpretation of the Basel model. In Figure 3.47, we report the mapping between the economic cycle and the common risk factor X. In this case, negative values of X correspond to bad economic times whereas positive values of X correspond to good economic times. We notice that the factor model does not encompass the dynamics of the economic cycle. The Basel model is typically a through-the-cycle approach, and not a point-in-time approach, meaning that the time horizon is the long-run (typically an economic cycle of 7 years).

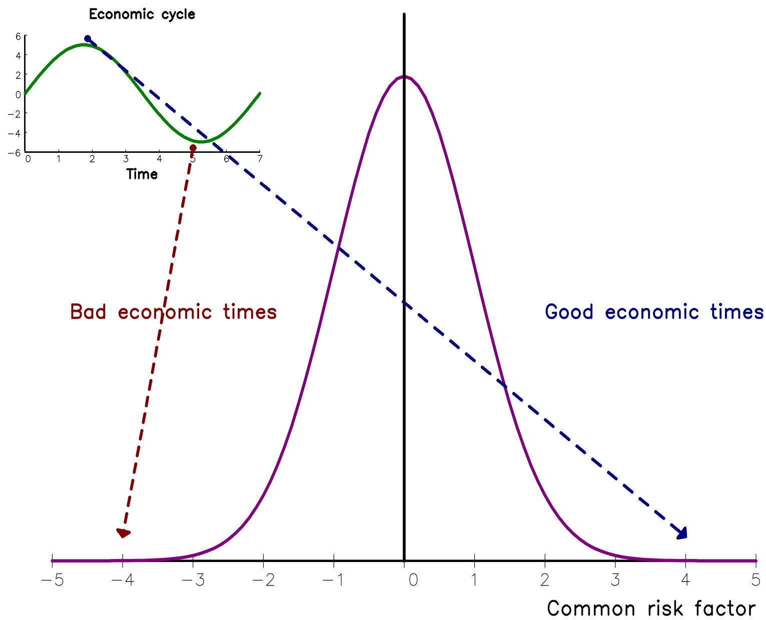

FIGURE 3.47: Economic interpretation of the common factor X

We recall that the loss function is $L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$. Let A and D be the attachment and detachment points of the tranche. We have:

$$\mathbb{E}\left[L \mid A \leq L < D\right] = \mathbb{E}_X\left[L\left(X\right) \mid A \leq L < D\right]$$

where $L\left(X\right)$ is the conditional loss with respect to the common factor X. With this model, the pricing of a CDO tranche uses all the economic scenarios, which are equally weighted. In practice, we know that market participants are more sensitive to bad economic times and have a shorter time horizon than the duration of an economic cycle. From a mathematical point of view, this implies that the factor component $\sqrt{\rho}X$ is certainly not Gaussian and symmetric about 0. Two directions have then been investigated in order to introduce skewness in credit risk modeling. The first approach assumes that the copula correlation ρ is not constant but stochastic, while the second approach states that the copula correlation is a function of the common factor X. These two approaches are two visions of the link between default correlations and the economic cycle.

Stochastic correlation model We consider an extension of the Basel model:

$$Z_i = \sqrt{R_i}X + \sqrt{1 - R_i}\,\varepsilon_i$$

where $R_i \in [0, 1]$ is a random variable that represents the stochastic correlation (Andersen and Sidenius, 2005). We notice that the conditional process $Z_i \mid R_i = \rho$ remains Gaussian, whereas the conditional probability of default becomes:

$$p_i\left(X\right) = \int_0^1 \Phi\left(\frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \mathrm{d}\mathbf{G}\left(\rho\right)$$

where $\mathbf{G}$ is the probability distribution of R_i. Burtschell *et al.* (2007) propose to model the stochastic correlation R_i as a binary random variable:

$$R_i = (1 - Y_i)\sqrt{\rho_1} + Y_i\sqrt{\rho_2}$$

where Y_i is a Bernoulli random variable $\mathcal{B}(p)$. For example, if $p = 5\%$, $\rho_1 = 0\%$ and $\rho_2 = 100\%$, the defaults are uncorrelated most of the time and perfectly correlated in 5% of cases. The copula of default times is then a mixture of the copula functions $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ as shown in Figure 3.48. From an economic point of view, we obtain a two-regime model.

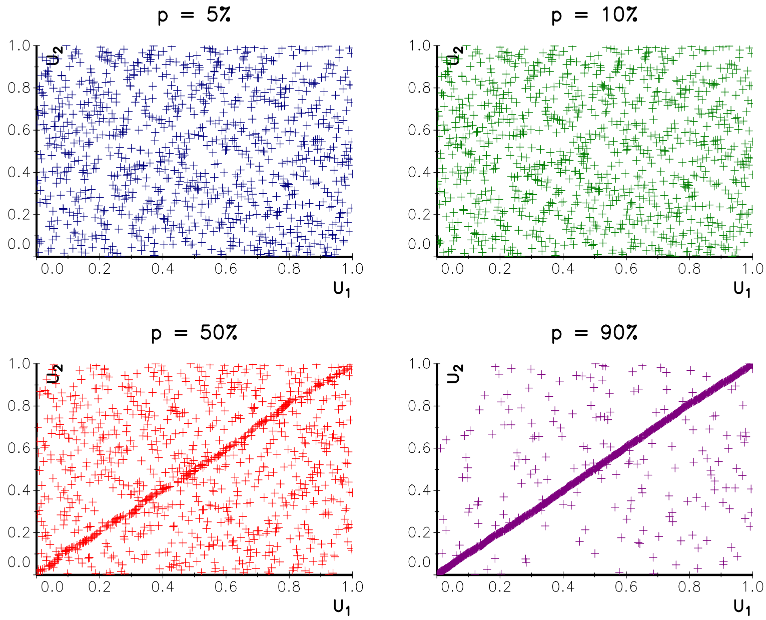

FIGURE 3.48: Dependogram of default times in the stochastic correlation model

Local correlation model In this model, we have:

$$Z_i = \beta(X)X + \sqrt{1 - \|\beta(X)\|_2^2}\,\varepsilon_i$$

where the factor loading $\beta(X)$ is a function of the factor X, meaning that $\beta(X)$ depends on the position in the economic cycle. In Figure 3.49, we consider two functions: $\beta_0(X)$ is constant (Basel model) and $\beta_1(X)$ decreases with the common factor. In this last case, the factor loading is high in bad economic times, meaning that the default correlation $\rho = \beta^2(X)$ is larger at the bottom of the economic cycle than at its top. This implies that the latent variable Z_i is not Gaussian and exhibits a skewness and an excess kurtosis. We verify this property on the normalized probability density function of the factor component $\beta(X)X$ (bottom/right panel in Figure 3.49). This specification has an impact of the joint distribution of defaults. For example, we report the empirical copula of default times in Figure 3.50 when the factor loading is $\beta_1(X)$. We notice that this copula function is not symmetric and the joint dependence of defaults is very high in bad economic times when the value of X is low. When $\beta(X)$ is a decreasing function of X, we observe a correlation skew. It is equivalent to change the probability measure in order to penalize the bad states of the economic cycle or to introduce a risk premium due to the misalignment between the time horizon of investors and the duration of the economic cycle.

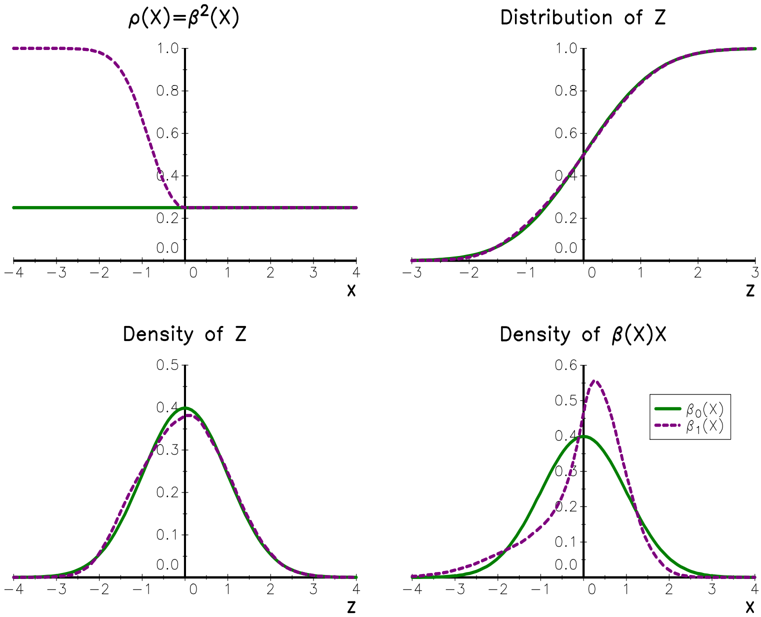

FIGURE 3.49: Distribution of the latent variable Z in the local correlation model

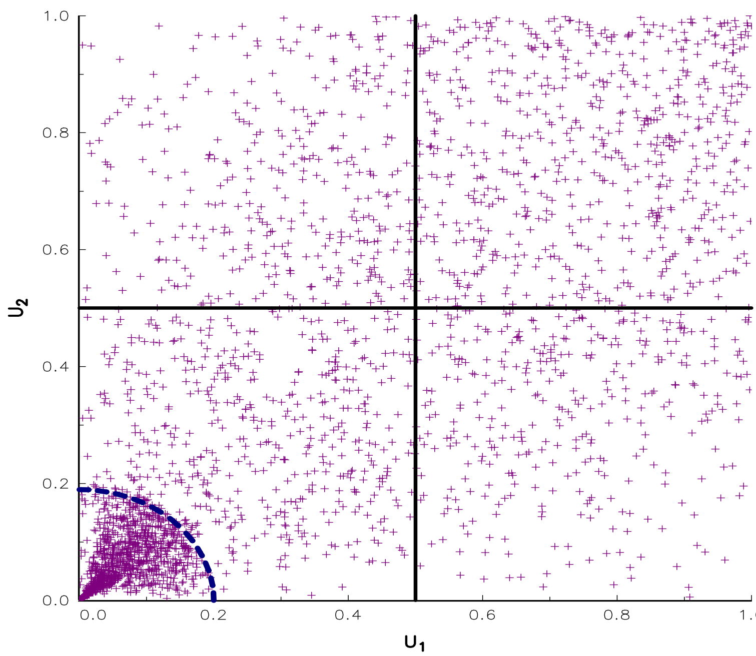

FIGURE 3.50: Dependogram of default times in the local correlation model

To implement this model, we consider the normalization $Z_i^\star = \sigma_Z^{-1}(Z_i - m_Z)$ where:

$$m_Z = \mathbb{E}[Z_i] = \int_{-\infty}^{+\infty} \beta(x)\, x\phi(x)\, \mathrm{d}x$$

and:

$$\sigma_Z^2 = \operatorname{var}(Z_i) = \int_{-\infty}^{+\infty} \left(1 - \beta^2(x) + \beta^2(x)\, x^2\right)\phi(x)\, \mathrm{d}x - m_Z^2$$

We notice that the probability distribution of the latent variable $Z_i^\star$ is equal to:

$$\begin{aligned}
\mathbf{F}^\star(z) &= \Pr\{Z_i^\star \leq z\} \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{m_z + \sigma_Z z - \beta(x)\, x}{\sqrt{1 - \|\beta(x)\|_2^2}}\right)\phi(x)\, \mathrm{d}x
\end{aligned}$$

To simulate correlated defaults[108], we use the inversion method such that $U_i = \mathbf{F}^\star(Z_i)$.

We consider the following parametrization:

$$\beta(x) = \begin{cases} 1 - \left(1 - \sqrt{\rho}\right) e^{-\frac{1}{2}\alpha x^2} & \text{if } x < 0 \\ \sqrt{\rho} & \text{if } x \geq 0 \end{cases}$$

The function $\beta(x)$ depends on two parameters ρ and α. The local correlation $\rho(x) = \beta^2(x)$ is given in Figure 3.51. The parameter ρ represents the default correlation when the economic cycle is good or the common factor X is positive. We also notice that the local correlation $\rho(x)$ tends to 1 when x tends to $-\infty$. This implies an absolute contagion of the default times when the economic situation is dramatic. The parameter α is then a measure of the contagion intensity when the economic cycle is unfavorable. Figure 3.52 shows the base correlation[109] which are generated by this model[110]. We observe that these concave skews are coherent with respect to those observed in the market.

In Figure 3.53, we report the base correlation of the 5Y European iTraxx index at the date of 14 June 2005. The estimation of the local correlation model gives $\rho = 0.5\%$ and $\alpha = 60\%$. We notice that the calibrated model fits well the correlation skew of the market. Moreover, the calibrated model implies an asymmetric distribution and a left fat tail of the factor component (top/right panel in Figure 3.54) and an implied economic cycle, which is more flattened than the economic cycle derived from a Gaussian distribution. In particular, we observe small differences within good economic times and large differences within bad economic times. If we consider the copula function, we find that defaults are generally weakly correlated except during deep economic crisis. Let us consider the ordinal sum of the two copula functions $\mathbf{C}^\perp$ and $\mathbf{C}^+$. This copula is represented in Figure 3.55. The 10% worst economic scenarios correspond to the perfect dependence (copula $\mathbf{C}^+$) whereas the remaining 90% economic scenarios correspond to the zero-correlation situation (copula $\mathbf{C}^\perp$). We notice that this copula function fits very well the correlation skew. We conclude that market participants underestimate default correlations in good times and overestimate default correlations in bad times.

[108] We calculate m_Z, σ_Z and $\mathbf{F}^\star(z)$. For $\mathbf{F}^\star(z)$, we consider a meshgrid (z_k). When $z \in (z_k, z_{k+1})$, we use the linear or the Gaussian interpolation.

[109] The base correlation is the implied correlation of an equity tranche, where the attachment point is equal to 0 and the detachment point is equal to the strike.

[110] We consider a CDO with a five-year maturity. The coupons are paid every quarter. The portfolio of underlying assets is homogenous with a spread of 100 bps and a recovery rate of 40%. The pricing is done with the method of Monte Carlo and one million simulations

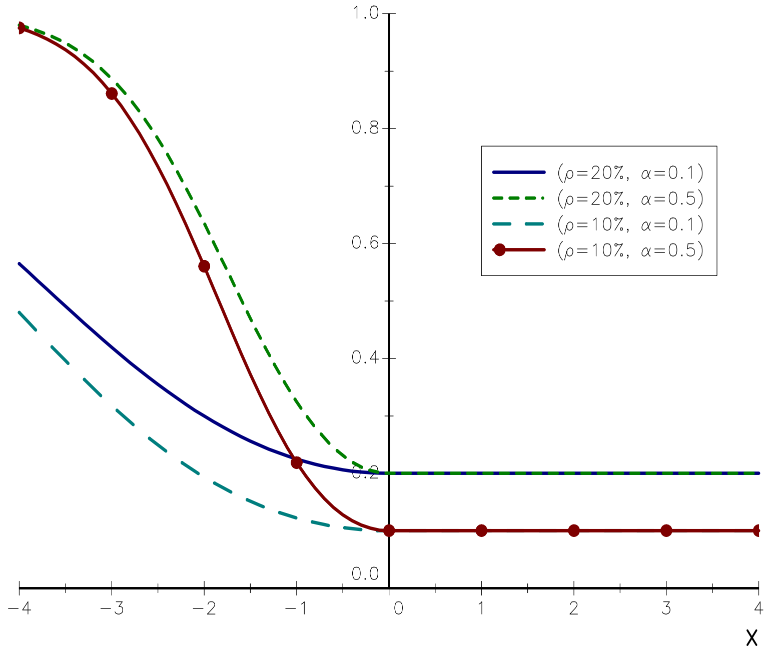

FIGURE 3.51: Local correlation $\rho(x) = \beta^2(x)$

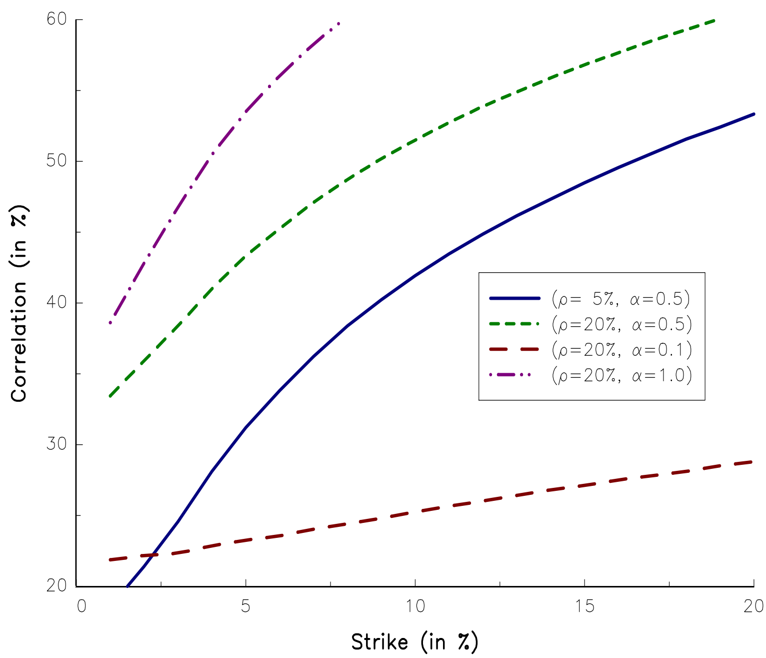

FIGURE 3.52: Correlation skew generated by the local correlation model

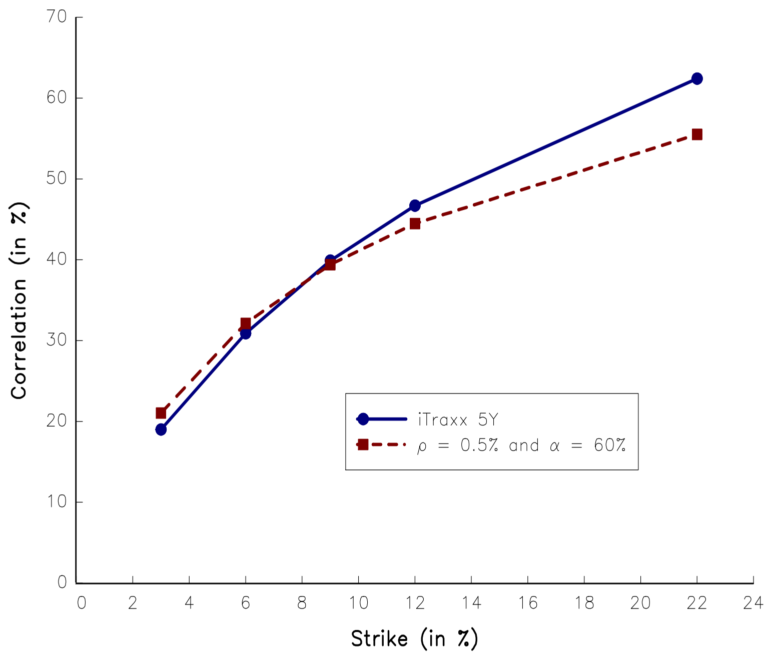

FIGURE 3.53: Calibration of the correlation skew (local correlation model)

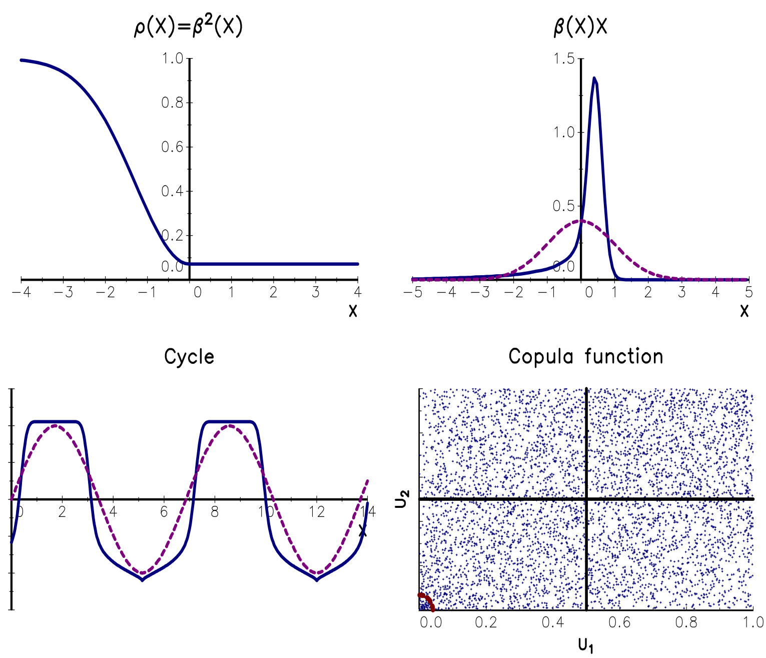

FIGURE 3.54: Implied local correlation model

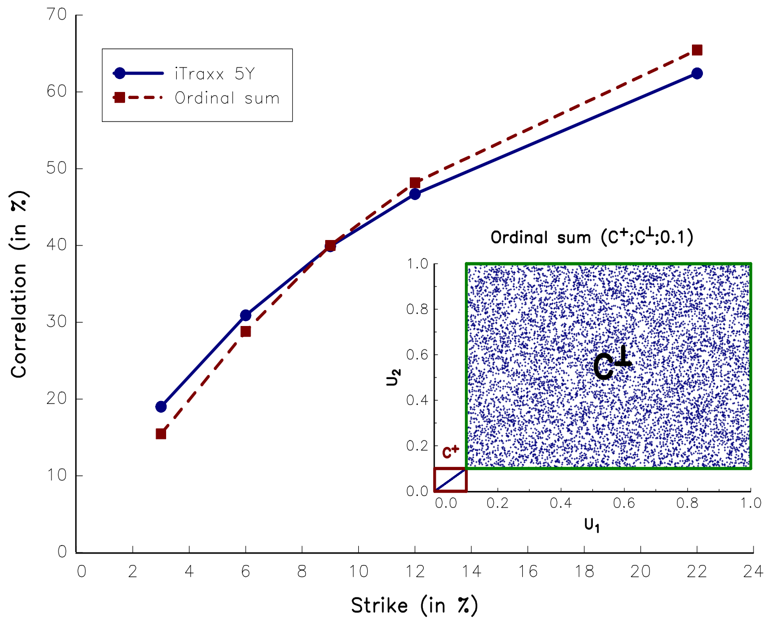

FIGURE 3.55: Calibration of the correlation skew (ordinal sum of $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$)

3.3.5 Granularity and concentration

The risk contribution of the Basel model has been obtained under the assumption that the portfolio is infinitely fine-grained. In this case, the common risk factor X largely dominates the specific risk factors ε_i. When the portfolio is concentrated in a few number of credits, the risk contribution formula, which has been derived on page 173, is not valid. In this case, the Basel regulation implies to calculate an additional capital. In the second consultative paper on the Basel II Accord (BCBS, 2001a), the Basel Committee suggested to complement the IRB-based capital by a '*granularity adjustment*' that captures the risk of concentration. Finally, the Basel Committee has abandoned the idea to calculate the additional capital in the first pillar. In fact, this granularity adjustment is today treated in the second pillar, and falls under the internal capital adequacy assessment process (ICAAP).

3.3.5.1 Difference between fine-grained and concentrated portfolios

Definition of the granularity adjustment We recall that the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\} \tag{3.58}$$

Under the assumption that the portfolio is infinitely fine-grained (IFG), we have shown that the one-year value-at-risk is given by[111]:

$$\mathrm{VaR}_{\alpha}\left(w_{\mathrm{IFG}}\right) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot \Phi\left(\frac{\Phi^{-1}\left(\mathrm{PD}_i\right) + \sqrt{\rho}\Phi^{-1}\left(\mathrm{PD}_i\right)}{\sqrt{1-\rho}}\right) \tag{3.59}$$

However, this assumption does not always hold, and the portfolio w cannot be fine-grained and present some concentration issues. In this case, the one-year value-at-risk is equal to

[111]Without any loss of generality, we assume that $T_i = 1$ in the sequel.

the quantile α of the loss distribution:

$$\mathrm{VaR}_\alpha(w) = \mathbf{F}_L^{-1}(\alpha)$$

The granularity adjustment GA is the difference between the two risk measures. In the case of the VaR and UL credit risk measures, we obtain:

$$\mathrm{GA} = \mathrm{VaR}_\alpha(w) - \mathrm{VaR}_\alpha(w_{\mathrm{IFG}})$$

In most cases, we expect that the granularity adjustment is positive, meaning that the IRB-based capital underestimates the credit risk of the portfolio.

The case of a perfectly concentrated portfolio Let us consider a portfolio that is composed of one credit. We have:

$$L = \mathrm{EAD}\cdot\mathrm{LGD}\cdot\mathbb{1}\{\boldsymbol{\tau} \leq T\}$$

Let $\mathbf{G}$ be the distribution function of the loss given default. It follows that:

$$\mathbf{F}_L(\ell) = \Pr\{\mathrm{EAD}\cdot\mathrm{LGD}\cdot\mathbb{1}\{\boldsymbol{\tau} \leq T\} \leq \ell\}$$

Since we have $\ell = 0 \Leftrightarrow \boldsymbol{\tau} > T$, we deduce that $\mathbf{F}_L(0) = \Pr\{\boldsymbol{\tau} > T\} = 1 - \mathrm{PD}$. If $\ell \neq 0$, this implies that the default has occurred and we have:

$$\begin{aligned} \mathbf{F}_L(\ell) &= \mathbf{F}_L(0) + \Pr\{\mathrm{EAD}\cdot\mathrm{LGD} \leq \ell \mid \boldsymbol{\tau} \leq T\} \\ &= (1-\mathrm{PD}) + \mathrm{PD}\cdot\mathbf{G}\left(\frac{\ell}{\mathrm{EAD}}\right) \end{aligned}$$

The value-at-risk of this portfolio is then equal to:

$$\mathrm{VaR}_\alpha(w) = \begin{cases} \mathrm{EAD}\cdot\mathbf{G}^{-1}\left(\dfrac{\alpha + \mathrm{PD} - 1}{\mathrm{PD}}\right) & \text{if } \alpha \geq 1 - \mathrm{PD} \\ 0 & \text{otherwise} \end{cases}$$

In figure 3.56, we consider an illustration when the exposure at default is equal to one. The first panel compares the value-at-risk $\mathrm{VaR}_\alpha(w)$ when $\mathrm{LGD} \sim \mathcal{U}[0,1]$ and $\mathrm{LGD} = 50\%$. Except for low default probabilities, $\mathrm{VaR}_\alpha(w)$ is larger when the loss given default is stochastic than when the loss given default is set to the mean $\mathbb{E}[\mathrm{LGD}]$. The next panels also shows that the IRB value-at-risk $\mathrm{VaR}_\alpha(w_{\mathrm{IFG}})$ underestimates the true value-at-risk $\mathrm{VaR}_\alpha(w)$ when PD is high. We conclude that the granularity adjustment depends on two main factors:

- the discrepancy between LGD and its expectation $\mathbb{E}[\mathrm{LGD}]$;
- the specific risk that can increase or decrease[112] the credit risk of the portfolio.

The diversification effect and the default correlation We also notice that the granularity adjustment is equal to zero when the default correlation tends to one:

$$\lim_{\rho\to 1} \mathrm{VaR}_\alpha(w) = \mathrm{VaR}_\alpha(w_{\mathrm{IFG}})$$

[112] For instance, the true value-at-risk can be lower than the sum of IRB contributions for well-rated portfolios.

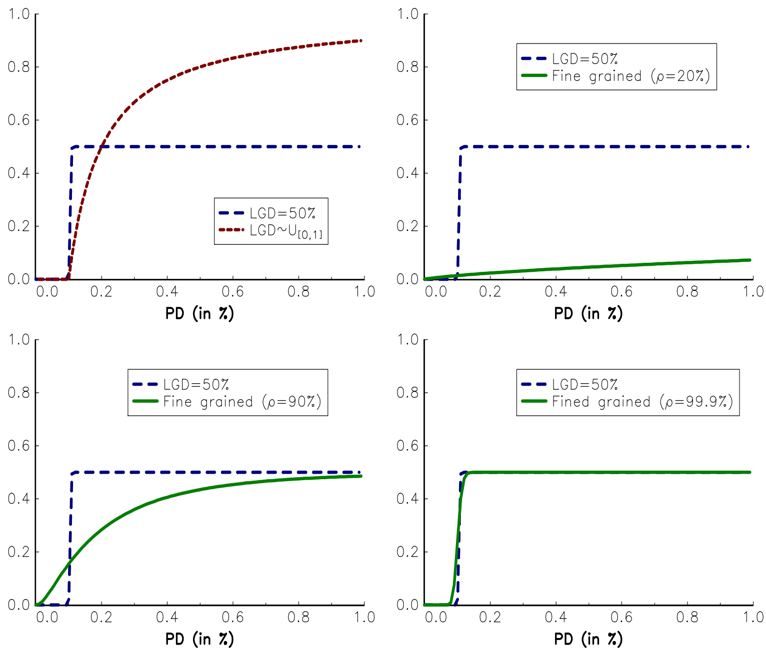

FIGURE 3.56: Comparison between the 99.9% value-at-risk of a loan and its risk contribution in an IFG portfolio

Indeed, when $\rho = 1$, there is no diversification effect. To illustrate this property, we report the loss distribution of an infinitely fine-grained portfolio[113] in Figure 3.57. When the correlation is equal to zero, the conditional expected loss does not depend on X and we have:

$$L = \mathbb{E}[L \mid X] = \mathrm{EAD} \cdot \mathrm{LGD} \cdot \mathrm{PD}$$

When the correlation is different from zero, we have:

$$\begin{cases} \mathbb{E}[L \mid X] > \mathbb{E}[L] & \text{for low values of } X \\ \mathbb{E}[L \mid X] < \mathbb{E}[L] & \text{for high values of } X \end{cases}$$

Since the value-at-risk considers a bad economic scenario, it is normal that the value-at-risk increases with respect to ρ because $\mathbb{E}[L \mid X]$ is an increasing function of ρ in bad economic times.

In Figure 3.58, we compare the normalized loss distribution[114] of non fined-grained, but homogenous portfolios. We notice that the loss distribution of the portfolio converges rapidly to the loss distribution of the IFG portfolio. It suffices that the number of credits is larger than 50. However, this result assumes that the portfolio is homogenous. In the case of non-homogenous portfolio, it is extremely difficult to define a rule to know if the portfolio is fine-grained or not.

[113] This is a homogeneous portfolio of 50 credits with the following characteristics: $\mathrm{EAD} = 1$, $\mathbb{E}[\mathrm{LGD}] = 50\%$ and $\mathrm{PD} = 10\%$.

[114] This is the loss of the portfolio divided by the number n of credits in the portfolio.

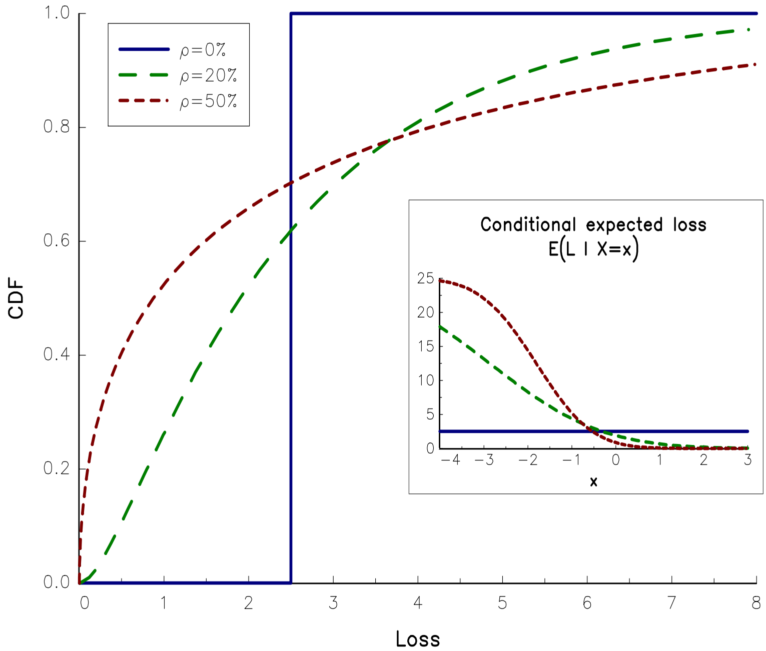

FIGURE 3.57: Loss distribution of an IFG portfolio

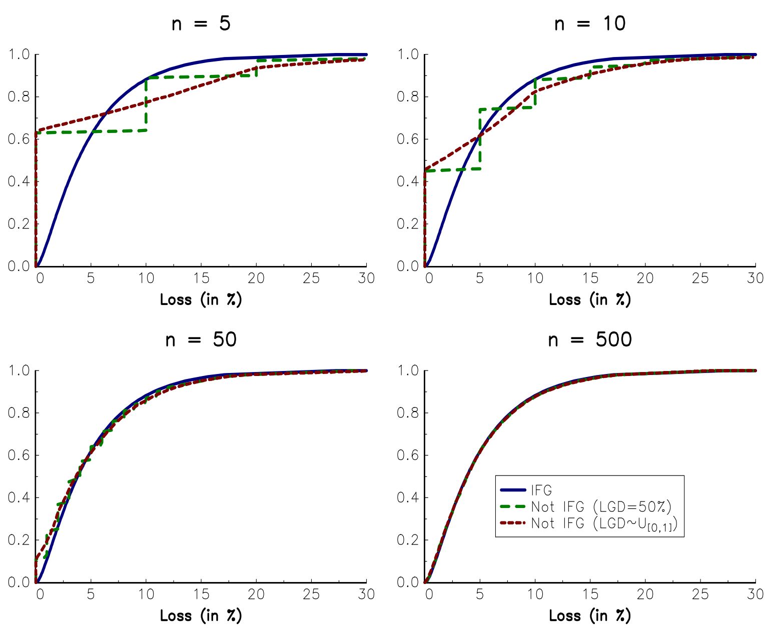

FIGURE 3.58: Comparison of the loss distribution of non-IFG and IFG portfolios

3.3.5.2 Granularity adjustment

Monte Carlo approach The first approach to compute the granularity adjustment is to estimate the quantile $\hat{\mathbf{F}}_L^{-1}(\alpha)$ of the portfolio loss using the Monte Carlo method. In Table 3.47, we have reported the (relative) granularity adjustment, which is defined as:

$$\mathrm{GA}^\star = \frac{\hat{\mathbf{F}}_L^{-1}(\alpha) - \mathrm{VaR}_\alpha(w_{\mathrm{IFG}})}{\mathrm{VaR}_\alpha(w_{\mathrm{IFG}})}$$

for different homogenous credit portfolios when EAD $= 1$. We consider different values of the default probability PD (1% and 10%), the size n of the portfolio (50, 100 and 500) and the confidence level α of the value-at-risk (90%, 99% and 99.9%). For the loss given default, we consider two cases: LGD $= 50\%$ and LGD $\sim \mathcal{U}[0,1]$. For each set of parameters, we use 10 million simulations for estimating the quantile $\hat{\mathbf{F}}_L^{-1}(\alpha)$ and the same seed for the random number generator[115] in order to compare the results. For example, when $n = 50$, PD $= 10\%$, $\rho = 10\%$, LGD $\sim \mathcal{U}[0,1]$ and $\alpha = 90\%$, we obtain $\mathrm{GA}^\star = 13.8\%$. This means that the capital charge is underestimated by 13.8% if we consider the IRB formula. We notice that the granularity adjustment is positive in the different cases we have tested. We verify that it decreases with respect to the portfolio size. However, it is difficult to draw other conclusions. For instance, it is not necessarily an increasing function of the confidence level.

TABLE 3.47: Granularity adjustment $\mathrm{GA}^\star$ (in %)

n		50	100	500	50	100	500
Parameters	α	LGD $\sim \mathcal{U}_{[0,1]}$			LGD $= 50\%$		
PD $= 10\%$, $\rho = 10\%$	90%	13.8	7.4	1.6	12.5	6.8	1.2
	99%	19.3	10.0	2.1	13.3	6.2	1.2
	99.9%	21.5	10.9	2.3	12.2	6.9	1.6
PD $= 10\%$, $\rho = 20\%$	90%	8.1	4.2	0.9	2.7	2.7	0.9
	99%	10.3	5.3	1.1	6.7	4.1	0.6
	99.9%	11.3	5.6	1.2	6.5	2.8	0.6
PD $= 1\%$, $\rho = 20\%$	90%	43.7	23.5	5.0	60.1	20.1	4.0
	99%	36.7	18.8	3.9	32.9	19.6	3.7
	99.9%	30.2	15.3	3.1	23.7	9.9	1.7

Analytical approach Let w be a credit portfolio. We have the following identity:

$$\mathrm{VaR}_\alpha(w) = \mathrm{VaR}_\alpha(w_{\mathrm{IFG}}) + \underbrace{\mathrm{VaR}_\alpha(w) - \mathrm{VaR}_\alpha(w_{\mathrm{IFG}})}_{\text{Granularity adjustment GA}} \tag{3.60}$$

The granularity adjustment is then the capital we have to add to the IRB value-at-risk in order to obtain the true value-at-risk (Wilde, 2001b; Gordy, 2003). Since we have seen that $\mathrm{VaR}_\alpha(w_{\mathrm{IFG}})$ is the conditional expected loss when the risk factor X corresponds to the quantile $1-\alpha$, we obtain:

$$\begin{aligned} \mathrm{VaR}_\alpha(w) &= \mathrm{VaR}_\alpha(w_{\mathrm{IFG}}) + \mathrm{GA} \\ &= \mathbb{E}[L \mid X = x_\alpha] + \mathrm{GA} \end{aligned}$$

[115]See Chapter 13 on page 787.

where $x_\alpha = \mathbf{H}^{-1}(1-\alpha)$ and $\mathbf{H}(x)$ is the cumulative distribution function of X. In order to derive the expression of the granularity adjustment, we rewrite Equation (3.60) in terms of portfolio loss:

$$L = \mathbb{E}[L \mid X] + (L - \mathbb{E}[L \mid X])$$

Since we have $\mathrm{VaR}_\alpha(w) = \mathbf{F}^{-1}(\alpha)$ where $\mathbf{F}(\ell)$ is the loss distribution, we deduce that:

$$\begin{aligned} \mathrm{VaR}_\alpha(w) &= \mathrm{VaR}_\alpha(L) \\ &= \mathrm{VaR}_\alpha\left(\mathbb{E}[L \mid X] + \eta\,(L - \mathbb{E}[L \mid X])\right)\big|_{\eta=1} \end{aligned}$$

Emmer and Tasche (2005) consider the second-order Taylor expansion of the value-at-risk:

$$\begin{aligned} \mathrm{VaR}_\alpha(w) &\approx \mathrm{VaR}_\alpha\left(\mathbb{E}[L \mid X]\right) + \left|\frac{\partial\ \mathrm{VaR}_\alpha\left(\mathbb{E}[L \mid X] + \eta\,(L - \mathbb{E}[L \mid X])\right)}{\partial\,\eta}\right|_{\eta=1} \\ &\quad + \frac{1}{2}\left|\frac{\partial^2\ \mathrm{VaR}_\alpha\left(\mathbb{E}[L \mid X] + \eta\,(L - \mathbb{E}[L \mid X])\right)}{\partial\,\eta^2}\right|_{\eta=1} \end{aligned}$$

Under some assumptions (homogeneous portfolio, regularity of the conditional expected loss, single factor model, etc.), Wilde (2001b) and Gordy (2003) show that the second-order Taylor expansion reduces to[116]:

$$\mathrm{VaR}_\alpha(w) \approx \mu(x_\alpha) - \frac{1}{2h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h(x)\,\upsilon(x)}{\partial_x \mu(x)}\right)\bigg|_{x=x_\alpha}$$

where $h(x)$ is the probability density function of X, $\mu(x)$ is the conditional expected loss function:

$$\mu(x) = \mathbb{E}[L \mid X = x]$$

and $\upsilon(x)$ is the conditional variance function:

$$\upsilon(x) = \sigma^2(L \mid X = x)$$

Since $\mu(x_\alpha) = \mathrm{VaR}_\alpha(w_{\mathrm{IFG}})$, we deduce that:

$$\mathrm{VaR}_\alpha(w) \approx \mathrm{VaR}_\alpha(w_{\mathrm{IFG}}) + \mathrm{GA}$$

where the granularity adjustment is equal to:

$$\begin{aligned} \mathrm{GA} &= -\frac{1}{2h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h(x)\,\upsilon(x)}{\partial_x \mu(x)}\right)\bigg|_{x=x_\alpha} \\ &= \frac{1}{2}\upsilon(x_\alpha)\frac{\partial_x^2 \mu(x_\alpha)}{\left(\partial_x \mu(x_\alpha)\right)^2} - \frac{1}{2}\frac{\partial_x \upsilon(x_\alpha)}{\partial_x \mu(x_\alpha)} - \frac{1}{2}\upsilon(x_\alpha)\frac{\partial_x \ln h(x_\alpha)}{\partial_x \mu(x_\alpha)} \end{aligned}$$

The granularity adjustment has been extensively studied[117]. Originally, the Basel Committee proposed to include the granularity adjustment in the first pillar (BCBS, 2001a), but it has finally preferred to move this issue into the second pillar[118].

[116] In fact, we can show that the first derivative vanishes (Gouriéroux *et al.*, 2000). If we remember the Euler allocation principle presented on page 105, this is not surprising since $\mathrm{VaR}_\alpha(\mathbb{E}[L \mid X])$ is the sum of risk contributions and already includes the first-order effects. In this case, it only remains the second-order effects.

[117] See for example Gordy (2003, 2004), Gordy and Marrone (2012), Gordy and Lütkebohmert (2013). The works of Wilde (2001a,b) and Emmer and Tasche (2005) are a good introduction to this topic.

[118] See Exercise 3.4.7 on page 253 for a derivation of the original Basel granularity adjustment.

3.4 Exercises

3.4.1 Single- and multi-name credit default swaps

1. We assume that the default time $\boldsymbol{\tau}$ follows an exponential distribution with parameter λ. Write the cumulative distribution function $\mathbf{F}$, the survival function $\mathbf{S}$ and the density function f of the random variable $\boldsymbol{\tau}$. How do we simulate this default time?

2. We consider a CDS 3M with two-year maturity and \$1 mn notional principal. The recovery rate $\mathcal{R}$ is equal to 40% whereas the spread s is equal to 150 bps at the inception date. We assume that the protection leg is paid at the default time.

   (a) Give the cash flow chart. What is the P&L of the protection seller A if the reference entity does not default? What is the P&L of the protection buyer B if the reference entity defaults in one year and two months?

   (b) What is the relationship between s, $\mathcal{R}$ and λ? What is the implied one-year default probability at the inception date?

   (c) Seven months later, the CDS spread has increased and is equal to 450 bps. Estimate the new default probability. The protection buyer B decides to realize his P&L. For that, he reassigns the CDS contract to the counterparty C. Explain the offsetting mechanism if the risky PV01 is equal to 1.189.

3. We consider the following CDS spread curves for three reference entities:

Maturity	#1	#2	#3
6M	130 bps	1 280 bps	30 bps
1Y	135 bps	970 bps	35 bps
3Y	140 bps	750 bps	50 bps
5Y	150 bps	600 bps	80 bps

   (a) Define the notion of credit curve. Comment the previous spread curves.

   (b) Using the Merton Model, we estimate that the one-year default probability is equal to 2.5% for #1, 5% for #2 and 2% for #3 at a five-year time horizon. Which arbitrage position could we consider about the reference entity #2?

4. We consider a basket of n single-name CDS.

   (a) What is a first-to-default (FtD), a second-to-default (StD) and a last-to-default (LtD)?

   (b) Define the notion of default correlation. What is its impact on the three previous spreads?

   (c) We assume that $n = 3$. Show the following relationship:

$$s_1^{\mathrm{CDS}} + s_2^{\mathrm{CDS}} + s_3^{\mathrm{CDS}} = s^{\mathrm{FtD}} + s^{\mathrm{StD}} + s^{\mathrm{LtD}}$$

   where s_i^{CDS} is the CDS spread of the i^{th} reference entity.

   (d) Many professionals and academics believe that the subprime crisis is due to the use of the Normal copula. Using the results of the previous question, what could you conclude?

3.4.2 Risk contribution in the Basel II model

1. We note L the portfolio loss of n credit and w_i the exposure at default of the i^{th} credit. We have:

$$L(w) = w^\top \varepsilon = \sum_{i=1}^{n} w_i \cdot \varepsilon_i \tag{3.61}$$

where ε_i is the unit loss of the i^{th} credit. Let $\mathbf{F}$ be the cumulative distribution function of $L(w)$.

   (a) We assume that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Compute the value-at-risk $\mathrm{VaR}_\alpha(w)$ of the portfolio when the confidence level is equal to α.

   (b) Deduce the marginal value-at-risk of the i^{th} credit. Define then the risk contribution $\mathcal{RC}_i$ of the i^{th} credit.

   (c) Check that the marginal value-at-risk is equal to:

$$\frac{\partial\, \mathrm{VaR}_\alpha(w)}{\partial\, w_i} = \mathbb{E}\left[\varepsilon_i \mid L(w) = \mathbf{F}^{-1}(\alpha)\right]$$

   Comment on this result.

2. We consider the Basel II model of credit risk and the value-at-risk risk measure. The expression of the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbb{1}\{\boldsymbol{\tau}_i < T_i\} \tag{3.62}$$

   (a) Define the different parameters EAD_i, LGD_i, $\boldsymbol{\tau}_i$ and T_i. Show that Model (3.62) can be written as Model (3.61) by identifying w_i and ε_i.

   (b) What are the necessary assumptions $\mathcal{H}$ to obtain this result:

$$\mathbb{E}\left[\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}[\mathrm{LGD}_i] \cdot \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

   with $D_i = \mathbb{1}\{\boldsymbol{\tau}_i < T_i\}$.

   (c) Deduce the risk contribution $\mathcal{RC}_i$ of the i^{th} credit and the value-at-risk of the credit portfolio.

   (d) We assume that the credit i defaults before the maturity T_i if a latent variable Z_i goes below a barrier B_i:

$$\boldsymbol{\tau}_i \leq T_i \Leftrightarrow Z_i \leq B_i$$

   We consider that $Z_i = \sqrt{\rho} \cdot X + \sqrt{1-\rho} \cdot \varepsilon_i$ where Z_i, X and ε_i are three independent Gaussian variables $\mathcal{N}(0,1)$. X is the factor (or the systematic risk) and ε_i is the idiosyncratic risk.

      i. Interpret the parameter ρ.

      ii. Calculate the unconditional default probability:

$$p_i = \Pr\{\boldsymbol{\tau}_i \leq T_i\}$$

      iii. Calculate the conditional default probability:

$$p_i(x) = \Pr\{\boldsymbol{\tau}_i \leq T_i \mid X = x\}$$

(e) Show that, under the previous assumptions $\mathcal{H}$, the risk contribution $\mathcal{RC}_i$ of the i^{th} credit is:

$$\mathcal{RC}_i = \text{EAD}_i \cdot \mathbb{E}\left[\text{LGD}_i\right] \cdot \Phi\left(\frac{\Phi^{-1}\left(p_i\right) + \sqrt{\rho}\Phi^{-1}\left(\alpha\right)}{\sqrt{1-\rho}}\right) \tag{3.63}$$

when the risk measure is the value-at-risk.

3. We now assume that the risk measure is the expected shortfall:

$$\text{ES}_\alpha\left(w\right) = \mathbb{E}\left[L \mid L \geq \text{VaR}_\alpha\left(w\right)\right]$$

(a) In the case of the Basel II framework, show that we have:

$$\text{ES}_\alpha\left(w\right) = \sum_{i=1}^{n} \text{EAD}_i \cdot \mathbb{E}\left[\text{LGD}_i\right] \cdot \mathbb{E}\left[p_i\left(X\right) \mid X \leq \Phi^{-1}\left(1-\alpha\right)\right]$$

(b) By using the following result:

$$\int_{-\infty}^{c} \Phi(a+bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

where $\Phi_2\left(x, y; \rho\right)$ is the cdf of the bivariate Gaussian distribution with correlation ρ on the space $\left[-\infty, x\right] \cdot \left[-\infty, y\right]$, deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} credit in the Basel II model is:

$$\mathcal{RC}_i = \text{EAD}_i \cdot \mathbb{E}\left[\text{LGD}_i\right] \cdot \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} \tag{3.64}$$

where $\mathbf{C}\left(u_1, u_2; \theta\right)$ is the Normal copula with parameter θ.

(c) What do the results (3.63) and (3.64) become if the correlation ρ is equal to zero? Same question if $\rho = 1$.

4. The risk contributions (3.63) and (3.64) were obtained by considering the assumptions $\mathcal{H}$ and the default model defined in Question 2(d). What are the implications in terms of Pillar 2?

3.4.3 Calibration of the piecewise exponential model

1. We denote by $\mathbf{F}$ and $\mathbf{S}$ the distribution and survival functions of the default time $\boldsymbol{\tau}$. Define the function $\mathbf{S}\left(t\right)$ and deduce the expression of the associated density function $f\left(t\right)$.

2. Define the hazard rate $\lambda\left(t\right)$. Deduce that the exponential model corresponds to the particular case $\lambda\left(t\right) = \lambda$.

3. We assume that the interest rate r is constant. In a continuous-time model, we recall that the CDS spread is given by the following expression:

$$\mathcal{s}\left(T\right) = \frac{\left(1-\boldsymbol{\mathcal{R}}\right) \cdot \int_0^T e^{-rt} f\left(t\right)\,\mathrm{d}t}{\int_0^T e^{-rt}\mathbf{S}\left(t\right)\,\mathrm{d}t} \tag{3.65}$$

where $\boldsymbol{\mathcal{R}}$ is the recovery rate and T is the maturity of the CDS. Find the triangle relationship when $\boldsymbol{\tau} \sim \mathcal{E}\left(\lambda\right)$.

4. Let us assume that:

$$\lambda(t) = \begin{cases} \lambda_1 & \text{if } t \leq 3 \\ \lambda_2 & \text{if } 3 < t \leq 5 \\ \lambda_3 & \text{if } t > 5 \end{cases}$$

   (a) Give the expression of the survival function $\mathbf{S}(t)$ and calculate the density function $f(t)$. Verify that the hazard rate $\lambda(t)$ is a piecewise constant function.

   (b) Find the expression of the CDS spread using Equation (3.65).

   (c) We consider three credit default swaps, whose maturities are respectively equal to 3, 5 and 7 years. Show that the calibration of the piecewise exponential model implies to solve a set of 3 equations with the unknown variables λ_1, λ_2 and λ_3. What is the name of this calibration method?

   (d) Find an approximated solution when r is equal to zero and λ_m is small. Comment on this result.

   (e) We consider the following numerical application: $r = 5\%$, $\mathcal{s}(3) = 100$ bps, $\mathcal{s}(5) = 150$ bps, $\mathcal{s}(7) = 160$ bps and $\mathcal{R} = 40\%$. Estimate the implied hazard function.

   (f) Using the previous numerical results, simulate the default time with the uniform random numbers 0.96, 0.23, 0.90 and 0.80.

3.4.4 Modeling loss given default

1. What is the difference between the recovery rate and the loss given default?

2. We consider a bank that grants 250 000 credits per year. The average amount of a credit is equal to \$50 000. We estimate that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of the litigation department is equal to \$12.5 mn. Give an estimation of the loss given default?

3. The probability density function of the beta probability distribution $\mathcal{B}(\alpha, \beta)$ is:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathfrak{B}(\alpha, \beta)}$$

   where $\mathfrak{B}(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u$.

   (a) Why is the beta probability distribution a good candidate to model the loss given default? Which parameter pair (α, β) does correspond to the uniform probability distribution?

   (b) Let us consider a sample $(x_1, \ldots, x_n)$ of n losses in case of default. Write the log-likelihood function. Deduce the first-order conditions of the maximum likelihood estimator.

   (c) We recall that the first two moments of the beta probability distribution are:

$$\begin{aligned} \mathbb{E}[X] &= \frac{\alpha}{\alpha+\beta} \\ \sigma^2(X) &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \end{aligned}$$

   Find the method of moments estimator.

4. We consider a risk class $\mathcal{C}$ corresponding to a customer/product segmentation specific to retail banking. A statistical analysis of 1 000 loss data available for this risk class gives the following results:

LGD_k	0%	25%	50%	75%	100%
n_k	100	100	600	100	100

where n_k is the number of observations corresponding to LGD_k.

   (a) We consider a portfolio of 100 homogeneous credits, which belong to the risk class $\mathcal{C}$. The notional is \$10 000 whereas the annual default probability is equal to 1%. Calculate the expected loss of this credit portfolio with a one-year time horizon if we use the previous empirical distribution to model the LGD parameter.

   (b) We assume that the LGD parameter follows a beta distribution $\mathcal{B}(\alpha, \beta)$. Calibrate the parameters α and β with the method of moments.

   (c) We assume that the Basel II model is valid. We consider the portfolio described in Question 4(a) and calculate the unexpected loss. What is the impact if we use a uniform probability distribution instead of the calibrated beta probability distribution? Why does this result hold even if we consider different factors to model the default time?

3.4.5 Modeling default times with a Markov chain

We consider a rating system with 4 risk classes (A, B, C and D), where rating D represents the default. The transition probability matrix with a two-year time horizon is equal to:

$$P(2) = \begin{pmatrix} 94\% & 3\% & 2\% & 1\% \\ 10\% & 80\% & 5\% & 5\% \\ 10\% & 10\% & 60\% & 20\% \\ 0\% & 0\% & 0\% & 100\% \end{pmatrix}$$

We also have:

$$P(4) = \begin{pmatrix} 88.860\% & 5.420\% & 3.230\% & 2.490\% \\ 17.900\% & 64.800\% & 7.200\% & 10.100\% \\ 16.400\% & 14.300\% & 36.700\% & 32.600\% \\ 0.000\% & 0.000\% & 0.000\% & 100.000\% \end{pmatrix}$$

and:

$$P(6) = \begin{pmatrix} 84.393\% & 7.325\% & 3.986\% & 4.296\% \\ 24.026\% & 53.097\% & 7.918\% & 14.959\% \\ 20.516\% & 15.602\% & 23.063\% & 40.819\% \\ 0.000\% & 0.000\% & 0.000\% & 100.000\% \end{pmatrix}$$

Let us denote by $\mathbf{S}_A(t)$, $\mathbf{S}_B(t)$ and $\mathbf{S}_C(t)$ the survival functions of each risk class A, B and C.

1. How are the matrices $P(4)$ and $P(6)$ calculated?

2. Assuming a piecewise exponential model, calibrate the hazard function of each risk class for $0 < t \leq 2$, $2 < t \leq 4$ and $4 < t \leq 6$.

3. Give the definition of a Markovian generator. How can we estimate the generator Λ associated to the transition probability matrices? Verify numerically that the direct estimator is equal to:

$$\hat{\Lambda} = \begin{pmatrix} -3.254 & 1.652 & 1.264 & 0.337 \\ 5.578 & -11.488 & 3.533 & 2.377 \\ 6.215 & 7.108 & -25.916 & 12.593 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix} \times 10^{-2}$$

4. In Figure 3.59, we show the hazard function $\lambda(t)$ deduced from Questions 2 and 3. Explain how do we calculate $\lambda(t)$ in both cases. Why do we obtain an increasing curve for rating A, a decreasing curve for rating C and an inverted U-shaped curve for rating B?

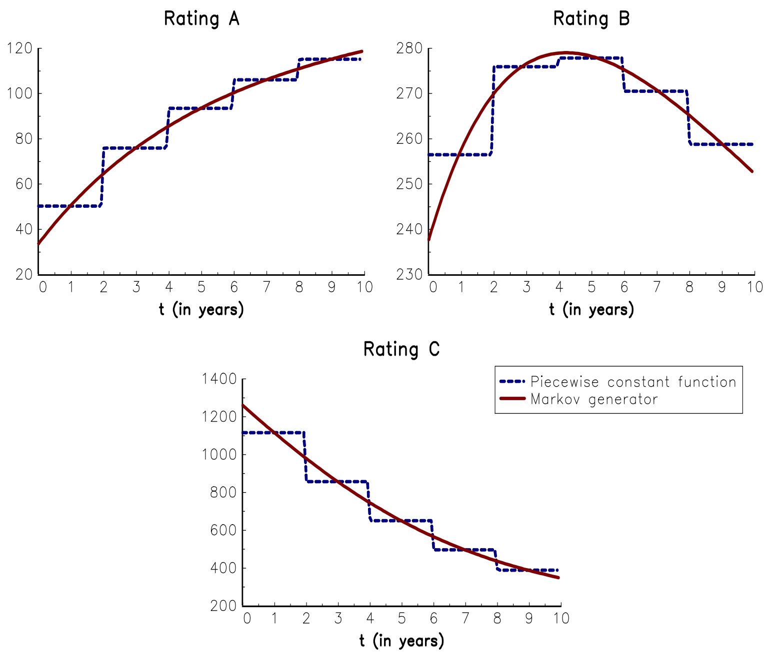

FIGURE 3.59: Hazard function $\lambda(t)$ (in bps) estimated respectively with the piecewise exponential model and the Markov generator

3.4.6 Continuous-time modeling of default risk

We consider a credit rating system with four risk classes (A, B, C and D), where rating D represents the default. The one-year transition probability matrix is equal to:

$$P = P(1) = \begin{pmatrix} 94\% & 3\% & 2\% & 1\% \\ 10\% & 80\% & 7\% & 3\% \\ 5\% & 15\% & 60\% & 20\% \\ 0\% & 0\% & 0\% & 0\% \end{pmatrix}$$

We denote by $\mathbf{S}_A(t)$, $\mathbf{S}_B(t)$ and $\mathbf{S}_C(t)$ the survival functions of each risk class A, B and C.

1. Explain how we can calculate the n-year transition probability matrix $P(n)$? Find the transition probability matrix $P(10)$.

2. Let $V = \left(V_1 \vdots V_2 \vdots V_3 \vdots V_4\right)$ and $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the matrices of eigenvectors and eigenvalues associated to P.

   (a) Show that:
   $$P(n) V = V D^n$$
   Deduce a second approach for calculating the n-year transition probability matrix $P(n)$.

   (b) Calculate the eigendecomposition of the transition probability matrix P. Deduce the transition probability matrix $P(10)$.

3. We assume that the default time follows a piecewise exponential model. Let $\mathbf{S}_i(n)$ and $\lambda_i(n)$ be the survival function and the hazard rate of a firm whose initial rating is the state i (A, B or C). Give the expression of $\mathbf{S}_i(n)$ and $\lambda_i(n)$. Show that:
   $$\lambda_i(1) = -\ln\left(1 - \mathbf{e}_i^\top P^n \mathbf{e}_4\right)$$
   Calculate $\mathbf{S}_i(n)$ and $\lambda_i(n)$ for $n \in \{0, \ldots, 10, 50, 100\}$.

4. Give the definition of a Markov generator. How can we estimate the generator Λ associated to the transition probability matrices? Give an estimate $\hat{\Lambda}$.

5. Explain how we can calculate the transition probability matrix $P(t)$ for the time horizon $t \geq 0$. Give the theoretical approximation of $P(t)$ based on Taylor expansion. Calculate the 6-month transition probability matrix.

6. Deduce the expression of $\mathbf{S}_i(t)$ and $\lambda_i(t)$.

3.4.7 Derivation of the original Basel granularity adjustment

In this exercise, we derive the formula of the granularity adjustment that was proposed by the Basel Committee in 2001. The mathematical proof follows Chapter 8 (§422 to §457) of BCBS (2001a) and the works of Wilde (2001a,b) and Gordy (2003, 2004). We encourage the reader to consult carefully these references. Most of the time, we use the notations of the Basel Committee[119]. We consider the Basel model that has been presented in Section 3.2.3.2 on page 169.

1. We consider the normalized loss:
   $$L_i = \mathrm{LGD}_i \cdot D_i$$
   We assume that the conditional probability of default is given by the CreditRisk+ model (Gordy, 2000):
   $$p_i(X) = p_i\left(1 + \varpi_i (X - 1)\right)$$
   where $\varpi_i \in [0, 1]$ is the factor weight and X is the systematic risk factor, which follows the gamma distribution $\mathcal{G}(\alpha_g, \beta_g)$. Calculate the conditional expected loss[120]:
   $$\mu(x) = \mathbb{E}[L_i \mid X = x]$$

[119] When they are different, we indicate the changes in footnotes.

[120] We use the notation $E_i = \mathbb{E}[\mathrm{LGD}_i]$.

and the conditional variance:

$$\upsilon(x) = \sigma^2(L_i \mid X = x)$$

The Basel Committee assumes that (BCBS, 2001a, §447):

$$\sigma(\mathrm{LGD}_i) = \frac{1}{2}\sqrt{E_i(1 - E_i)}$$

Deduce that we have the following approximation:

$$\upsilon(x) \approx E_i\left(\frac{1}{4} + \frac{3}{4}E_i\right)p_i(1 + \varpi_i(x - 1))$$

2. Calculate the granularity adjustment function:

$$\beta(x) = \frac{1}{2h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h(x)\,\upsilon(x)}{\partial_x \mu(x)}\right)$$

3. In order to maintain the coherency with the IRB formula, the Basel Committee imposes that the conditional probabilities are the same for the IRB formula (Vasicek model) and the granularity formula (CreditRisk+ model). Show that:

$$\varpi_i = \frac{1}{(x - 1)}\frac{F_i}{p_i}$$

where:

$$F_i = \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1 - \rho}}\right) - p_i$$

Deduce the expression of $\beta(x)$.

4. The calibration has been done by assuming that $\mathbb{E}[X] = 1$ and $\sigma(X) = 2$ (BCBS, 2001a, §445). Show that:

$$\beta(x_\alpha) = (0.4 + 1.2 \cdot E_i)\left(0.76229640 + 1.0747964 \cdot \frac{p_i}{F_i}\right)$$

We recall that the Basel Committee finds the following expression of $\beta(x_\alpha)$:

$$\beta(x_\alpha) = (0.4 + 1.2 \cdot E_i)\left(0.76 + 1.10 \cdot \frac{p_i}{F_i}\right)$$

How to obtain exactly this formula?

5. In order to transform the granularity adjustment function $\beta(x_\alpha)$ into risk-weighted assets, the Basel Committee indicates that it uses a scaling factor $c = 1.5$ (BCBS, 2001a, §457). Moreover, the Basel Committee explains that the "*the baseline IRB risk-weights for non-retail assets (i.e. the RWA before granularity adjustment) incorporate a margin of 4% to cover average granularity*". Let $w^\star$ be the equivalent homogenous portfolio of the current portfolio w. Show that the granularity adjustment is equal to[121]:

$$\mathrm{GA} = \frac{\mathrm{EAD}^\star}{n^\star} \cdot \mathrm{GSF} - 0.04 \cdot \mathrm{RWA}_{\mathrm{NR}}$$

[121]The Basel Committee uses the notation $\square_{\mathrm{AG}}$ instead of $\square^\star$ for the equivalent homogeneous portfolio. The global exposure $\mathrm{EAD}^\star$ corresponds to the variable TNRE (total non-retail exposure) of the Basel Committee.

where RWA_{NR} are the risk-weighted assets for non-retail assets and:

$$\text{GSF} = (0.6 + 1.8 \cdot E^\star)\left(9.5 + 13.75 \cdot \frac{p^\star}{F^\star}\right)$$

6. The Basel Committee considers the following definition of the portfolio loss:

$$L = \sum_{j=1}^{n_{\mathcal{C}}} \sum_{i \in \mathcal{C}_j} \text{EAD}_i \cdot \text{LGD}_i \cdot D_i$$

where $\mathcal{C}_j$ is the j^{th} class of risk. Find the equivalent homogeneous portfolio $w^\star$ of size $n^\star$ and exposure $\text{EAD}^\star$. Calibrate the parameters $p^\star$, $E^\star$ and $\sigma\left(\text{LGD}^\star\right)$.

7. Using the notations of BCBS (2001a), summarize the different steps for computing the original Basel granularity adjustment.

3.4.8 Variance of the conditional portfolio loss

The portfolio loss is given by:

$$L = \sum_{i=1}^{n} w_i \cdot \text{LGD}_i \cdot D_i$$

where w_i is the exposure at default of the i^{th} credit, LGD_i is the loss given default, T_i is the residual maturity and $D_i = \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$ is the default indicator function. We suppose the assumptions of the Basel II model are satisfied. We note $D_i(X)$ and $p_i(X)$ the conditional default indicator function and the conditional default probability with respect to the risk factor X.

1. Define $D_i(X)$. Calculate $\mathbb{E}\left[D_i(X)\right]$, $\mathbb{E}\left[D_i^2(X)\right]$ and $\mathbb{E}\left[D_i(X)\, D_j(X)\right]$.

2. Define the conditional portfolio loss $L(X)$.

3. Calculate the expectation of $L(X)$.

4. Show that the variance of $L(X)$ is equal to:

$$\sigma^2\left(L(X)\right) = \sum_{i=1}^{n} w_i^2 \left(\mathbb{E}\left[D_i(X)\right]\sigma^2\left(\text{LGD}_i\right) + \mathbb{E}^2\left[\text{LGD}_i\right]\sigma^2\left(D_i(X)\right)\right)$$