Chapter 7

Asset Liability Management Risk

Asset liability management (ALM) corresponds to the processes that address the mismatch risk between assets and liabilities. These methods concern financial institutions, which are mainly defined by a balance sheet. For example, this is the case of pension funds and insurance companies. In this chapter, we focus on ALM risks in banks, and more precisely ALM risks of the banking book. Previously, we have already seen some risks that impact the banking book such as credit or operational risk. In what follows, we consider the four specific ALM risks: liquidity risk[1], interest rate risk, option risk and currency risk.

Generally, ALM risks are little taught in university faculties because they are less known by academics. In fact, asset liability management is a mix of actuarial science, accounting and statistical modeling, and seems at first sight less mathematical than risk management. Another difference is that the ALM function is generally within the finance department and not within the risk management department. This is because ALM implies to take decisions that are not purely related to risk management considerations, but also concerns commercial choices and business models.

7.1 General principles of the banking book risk management

Before presenting the tools to manage the ALM risks, we define the outlines of the asset and liability management. In particular, we show why ALM risks are so specific if we compare them to market and credit risks. In fact, asset and liability management has two components. The first component is well-identified and corresponds to the risk measurement of ALM operations. The second component is much more vague, because it concerns both risk management and business development. Indeed, banking business is mainly a financial intermediation business, since banks typically tend to borrow short term and lend long term. The mismatch between assets and liabilities is then inherent to banking activities. Similarly, the balance sheet of a bank and its income statement are highly related, implying that future income may be explained by the current balance sheet. The debate on whether the ALM department is a profit center summarizes this duality between risk and business management.

[1]Liquidity risk was the subject of the previous chapter. However, we have discussed this topic from a risk management point of view by focusing on the regulatory ratios (LCR and NSFR). In this chapter, we tackle the issue of liquidity risk from an ALM perspective.

7.1.1 Definition

7.1.1.1 Balance sheet and income statement

The ALM core function is to measure the asset liability mismatch of the balance sheet of the bank. In Table 7.1, we report the 2018 balance sheet of FDIC-insured commercial banks and savings institutions as provided by FDIC (2019). It concerns 5 406 financial institutions in the US. We notice that the total assets and liabilities are equal to \$17.9 tn. The most important items are loans and leases, investment securities and cash & due from depository institutions on the asset side, deposits and equity capital on the liability

TABLE 7.1: Assets and liabilities of FDIC-insured commercial banks and savings institutions (Amounts in \$ bn)

Total Assets	**17 943**
Loans secured by real estate	**4 888**
1-4 Family residential mortgages	2 119
Nonfarm nonresidential	1 446
Construction and development	350
Home equity lines	376
Multifamily residential real estate	430
Farmland	105
Real estate loans in foreign offices	62
Commercial & industrial loans	**2 165**
Loans to individuals	**1 743**
Credit cards	903
Other loans to individuals	839
Auto loans	455
Farm loans	82
Loans to depository institutions	84
Loans to foreign gov. & official inst.	11
Obligations of states in the U.S.	188
Other loans	862
Lease financing receivables	133
Gross total loans and leases	10 155
Less: Unearned income	2
Total loans and leases	10 152
Less: Reserve for losses	125
Net loans and leases	**10 028**
Securities	**3 723**
Available for sale (fair value)	2 590
Held to maturity (amortized cost)	1 129
U.S. Treasury securities	549
Mortgage-backed securities	2 187
State and municipal securities	330
Equity securities	3
Cash & due from depos. instit.	**1 694**
Fed. funds sold and reverse repos	622
Bank premises and fixed assets	130
Other real estate owned	7
Trading account assets	572
Intangible assets	399
Goodwill	334
Other Assets	769

Total liabilities and capital	**17 943**
Deposits	**13 866**
Foreign office deposits	1 253
Domestic office deposits	12 613
Interest-bearing deposits	9 477
Noninterest-bearing deposits	3 136
Estimated insured deposits	7 483
Time deposits	1 971
Brokered deposits	1 071
Federal funds purchased & repos	240
FHLB advances	571
Other borrowed money	557
Subordinated debt	69
Trading account liabilities	236
Other liabilities	381
Total liabilities	**15 921**
Total equity capital	**2 023**
Total bank equity capital	2 019
Perpetual preferred stock	9
Common stock	43
Surplus	1 277
Undivided profits	759
Other comprehensive income	-68
Net unrealized P&L on AFS	0

Source: Federal Deposit Insurance Corporation (2019), www.fdic.gov/bank/analytical/qbp.

side. Table 7.2 shows a simplified version of the balance sheet. The bank collects retail and corporate deposits and lends money to households and firms.

TABLE 7.2: A simplified balance sheet

Assets	Liabilities
Cash	Due to central banks
Loans and leases	Deposits
Mortgages	Deposit accounts
Consumer credit	Savings
Credit cards	Term deposits
Interbank loans	Interbank funding
Investment securities	Short-term debt
Sovereign bonds	Subordinated debt
Corporate bonds	Reserves
Other assets	Equity capital

Some deposits have a fixed maturity (e.g. a certificate of deposit), while others have an undefined maturity. This is for example the case of demand deposits or current accounts. These liabilities are then called non-maturity deposits (NMD), and include transaction deposits, NOW (negotiable order of withdrawal) accounts, money market deposit accounts and savings deposits. Term deposits (also known as time deposits or certificates of deposit) are deposits with a fixed maturity, implying that the customer cannot withdraw his funds before the term ends. Generally, the bank considers that the core deposits correspond to deposits of the retail customers and are a stable source of its funding. On the asset side, the bank proposes credit, loans and leases, and holds securities and other assets such as real estate, intangible assets[2] and goodwill[3]. In Chapter 3 on page 125, we have seen that loans concern both individuals, corporates and sovereigns. We generally distinguish loans secured by real estate, consumer loans, commercial and industrial loans. Leases correspond to contract agreements, where the bank purchases the asset on behalf of the customer, and the customer uses the asset in return and pays to the bank a periodic lease payment for the duration of the agreement[4]. Investment securities include repos, sovereign bonds, asset-backed securities, debt instruments and equity securities. We reiterate that the balance sheet does not concern off-balance sheet items. Indeed, the risk of credit lines (e.g. commitments, standby facilities or letters of credit) is measured by the credit risk[5], while derivatives (swaps, forwards, futures and options) are mainly managed within the market risk and the counterparty credit risk.

Another difference between assets and liabilities is that they are not '*priced*' at the same interest rate since the primary business of the bank is to capture the interest rate spread between its assets and its liabilities. The bank receives income from the loans and its investment portfolio, whereas the expenses of the bank concern the interest it pays on deposits and its debt, and the staff and operating costs. In Table 7.3, we report the 2018 income statement of FDIC-insured commercial banks and savings institutions. We can simplify the computation of this income statement and obtain the simplified version

[2] Intangible assets are non-physical assets that have a multi-period useful life such as servicing rights or customer lists. They are also intellectual assets (patents, copyrights, softwares, etc).

[3] Goodwill is the excess of the purchase price over the fair market value of the net assets acquired. The difference can be explained because of the brand name, good customer relations, etc.

[4] At the end of the contract, the customer may have the option to buy the asset.

[5] In this case, the difficult task is to estimate the exposure at default and the corresponding CCF parameter.

TABLE 7.3: Annual income and expense of FDIC-insured commercial banks and savings institutions (Amounts in $ mn)

Total interest income	**660 988**
Domestic office loans	492 201
Foreign office loans	21 965
Lease financing receivables	5 192
Balances due from depository institutions	24 954
Securities	92 908
Trading accounts	11 025
Federal funds sold	8 347
Other interest income	4 397
Total interest expense	**119 799**
Domestic office deposits	74 781
Foreign office deposits	8 877
Federal funds purchased	4 108
Trading liabilities and other borrowed money	28 629
Subordinated notes and debentures	2 780
Net interest income	**541 189**
Provision for loan and lease losses	**49 998**
Total noninterest income	**266 165**
Fiduciary activities	37 525
Service charges on deposit accounts	35 745
Trading account gains and fees	26 755
Interest rate exposures	7 148
Foreign exchange exposures	12 666
Equity security and index exposures	4 750
Commodity and other exposures	1 299
Credit exposures	367
Investment banking, advisory, brokerage and underwriting fees and commissions	12 522
Venture capital revenue	60
Net servicing fees	10 680
Net securitization income	230
Insurance commission fees and income	4 574
Net gains (losses) on sales of loans	12 593
Net gains (losses) on sales of other real estate owned	-99
Net gains (losses) on sales of other assets (except securities)	1 644
Other noninterest income	123 938
Total noninterest expense	**459 322**
Salaries and employee benefits	217 654
Premises and equipment expense	45 667
Other noninterest expense	190 944
Amortization expense and goodwill impairment losses	5 058
Securities gains (losses)	328
Income (loss) before income taxes and extraordinary items	**298 362**
Applicable income taxes	61 058
Extraordinary gains (losses), net	-267
Net charge-offs	47 479
Cash dividends	164 704
Retained earnings	72 045
Net operating income	**237 059**

Source: Federal Deposit Insurance Corporation (2019), www.fdic.gov/bank/analytical/qbp.

given in Table 7.4. Net interest income corresponds to the income coming from interest rates, whereas non-interest income is mainly generated by service fees and commissions. The income statement depends of course on the balance sheet items, but also on off-balance sheet items. Generally, loans, leases and investment securities are called the earning assets, whereas deposits are known as interest bearing liabilities.

TABLE 7.4: A simplified income statement

	Interest income
−	Interest expenses
=	Net interest income
+	Non-interest income
=	Gross income
−	Operating expenses
=	Net income
−	Provisions
=	Earnings before tax
−	Income tax
=	Profit after tax

7.1.1.2 Accounting standards

We understand that the goal of ALM is to control the risk of the balance sheet in order to manage and secure the future income of the bank. However, the ALM policy is constrained by accounting standards since the bank must comply with some important rules that distinguish banking and trading books. Accounting systems differ from one country to another country, but we generally distinguish four main systems: US GAAP[6], Japanese combined system[7], Chinese accounting standards and International Financial Reporting Standards (or IFRS). IFRS are standards issued by the IFRS Foundation and the International Accounting Standards Board (IASB) to provide a global accounting system for business affairs and capital markets. In March 2019, there were 144 jurisdictions that required the use of IFRS Standards for publicly listed companies and 12 jurisdictions that permitted its use. IFRS is then the world's most widely used framework. For example, it is implemented in European Union, Australia, Middle East, Russia, South Africa, etc. Since January 2018, IFRS 9 has replaced IAS 39 that was considered excessively complicated and inappropriate.

Financial instruments IAS 39 required financial assets to be classified in the four following categories:

- financial assets at fair value through profit and loss (FVTPL);
- available-for-sale financial assets (AFS);
- loans and receivables (L&R);
- held-to-maturity investments (HTM).

[6]GAAP stands for Generally Accepted Accounting Principles.

[7]Companies may choose one of the four accepted financial reporting frameworks: Japanese GAAP (which is the most widespread system), IFRS standards, Japan's modified international standards (JMIS) and US GAAP.

The FVTPL category had two subcategories. The first category (designated) included any financial asset that was designated on initial recognition as one to be measured at fair value with fair value changes in profit and loss. The second category (held-for-trading or HFT) included financial assets that were held for trading. Depending on the category, the bank measured the financial asset using the fair value approach[8] (AFS and FVTPL) or the amortized cost approach (L&R and HTM). In IFRS 9, the financial assets are divided into two categories:

- amortized cost (AC);
- fair value (FV).

For FV assets, we distinguish fair value through profit and loss (FVTPL) and fair value through other comprehensive income (FVOCI). Category changes between AC, FVTPL and FVOCI are recognized when the asset is derecognized or reclassified. In fact, the classification of an asset depends on two tests: the business model (BM) test and the solely payments of principal and interest (SPPI) test. In the BM test, the question is to know "*if the objective of the bank is to hold the financial asset to collect the contractual cash flows*" or not. In the SPPI test, the question is rather to understand if "*the contractual terms of the financial asset give rise on specified dates to cash flows that are solely payments of principal and interest on the principal amount outstanding*". It is obvious that the classification of an asset affects the ALM policy because it impacts differently the income statement.

On the liability side, there is little difference between IAS 39 and IFRS 9. All equity investments are measured at fair value, HFT financial liabilities are measured at FVTPL and all other financial liabilities are measured at amortized cost if the fair value option is applied.

Remark 69 *The main revision of IFRS 9 concerns impairment of financial assets since it establishes new models of expected credit loss for receivables and loans. This implies that banks can calculate loss provisioning as soon as the loan is entered the banking book.*

Hedging instruments Hedge accounting is an option and not an obligation. It considers that some financial assets are not held for generating P&L, but are used in order to offset a given risk. This implies that the hedging instrument is fully related to the hedged item. IAS 39 and IFRS 9 recognize three hedging strategies:

- a fair value hedge (FVH) is a hedge of the exposure to changes in fair value of a recognized asset or liability;
- a cash flow hedge (CFH) is a hedge of the exposure to variability in cash flows that is attributable to a particular risk;
- a net investment hedge (NIH) concerns currency risk hedging.

In the case of FVH, fair value of both the hedging instrument and the hedged item are recognized in profit and loss. In the case of CFH or NIH, the effective portion of the gain or loss on the hedging instrument is recognized in equity (other comprehensive income[9] or OCI), while the ineffective portion of the gain or loss on the hedging instrument is recognized in profit and loss.

[8]In the AFS case, gains and losses impact the equity capital and then the balance sheet, whereas gains and losses of FVTPL assets directly concerns the income statement.

[9]See Table 7.1 on page 370.

7.1.1.3 Role and importance of the ALCO

Remark 69 shows that IFRS 9 participates to the convergence of risk, finance and accounting that we recently observe. In fact, ALM is at the junction of these three concepts. This is why we could discuss how to organize the ALM function. Traditionally, it is located in the finance department because the ALM committee (ALCO) is in charge of both risk management and income management. In particular, it must define the funds transfer pricing (FTP) policy. Indeed, resources concerning interest and liquidity risks are transferred from business lines to the ALM portfolio. The ALCO and the ALM unit is in charge to manage the risks of this portfolio, and allocate the P&L across business lines:

> "*A major purpose of internal prices is to determine the P&L of the business lines. Transfer prices are internal prices of funds charged to business units or compensating cheap resources such as deposits. [...] Transfer pricing systems are notably designed for the banking book, for compensating resources collected from depositors and for charging funds used for lending. Internal prices also serve for exchanging funds between units with deficits of funds and units with excesses of funds. As they are used for calculating the P&L of a business line, they perform income allocation across business lines*" (Bessis, 2015, pages 109-110).

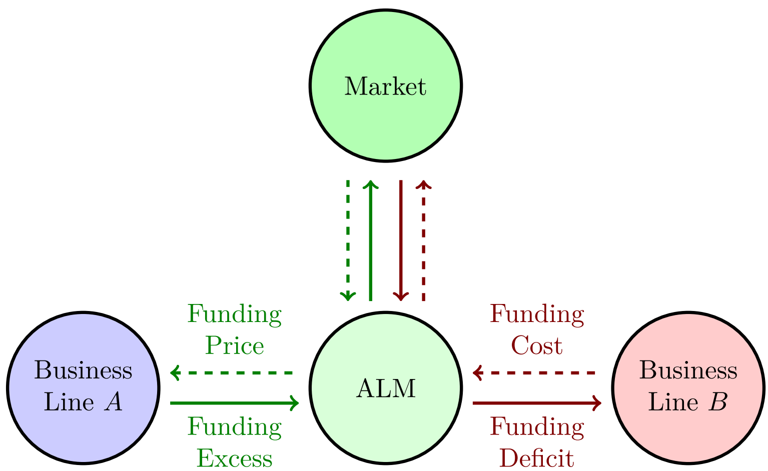

FIGURE 7.1: Internal and external funding transfer

This means that business lines with a funding excess will provide the liquidity to business lines with a funding deficit. For example, Figure 7.1 shows the relationships between the ALM unit and two business lines A and B. In this case, the business line A must be rewarded and receives the funding price, whereas the business B pays the funding cost. Internal funds transfer system avoids that business lines A and B directly go to the market. However, the ALM unit has access to the market for both lending the funding liquidity excess or borrowing the funding liquidity deficit of the bank. At first sight, we can assume that the internal funding price is equal to the external funding price and the internal funding cost is equal to the external funding cost. In this case, the ALM unit captures the bid/ask spread of the funding liquidity. In the real life, it is not possible and it is not necessarily desirable. Indeed, we reiterate that the goal of a bank is to perform liquidity transformation. This means that the liquidity excess of the business line A does not match necessarily the liquidity deficit of the business line B. Second, the role of the ALM is also to be sure that business lines can pilot their commercial development. In this situation, it is important that internal funding prices and costs are less volatile than external funding prices and costs in order to better stabilize commercial margins. Since the funds transfer pricing policy is decided by the ALCO, we notice that the role of ALM cannot be reduced to a risk management issue. Even if the risk transfer is intentionally rational and fair, meaning that internal prices are related to market prices, the ALM remains a business issue because the assets and liabilities are generally not tradable and there are not always real market prices for these items. For example, what is the price of a \$100 deposit? It depends on the behavior of the customer, but also on the risk appetite of the bank. What is the margin of a \$100 loan? It is not the spread between the loan interest rate and the market interest rate, because there is no perfect matching between the two interest rates. In this case, the margin will depend on the risk management policy. This duality between income generation and risk management is the specificity of asset liability management. Therefore, the role of the ALCO is essential for a bank, because it impacts the risk management of its balance sheet, but also the income generated by its banking book.

7.1.2 Liquidity risk

In this section, we define the concept of liquidity gap, which is the main tool for measuring the ALM liquidity risk. In particular, we make the distinction between static and dynamic liquidity gap when we consider the new production and future projections. In order to calculate liquidity gaps, we also need to understand asset and liability amortization, and liquidity cash flow schedules. Finally, we present liquidity hedging tools, more precisely the standard instruments for managing the ALM liquidity risk.

7.1.2.1 Definition of the liquidity gap

Basel III uses two liquidity ratios (LCR and NSFR), which are related to the ALM liquidity risk. More generally, financial institutions (banks, insurance companies, pension funds and asset managers) manage funding risks by considering funding ratios or funding gaps. The general expression of a funding ratio is:

$$\mathcal{FR}(t) = \frac{A(t)}{L(t)} \tag{7.1}$$

where $A(t)$ is the value of assets and $L(t)$ is the value of liabilities at time t, while the funding gap is defined as the difference between asset value and liability value:

$$\mathcal{FG}(t) = A(t) - L(t) \tag{7.2}$$

If $\mathcal{FR}(t) > 1$ or $\mathcal{FG}(t) > 0$, the financial institution does not need funding because the selling of the assets covers the repayment of the liabilities. Equations (7.1) and (7.2) correspond to the bankruptcy or the liquidation point of view: if we stop the activity, are there enough assets to meet the liability requirements of the financial institution? Another point of view is to consider that the case $A(t) > L(t)$ requires financing the gap $A(t) - L(t)$, implying that the financial institution has to raise liability funding to match the assets. From that point of view, Equations (7.1) and (7.2) becomes[10]:

$$\mathcal{LR}(t) = \frac{L(t)}{A(t)} \tag{7.3}$$

[10]We use the letter $\mathcal{L}$ (liquidity) instead of $\mathcal{F}$ (funding) in order to make the difference between the two definitions.

and:

$$\mathcal{LG}(t) = L(t) - A(t) \tag{7.4}$$

In what follows, we consider the liquidity gap $\mathcal{LG}(t)$ instead of the funding gap $\mathcal{FG}(t)$, meaning that a positive (resp. negative) gap corresponds to a liquidity excess (resp. liquidity deficit).

Example 66 *We consider a simplified balance sheet with few items. The assets $A(t)$ are composed of loans that are linearly amortized in a monthly basis during the next year. Their values are equal to* 120*. The liabilities $L(t)$ are composed of three short-term in fine debt instruments, and the capital. The corresponding debt notional is respectively equal to* 65*,* 10 *and* 5 *whereas the associated remaining maturity is equal to two, seven and twelve months. The amount of capital is stable for the next twelve months and is equal to* 40*.*

In Table 7.5, we have reported the asset and liability values $A(t)$ and $L(t)$. Since the loans are linearly amortized in a monthly basis, $A(t)$ is equal to 110 after one month, 100 after two months, etc. The value of the first debt instrument remains 65 for the first and second months, and is then equal to zero because the maturity has expired. It follows that the value of the total debt is a piecewise constant function. It is equal to 80 until two months, 15 between three and seven months and 5 after. We can then calculate the liquidity gap. At the initial date, it is equal to zero by definition. At time $t = 1$, we deduce that $\mathcal{LG}(1) = +10$ because we have $A(1) = 110$ and $L(1) = 120$.

TABLE 7.5: Computation of the liquidity gap

Period	0	1	2	3	4	5	6	7	8	9	10	11	12
Loans	120	110	100	90	80	70	60	50	40	30	20	10	0
Assets	120	110	100	90	80	70	60	50	40	30	20	10	0
Debt #1	65	65	65										
Debt #2	10	10	10	10	10	10	10	10					
Debt #3	5	5	5	5	5	5	5	5	5	5	5	5	5
Debt (total)	80	80	80	15	15	15	15	15	5	5	5	5	5
Equity	40	40	40	40	40	40	40	40	40	40	40	40	40
Liabilities	120	120	120	55	55	55	55	55	45	45	45	45	45
$\mathcal{LG}(t)$	0	10	20	−35	−25	−15	−5	5	5	15	25	35	45

The time profile of the liquidity gap is given in Figure 7.2. We notice that it is positive at the beginning, implying that the bank has an excess of liquidity funding in the short-run. Then, we observe that the liquidity gap is negative and the bank needs liquidity funding. From the seventh month, the liquidity gap becomes again positive. At the end, the liquidity gap is always positive since assets and liabilities are fully amortized, implying that the balance sheet is only composed of the capital.

7.1.2.2 Asset and liability amortization

In order to calculate liquidity gaps, we need to understand the amortization of assets and liabilities, in particular the amortization of loans, mortgages, bonds and other debt instruments. The general rules applied to debt payment are the following:

- The annuity amount $A(t)$ at time t is composed of the interest payment $I(t)$ and the principal payment $P(t)$:

$$A(t) = I(t) + P(t)$$

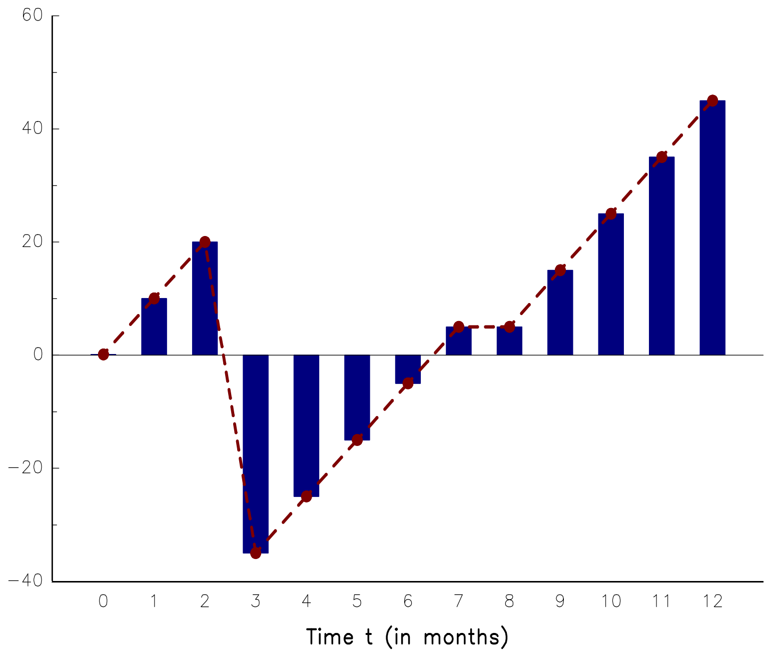

FIGURE 7.2: An example of liquidity gap

This implies that the principal payment at time t is equal to the annuity $A(t)$ minus the interest payment $I(t)$:

$$P(t) = A(t) - I(t)$$

It corresponds to the principal or the capital which is amortized at time t.

- The interest payment at time t is equal to the interest rate $i(t)$ times the outstanding principal balance (or the remaining principal) at the end of the previous period $N(t-1)$:

$$I(t) = i(t)\,N(t-1)$$

- The outstanding principal balance $N(t)$ is the remaining amount due. It is equal to the previous outstanding principal balance $N(t-1)$ minus the principal payment $P(t)$:

$$N(t) = N(t-1) - P(t) \tag{7.5}$$

  At the initial date $t = 0$, the outstanding principal balance is equal to the notional of the debt instrument. At the maturity $t = n$, we must verify that the remaining amount due is equal to zero.

- The outstanding principal balance $N(t)$ is equal to the present value $C(t)$ of forward annuity amounts:

$$N(t) = C(t)$$

We can distinguish different types of debt instruments. For instance, we can assume that the capital is linearly amortized meaning that the principal payment $P(t)$ is constant over time (constant amortization debt). We can also assume that the annuity amount $A(t)$ is constant during the life of the debt instrument (constant payment debt). In this case, the principal payment $P(t)$ is an increasing function with respect to the time t. Another amortization scheme corresponds to the case where the notional is fully repaid at the time of maturity (bullet repayment debt). This is for example the case of a zero-coupon bond.

Let us consider the case where the interest rate $i(t)$ is constant. For the constant amortization debt, we have:

$$P(t) = \frac{1}{n} N_0$$

where n is the number of periods and N_0 is the notional of the mortgage. The cumulative principal payment $Q(t)$ is equal to:

$$Q(t) = \sum_{s \leq t} P(s) = \frac{t}{n} N_0$$

We deduce that the outstanding principal balance $N(t)$ verifies:

$$N(t) = N_0 - Q(t) = \left(1 - \frac{t}{n}\right) N_0$$

We also have $I(t) = iC(t-1)$ where $C(t-1) = N(t-1)$ and:

$$A(t) = I(t) + P(t) = \left(\frac{1}{n} + i\left(1 - \frac{t-1}{n}\right)\right) N_0$$

In Exercise 7.4.1 on page 449, we derive the formulas of the constant payment debt. The constant annuity is equal to:

$$A(t) = A = \frac{i}{1 - (1+i)^{-n}} N_0$$

It is composed of the interest payment:

$$I(t) = \left(1 - \frac{1}{(1+i)^{n-t+1}}\right) A$$

and the principal payment:

$$P(t) = \frac{1}{(1+i)^{n-t+1}} A$$

Moreover, we show that the outstanding principal balance $N(t)$ verifies:

$$N(t) = \left(\frac{1 - (1+i)^{-(n-t)}}{i}\right) A$$

Finally, in the case of the bullet repayment debt, we have $I(t) = iN_0$, $P(t) = \mathbb{1}\{t = n\} \cdot N_0$, $A(t) = I(t) + P(t)$ and $N(t) = \mathbb{1}\{t \neq n\} \cdot N_0$.

Example 67 *We consider a 10-year mortgage, whose notional is equal to* \$100. *The annual interest rate i is equal to* 5%, *and we assume annual principal payments.*

Results are given in Tables 7.6, 7.7 and 7.8. For each payment structure, we have reported the value of the remaining capital $C(t-1)$ at the beginning of the period, the annuity paid at time t, the split between the interest payment $I(t)$ and the principal payment $P(t)$, the cumulative principal payment $Q(t)$. When calculating liquidity gaps, the most important quantity is the outstanding principal balance $N(t)$ given in the last column, because it corresponds to the amortization of the debt.

TABLE 7.6: Repayment schedule of the constant amortization mortgage

t	$C(t-1)$	$A(t)$	$I(t)$	$P(t)$	$Q(t)$	$N(t)$
1	100.00	15.00	5.00	10.00	10.00	90.00
2	90.00	14.50	4.50	10.00	20.00	80.00
3	80.00	14.00	4.00	10.00	30.00	70.00
4	70.00	13.50	3.50	10.00	40.00	60.00
5	60.00	13.00	3.00	10.00	50.00	50.00
6	50.00	12.50	2.50	10.00	60.00	40.00
7	40.00	12.00	2.00	10.00	70.00	30.00
8	30.00	11.50	1.50	10.00	80.00	20.00
9	20.00	11.00	1.00	10.00	90.00	10.00
10	10.00	10.50	0.50	10.00	100.00	0.00

TABLE 7.7: Repayment schedule of the constant payment mortgage

t	$C(t-1)$	$A(t)$	$I(t)$	$P(t)$	$Q(t)$	$N(t)$
1	100.00	12.95	5.00	7.95	7.95	92.05
2	92.05	12.95	4.60	8.35	16.30	83.70
3	83.70	12.95	4.19	8.77	25.06	74.94
4	74.94	12.95	3.75	9.20	34.27	65.73
5	65.73	12.95	3.29	9.66	43.93	56.07
6	56.07	12.95	2.80	10.15	54.08	45.92
7	45.92	12.95	2.30	10.65	64.73	35.27
8	35.27	12.95	1.76	11.19	75.92	24.08
9	24.08	12.95	1.20	11.75	87.67	12.33
10	12.33	12.95	0.62	12.33	100.00	0.00

TABLE 7.8: Repayment schedule of the bullet repayment mortgage

t	$C(t-1)$	$A(t)$	$I(t)$	$P(t)$	$Q(t)$	$N(t)$
1	100.00	5.00	5.00	0.00	0.00	100.00
2	100.00	5.00	5.00	0.00	0.00	100.00
3	100.00	5.00	5.00	0.00	0.00	100.00
4	100.00	5.00	5.00	0.00	0.00	100.00
5	100.00	5.00	5.00	0.00	0.00	100.00
6	100.00	5.00	5.00	0.00	0.00	100.00
7	100.00	5.00	5.00	0.00	0.00	100.00
8	100.00	5.00	5.00	0.00	0.00	100.00
9	100.00	5.00	5.00	0.00	0.00	100.00
10	100.00	105.00	5.00	100.00	100.00	0.00

Previously, we have assumed that the payment type is annual, but we can consider other periods for the amortization schedule. The most common frequencies are monthly, quarterly, semi-annually and annually[11]. Let i be the annual interest rate and p the frequency or the number of compounding periods per year. The consistency principle of the accumulation factor implies the following identity:

$$(1+i) = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

where $i^{(p)}$ is the nominal interest rate expressed in a yearly basis. For example, if the nominal interest rate $i^{(\text{monthly})}$ is equal to 12%, the borrower pays a monthly interest rate of 1%, which corresponds to an annual interest rate of 12.6825%.

Remark 70 *The interest rate i is also called the annual equivalent rate (AER) or the effective annual rate (EAR).*

Example 68 *We consider a 30-year mortgage, whose notional is equal to* \$100*. The annual interest rate i is equal to* 5%*, and we assume monthly principal payments.*

This example is a variant of the previous example, since the maturity is higher and equal to 30 years, and the payment schedule is monthly. This implies that the number n of periods is equal to 360 months and the monthly interest rate is equal to 5%/12 or 41.7 bps. In Figure 7.3, we show the amortization schedule of the mortgage for the three cases: constant (or linear[12]) amortization, constant payment or annuity and bullet repayment. We notice that the constant annuity case is located between the constant amortization and the bullet repayment. We have also reported the constant annuity case when the interest rate is equal to 10%. We notice that we obtain the following ordering:

$$i_1 \geq i_2 \Rightarrow N(t \mid i_1) \geq N(t \mid i_2)$$

where $N(t)(i)$ is the outstanding principal balance given the interest rate i. In fact, constant annuity and constant amortization coincide when the interest rate goes to zero whereas constant annuity and bullet repayment coincide when the interest rate goes to infinity.

Example 69 *We consider the following simplified balance sheet:*

Assets				*Liabilities*			
Items	*Notional*	*Rate*	*Mat.*	*Items*	*Notional*	*Rate*	*Mat.*
Loan #1	100	5%	10	*Debt #1*	120	5%	10
Loan #2	50	8%	16	*Debt #2*	80	3%	5
Loan #3	40	3%	8	*Debt #3*	70	4%	10
Loan #4	110	2%	7	*Capital #4*	30		

The balance sheet is composed of four asset items and four liability items. Asset items correspond to different loans, whose remaining maturity is respectively equal to 10, 16, 8 *and* 7 *years. Liabilities contain three debt instruments and the capital, which is not amortized by definition. All the debt instruments are subject to monthly principal payments.*

In Figure 7.4, we have calculated the liquidity gap for different amortization schedule: constant payment, constant annuity and bullet repayment at maturity. We notice that constant payment and constant annuity give similar amortization schedule. This is not the

[11] Monthly is certainly the most used frequency for debt instruments.

[12] The two terms constant and linear can be used interchangeably.

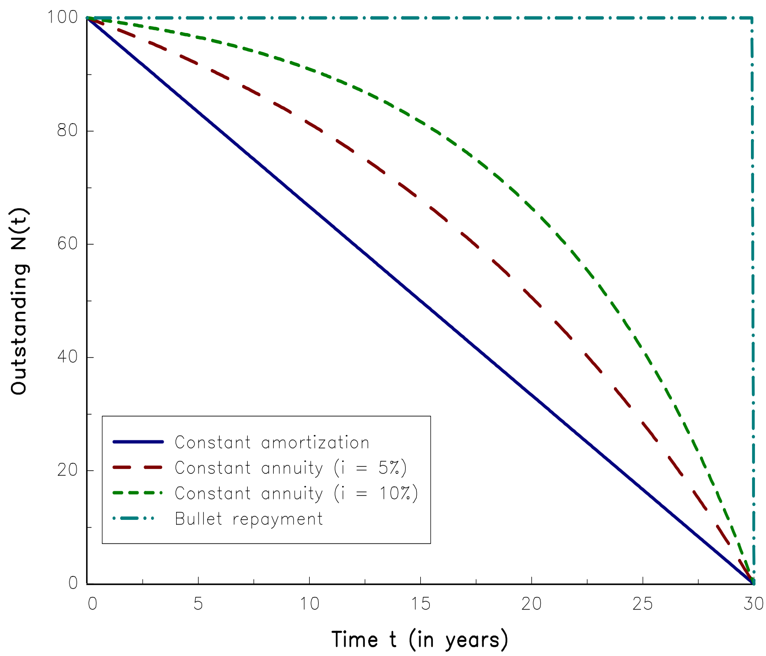

FIGURE 7.3: Amortization schedule of the 30-year mortgage

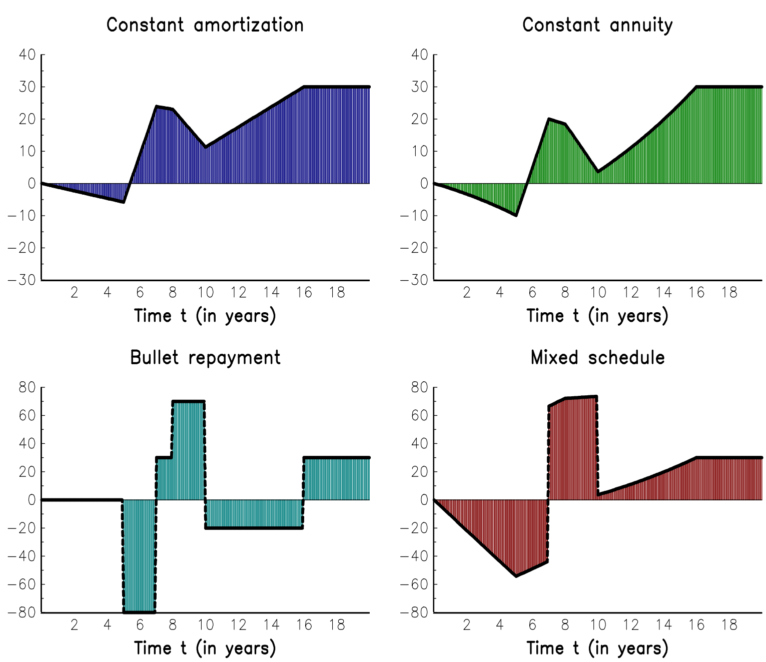

FIGURE 7.4: Impact of the amortization schedule on the liquidity gap

TABLE 7.9: Computation of the liquidity gap (mixed schedule)

t	Assets					Liabilities					$\mathcal{LG}_t$
	#1	#2	#3	#4	A_t	#1	#2	#3	#4	L_t	
1	99.4	49.9	39.6	110	298.8	119.2	78.7	70	30	297.9	−0.92
2	98.7	49.7	39.2	110	297.6	118.5	77.3	70	30	295.8	−1.83
3	98.1	49.6	38.8	110	296.4	117.7	76.0	70	30	293.7	−2.75
4	97.4	49.5	38.3	110	295.2	116.9	74.7	70	30	291.6	−3.66
5	96.8	49.3	37.9	110	294.0	116.1	73.3	70	30	289.4	−4.58
6	96.1	49.2	37.5	110	292.8	115.3	72.0	70	30	287.3	−5.49
7	95.4	49.1	37.1	110	291.6	114.5	70.7	70	30	285.2	−6.41
8	94.8	48.9	36.7	110	290.4	113.7	69.3	70	30	283.1	−7.32
9	94.1	48.8	36.3	110	289.2	112.9	68.0	70	30	280.9	−8.24
10	93.4	48.7	35.8	110	287.9	112.1	66.7	70	30	278.8	−9.15
11	92.8	48.5	35.4	110	286.7	111.3	65.3	70	30	276.7	−10.06
12	92.1	48.4	35.0	110	285.5	110.5	64.0	70	30	274.5	−10.97
0	100.0	50.0	40.0	110	300.0	120.0	80.0	70	30	300.0	0.00
1	92.1	48.4	35.0	110	285.5	110.5	64.0	70	30	274.5	−10.97
2	83.8	46.7	30.0	110	270.4	100.5	48.0	70	30	248.5	−21.90
3	75.0	44.8	25.0	110	254.8	90.1	32.0	70	30	222.1	−32.76
4	65.9	42.7	20.0	110	238.6	79.0	16.0	70	30	195.0	−43.55
5	56.2	40.5	15.0	110	221.7	67.4		70	30	167.4	−54.27
6	46.1	38.1	10.0	110	204.2	55.3		70	30	155.3	−48.91
7	35.4	35.5	5.0		75.9	42.5		70	30	142.5	66.56
8	24.2	32.7			56.9	29.0		70	30	129.0	72.12
9	12.4	29.7			42.1	14.9		70	30	114.9	72.81
10		26.4			26.4				30	30.0	3.62
11		22.8			22.8				30	30.0	7.19
12		18.9			18.9				30	30.0	11.06
13		14.8			14.8				30	30.0	15.24
14		10.2			10.2				30	30.0	19.77
15		5.3			5.3				30	30.0	24.68
16					0.0				30	30.0	30.00

case of bullet repayment. In the fourth panel, we consider a more realistic situation where we have both constant principal (loan #3 and debt #2), constant annuity (loan #1, loan #2 and debt #1) and bullet repayment (loan #4 and debt #2). Computation details for this last mixed schedule are given in Table 7.9. The top panel presents the liquidity gap $\mathcal{LG}(t)$ of the first twelve months while the bottom panel corresponds to the annual schedule. The top panel is very important since it corresponds to the first year, which is the standard horizon used by the ALCO for measuring liquidity requirements. We see that the bank will face a liquidity deficit during the first year.

The previous analysis does not take into account two important phenomena. The first one concerns customer behaviorial options such as prepayment decisions. We note $N^c(t)$ the conventional outstanding principal balance that takes into account the prepayment risk. We have:

$$N^c(t) = N(t) \cdot \mathbb{1}\{\boldsymbol{\tau} > t\}$$

where $N(t)$ is the theoretical outstanding principal balance and $\boldsymbol{\tau}$ is the prepayment time of the debt instrument. The prepayment time in ALM modeling is equivalent to the survival or default time that we have seen in credit risk modeling. Then $\boldsymbol{\tau}$ is a random variable, which is described by its survival function $\mathbf{S}(t)$. Let $p(t)$ be the probability that the debt instrument has not been repaid at time t. We have:

$$p(t) = \mathbb{E}[\mathbb{1}\{\boldsymbol{\tau} > t\}] = \mathbf{S}(t)$$

By construction, $N^c(t)$ is also random. Therefore, we can calculate its mathematical expectation, and we have $\bar{N}^c(t) = \mathbb{E}[N^c(t)] = p(t) \cdot N(t)$ For example, if we assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ where λ is the prepayment intensity, we obtain $\bar{N}^c(t) = e^{-\lambda t} \cdot N(t)$. By definition, we always have $N^c(t) \leq N(t)$ and $\bar{N}^c(t) \leq N(t)$.

In Figure 7.5, we consider the constant payment mortgage given in Example 68 on page 381. The first panel shows the theoretical or contractual outstanding principal balance. In the second and third panels, we consider that there is a prepayment at time $\boldsymbol{\tau} = 10$ and $\boldsymbol{\tau} = 20$. This conventional schedule coincides with the contractual schedule, but is equal to zero once the prepayment time occurs. Finally, the fourth panel presents the conventional amortization schedule $\bar{N}^c(t)$ when the prepayment time is exponentially distributed. When λ is equal to zero, we retrieve the previous contractual schedule $N(t)$. Otherwise, the mortgage amortization is quicker.

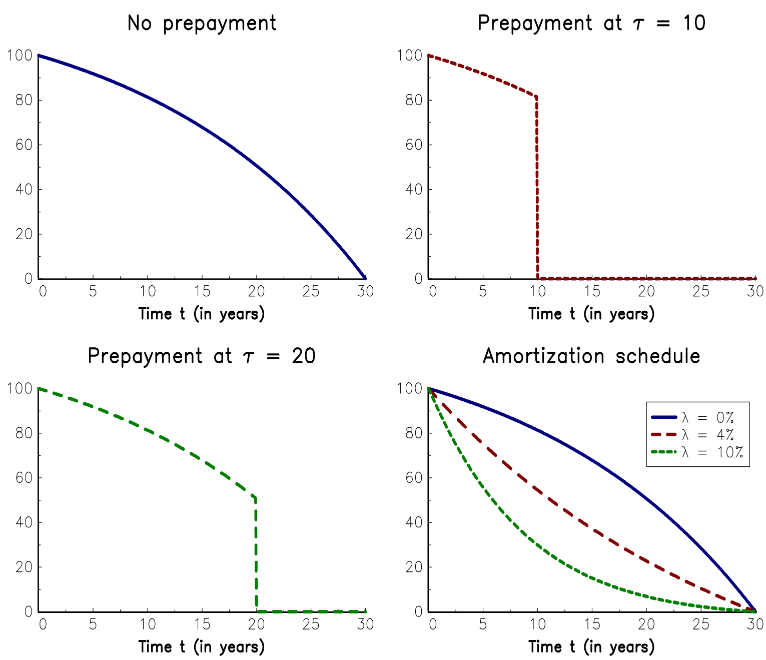

FIGURE 7.5: Conventional amortization schedule with prepayment risk

The second important phenomenon that impacts amortization schedule is the new production of assets and liabilities. If we consider a balance sheet item, its outstanding amount at time t is equal to the outstanding amount at time $t-1$ minus the amortization between t and $t-1$ plus the new production at time t:

$$N(t) = N(t-1) - \mathrm{AM}(t) + \mathrm{NP}(t) \tag{7.6}$$

This relationship is illustrated in Figure 7.6 and can be considered as an accounting identity (Demey *et al.*, 2003). In the case where there is no prepayment, the amortization $\mathrm{AM}(t)$ is exactly equal to the principal payment $P(t)$ and we retrieve Equation (7.5) except the term $\mathrm{NP}(t)$. However, there is a big difference between Equations (7.6) and (7.5). The first one describes the amortization of a debt instrument, for example a loan or a mortgage. The

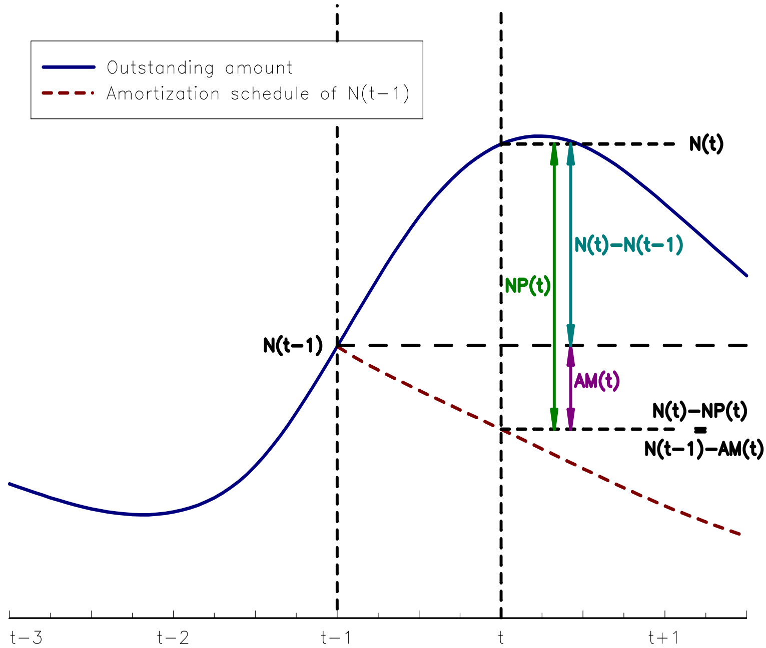

FIGURE 7.6: Impact of the new production on the outstanding amount

Source: Demey *et al.* (2003).

second one describes the amortization of a balance sheet item, that is the aggregation of several debt instruments. The new production $\mathrm{NP}(t)$ corresponds to the financial transactions that appear in the balance sheet between t and $t-1$. They concern the new credit lines, customer loans, mortgages, deposits, etc. that have been traded by the bank during the last period $[t-1, t]$. The introduction of the new production leads to the concept of dynamic liquidity gap, in contrast to the static liquidity gap.

Remark 71 *As we will see in the next section, dynamic liquidity analysis is then more complex since the function* $\mathrm{NP}(t)$ *is not always known and depends on many parameters. Said differently,* $\mathrm{NP}(t)$ *is more a random variable. However, it is more convenient to treat* $\mathrm{NP}(t)$ *as a deterministic function than a stochastic function in order to obtain closed-form formula and not to use Monte Carlo methods*[13]*.*

7.1.2.3 Dynamic analysis

According to BCBS (2016d) and EBA (2018a), we must distinguish three types of analysis:

- Run-off balance sheet
  A balance sheet where existing non-trading book positions amortize and are not replaced by any new business.

[13] Equation (7.6) can also be written as follows:

$$\mathrm{NP}(t) = N(t) - (N(t-1) - \mathrm{AM}(t))$$

Written in this form, this equation indicates how to calculate the new production. In particular, this relationship can be used to define an estimator of $\mathrm{NP}(t)$.

- Constant balance sheet
  A balance sheet in which the total size and composition are maintained by replacing maturing or repricing cash flows with new cash flows that have identical features.

- Dynamic balance sheet
  A balance sheet incorporating future business expectations, adjusted for the relevant scenario in a consistent manner.

The run-off balance sheet analysis has been exposed in the previous section. The constant or dynamic balance sheet analysis assumes that we include the new production when calculating the liquidity gap. For the constant analysis, this task is relatively easy since we consider a like-for-like replacement of assets and liabilities. The dynamic analysis is more difficult to implement because it highly depends "*on key variables and assumptions that are extremely difficult to project with accuracy over an extended period and can potentially hide certain key underlying risk exposures*" (BCBS, 2016d, page 8).

Stock-flow analysis According to Demey *et al.* (2003), the non-static analysis requires a mathematical framework in order to distinguish stock and flow streams. We follow these authors, and more particularly we present the tools introduced in Chapter 1 of their book. We note $\mathrm{NP}(t)$ the new production at time t and $\mathrm{NP}(t,u)$ the part of this production[14] that is always reported in the balance sheet at time $u \geq t$. The amortization function $\mathbf{S}(t,u)$ is defined by the following equation:

$$\mathrm{NP}(t,u) = \mathrm{NP}(t) \cdot \mathbf{S}(t,u)$$

The amortization function is in fact a survival function, implying that the following properties hold: $\mathbf{S}(t,t) = 1$, $\mathbf{S}(t,\infty) = 0$ and $\mathbf{S}(t,u)$ is a decreasing function with respect to u. The amortization function is homogeneous if we have $\mathbf{S}(t,u) = \mathbf{S}(u-t)$ for all $u \geq t$. Otherwise, amortization function is non-homogeneous and may depend on the information $\mathcal{I}_{t:u}$ between t and u. In this case, we can write $\mathbf{S}(t,u) = \mathbf{S}(t,u;\mathcal{I}_{t:u})$ where $\mathcal{I}_{t:u}$ may contain the trajectory of interest rates, the history of prepayment times, etc. We define the amortization rate as the hazard rate associated to the survival function $\mathbf{S}(t,u)$:

$$\lambda(t,u) = -\frac{\partial \ln \mathbf{S}(t,u)}{\partial u}$$

In management, we generally make the distinction between stock and flow streams, but we know that the stock at time t is the sum of past flows. In the case of ALM, the outstanding amount plays the role of stock while the new production corresponds to a flow. Therefore, the outstanding amount at time t is the sum of past productions that are always present in the balance sheet at time t:

$$N(t) = \int_0^\infty \mathrm{NP}(t-s,t)\,\mathrm{d}s$$

If follows that:

$$\begin{aligned} N(t) &= \int_0^\infty \mathrm{NP}(t-s)\,\mathbf{S}(t-s,t)\,\mathrm{d}s \\ &= \int_{-\infty}^t \mathrm{NP}(s)\,\mathbf{S}(s,t)\,\mathrm{d}s \end{aligned} \tag{7.7}$$

[14] We have $\mathrm{NP}(t) = \mathrm{NP}(t,t)$ and $\mathrm{NP}(t,\infty) = 0$.

TABLE 7.10: Relationship between the new production and the outstanding amount

s	$\text{NP}(s)$	$\mathbf{S}(s,7)$	$\text{NP}(s,7)$	$\mathbf{S}(s,10)$	$\text{NP}(s,10)$	$\mathbf{S}(s,12)$	$\text{NP}(s,12)$
1	110	0.301	33.13	0.165	18.18	0.111	12.19
2	125	0.368	45.98	0.202	25.24	0.135	16.92
3	95	0.449	42.69	0.247	23.43	0.165	15.70
4	75	0.549	41.16	0.301	22.59	0.202	15.14
5	137	0.670	91.83	0.368	50.40	0.247	33.78
6	125	0.819	102.34	0.449	56.17	0.301	37.65
7	115	1.000	115.00	0.549	63.11	0.368	42.31
8	152			0.670	101.89	0.449	68.30
9	147			0.819	120.35	0.549	80.68
10	159			1.000	159.00	0.670	106.58
11	152					0.819	124.45
12	167					1.000	167.00
$N(t)$			472.14		640.36		720.69

In the discrete-time analysis, the previous relationship becomes:

$$\begin{aligned} N(t) &= \sum_{s=0}^{\infty} \text{NP}(t-s,t) \\ &= \sum_{s=-\infty}^{t} \text{NP}(s) \cdot \mathbf{S}(s,t) \end{aligned}$$

The outstanding amount $N(t)$ at time t is then the sum of each past production $\text{NP}(s)$ times its amortization function $\mathbf{S}(s,t)$. In Table 7.10, we provide an example of calculating the outstanding amount using the previous convolution method. In the second column, we report the production of each year s. We assume that the amortization function is homogeneous and is an exponential distribution with an intensity λ equal to 20%. The third and fourth columns give the values of the amortization function and the production that is present in the balance sheet at time $t = 7$. We obtain $N(7) = 472.14$. The four last columns correspond to the cases $t = 10$ and $t = 12$.

Demey *et al.* (2003) introduce the concept of stock amortization. We recall that the amortization function $\mathbf{S}(t,u)$ indicates the proportion of \$1 entering in the balance sheet at time t that remains present at time $u \geq t$. Similarly, the stock amortization function $\mathbf{S}^{\star}(t,u)$ measures the proportion of \$1 of outstanding amount at time t that remains present at time $u \geq t$. In order to obtain an analytical and tractable function $\mathbf{S}^{\star}(t,u)$, we must assume that the new production is equal to zero after time t. This corresponds to the run-off balance sheet analysis. Demey *et al.* (2003) show that the non-amortized outstanding amount is equal to:

$$N(t,u) = \int_{-\infty}^{t} \text{NP}(s)\, \mathbf{S}(s,u)\, \mathrm{d}s$$

where t is the current time and u is the future date. For instance, $N(5,10)$ indicates the outstanding amount that is present in the balance sheet at time $t = 5$ and will remain in the balance sheet five years after. It follows that:

$$N(t,u) = N(t) \cdot \mathbf{S}^{\star}(t,u)$$

and we deduce that:

$$\begin{aligned} \mathbf{S}^{\star}(t,u) &= \frac{N(t,u)}{N(t)} \\ &= \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\,\mathbf{S}(s,u)\,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\,\mathbf{S}(s,t)\,\mathrm{d}s} \end{aligned}$$

Dynamics of the outstanding amount Using Equation (7.7), we obtain[15]:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\int_{-\infty}^{t} \mathrm{NP}(s)\, f(s,t)\,\mathrm{d}s + \mathrm{NP}(t) \tag{7.8}$$

where $f(t,u) = -\partial_u \mathbf{S}(t,u)$ is the density function of the amortization. This is the continuous-time version of the amortization schedule given by Equation (7.6):

$$N(t) - N(t-1) = -\mathrm{AM}(t) + \mathrm{NP}(t)$$

where:

$$\mathrm{AM}(t) = \int_{-\infty}^{t} \mathrm{NP}(s)\, f(s,t)\,\mathrm{d}s$$

As already said, we notice the central role of the new production when building a dynamic gap analysis. It is obvious that the new production depends on several parameters, for example the commercial policy of the bank, the competitive environment, etc.

Estimation of the dynamic liquidity gap We can then define the dynamic liquidity gap at time t for a future date $u \geq t$ as follows[16]:

$$\begin{aligned} \mathcal{LG}(t,u) = & \sum_{k \in \mathcal{Liabilities}} \left(N_k(t,u) + \int_t^u \mathrm{NP}_k(s)\,\mathbf{S}_k(s,u)\,\mathrm{d}s \right) - \\ & \sum_{k \in \mathcal{Assets}} \left(N_k(t,u) + \int_t^u \mathrm{NP}_k(s)\,\mathbf{S}_k(s,u)\,\mathrm{d}s \right) \end{aligned}$$

where k represents a balance sheet item. This is the difference between the liability outstanding amount and the asset outstanding amount. For a given item k, the dynamic outstanding amount is composed of the outstanding amount $N_k(t,u)$ that will be non-amortized at time u plus the new production between t and u that will be in the balance sheet at time u. The difficulty is then to estimate the new production and the amortization function. As said previously, the new production generally depends on the business strategy of the bank.

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[15] We have:

$$\begin{aligned} \mathrm{d}\left(\int_{-\infty}^{t} \mathrm{NP}(s)\,\mathbf{S}(s,t)\,\mathrm{d}s \right) &= \mathrm{NP}(t)\,\mathbf{S}(t,t)\,\mathrm{d}t + \int_{-\infty}^{t} \mathrm{NP}(s)\,\frac{\partial\,\mathbf{S}(s,t)}{\partial\,t}\,\mathrm{d}s \\ &= \mathrm{NP}(t)\,\mathrm{d}t - \int_{-\infty}^{t} \mathrm{NP}(s)\left(-\frac{\partial\,\mathbf{S}(s,t)}{\partial\,t}\right)\mathrm{d}s \end{aligned}$$

[16] In the case of the run-off balance sheet, we set $\mathrm{NP}_k(s) = 0$ and we obtain the following formula:

$$\mathcal{LG}(t,u) = \sum_{k \in \mathcal{Liabilities}} N_k(t,u) - \sum_{k \in \mathcal{Assets}} N_k(t,u)$$

Concerning the amortization function, we can calibrate $\mathbf{S}_k(t,u)$ using a sample of new productions if we assume that the amortization function is homogenous: $\mathbf{S}_k(t,u) = \mathbf{S}_k(u-t)$. It follows that:

$$\hat{\mathbf{S}}_k(u-t) = \frac{\sum_{j\in k} \mathrm{NP}_j(t,u)}{\sum_{j\in k} \mathrm{NP}_j(t)}$$

Moreover, we can show that $\hat{\mathbf{S}}_k(u-t)$ is a convergent estimator and its asymptotic distribution is given by:

$$\hat{\mathbf{S}}_k(u-t) - \mathbf{S}_k(u-t) \to \mathcal{N}\left(0, H \cdot \mathbf{S}_k(u-t) \cdot (1 - \mathbf{S}_k(u-t))\right)$$

where H is the Herfindahl index associated to the sample of new productions[17].

Remark 72 *This result can be deduced from the empirical estimation theory. Let $\mathbf{S}(t)$ be the survival function of the survival time $\boldsymbol{\tau}$. The empirical survival function of the weighted sample $\{(w_j, \boldsymbol{\tau}_j), j = 1, \ldots, n\}$ is equal to:*

$$\hat{\mathbf{S}}(t) = \frac{\sum_{j=1}^n w_j \cdot D_j}{\sum_{j=1}^n w_j}$$

where $D_j = \mathbb{1}\{\boldsymbol{\tau}_j > t\}$ is a Bernoulli random variable with parameter $p = \mathbf{S}(t)$. If we assume that the sample observations are independent, we deduce that:

$$\mathrm{var}\left(\hat{\mathbf{S}}(t)\right) = \frac{\sum_{j=1}^n w_j^2 \cdot \mathrm{var}(D_j)}{\left(\sum_{j=1}^n w_j\right)^2} = \sum_{j=1}^n \frac{w_j^2}{\left(\sum_{j'=1}^n w_{j'}\right)^2} \cdot \mathbf{S}(t) \cdot (1 - \mathbf{S}(t))$$

Example 70 *We consider a sample of five loans that belong to the same balance sheet item. Below, we have reported the value taken by $\mathrm{NP}_j(t,u)$:*

$u-t$	0	1	2	3	4	5	6	7	8	9	10	11
#1	100	90	80	70	60	50	40	30	20	10	5	0
#2	70	65	55	40	20	10	5	0				
#3	100	95	85	80	60	40	20	10	0			
#4	50	47	44	40	37	33	27	17	10	7	0	
#5	20	18	16	14	10	8	5	3	0			

In Figure 7.7, we have estimated the amortization function $\hat{\mathbf{S}}(u-t)$. We have also computed the variance of the estimator and reported the 95% confidence interval[18].

Liquidity duration Another important tool to measure the mismatch between assets and liabilities is to calculate the liquidity duration, which is defined as the average time of the amortization of the new production $\mathrm{NP}(t)$. In a discrete-time analysis, the amortization value between two consecutive dates is equal to $\mathrm{NP}(t,u) - \mathrm{NP}(t,u+1)$. Therefore, the liquidity duration is the weighted average life (WAL) of the principal repayments:

$$\mathcal{D}(t) = \frac{\sum_{u=t}^{\infty} (\mathrm{NP}(t,u) - \mathrm{NP}(t,u+1)) \cdot (u-t)}{\sum_{u=t}^{\infty} (\mathrm{NP}(t,u) - \mathrm{NP}(t,u+1))}$$

[17] We have $H = \sum_{j\in k} w_j^2$ where:

$$w_j = \frac{\mathrm{NP}_j(t)}{\sum_{j'\in k} \mathrm{NP}_{j'}(t)}$$

[18] We have assumed that the sample is composed of 100 loans or 20 copies of the five previous loans. Otherwise, the confidence interval is too large because the sample size is small.

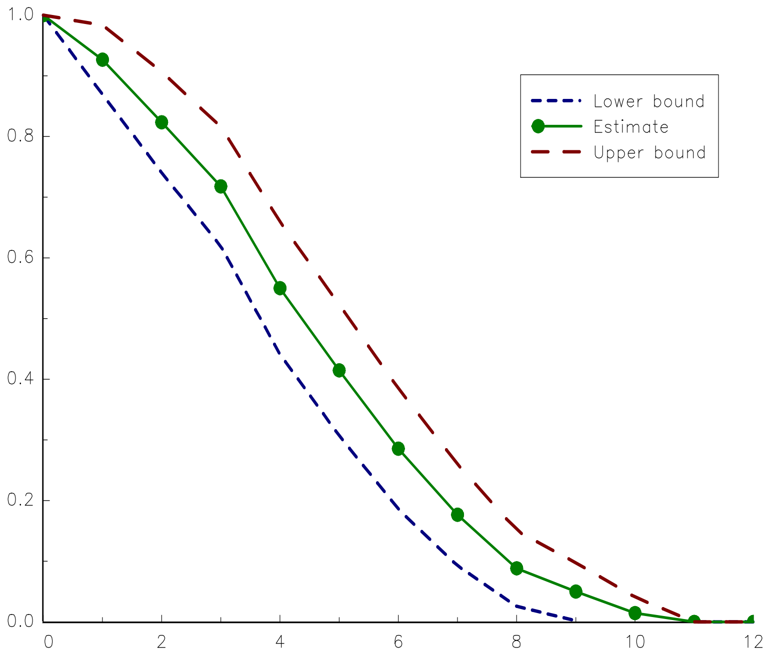

FIGURE 7.7: Estimation of the amortization function $\hat{\mathbf{S}}(u-t)$

Since we have:

$$\mathrm{NP}(t,u) - \mathrm{NP}(t,u+1) = -\mathrm{NP}(t) \cdot (\mathbf{S}(t,u+1) - \mathbf{S}(t,u))$$

and[19]:

$$\begin{aligned} \sum_{u=t}^{\infty} (\mathrm{NP}(t,u) - \mathrm{NP}(t,u+1)) &= -\mathrm{NP}(t) \cdot \sum_{u=t}^{\infty} (\mathbf{S}(t,u+1) - \mathbf{S}(t,u)) \\ &= \mathrm{NP}(t) \end{aligned}$$

we obtain the following formula:

$$\mathcal{D}(t) = -\sum_{u=t}^{\infty} (\mathbf{S}(t,u+1) - \mathbf{S}(t,u)) \cdot (u-t)$$

In the continuous-time analysis, the liquidity duration is equal to:

$$\begin{aligned} \mathcal{D}(t) &= -\int_t^{\infty} \frac{\partial\,\mathbf{S}(t,u)}{\partial\,u}(u-t)\,\mathrm{d}u \\ &= \int_t^{\infty} (u-t)\,f(t,u)\,\mathrm{d}u \end{aligned}$$

where $f(t,u)$ is the density function associated to the survival function $\mathbf{S}(t,u)$.

Remark 73 *If we consider the stock approach of the liquidity duration, we have:*

$$\mathcal{D}^{\star}(t) = \int_t^{\infty} (u-t)\,f^{\star}(t,u)\,\mathrm{d}u$$

[19]Because we have $\mathbf{S}(t,t) = 1$ and $\mathbf{S}(t,\infty) = 0$.

where $f^{\star}(t,u)$ is the density function associated to the survival function $\mathbf{S}^{\star}(t,u)$:

$$\begin{aligned} f^{\star}(t,u) &= -\frac{\partial\,\mathbf{S}^{\star}(t,u)}{\partial\,u} \\ &= \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, f(s,u)\, \mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, \mathbf{S}(s,t)\, \mathrm{d}s} \end{aligned}$$

Some examples We consider the three main amortization schemes: bullet repayment, constant (or linear) amortization and exponential amortization. In Exercise 7.4.3 on page 450, we have calculated the survival functions $\mathbf{S}(t,u)$ and $\mathbf{S}^{\star}(t,u)$, the liquidity duration $\mathcal{D}(t)$ and $\mathcal{D}^{\star}(t)$ and the outstanding dynamics $\mathrm{d}N(t)$ where m is the debt maturity and λ is the exponential parameter. Their expression is reported below in Table 7.11.

TABLE 7.11: Amortization function and liquidity duration of the three amortization schemes

Amortization	$\mathbf{S}(t,u)$	$\mathcal{D}(t)$
Bullet	$\mathbb{1}\{t \le u < t+m\}$	m
Constant	$\mathbb{1}\{t \le u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)$	$\frac{m}{2}$
Exponential	$e^{-\lambda(u-t)}$	$\frac{1}{\lambda}$
Amortization	$\mathbf{S}^{\star}(t,u)$	$\mathcal{D}^{\star}(t)$
Bullet	$\mathbb{1}\{t \le u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)$	$\frac{m}{2}$
Constant	$\mathbb{1}\{t \le u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)^2$	$\frac{m}{3}$
Exponential	$e^{-\lambda(u-t)}$	$\frac{1}{\lambda}$
Amortization	$\mathrm{d}N(t)$	
Bullet	$\mathrm{d}N(t) = (\mathrm{NP}(t) - \mathrm{NP}(t-m))\, \mathrm{d}t$	
Constant	$\mathrm{d}N(t) = \left(\mathrm{NP}(t) - \frac{1}{m}\int_{t-m}^{t} \mathrm{NP}(s)\, \mathrm{d}s\right) \mathrm{d}t$	
Exponential	$\mathrm{d}N(t) = (\mathrm{NP}(t) - \lambda N(t))\, \mathrm{d}t$	

We have represented these amortization functions $\mathbf{S}(t,u)$ and $\mathbf{S}^{\star}(t,u)$ in Figure 7.8. The maturity m is equal to 10 years and the exponential parameter λ is set to 30%. Besides the three previous amortization schemes, we also consider the constant payment mortgage (CPM), whose survival functions are equal to[20]:

$$\mathbf{S}(t,u) = \mathbb{1}\{t \le u < t+m\} \cdot \frac{1 - e^{-i(t+m-u)}}{1 - e^{-im}}$$

and:

$$\mathbf{S}^{\star}(t,u) = \frac{i(t+m-u) + e^{-i(t+m-u)} - 1}{im + e^{-im} - 1}$$

[20]These expressions are derived in Exercise 7.4.3 on page 450.

where i is the interest rate and m is the debt maturity. The CPM amortization scheme corresponds to the bottom/right panel[21] in Figure 7.8.

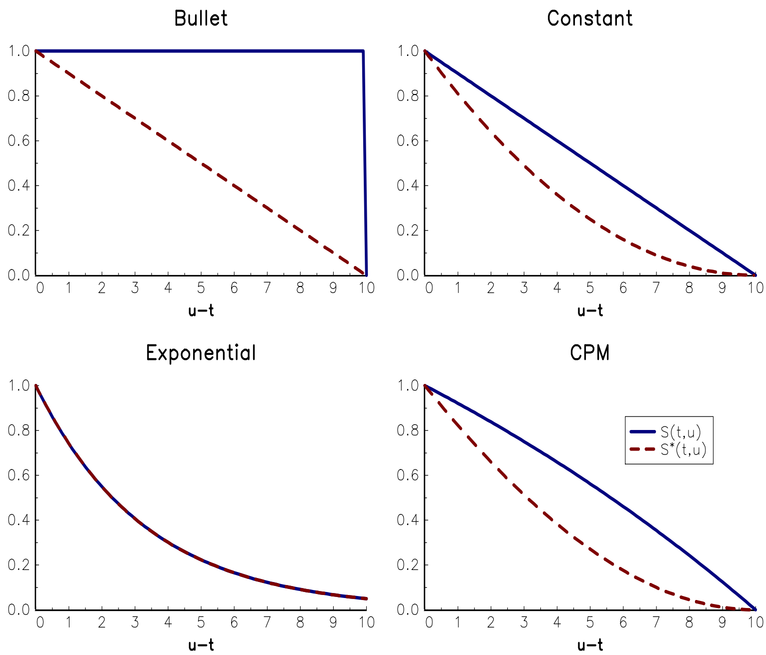

FIGURE 7.8: Amortization functions $\mathbf{S}(t,u)$ and $\mathbf{S}^{\star}(t,u)$

Remark 74 *We notice the convex profile of the constant and exponential amortization schemes, whereas the profile is concave for the CPM amortization scheme. Moreover, when the interest rate i goes to zero, the CPM profile corresponds to the constant profile.*

7.1.2.4 Liquidity hedging

When we face a risk that is not acceptable, we generally hedge it. In the case of the liquidity, the concept of hedging is unclear. Indeed, at first sight, it seems that there are no liquidity forwards, swaps or options in the market. On the other hand, liquidity hedging seems to be trivial. Indeed, the bank can lend to other market participants when having an excess of funding, or the bank can borrow when having a deficit of funding. For that, it may use the interbank market or the bond market. Nevertheless, there is generally an uncertainty about the liquidity gap, because the amortization schedule and the new production are not known for sure. This is why banks must generally adopt a conservative approach. For instance, they must not lend (or buy bonds) too much. In a similar way, they must not borrow too short. The liquidity gap analysis is particularly important in order to split the decision between daily, weekly, monthly and quarterly adjustments. Let us assume that the bank anticipates a liquidity deficit of \$10 mn for the next three months. It can borrow exactly \$10 mn for three months. One month later, the bank has finally an excess of liquidity. It is obvious that the previous lending is not optimal because the bank must pay a three-month interest rate while it could have paid a one-month interest rate.

The previous example shows that the management of the liquidity consists in managing interbank and bond operations. It is obvious that the funding program depends on the

[21] The interest rate i is set to 5%.

liquidity gap but also on the risk appetite of the bank. Some banks prefer to run a long-term liquidity program, others prefer to manage the liquidity on a shorter-term basis. The ALCO decisions may have therefore a big impact on the risk profile of the bank. The 2008 Global Financial Crisis has demonstrated that liquidity management is key during periods of stress. For instance, a bank, which has a structural liquidity excess, may stop to lend to the other participants in order to keep this liquidity for itself, while a bank, which has a structural liquidity need, may issue long-term debt in order to reduce day-to-day funding requirements. It is clear that ALCO decisions are beyond the scope of risk management and fall within strategic and business issues.

7.1.3 Interest rate risk in the banking book

The ALM of interest rate risk is extensively developed in the next section. However, we give here the broad lines, notably the regulation framework, which has been elaborated by the Basel Committee in April 2016 (BCBS, 2016d) and which is known as IRRBB (or interest rate risk in the banking book). IRRBB can be seen as the revision of the 2004 publication (BCBS, 2004b), but not solely. Indeed, this 2016 publication is relatively precise in terms of risk framework and defines a standardized framework, which was not the case in 2004. In particular, capital requirements are more closely supervised than previously, even if IRRBB continues to be part of the Basel capital framework's Pillar 2.

7.1.3.1 Introduction on IRRBB

Definition of IRRBB According to BCBS (2016d), "*IRRBB refers to the current or prospective risk to the bank' capital and earnings arising from adverse movements in interest rates that affect the bank's banking book positions. When interest rates change, the present value and timing of future cash flows change. This in turn changes the underlying value of a bank's assets, liabilities and off-balance sheet items and hence its economic value. Changes in interest rates also affect a bank's earnings by altering interest rate-sensitive income and expenses, affecting its net interest income*". We notice that the Basel Committee considers both economic value (EV) and earnings-based risk measures. EV measures reflect changes in the net present value of the balance sheet resulting from IRRBB, whereas earnings-based measures reflect changes in the expected future profitability of the bank. Since EV measures are generally used by supervisors[22] and earnings-based measures are more widely used by commercial banks[23], the Basel Committee thinks that the bank must manage these two risks because they capture two different time horizons. Economic value is calculated over the remaining life of debt instruments, implying a run-off balance sheet assumption. The earnings-based measure is calculated for a given time horizon, typically the next 12 month period. In this case, a constant or dynamic balance sheet assumption is more appropriate.

Categories of IRR For the Basel Committee, there are three main sources of interest rate risk: gap risk, basis risk and option risk. Gap risk refers to the mismatch risk arising from the term structure of banking book instruments. It includes repricing risk and yield curve risk. Repricing risk corresponds to timing differences in the maturity or the risk of changes in interest rates between assets and liabilities. For example, if the bank funds a long-term fixed-rate loan with a short-term floating-rate deposit, the future income may decrease if interest rates increases. Therefore, repricing risk has two components. The first one is the maturity difference between assets and liabilities. The second one is the change in

[22] Because they are more adapted for comparing banks.
[23] Because banks want to manage the volatility of earnings.

floating interest rates. Yield curve risk refers to non-parallel changes in the term structure of interest rates. A typical example concerns flattening, when short-term interest rates rise faster than long-term interest rates.

Basis risk occurs when changes in interest rates impact differently financial instruments with similar repricing tenors, because they are priced using different interest rate indices. Therefore, basis risk corresponds to the correlation risk of interest rate indices with the same maturity. For example, the one-month Libor rate is not perfectly correlated to the one-month Treasury rate. Thus, there is a basis risk when a one-month Treasury-based asset is funded with a one-month Libor-based liability, because the margin can change from one month to another month.

Option risk arises from option derivative positions or when the level or the timing of cash flows may change due to embedded options. A typical example is the prepayment risk. The Basel Committee distinguishes automatic option risk and behavioral option risk. Automatic options concern caps, floors, swaptions and other interest rate derivatives that are located in the banking book, while behavioral option risk includes fixed rate loans subject to prepayment risk, fixed rate loan commitments, term deposits subject to early redemption risk and non-maturity deposits (or NMDs).

Risk measures The economic value of a series of cash flows $\mathrm{CF} = \{\mathrm{CF}(t_k), t_k \geq t\}$ is the present value of these cash flows:

$$\begin{aligned} \mathrm{EV} &= PV_t(\mathrm{CF}) \\ &= \mathbb{E}\left[\sum_{t_k \geq t} \mathrm{CF}(t_k) \cdot e^{-\int_t^{t_k} r(s)\,\mathrm{d}s}\right] \\ &= \sum_{t_k \geq t} \mathrm{CF}(t_k) \cdot B(t, t_k) \end{aligned}$$

where $B_t(t_k)$ is the discount factor (e.g. the zero-coupon bond) for the maturity date t_k. To calculate the economic value of the banking book, we slot all notional repricing cash flows of assets and liabilities into a set of time buckets. Then, we calculate the net cash flows, which are equal to $\mathrm{CF}(t_k) = \mathrm{CF}_A(t_k) - \mathrm{CF}_L(t_k)$ where $\mathrm{CF}_A(t_k)$ and $\mathrm{CF}_L(t_k)$ are the cash flows of assets and liabilities for the time bucket t_k. Finally, the economic value is given by:

$$\begin{aligned} \mathrm{EV} &= \sum_{t_k \geq t} \mathrm{CF}(t_k) \cdot B(t, t_k) \\ &= \sum_{t_k \geq t} \mathrm{CF}_A(t_k) \cdot B(t, t_k) - \sum_{t_k \geq t} \mathrm{CF}_L(t_k) \cdot B(t, t_k) \\ &= \mathrm{EV}_A - \mathrm{EV}_L \end{aligned}$$

It is equal to the present value of assets minus the present value of liabilities. By construction, the computation of EV depends on the yield curve. We introduce the notation s in order to take into account a stress scenario of the yield curve. Then, we define the EV change as the difference between the EV for the base scenario and the EV for the given scenario s:

$$\begin{aligned} \Delta \mathrm{EV}_s &= \mathrm{EV}_0 - \mathrm{EV}_s \\ &= \sum_{t_k \geq t} \mathrm{CF}_0(t_k) \cdot B_0(t, t_k) - \sum_{t_k \geq t} \mathrm{CF}_s(t_k) \cdot B_s(t, t_k) \end{aligned}$$

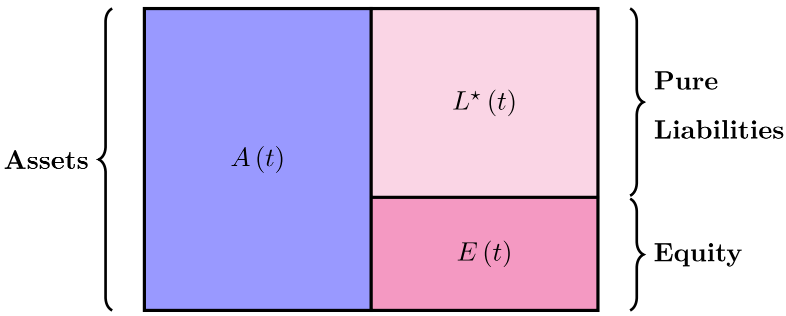

FIGURE 7.9: Relationship between $A(t)$, $L^{\star}(t)$ and $E(t)$

In this equation, the base scenario is denoted by 0 and corresponds to the current term structure of interest rates. The stress scenario s of the yield curve impacts the discount factors, but also the cash flows that depend on the future interest rates. $\Delta\,\mathrm{EV}_s > 0$ indicates then a loss if the stress scenario s occurs. The Basel Committee defines the concept of economic value of equity (EVE or EV_E) as a specific form of EV where equity is excluded from the cash flows. We recall that the value of assets is equal to the value of liabilities at the current time t. If we distinguish pure liabilities $L^{\star}(t)$ from the bank equity capital $E(t)$, we obtain the balance sheet given in Figure 7.9. Since there is a perfect match between assets and liabilities, the value of the capital is equal to[24] $E(t) = A(t) - L^{\star}(t)$. It follows that:

$$\mathrm{EVE} = \mathrm{EV}_A - \mathrm{EV}_{L^{\star}}$$

We can then define $\Delta\,\mathrm{EVE}_s$ as the loss $\Delta\,\mathrm{EV}_s$ where we have excluded the equity from the computation of the cash flows. Said differently, $\Delta\,\mathrm{EVE}_s$ is the capital loss if the stress scenario s occurs.

Remark 75 *Changes in economic value can also be measured with the PV01 metric or the economic value-at-risk (EVaR). PV01 is calculated by assuming a single basis point change in interest rates. EVaR is the value-at-risk measure applied to the economic value of the banking book. Like the VaR, it requires specifying the holding period and the confidence level. The Basel Committee motivates the choice of EVE instead of PV01 and EVaR, because they would like to measure the impact of losses on the capital in a stress testing framework. In particular, PV01 ignores basis risks whereas EVaR is designed for normal market circumstances.*

Earnings-based measures are computed using the net interest income (NII), which is the difference between the interest payments on assets and the interest payments of liabilities. Said differently, NII is the difference between interest rate revenues received by the bank and interest rate costs paid by the bank. For a given scenario s, we define the change in net interest income as follows:

$$\Delta\,\mathrm{NII}_s = \mathrm{NII}_0 - \mathrm{NII}_s$$

Like for the risk measure $\Delta\,\mathrm{EVE}_s$, $\Delta\,\mathrm{NII}_s > 0$ indicates a loss if the stress scenario s occurs.

[24] We have:

$$\begin{aligned} A(t) &= L(t) \\ &= L^{\star}(t) + E(t) \end{aligned}$$

Finally, the economic value and earnings-based risk measures are equal to the maximum of losses by considering the different scenarios:

$$\mathcal{R}(\text{EVE}) = \max_s (\Delta \text{EVE}_s, 0)$$

and:

$$\mathcal{R}(\text{NII}) = \max_s (\Delta \text{NII}_s, 0)$$

Since IRRBB is part of the second pillar, there are no minimum capital requirements $\boldsymbol{\mathcal{K}}$. Nevertheless, the Basel Committee imposes that $\mathcal{R}(\text{EVE})$ must be lower than 15% of the bank's tier 1 capital.

7.1.3.2 Interest rate risk principles

The Basel Committee defines nine IRR principles for banks and three IRR principles for supervisors. The first and second principles recall that banks must specifically manage IRRBB (and also CSRBB[25]) and have a governing body that oversights IRRBB. The third and fourth principles explain that the risk appetite of the bank for IRRBB must be defined with respect to both economic value and earnings-based risk measures arising from interest rate shocks and stress scenarios. The objective is to measure the change in the net present value of the banking book and the future profitability. To compute Δ EVE, banks must consider a run-off balance sheet assumption, whereas they must use a constant or dynamic balance sheet and a rolling 12-month period for computing Δ NII. For that, they have to consider multiple interest rate scenarios, for example historical and hypothetical scenarios. Besides these internal scenarios, six external scenarios are defined by the Basel Committee[26]: (1) parallel shock up; (2) parallel shock down; (3) steepener shock (short rates down and long rates up); (4) flattener shock (short rates up and long rates down); (5) short rates shock up; and (6) short rates shock down. The fifth principle deals with behaviorial and modeling assumptions, in particular embedded optionalities. The last three principles deals with risk management and model governance process, the disclosure of the information and the capital adequacy policy.

The role of supervisors is strengthened. They should collect on a regular basis sufficient information from the bank to assess its IRRBB exposure. This concerns modeling assumptions, interest rate and option exposures, yield curve parameters, statistical methodologies, etc. An important task is also the identification of outlier banks. The outlier/materiality test compares the bank's maximum Δ EVE (or $\mathcal{R}(\text{EVE})$) with 15% of its tier 1 capital. If this threshold is exceeded, supervisors must require mitigation actions, hedging programs and/or additional capital.

7.1.3.3 The standardized approach

Overview of the standardized framework There are five steps for measuring the bank's IRRBB:

1. The first step consists in allocating the interest rate sensitivities of the banking book to three categories:
   (a) amenable to standardization[27];
   (b) less amenable to standardization[28];
   (c) not amenable to standardization[29].

[25] Credit spread risk in the banking book.
[26] These scenarios are described in the next paragraph on page 397.
[27] The Basel Committee distinguish two main categories: fixed rate positions and floating rate positions.
[28] They concern explicit automatic interest rate options.
[29] This category is composed of NMDs, fixed rate loans subject to prepayment risk and term deposits subject to early redemption risk.

2. Then, the bank must slot cash flows (assets, liabilities and off-balance sheet items) into 19 predefined time buckets[30]: overnight (O/N), O/N–1M, 1M–3M, 3M–6M, 6M–9M, 9M–1Y, 1Y–18M, 1.5Y–2Y, 2Y–3Y, 3Y–4Y, 4Y–5Y, 5Y–6Y, 6Y–7Y, 7Y–8Y, 8Y–9Y, 9Y–10Y, 10Y–15Y, 15Y–20Y, 20Y+. This concerns positions amenable to standardization. For positions less amenable to standardization, they are excluded from this step. For positions with embedded automatic interest rate options, the optionality is ignored.

3. The bank determines $\Delta\,\mathrm{EVE}_{s,c}$ for each interest rate scenario s and each currency c.

4. In the fourth step, the bank calculates the total measure for automatic interest rate option risk $\mathrm{KAO}_{s,c}$.

5. Finally, the bank calculates the EVE risk measure for each interest rate shock s:

$$\mathcal{R}\left(\mathrm{EVE}_s\right) = \max\left(\sum_c \left(\Delta\,\mathrm{EVE}_{s,c} + \mathrm{KAO}_{s,c}\right)^+ ; 0\right)$$

The standardized EVE risk measure is the maximum loss across all the interest rate shock scenarios:

$$\mathcal{R}\left(\mathrm{EVE}\right) = \max_s \mathcal{R}\left(\mathrm{EVE}_s\right)$$

The supervisory interest rate shock scenarios The six stress scenarios are based on three shock sizes that the Basel Committee has calibrated using the period 2010 – 2015: the parallel shock size $\mathbb{S}_0$, the short shock size $\mathbb{S}_1$ and the long shock size $\mathbb{S}_2$. In the table below, we report their values for some currencies[31]:

Shock size		USD/CAD/SEK	EUR/HKD	GBP	JPY	EM
$\mathbb{S}_0$	(parallel)	200	200	250	100	400
$\mathbb{S}_1$	(short)	300	250	300	100	500
$\mathbb{S}_2$	(long)	150	100	150	100	300

where EM is composed of ARS, BRL, INR, MXN, RUB, TRY and ZAR. Given $\mathbb{S}_0$, $\mathbb{S}_1$ and $\mathbb{S}_2$, we calculate the following generic shocks for a given maturity t:

	Parallel shock $\Delta R^{(\text{parallel})}(t)$	Short rates shock $\Delta R^{(\text{short})}(t)$	Long rates shock $\Delta R^{(\text{long})}(t)$
Up	$+\mathbb{S}_0$	$+\mathbb{S}_1 \cdot e^{-t/\tau}$	$+\mathbb{S}_2 \cdot \left(1 - e^{-t/\tau}\right)$
Down	$-\mathbb{S}_0$	$-\mathbb{S}_1 \cdot e^{-t/\tau}$	$-\mathbb{S}_2 \cdot \left(1 - e^{-t/\tau}\right)$

where τ is equal to four years. Finally, the five standardized interest rate shock scenarios are defined as follows:

1. Parallel shock up:

$$\Delta R^{(\text{parallel})}(t) = +\mathbb{S}_0$$

[30] The buckets are indexed by k from 1 to 19. For each bucket, the midpoint is used for defining the corresponding maturity t_k. We have $t_1 = 0.0028$, $t_2 = 0.0417$, $t_3 = 0.1667$, $t_4 = 0.375$, $t_5 = 0.625$, ..., $t_{17} = 12.5$, $t_{18} = 17.5$ and $t_{19} = 25$.

[31] The values for a more comprehensive list of currencies are given in BCBS (2016d) on page 44.

2. Parallel shock down:
$$\Delta R^{(\text{parallel})}(t) = -\mathbb{S}_0$$

3. Steepener shock (short rates down and long rates up):
$$\Delta R^{(\text{steepnener})}(t) = 0.90 \cdot \left|\Delta R^{(\text{long})}(t)\right| - 0.65 \cdot \left|\Delta R^{(\text{short})}(t)\right|$$

4. Flattener shock (short rates up and long rates down):
$$\Delta R^{(\text{flattener})}(t) = 0.80 \cdot \left|\Delta R^{(\text{short})}(t)\right| - 0.60 \cdot \left|\Delta R^{(\text{long})}(t)\right|$$

5. Short rates shock up:
$$\Delta R^{(\text{short})}(t) = +\mathbb{S}_1 \cdot e^{-t/\tau}$$

6. Short rates shock down:
$$\Delta R^{(\text{short})}(t) = -\mathbb{S}_1 \cdot e^{-t/\tau}$$

Example 71 *We assume that $\mathbb{S}_0 = 100$ bps, $\mathbb{S}_1 = 150$ bps and $\mathbb{S}_2 = 200$ bps. We would like to calculate the standardized shocks for the one-year maturity.*

The parallel shock up is equal to +100 bps, while the parallel shock down is equal to −100 bps. For the short rates shock, we obtain:
$$\Delta R^{(\text{short})}(t) = 150 \times e^{-1/4} = 116.82 \text{ bps}$$
for the up scenario and −116.82 bps for the down scenario. Since we have $\left|\Delta R^{(\text{short})}(t)\right| = 116.82$ and $\left|\Delta R^{(\text{long})}(t)\right| = 44.24$, the steepener shock is equal to:
$$\Delta R^{(\text{steepnener})}(t) = 0.90 \times 44.24 - 0.65 \times 116.82 = -36.12 \text{ bps}$$
For the flattener shock, we have:
$$\Delta R^{(\text{flattener})}(t) = 0.80 \times 116.82 - 0.60 \times 44.24 = 66.91 \text{ bps}$$
In Figure 7.10, we have represented the six interest rate shocks $\Delta R(t)$ for the set of parameters $(\mathbb{S}_0 = 100, \mathbb{S}_1 = 150, \mathbb{S}_2 = 200)$.

In Figure 7.11, we consider the yield curve generated by the Nelson-Siegel model[32] with the following parameters $\theta_1 = 8\%$, $\theta_2 = -7\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. Then, we apply the standardized interest rate shocks by considering EUR and EM currencies. We verify that the parallel shock moves uniformly the yield curve, the steepener shock increases the slope of the yield curve, the flattener shock reduces the spread between long and short interest rates, and the short rates shock has no impact on the long maturities after 10 years. We also notice that the deformation of the yield curve is more important for EM currencies than for the EUR currency.

Treatment of NMDs NMDs are segmented into three categories: retail transactional, retail non-transactional and wholesale Then, the bank must estimate the stable and non-stable part of each category[33]. Finally, the stable part of NMDs must be split between core and non-core deposits. However, the Basel Committee imposes a cap ω^+ on the proportion of core deposits (see Table 7.12). For instance, core deposits cannot exceed 70% of the retail non-transactional stable deposits. The time bucket for non-core deposits is set to overnight (or the first time bucket), meaning that the corresponding time bucket midpoint is equal to $t_1 = 0.0028$. For core deposits, the bank determines the appropriate cash flow slotting, but the average maturity cannot exceed the cap t^+, which is given in Table 7.12.

[32] We recall that it is defined in Footnote 8 on page 131.
[33] This estimation must be based on the historical data of the last 10 years.

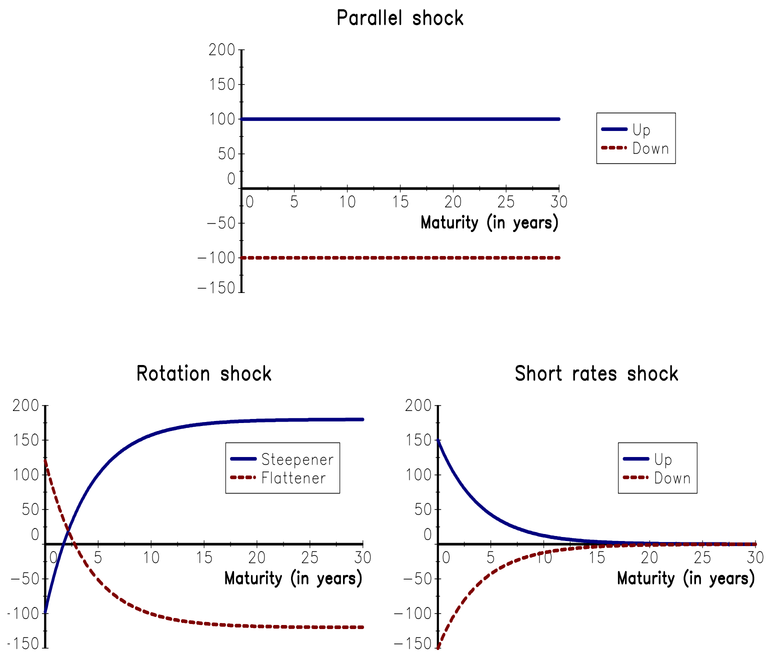

FIGURE 7.10: Interest rate shocks (in bps)

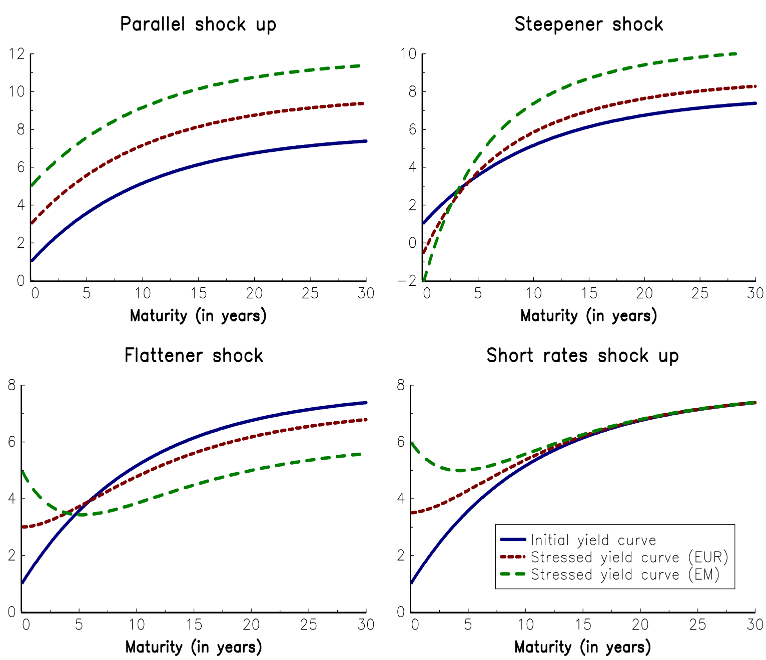

FIGURE 7.11: Stressed yield curve (in %)

TABLE 7.12: Cap on core deposits and maximum average maturity

Category	Cap ω^+	Cap t^+
Retail transactional	90%	5.0
Retail non-transactional	70%	4.5
Wholesale	50%	4.0

Behavioral options of retail customers This category mainly concerns fixed rate loans because of the prepayment risk, and fixed-term deposits because of the early redemption risk. The Basel Committee proposes to use a two-step procedure. First, the bank determine the baseline estimate of each category given the current yield curve. Then, the baseline estimate is multiplied according to the standardized interest rate scenarios. In the case of fixed rate loans subject to prepayment risk, the bank establishes the different homogenous prepayment categories. For each category, the bank estimates the baseline conditional prepayment rate CPR_0 and calculates the stressed conditional prepayment rate as follows:

$$\text{CPR}_s = \min\left(1, \gamma_s \cdot \text{CPR}_0\right)$$

where γ_s is the multiplier for the scenario s. The coefficient γ_s takes two values:

- $\gamma_s = 0.8$ for the scenarios 1, 3 and 5 (parallel up, steepener and short rates up);
- $\gamma_s = 1.2$ for the scenarios 2, 4 and 6 (parallel down, flattener and short rates down).

The cash flow for the time bucket t_k is the sum of two components:

$$\text{CF}_s\left(t_k\right) = \text{CF}_s^1\left(t_k\right) + \text{CF}_s^2\left(t_k\right)$$

where $\text{CF}_s^1\left(t_k\right)$ refers to the scheduled interest and principal repayment (without prepayment) and $\text{CF}_s^2\left(t_k\right)$ refers to the prepayment cash flow:

$$\text{CF}_s^2\left(t_k\right) = \text{CPR}_s \cdot N_s\left(t_{k-1}\right)$$

where $N_s\left(t_{k-1}\right)$ is the notional outstanding at time bucket t_{k-1} calculated with the stress scenario s.

The methodology for term deposits subject to early redemption risk is similar to the one of the fixed rate loans subject to prepayment risk. First, the bank estimates the baseline term deposit redemption ratio TDRR_0 for each homogeneous portfolio. Second, the stressed term deposit redemption ratio is equal to:

$$\text{TDRR}_s = \min\left(1, \gamma_s \cdot \text{TDRR}_0\right)$$

where γ_s is the multiplier for the scenario s. The coefficient γ_s takes two values:

- $\gamma_s = 1.2$ for the scenarios 1, 4 and 5 (parallel up, flattener and short rates up);
- $\gamma_s = 0.8$ for the scenarios 2, 3 and 6 (parallel down, steepener and short rates down).

Third, the term deposits which are expected to be redeemed early are slotted into the overnight time bucket, implying that the corresponding cash flows are given by:

$$\text{CF}_s\left(t_1\right) = \text{TDRR}_s \cdot N_0$$

where N_0 is the outstanding amount of term deposits.

Remark 76 *Fixed rate loans subject to prepayment risk and term deposits subject to early redemption risk follow the same methodology, but with two main differences. The first one concerns the impact of the stress scenario on the stress ratios* CPR_s *and* TDRR_s*. In the case of prepayment risk, the conditional prepayment rate generally increases when interest rates are falling and decreases when interest rates are rising. This is why we have* $\mathrm{CPR}_s > \mathrm{CPR}_0$ *for the scenarios where interest rates or the slope of the yield curve decrease (scenarios 1, 3 and 5). In the case of early redemption risk, the term deposit redemption ratio mainly depends on the short term interest rates. In particular, the ratio* TDRR_s *must increase when short rates increase, because this creates an incentive to negotiate a term deposit contract with a higher interest rate.*

Automatic interest rate options The computation of the automatic interest rate option risk KAO_s is given by:

$$\mathrm{KAO}_s = \sum_{i \in \mathcal{S}} \Delta \, \mathrm{FVAO}_{s,i} - \sum_{i \in \mathcal{B}} \Delta \, \mathrm{FVAO}_{s,i}$$

where:

- $i \in \mathcal{S}$ denotes an automatic interest rate option which is sold by the bank;
- $i \in \mathcal{B}$ denotes an automatic interest rate option which is bought by the bank;
- $\mathrm{FVAO}_{0,i}$ is the fair value of the automatic option i given the current yield curve and the current implied volatility surface;
- $\mathrm{FVAO}_{s,i}$ is the fair value of the automatic option i given the stressed yield curve and a relative increase in the implied volatility of 25%;
- $\Delta \, \mathrm{FVAO}_{s,i}$ is the change in the fair value of the option:

$$\Delta \, \mathrm{FVAO}_{s,i} = \mathrm{FVAO}_{s,i} - \mathrm{FVAO}_{0,i}$$

An example We consider a simplified USD-denominated balance sheet. The assets are composed of loans with the following cash flow slotting:

Instruments	Loans	Loans	Loans
Maturity	1Y	5Y	13Y
Cash flows	200	700	200

The loans are then slotted into three main buckets (short-term, medium-term and long-term loans). The average maturity is respectively equal to one-year, five-year and thirteen-year. The liabilities are composed of retail deposit accounts, term deposits, debt and tier-one equity capital. The cash flow slotting is given below:

Instruments	Non-core deposits	Term deposits	Core deposits	Debt ST	Debt LT	Equity capital
Maturity	O/N	7M	3Y	4Y	8Y	
Cash flows	100	50	450	100	100	200

The non-maturity deposits are split into non-core and core deposits. The maturity of non-core deposits is assumed to be overnight (O/N), whereas the estimated maturity of core deposits is around three years. We also have two debt instruments: one with a remaining

TABLE 7.13: Economic value of the assets

Bucket	t_k	$\mathrm{CF}_0(t_k)$	$R_0(t_k)$	$\mathrm{EV}_0(t_k)$
6	0.875	200	1.55%	197.31
11	4.50	700	3.37%	601.53
17	12.50	100	5.71%	48.98
EV_0				847.82

TABLE 7.14: Economic value of the pure liabilities

Bucket	t_k	$\mathrm{CF}_0(t_k)$	$R_0(t_k)$	$\mathrm{EV}_0(t_k)$
1	0.0028	100	1.00%	100.00
5	0.625	50	1.39%	49.57
9	2.50	450	2.44%	423.35
10	3.50	100	2.93%	90.26
14	7.50	100	4.46%	71.56
EV_0				734.73

maturity of four years and another one with a remaining maturity of eight years. The term deposits are slotted in a single bucket corresponding to a seven-month maturity.

We assume that the current yield curve is given by the Nelson-Siegel model with $\theta_1 = 8\%$, $\theta_2 = -7\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. In Table 7.13, we have reported the current economic value of the assets. It is respectively equal to 197.31, 601.53 and 48.98 for the three buckets and 847.82 for the total of assets. We have done the same exercise for the pure liabilities (Table 7.14). We obtain an economic value equal to 734.73. We deduce that the current economic value of equity is $\mathrm{EVE}_0 = 847.82 - 734.73 = 113.09$. Since the balance sheet is expressed in USD, we use the USD shocks for the interest rates scenarios: $\mathbb{S}_0 = 200$ bps, $\mathbb{S}_1 = 300$ bps and $\mathbb{S}_2 = 150$ bps. In Table 7.15, we have reported the stressed values of interest rates $R_s(t_k)$ and economic value $\mathrm{EV}_s(t_k)$ for every bucket of the balance sheet. By computing the stressed economic value of assets and pure liabilities, we deduce the stressed economic value of equity. For instance, in the case of the first stress scenario, we have $\mathrm{EVE}_1 = 781.79 - 697.39 = 84.41$. It follows that the economic value of equity will be reduced if the standardized parallel shock up occurs: $\Delta\,\mathrm{EVE}_1 = 113.10 - 84.41 = 28.69$. We observe that the economic value of equity decreases for scenarios 1, 3 and 5, and increases for scenarios 2, 4 and 6. Finally, we deduce that the risk measure $\mathcal{R}(\mathrm{EVE}) = \max_s(\Delta\,\mathrm{EVE}_s, 0) = 28.69$ represents 14.3% of the equity. This puts under the threshold 15% of the materiality test.

7.1.4 Other ALM risks

Even if liquidity and interest rate risks are the main ALM risks, there are other risks that impact the banking book of the balance sheet, in particular currency risk and credit spread risk.

7.1.4.1 Currency risk

We recall that the standardized approach for implementing IRRBB considers each currency separately. Indeed, the risk measures $\Delta\,\mathrm{EVE}_{s,c}$ and $\mathrm{KAO}_{s,c}$ are calculated for each interest rate scenario s and each currency c. Then, the aggregated value $\sum_c (\Delta\,\mathrm{EVE}_{s,c} + \mathrm{KAO}_{s,c})^+$ is calculated across the different currencies and the maximum is selected for the global risk measure of the bank.

TABLE 7.15: Stressed economic value of equity

Bucket	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$
			Assets			
$R_s(t_6)$	3.55%	−0.45%	0.24%	3.30%	3.96%	−0.87%
$R_s(t_{11})$	5.37%	1.37%	3.65%	3.54%	4.34%	2.40%
$R_s(t_{17})$	7.71%	3.71%	6.92%	4.96%	5.84%	5.58%
$\mathrm{EV}_s(t_6)$	193.89	200.80	199.57	194.31	193.20	201.52
$\mathrm{EV}_s(t_{11})$	549.76	658.18	594.03	596.91	575.74	628.48
$\mathrm{EV}_s(t_{17})$	38.15	62.89	42.13	53.83	48.18	49.79
EV_s	781.79	921.87	835.74	845.05	817.11	879.79
			Pure liabilities			
$R_s(t_1)$	3.00%	−1.00%	−0.95%	3.40%	4.00%	−2.00%
$R_s(t_5)$	3.39%	−0.61%	−0.08%	3.32%	3.96%	−1.17%
$R_s(t_9)$	4.44%	0.44%	2.03%	3.31%	4.05%	0.84%
$R_s(t_{10})$	4.93%	0.93%	2.90%	3.40%	4.18%	1.68%
$R_s(t_{14})$	6.46%	2.46%	5.31%	4.07%	4.92%	4.00%
$\mathrm{EV}_s(t_1)$	99.99	100.00	100.00	99.99	99.99	100.01
$\mathrm{EV}_s(t_5)$	48.95	50.19	50.02	48.97	48.78	50.37
$\mathrm{EV}_s(t_9)$	402.70	445.05	427.77	414.27	406.69	440.69
$\mathrm{EV}_s(t_{10})$	84.16	96.80	90.34	88.77	86.39	94.30
$\mathrm{EV}_s(t_{14})$	61.59	83.14	67.17	73.70	69.13	74.07
EV_s	697.39	775.18	735.31	725.71	710.98	759.43
			Equity			
EVE_s	84.41	146.68	100.43	119.34	106.13	120.37
$\Delta\,\mathrm{EVE}_s$	28.69	−33.58	12.67	−6.24	6.97	−7.27

One of the issues concerns currency hedging. Generally, it is done by rolling reverse FX forward contracts, implying that the hedging cost is approximatively equal to $i^\star - i$, where i is the domestic interest rate and $i^\star$ is the foreign interest rate. This relationship comes from the covered interest rate parity (CIP). We deduce that the hedging cost can be large when $i^\star \gg i$. This has been particularly true for European and Japanese banks, because these regions have experienced some periods of low interest rates. The question of full hedging, partial hedging or no hedging has then been readdressed after the 2008 Global Financial Crisis. Most of banks continue to fully hedge the banking book including the equity capital, but it is not obvious that it is optimal. Another issue has concerned the access to dollar funding of non-US banks. Traditionally, "*their branches and subsidiaries in the United States were a major source of dollar funding, but the role of these affiliates has declined*" (Aldasoro and Ehlers, 2018, page 15). Today, we notice that a lot of non-US banks issue many USD-denominated debt instruments in order to access dollar funding[34]. Banks must now manage a complex multi-currency balance sheet, implying that currency management has become an important topic in ALM.

7.1.4.2 Credit spread risk

According to BCBS (2016d), credit spread risk in the banking book (CSRBB) is driven "*by changes in market perception about the credit quality of groups of different credit-risky instruments, either because of changes to expected default levels or because of changes to*

[34] See for instance annual reports of European and Japanese banks.

market liquidity". In Figure 7.12, we have reproduced the scheme provided by the Basel Committee in order to distinguish IRRBB and CSRBB. Therefore, CSRBB can be seen as the ALM spread risk of credit-risky instruments which is not explained by IRRBB and idiosyncratic credit risk. However, the definition provided by the Basel Committee is too broad, and does not avoid double counting with credit and jump-to-default risk[35]. At of the date of the publication of this book, the debate on CSRBB is far from finished, even if CSRBB must be monitored and assessed since 2018.

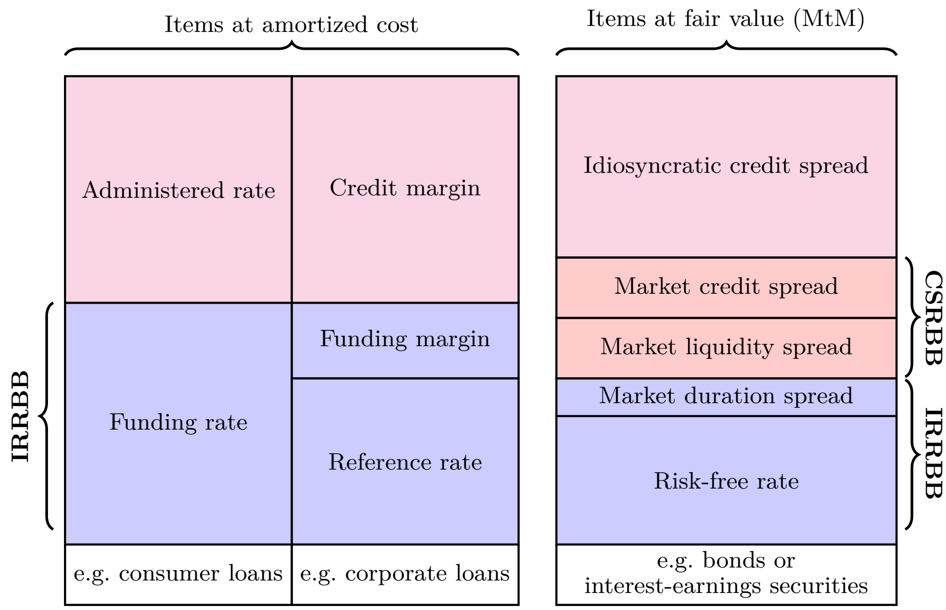

FIGURE 7.12: Components of interest rates

Source: BCBS (2016d, page 34).

7.2 Interest rate risk

In this section, we focus on the ALM tools that are related to the interest rate risk in the banking book. We first introduce the concept of duration gap and show how it is related to the economic value risk of the banking book. Then, we present the different ways to calculate earnings-at-risk (EaR) measures and focus more particularly on the net interest income sensitivity and the interest rate hedging strategies. The third part is dedicated to funds transfer pricing, whose objective is to centralize all interest rate risks, to manage them and to allocate profit between the different business units. Finally, we present an econometric model for simulating and evaluating interest rate scenarios.

[35]See for example the position of the European Banking Federation (EBF): `www.ebf.eu/regulation-supervision/credit-spread-risk-in-the-banking-book-ebf-position`.

7.2.1 Duration gap analysis

In this section, we focus on the duration gap analysis, which is an approximation of the repricing gap analysis we have previously presented. The idea is to obtain an estimation of $\Delta\,\mathrm{EVE}$. Although this approach is only valid in the case of parallel interest rate shocks[36], it is an interesting method because we obtain closed-form formulas. In the case of non-parallel interest rate scenarios or if we want to obtain more accurate results, it is better to implement the repricing gap analysis, which consists in computing the stressed economic value of assets and liabilities in order to deduce the impact on the economic value of equity.

7.2.1.1 Relationship between Macaulay duration and modified duration

We consider a financial asset, whose price is given by the present value of cash flows:

$$V = \sum_{t_k \geq t} B(t, t_k) \cdot \mathrm{CF}(t_k)$$

where $\mathrm{CF}(t_k)$ is the cash flow at time t_k and $B(t, t_k)$ is the associated discount factor. The Macaulay duration $\mathcal{D}$ is the weighted average of the cash flow maturities:

$$\mathcal{D} = \sum_{t_k \geq t} w(t, t_k) \cdot (t_k - t)$$

where $w(t, t_k)$ is the weight associated to the cash flow at time t_k:

$$w(t, t_k) = \frac{B(t, t_k) \cdot \mathrm{CF}(t_k)}{V}$$

In the case of a zero-coupon bond whose maturity date is T, the Macaulay duration is equal to the remaining maturity $T - t$.

Let us define the yield to maturity y as the solution of the following equation:

$$V = \sum_{t_k \geq t} \frac{\mathrm{CF}(t_k)}{(1 + y)^{(t_k - t)}}$$

We have:

$$\begin{aligned}
\frac{\partial V}{\partial y} &= \sum_{t_k \geq t} -(t_k - t) \cdot (1 + y)^{-(t_k - t) - 1} \cdot \mathrm{CF}(t_k) \\
&= -\frac{\mathcal{D}}{(1 + y)} \cdot V \\
&= -\mathfrak{D} \cdot V
\end{aligned}$$

where $\mathfrak{D}$ is the modified duration:

$$\mathfrak{D} = \frac{\mathcal{D}}{1 + y}$$

We deduce that the modified duration is the price sensitivity measure:

$$\mathfrak{D} = \frac{1}{V} \cdot \frac{\partial V}{\partial y} = -\frac{\partial \ln V}{\partial y}$$

[36]The duration gap analysis can be viewed as the first-order approximation of the repricing gap analysis.

If the yield to maturity is low, we have $\mathfrak{D} \approx \mathcal{D}$. Since the Macaulay duration is easier to interpret, the modified duration is more relevant to understand the impact of an interest rate stress scenario. Indeed, we have:

$$\Delta V \approx -\mathfrak{D} \cdot V \cdot \Delta y$$

Nevertheless, we can use the following alternative formula to evaluate the impact of an interest rate parallel shift:

$$\Delta V \approx -\mathcal{D} \cdot V \cdot \frac{\Delta y}{1 + y}$$

Remark 77 *Using a continuous-time framework, the yield to maturity is defined as the root of the following equation:*

$$V = \sum_{t_k \geq t} e^{-y(t_k - t)} \cdot \mathrm{CF}(t_k)$$

We deduce that:

$$\begin{aligned}
\frac{\partial V}{\partial y} &= \sum_{t_k \geq t} -(t_k - t) \cdot e^{-y(t_k - t)} \cdot \mathrm{CF}(t_k) \\
&= -\mathcal{D} \cdot V
\end{aligned}$$

It follows that the modified duration $\mathfrak{D}$ is defined as the Macaulay duration $\mathcal{D}$ in continuous-time modeling.

Example 72 *We consider the following cash flows stream $\{t_k, \mathrm{CF}(t_k)\}$:*

t_k	1	4	7	11
$\mathrm{CF}(t_k)$	200	500	200	100

The current zero-coupon interest rates are: $R(1) = 2\%$, $R(4) = 3\%$, $R(7) = 4\%$, and $R(11) = 5\%$.

If we consider the discrete-time modeling framework, we obtain $V = 850.77$, $y = 3.61\%$, $\mathcal{D} = 4.427$ and $\mathfrak{D} = 4.273$. A parallel shock of $+1\%$ decreases the economic value since we obtain $V(R + \Delta R) = 816.69$. It follows that $\Delta V = -34.38$. Using the duration-based approximation, we have[37]:

$$\begin{aligned}
\Delta V &\approx -\mathfrak{D} \cdot V \cdot \Delta R \\
&= -4.273 \times 850.77 \times 1\% \\
&= -36.35
\end{aligned}$$

In the case of the continuous-time modeling framework, the results become $V = 848.35$, $y = 3.61\%$ and $\mathcal{D} = \mathfrak{D} = 4.422$. If we consider a parallel shock of $+1\%$, the exact value of ΔV is equal to -35.37, whereas the approximated value is equal to -37.51. In Table 7.16, we also report the results for a parallel shock of -1%. Moreover, we indicate the case where we stress the yield to maturity and not the yield curve. Because $V(y + \Delta R) \neq V(R + \Delta R)$, we observe a small difference between the approximation and the true value of ΔV.

[37]This approximation is based on the assumption that the yield curve is flat. However, numerical experiments show that it is also valid when the term structure of interest rates is increasing or decreasing.

TABLE 7.16: Impact of a parallel shift of the yield curve

	Discrete-time		Continuous-time	
ΔR	$+1\%$	-1%	$+1\%$	-1%
$V(R+\Delta R)$	816.69	887.52	812.78	886.09
ΔV	-34.38	36.75	-35.57	37.74
$V(\mathcal{y}+\Delta R)$	815.64	888.42	811.94	887.02
ΔV	-35.13	37.64	-36.41	38.67
Approximation	-36.35	36.35	-37.51	37.51

Remark 78 *From a theoretical point of view, duration analysis is valid under the assumption that the term structure of interest rates is flat and the change in interest rates is a parallel shift. This framework can be extended by considering the convexity:*

$$\mathfrak{C} = \frac{1}{V} \cdot \frac{\partial^2 V}{\partial \mathcal{y}^2}$$

In this case, we obtain the following second-order approximation:

$$\Delta V \approx -\mathfrak{D} \cdot V \cdot \Delta \mathcal{y} + \frac{1}{2}\mathfrak{C} \cdot V \cdot (\Delta \mathcal{y})^2$$

7.2.1.2 Relationship between the duration gap and the equity duration

Let V_j and $\mathcal{D}_j$ be the market value and the Macaulay duration associated to the j^{th} cash flow stream. Then, the market value of a portfolio that is composed of m cash flow streams is equal to the sum of individual market values:

$$V = \sum_{j=1}^{m} V_j$$

while the duration of the portfolio is the average of individual durations:

$$\mathcal{D} = \sum_{j=1}^{m} w_j \cdot \mathcal{D}_j$$

where:

$$w_j = \frac{V_j}{V}$$

This result is obtained by considering a common yield to maturity.

We recall that $E(t) = A(t) - L^\star(t)$ and $\mathrm{EV}_E = \mathrm{EV}_A - \mathrm{EV}_{L^\star}$. Using the previous result, we deduce that the duration of equity is equal to:

$$\begin{aligned}\mathcal{D}_E &= \frac{\mathrm{EV}_A}{\mathrm{EV}_A - \mathrm{EV}_{L^\star}} \cdot \mathcal{D}_A - \frac{\mathrm{EV}_{L^\star}}{\mathrm{EV}_A - \mathrm{EV}_{L^\star}} \cdot \mathcal{D}_{L^\star} \\ &= \frac{\mathrm{EV}_A}{\mathrm{EV}_A - \mathrm{EV}_{L^\star}} \cdot \mathcal{D}_{\mathcal{G}ap} \qquad (7.9)\end{aligned}$$

where the duration gap (also called DGAP) is defined as the difference between the duration of assets and the duration of pure liabilities scaled by the ratio $\mathrm{EV}_{L^\star} / \mathrm{EV}_A$:

$$\mathcal{D}_{\mathcal{G}ap} = \mathcal{D}_A - \frac{\mathrm{EV}_{L^\star}}{\mathrm{EV}_A} \cdot \mathcal{D}_{L^\star} \qquad (7.10)$$

Another expression of the equity duration is:

$$\mathcal{D}_E = \frac{\mathrm{EV}_A}{\mathrm{EV}_E} \cdot \mathcal{D}_{\mathcal{G}ap} = \mathcal{L}_{A/E} \cdot \frac{\mathcal{D}_{\mathcal{G}ap}}{\mathcal{L}_{E/A}} \tag{7.11}$$

We notice that the equity duration is equal to the duration gap multiplied by the leverage ratio, where $\mathcal{L}_{A/E}$ is the ratio between the economic value of assets and the economic value of equity.

By definition of the modified duration, we have[38]:

$$\begin{aligned} \Delta\,\mathrm{EVE} &= \Delta\,\mathrm{EV}_E \\ &\approx -\mathfrak{D}_E \cdot \mathrm{EV}_E \cdot \Delta y \\ &= -\mathcal{D}_E \cdot \mathrm{EV}_E \cdot \frac{\Delta y}{1+y} \end{aligned}$$

Using Equation (7.11), we deduce that:

$$\Delta\,\mathrm{EVE} \approx -\mathcal{D}_{\mathcal{G}ap} \cdot \mathrm{EV}_A \cdot \frac{\Delta y}{1+y} \tag{7.12}$$

Formulas (7.10) and (7.12) are well-known and are presented in many handbooks of risk management (Crouhy *et al.*, 2013; Bessis, 2015).

7.2.1.3 An illustration

We consider the following balance sheet:

Assets	V_j	$\mathcal{D}_j$	Liabilities	V_j	$\mathcal{D}_j$
Cash	5	0.0	Deposits	40	3.2
Loans	40	1.5	CDs	20	0.8
Mortgages	40	6.0	Debt	30	1.7
Securities	15	3.8	Equity capital	10	
Total	100		Total	100	

We have $\mathrm{EV}_A = 100$, $\mathrm{EV}_{L^\star} = 90$ and $\mathrm{EV}_E = 10$. We deduce that the leverage is equal to:

$$\mathcal{L}_{A/E} = \frac{\mathrm{EV}_A}{\mathrm{EV}_E} = \frac{100}{10} = 10$$

The duration of assets is equal to:

$$\mathcal{D}_A = \frac{5}{100} \times 0 + \frac{40}{100} \times 1.5 + \frac{40}{100} \times 6.0 + \frac{15}{100} \times 3.8 = 3.57 \text{ years}$$

For the pure liabilities, we obtain:

$$\mathcal{D}_{L^\star} = \frac{40}{90} \times 3.2 + \frac{20}{90} \times 0.8 + \frac{30}{90} \times 1.7 = 2.17 \text{ years}$$

It follows that the duration gap is equal to:

$$\mathcal{D}_{\mathcal{G}ap} = 3.57 - \frac{90}{100} \times 2.17 = 1.62 \text{ years}$$

[38] We recall that EVE is an alternative expression for designating EV_E.

while the value $\mathcal{D}_E$ of the equity duration is 16.20 years. Since $\mathcal{D}_{\mathcal{G}ap}$ is equal to 1.62 years, the average duration of assets exceeds the average duration of liabilities. This is generally the normal situation because of the bank's liquidity transformation (borrowing short and lending long). In the table below, we have reported the impact of an interest rate shift on the economic value of equity when the yield to maturity is equal to 3%:

Δy	-2%	-1%	$+1\%$	$+2\%$
$\Delta\,\mathrm{EVE}$	3.15	1.57	-1.57	-3.15
$\dfrac{\Delta\,\mathrm{EVE}}{\mathrm{EVE}}$	31.46%	15.73%	-15.73%	-31.46%

Since the duration gap is positive, the economic value of equity decreases when interest rates increase, because assets will fall more than liabilities. For instance, an interest rate rise of 1% induces a negative variation of 1.57 in EVE. This impact is large and represents a relative variation of -15.73%.

7.2.1.4 Immunization of the balance sheet

In order to reduce the sensitivity of the bank balance sheet to interest rate changes, we have to reduce the value of $|\Delta\,\mathrm{EVE}|$. Using Equation (7.12), this is equivalent to control the value of the duration gap. In particular, a full immunization implies that:

$$\begin{aligned}\Delta\,\mathrm{EVE} = 0 \quad &\Leftrightarrow \quad \mathcal{D}_{\mathcal{G}ap} = 0 \\ &\Leftrightarrow \quad \mathcal{D}_A - \frac{\mathrm{EV}_{L^\star}}{\mathrm{EV}_A} \cdot \mathcal{D}_{L^\star} = 0 \end{aligned} \tag{7.13}$$

If we consider the normal situation where the duration gap is positive, we have three solutions:

1. we can reduce the duration of assets;
2. we can increase the relative weight of the liabilities with respect to the assets;
3. we can increase the duration of liabilities.

Generally, it takes time to implement the first two solutions. For instance, reducing the duration of assets implies redefining the business model by reducing the average maturity of loans. It can be done by decreasing the part of mortgages and increasing the part of short-term loans (e.g. consumer credit or credit cards). In fact, the third solution is the easiest way to immunize the bank balance sheet to interest rate changes. For example, the bank can issue a long-term debt instrument. Therefore, hedging the balance sheet involves managing the borrowing program of the bank.

Let us consider the previous example. We found $\mathcal{D}_A = 3.57$ and $\dfrac{\mathrm{EV}_{L^\star}}{\mathrm{EV}_A} = \dfrac{90}{100}$. It follows that the optimal value of the liability duration must be equal to 3.97 years:

$$\mathcal{D}_{\mathcal{G}ap} = 0 \Leftrightarrow \mathcal{D}_{L^\star} = \frac{100}{90} \times 3.57 = 3.97 \text{ years}$$

We assume that the bank issues a 10-year zero-coupon bond by reducing its current debt amount. The notional of the zero-coupon bond must then satisfy this equation:

$$\frac{40}{90} \times 3.2 + \frac{20}{90} \times 0.8 + \frac{30 - N}{90} \times 1.7 + \frac{N}{90} \times 10 = 3.97$$

or:

$$N = \frac{3.97 \times 90 - (40 \times 3.2 + 20 \times 0.8 + 30 \times 1.7)}{10 - 1.7} = 19.52$$

TABLE 7.17: Bank balance sheet after immunization of the duration gap

Assets	V_j	$\mathcal{D}_j$	Liabilities	V_j	$\mathcal{D}_j$
Cash	5	0.0	Deposits	40	3.2
Loans	40	1.5	CDs	20	0.8
Mortgages	40	6.0	Debt	10.48	1.7
Securities	15	3.8	Zero-coupon bond	19.52	10.0
			Equity capital	10	0.0
Total	100		Total	100	

After immunization, the duration of equity is equal to zero and we obtain the balance sheet given in Table 7.17.

Remark 79 *The duration gap analysis covers the gap risk, which is the first-order source of interest rate risk. It is not adapted for measuring basis and option risks. For these two risks, we need to use the repricing analysis.*

7.2.2 Earnings-at-risk

Earnings-at-risk assesses potential future losses due to a change in interest rates over a specified period. Several measures of earnings can be used: accounting earnings, interest margins, commercial margins, etc. For interest rate scenarios, we can use predefined[39], historical or Monte Carlo scenarios. Once earnings distributions are obtained, we can analyze the results for each scenario, derive the most severe scenarios, compute a value-at-risk, etc. In this section, we first focus on the income gap analysis, which is the equivalent of the duration gap analysis when analyzing interest rate income risks. Then we present the tools for calculating the net interest income (NII). Finally, we consider hedging strategies in the context where both Δ EVE and NII risk measures are managed.

7.2.2.1 Income gap analysis

Definition of the gap Since Δ EVE measures the price risk of the balance sheet, Δ NII measures the earnings risk of the income statement. It refers to the risk of changes in the interest rates on assets and liabilities from the point of view of the net income. Indeed, if interest rates change, this induces a gap (or repricing) risk because the bank will have to reinvest assets and refinance liabilities at a different interest rate level in the future. The gap is defined as the difference between rate sensitive assets (RSA) and rate sensitive liabilities (RSL):

$$\mathrm{GAP}(t,u) = \mathrm{RSA}(t,u) - \mathrm{RSL}(t,u) \tag{7.14}$$

where t is the current date and u is the time horizon of the gap[40]. While Δ EVE considers all the cash flows, Δ NII is generally calculated using a short-term time horizon, for example the next quarter or the next year. Therefore, rate sensitive assets/liabilities correspond to assets/liabilities that will mature or reprice before the time horizon of the gap. This is why the interest rate gap risk is also called the repricing risk or the reset risk.

In order to calculate the interest rate gap, the bank must decide which items are rate sensitive. This includes two main categories. The first one corresponds to items that mature

[39] Such as the six scenarios of the standardized IRRBB approach.
[40] This means that $h = u - t$ is the maturity of the gap.

before the time horizon $t + h$, whereas the second one corresponds to floating rate instruments. For example, consider the following balance sheet expressed in millions of dollars:

Assets	Amount	Liabilities	Amount
Loans		Deposits	
Less than 1 year	200	Non-maturity deposits	150
1 to 2 years	100	Money market deposits	250
Greater than 2 years	100	Term deposits	
Mortgages		Fixed rate	250
Fixed rate	100	Variable rate	100
Variable rate	350	Borrowings	
Securities		Less than 1 year	50
Fixed rate	50	Greater than 1 year	100
Physical assets	100	Capital	100
Total	1000	Total	1000

If the time horizon of the gap is set to one year, the rate sensitive assets are loans with maturities of less than one year (200) and variable rate mortgages (350), while the rate sensitive liabilities are money market deposits (250), variable rate term deposits (100) and borrowings with maturities of less than one year (50). Therefore, we can split the balance sheet between rate sensitive, fixed rate and non-earning assets:

Assets	Amount	Liabilities	Amount
Rate sensitive	550	Rate sensitive	400
Fixed rate	350	Fixed rate	600
Non-earning	100	Non-earning	100

We deduce that the one-year gap is equal to \$150 million:

$$\mathrm{GAP}(t, t+1) = 550 - 400 = 150$$

Approximation of Δ NII We consider the following definition of the net interest income:

$$\begin{aligned}\mathrm{NII}(t,u) &= \mathrm{RSA}(t,u)\cdot R_{\mathrm{RSA}}(t,u) + \mathrm{NRSA}(t,u)\cdot R_{\mathrm{NRSA}}(t,u) - \\ &\quad \mathrm{RSL}(t,u)\cdot R_{\mathrm{RSL}}(t,u) - \mathrm{NRSL}(t,u)\cdot R_{\mathrm{NRSL}}(t,u)\end{aligned}$$

where RNSA and RNSL denote assets and liabilities that are not rate sensitive and $R_{\mathcal{C}}(t,u)$ is the average interest rate for the category $\mathcal{C}$ and the maturity date u. We have:

$$\Delta\,\mathrm{NII}(t,u) = \mathrm{NII}(t+h, u+h) - \mathrm{NII}(t,u)$$

By considering a static gap[41], we deduce that:

$$\begin{aligned}\Delta\,\mathrm{NII}(t,u) &= \mathrm{RSA}(t,u)\cdot\left(R_{\mathrm{RSA}}(t+h,u+h) - R_{\mathrm{RSA}}(t,u)\right) + \\ &\quad \mathrm{NRSA}(t,u)\cdot\left(R_{\mathrm{NRSA}}(t+h,u+h) - R_{\mathrm{NRSA}}(t,u)\right) - \\ &\quad \mathrm{RSL}(t,u)\cdot\left(R_{\mathrm{RSL}}(t+h,u+h) - R_{\mathrm{RSL}}(t,u)\right) - \\ &\quad \mathrm{NRSL}(t,u)\cdot\left(R_{\mathrm{NRSL}}(t+h,u+h) - R_{\mathrm{NRSL}}(t,u)\right)\end{aligned}$$

Since interest income and interest expense do not change for fixed rate assets and liabilities between t and $t+h$ — $R_{\mathrm{NRSA}}(t+h,u+h) = R_{\mathrm{NRSA}}(t,u)$ and $R_{\mathrm{NRSL}}(t+h,u+h) - R_{\mathrm{NRSL}}(t,u)$, we have:

$$\Delta\,\mathrm{NII}(t,u) = \mathrm{RSA}(t,u)\cdot\Delta R_{\mathrm{RSA}}(t,u) - \mathrm{RSL}(t,u)\cdot\Delta R_{\mathrm{RSL}}(t,u)$$

[41] This means that $\mathrm{RSA}(t+h,u+h) = \mathrm{RSA}(t,u)$, $\mathrm{NRSA}(t+h,u+h) = \mathrm{NRSA}(t,u)$, $\mathrm{RSL}(t+h,u+h) = \mathrm{RSL}(t,u)$ and $\mathrm{NRSL}(t+h,u+h) = \mathrm{NRSL}(t,u)$ where $h = u - t$.

By assuming that the impact of interest rate changes is the same for rate sensitive assets and liabilities, we finally obtain:

$$\Delta \, \mathrm{NII}\,(t,u) \approx \mathrm{GAP}\,(t,u) \cdot \Delta R \tag{7.15}$$

where ΔR is the parallel shock of interest rates. Income gap analysis is then described by Equations (7.14) and (7.15).

For instance, if we consider the previous example, the one-year gap is equal to \$150 million and we have the following impact on the income:

ΔR	-2%	-1%	0%	$+1\%$	$+2\%$
Δ NII	$-$\$3 mn	$-$\$1.5 mn	0	+\$1.5 mn	+\$3 mn

If interest rates rise by 2%, the bank expects that its income increases by \$3 million. On the contrary, the loss can be equal to \$3 million if interest rates fall by 2%.

Remark 80 *The previous analysis is valid for a given maturity $h = u - t$. For example, $\Delta \, \mathrm{NII}\,(t, t + 0.25)$ measures the impact for the next three months while $\Delta \, \mathrm{NII}\,(t, t + 1)$ measures the impact for the next year. It is common to consider the change in income for a given time period $[u_1, u_2[$ where $u_1 = t + h_1$ and $u_2 = t + h_2$. We notice that:*

$$\begin{aligned} \Delta \, \mathrm{NII}\,(t, u_1, u_2) &= \Delta \, \mathrm{NII}\,(t, u_2) - \Delta \, \mathrm{NII}\,(t, u_1) \\ &= (\mathrm{GAP}\,(t, u_2) - \mathrm{GAP}\,(t, u_1)) \cdot \Delta R \\ &= \mathrm{GAP}\,(t, u_1, u_2) \cdot \Delta R \\ &= (\mathrm{RSA}\,(t, u_1, u_2) - \mathrm{RSL}\,(t, u_1, u_2)) \cdot \Delta R \end{aligned}$$

where $\mathrm{GAP}\,(t, u_1, u_2)$, $\mathrm{RSA}\,(t, u_1, u_2)$ and $\mathrm{RSL}\,(t, u_1, u_2)$ are respectively the static gap, rate sensitive assets and rate sensitive liabilities for the period $[u_1, u_2[$.

7.2.2.2 Net interest income

Definition We recall that the net interest income of the bank is the difference between interest rate revenues of its assets and interest rate expenses of its liabilities:

$$\mathrm{NII}\,(t,u) = \sum_{i \in \mathcal{A}ssets} N_i\,(t,u) \cdot R_i\,(t,u) - \sum_{j \in \mathcal{L}iabilities} N_j\,(t,u) \cdot R_j\,(t,u) \tag{7.16}$$

where $\mathrm{NII}\,(t,u)$ is the net interest income at time t for the maturity date u, $N_i\,(t,u)$ is the notional outstanding at time u for the instrument i and $R_i\,(t,u)$ is the associated interest rate. This formula is similar to the approximated equation presented above, but it is based on a full repricing model. However, this formula is static and assumes a run-off balance sheet. In order to be more realistic, we can assume a dynamic balance sheet. However, the computation of the net interest income is then more complex because it requires modeling the liquidity gap and also behavioral options.

An example We consider a simplified balance sheet with the following asset and liability positions:

- The asset position is made up of two bullet loans A and B, whose remaining maturity is respectively equal to 18 months and 2 years. The outstanding notional of each loan is equal to 500. Moreover, we assume that the interest rate is equal to 6% for the first loan and 5% for the second loan.

TABLE 7.18: Interest income schedule and liquidity gap

$u-t$	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Loan A	7.50	7.50	7.50	7.50	7.50	7.50		
Loan B	6.25	6.25	6.25	6.25	6.25	6.25	6.25	6.25
IR revenues	13.25	13.25	13.25	13.25	13.25	13.25	6.25	6.25
Debt C	6.00	6.00	6.00	6.00				
Equity	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
IR expenses	6.00	6.00	6.00	6.00	0.00	0.00	0.00	0.00
$\text{NII}(t,u)$	7.25	7.25	7.25	7.25	13.25	13.25	6.25	6.25
$\mathcal{LG}(t,u)$	0	0	0	0	-800	-800	-300	-300

- The liability position is made up of a bullet debt instrument C, whose remaining maturity is 1 year and outstanding notional is 800. We assume that the interest rate is equal to 3%.
- The equity capital is equal to 200.

To calculate the net interest income, we calculate the interest rate revenues and costs. By assuming a quarterly pricing, the quarterly income of the instruments are:

$$\begin{aligned} I_A &= \frac{1}{4} \times 6\% \times 500 = 7.50 \\ I_B &= \frac{1}{4} \times 5\% \times 500 = 6.25 \\ I_C &= \frac{1}{4} \times 3\% \times 800 = 6.00 \end{aligned}$$

We obtain the interest income schedule given in Table 7.18. However, calculating the net interest income as the simple difference between interest rate revenues and expenses ignores the fact that the balance sheet is unbalanced. In the last row in Table 7.18, we have reported the liquidity gap. At time $u = t + 1.25$, the value of the liabilities is equal to 200 because the borrowing has matured. It follows that the liquidity gap is equal to -800. At time $u = t + 1.75$, the loan A will mature. In this case, the liabilities is made up of the equity capital whereas the assets is made up of the loan B. We deduce that the liquidity gap is equal to $200 - 500 = -300$.

TABLE 7.19: Balance sheet under the constraint of a zero liquidity gap

	$u-t$	1.25	1.50	1.75	2.00
Approach #1	Debt D	500	500		
	Debt E	300	300	300	300
Approach #2	Loan F			500	500
	Debt G	800	800	800	800

At this stage, we can explore several approaches to model the net interest income, and impose a zero liquidity gap. In the first approach, the bank borrows 500 for the period $[t+1, t+1.50]$ and 300 for the period $[t+1, t+2]$. This corresponds to debt instruments D and E in Table 7.19. We note $\tilde{R}_L(t,u)$ the interest rate for these new liabilities. We notice that $\tilde{R}_L(t,u)$ is a random variable at time t, because it will be known at time $t+1$.

We have:

$$\begin{aligned}
\operatorname{NII}(t,u) &= 13.25 - \frac{1}{4}\times 800 \times \tilde{R}_L(t,u) \\
&= 13.25 - \frac{1}{4}\times 800 \times \left(\tilde{R}_L(t,u) - 3\%\right) - \frac{1}{4}\times 800 \times 3\% \\
&= 7.25 - 200 \times \left(\tilde{R}_L(t,u) - 3\%\right)
\end{aligned}$$

for $u = t + 0.25$ and $u = t + 0.5$, and:

$$\operatorname{NII}(t,u) = 6.25 - \frac{1}{4}\times 300 \times \tilde{R}_L(t,u) = 4.00 - 75 \times \left(\tilde{R}_L(t,u) - 3\%\right)$$

for $u = t + 1.75$ and $u = t + 2.0$.

The drawback of the previous approach is that the size of the balance sheet has been dramatically reduced for the two last dates. This situation is not realistic, because it assumes that the assets are not replaced by the new production. This is why it is better to consider that Loan A is rolled into Loan F, and the debt instrument C is replaced by the debt instrument G (see Table 7.19). In this case, we obtain:

$$\begin{aligned}
\operatorname{NII}(t,u) &= 6.25 + \frac{1}{4}\times 500 \times \tilde{R}_A(t,u) - \frac{1}{4}\times 800 \times \tilde{R}_L(t,u) \\
&= 6.25 + \frac{1}{4}\times 500 \times \left(\tilde{R}_A(t,u) - 6\%\right) + \frac{1}{4}\times 500 \times 6\% - \\
&\quad \frac{1}{4}\times 800 \times \left(\tilde{R}_L(t,u) - 3\%\right) - \frac{1}{4}\times 800 \times 3\% \\
&= 7.25 + \frac{1}{4}\times 500 \times \left(\tilde{R}_A(t,u) - 6\%\right) - \frac{1}{4}\times 800 \times \left(\tilde{R}_L(t,u) - 3\%\right)
\end{aligned}$$

If we note $\Delta R_L = \tilde{R}_L(t,u) - 3\%$ and $\Delta R_A = \tilde{R}_A(t,u) - 6\%$, we obtain the following figures[42]:

ΔR_A	-2%	-1%	0%	$+1\%$	$+2\%$	-2%	-2%	-1.5%
ΔR_L	-2%	-1%	0%	$+1\%$	$+2\%$	0%	-1%	0.0%
$t + 1.00$	7.25	7.25	7.25	7.25	7.25	7.25	7.25	7.25
$t + 1.25$	11.25	9.25	7.25	5.25	3.25	7.25	9.25	7.25
$t + 1.50$	11.25	9.25	7.25	5.25	3.25	7.25	9.25	7.25
$t + 1.75$	8.75	8.00	7.25	6.50	5.75	4.75	6.75	4.13
$t + 2.00$	8.75	8.00	7.25	6.50	5.75	4.75	6.75	4.13

The case $\Delta R_L = \Delta R_A$ is equivalent to use the income gap analysis. However, this approach is simple and approximative. It does not take into account the maturity of the instruments and the dynamics of the yield curve. Let us consider a period of falling interest rates. We assume that the yield of assets is equal to the short interest rate plus 2% on average while the cost of liabilities is generally equal to the short interest rate plus 1%. On average, the bank captures a net interest margin (NIM) of 1%. This means that the market interest rate was equal to 5% for Loan A, 4% for Loan B and 2% for Debt C. We can then think that Loan A has been issued a long time ago whereas Debt C is more recent. If the interest rate environment stays at 2%, we have $\tilde{R}_A(t,u) = 4\%$ and $\tilde{R}_L(t,u) = 3\%$, which implies that $\Delta R_A = 4\% - 6\% = -2\%$ and $\Delta R_L = 3\% - 3\% = 0\%$. We obtain the results given in the seventh column. We can also explore other interest rate scenarios or other business

[42] We have $\operatorname{NII}(t,t+1) = 7.25$, $\operatorname{NII}(t,t+1.25) = \operatorname{NII}(t,t+1.5) = 7.25 - 200 \times \Delta R_L$ and $\operatorname{NII}(t,t+1.75) = \operatorname{NII}(t,t+2) = 7.25 + 125 \times \Delta R_A - 200 \times \Delta R_L$.

scenarios. For instance, the bank may be safer than before, meaning that the spread paid to the market is lower (eight column) or the bank may have an aggressive loan issuing model, implying that the interest rate margin is reduced (ninth column).

Remark 81 *The previous analysis gives the impression that the net interest income is known for $u < t + 1.5$ and stochastic after. In fact, this is not true. Indeed, we notice that the interest rates of Loans A and B are equal to 6% and 5% whereas the current interest rates are around 2%. Therefore, we can anticipate that the bank will be subject to prepayment issues. Our analysis does not take into account the behavior of clients and the impact of embedded options in the net interest income*[43].

Mathematical formulation We reiterate that the net interest income is equal to:

$$\mathrm{NII}\left(t,u\right)=\sum_{i\in\mathcal{A}ssets}N_{i}\left(t,u\right)\cdot R_{i}\left(t,u\right)-\sum_{j\in\mathcal{L}iabilities}N_{j}\left(t,u\right)\cdot R_{j}\left(t,u\right)$$

If we consider a future date $t' > t$, we have:

$$\begin{aligned}\mathrm{NII}\left(t',u\right) &= \sum_{i\in\mathcal{A}ssets}N_{i}\left(t',u\right)\cdot R_{i}\left(t',u\right)-\sum_{j\in\mathcal{L}iabilities}N_{j}\left(t',u\right)\cdot R_{j}\left(t',u\right)- \\ &\quad \left(\sum_{i\in\mathcal{A}ssets}N_{i}\left(t',u\right)-\sum_{j\in\mathcal{L}iabilities}N_{j}\left(t',u\right)\right)\cdot R\left(t',u\right)\end{aligned}$$

The future NII requires the projection of the new production and the forecasting of asset and liability rates (or customer rates). The third term represents the liquidity gap that must be financed or placed[44]. In what follows, we assume that the future liquidity gap is equal to zero in order to obtain tractable formulas.

Since we have the identity $\Delta\,\mathrm{NII}\left(t',u\right)=\mathrm{GAP}\left(t',u\right)\cdot\Delta R$, we deduce that:

$$\begin{aligned}\mathrm{GAP}\left(t',u\right) &= \frac{\Delta\,\mathrm{NII}\left(t',u\right)}{\Delta R} \\ &= \sum_{i\in\mathcal{A}ssets}N_{i}\left(t',u\right)\cdot\left(\frac{\Delta R_{i}\left(t',u\right)}{\Delta R}-1\right)- \\ &\quad \sum_{j\in\mathcal{L}iabilities}N_{j}\left(t',u\right)\cdot\left(\frac{\Delta R_{j}\left(t',u\right)}{\Delta R}-1\right)\end{aligned}$$

If we consider a continuous-time analysis where $u = t' + \mathrm{d}t$, we obtain:

$$\begin{aligned}\mathrm{GAP}\left(t',u\right) &= \sum_{i\in\mathcal{A}ssets}N_{i}\left(t',u\right)\cdot\left(\frac{\partial\,R_{i}\left(t',u\right)}{\partial\,R}-1\right)- \\ &\quad \sum_{j\in\mathcal{L}iabilities}N_{j}\left(t',u\right)\cdot\left(\frac{\partial\,R_{j}\left(t',u\right)}{\partial\,R}-1\right)\end{aligned}$$

where R represents the market interest rate[45]. Demey *et al.* (2003) consider two opposite situations corresponding to two categories of asset/liability rates:

[43] This issue is analyzed in the third section of this chapter on page 427.
[44] The borrowing/lending interest rate is denoted by $R(t', u)$.
[45] We recall that the gap analysis assumes a flat yield curve.

$\mathcal{C}_1$ The asset/liability rates are deterministic and independent from market interest rates:

$$\frac{\partial R_i(t',u)}{\partial R} = \frac{\partial R_j(t',u)}{\partial R} = 0$$

This category corresponds to contractual rates that are generally fixed.

$\mathcal{C}_2$ The asset/liability rates depend on market interest rates:

$$\begin{cases} R_i(t',u) = R + m_A \\ R_j(t',u) = R + m_L \end{cases}$$

where m_A and m_L are the commercial margins for assets and liabilities. It follows that:

$$\frac{\partial R_i(t',u)}{\partial R} = \frac{\partial R_j(t',u)}{\partial R} = 1$$

This category generally concerns floating rates that are based on a market reference rate plus a spread.

We deduce that the gap is the difference between liabilities and assets that belong to the first category $\mathcal{C}_1$:

$$\text{GAP}(t',u) = \sum_{\substack{j \in \mathcal{L}iabilities \\ j \in \mathcal{C}_1}} N_j(t',u) - \sum_{\substack{i \in \mathcal{A}ssets \\ i \in \mathcal{C}_1}} N_i(t',u)$$

Modeling customer rates Until now, we have used the variable R for defining the general level of interest rates and ΔR for defining a parallel shock on the yield curve. However, this definition is not sufficiently precise to understand the real nature of R. In fact, the study of client rates is essential to understand which interest rate is important for calculating earnings-at-risk measures. In what follows, we introduce the notation $R(t) = R(t, t + \mathrm{d}t)$ and $R(u) = R(u, u + \mathrm{d}u)$. The current date or the agreement date is denoted by t while $u > t$ is a future date.

We have already distinguished fixed rates and floating (or variable) rates. By definition, a fixed rate must be known and constant when the agreement is signed between the customer and the bank:

$$R(u) = R^\star = R(t)$$

On the contrary, the customer rate is variable if:

$$\Pr\left\{\tilde{R}(u) = R(t)\right\} < 1$$

In this case, the customer rate is a random variable at time t and depends on a reference rate, which is generally a market rate. Mathematically, we can write:

$$\begin{aligned} \tilde{R}(u) &= R(t) \cdot \mathbb{1}\{u < \tau\} + \tilde{R}(\tau) \cdot \mathbb{1}\{u \geq \tau\} \\ &= R^\star \cdot \mathbb{1}\{u < \tau\} + \tilde{R}(\tau) \cdot \mathbb{1}\{u \geq \tau\} \end{aligned} \tag{7.17}$$

where τ is the time at which the customer rate will change. τ is also called the next repricing date. For some products, τ is known while it may be stochastic in some situations[46]. If $\tilde{R}(\tau)$ is a function of a market rate, we can write:

$$\tilde{R}(\tau) = f(\tau, r(\tau))$$

[46]When the repricing date is known, it is also called the reset date.

We use the notation $r(\tau)$, because the market rate is generally a short-term interest rate. If we assume a linear relationship (noted $\mathcal{H}_{\mathcal{L}inear}$), we have:

$$\tilde{R}(\tau) = \rho \cdot r(\tau) + \tilde{m} \tag{7.18}$$

where ρ is the correlation between the customer rate and the market rate and $\tilde{m}$ is related to the commercial margin[47]. This is the simplest way for modeling $\tilde{R}(\tau)$, but there are some situations where the relationship is more complex. For example, Demey *et al.* (2003) study the case where the customer rate has a cap:

$$\tilde{R}(\tau) = r(\tau) \cdot \mathbb{1}\left\{r(\tau) < r^+\right\} + r^+ \cdot \mathbb{1}\left\{r(\tau) \geq r^+\right\} + \tilde{m}$$

where $r^+ + \tilde{m}$ is the cap.

Another challenge for modeling $\tilde{R}(u)$ is the case where the next repricing date τ is unknown. We generally assume that τ is exponentially distributed with parameter λ. If we consider the linear relationship (7.18), it follows that the expected customer rate is:

$$\begin{aligned} R(u) &= \mathbb{E}\left[\tilde{R}(u)\right] \\ &= R^\star \cdot e^{-\lambda(u-t)} + (\rho \cdot r(u) + \tilde{m}) \cdot \left(1 - e^{-\lambda(u-t)}\right) \end{aligned} \tag{7.19}$$

Sometimes, the relationship between the customer rate and the market rate is not instantaneous. For instance, Demey *et al.* (2003) consider the case where the customer rate is an average of the market rate over a window period h. Therefore, Equation (7.19) becomes[48]:

$$R(u) = R^\star \cdot e^{-\lambda(u-t)} + \lambda \int_{u-h}^{u} (\rho \cdot r(s) + \tilde{m}) \cdot e^{-\lambda(s-t)} \, \mathrm{d}s$$

Let us go back to the problem of determining the parallel shock ΔR. Using Equation (7.17),we have:

$$\begin{aligned} \Delta R &= \tilde{R}(u) - R(t) \\ &= \begin{cases} 0 & \text{if } u < \tau \\ \tilde{R}(\tau) - R^\star & \text{otherwise} \end{cases} \end{aligned}$$

Under the assumption $\mathcal{H}_{\mathcal{L}inear}$, we deduce that:

$$\Delta R = \tilde{R}(\tau) - R^\star = \rho \cdot \Delta r \tag{7.20}$$

where $\Delta r = r(\tau) - r(t)$ is the shock on the market rate. We notice that modeling the net interest income variation requires determining ρ and Δr. In the case where $\rho = 0$, we retrieve the previous result that $\Delta\,\mathrm{NII}$ is not sensitive to fixed rate items. Otherwise, Equation (7.20) shows that interest rate gaps must be conducted on a contract by contract basis or at least for each reference rate:

> "*Floating-rate interest gaps can be defined for all floating-rate references (1-month Libor, 1-year Libor, etc.). These floating-rate gaps are not fungible: they cannot be aggregated unless assuming a parallel shift of all rates*" (Bessis, 2015, page 47).

[47] The commercial margin is equal to:

$$\begin{aligned} m &= \tilde{R}(\tau) - r(\tau) \\ &= \tilde{m} - (1 - \rho)\, r(\tau) \end{aligned}$$

When the correlation is equal to one, $\tilde{m}$ is equal to the commercial margin, otherwise it is greater.

[48] We assume that $u - h \geq t$.

Indeed, two contracts may have two different correlations with the same reference rate, and two contracts may have two different reference rates.

Equation (7.20) is valid only if we assume that the next repricing date is known. If τ is stochastic, Demey *et al.* (2003) obtain the following formula:

$$\begin{aligned}\Delta R(u) &= \mathbb{E}\left[\left(\tilde{R}(u) - R(t)\right) \cdot \mathbb{1}\{u \geq \tau\}\right] \\ &= \rho \cdot \Delta r \cdot \Pr\{\tau \leq u\}\end{aligned}$$

We conclude that the sensitivity of the customer rate to the market rate is equal to:

$$\rho(t,u) = \frac{\Delta R(u)}{\Delta r} = \rho \cdot \Pr\{\tau \leq u\}$$

It depends on two parameters: the correlation ρ between the two rates and the probability distribution of the repricing date τ. If τ follows an exponential distribution with parameter λ, we have $\rho(t,u) = \rho\left(1 - e^{-\lambda(u-t)}\right)$. We verify that $\rho(t,u) \leq \rho$. The upper limit case $\rho(t,u) = \rho$ is reached in the deterministic case (no random repricing), whereas the function $\rho(t,u)$ is equal to zero if ρ is equal to zero (no correlation). By definition of the exponential distribution, the average time between two repricing dates is equal to $1/\lambda$. In Figure 7.13, we have reported the function $\rho(t,u)$ for three values of the correlation : 0%, 50% and 100%. We show how λ impacts the sensitivity $\rho(t,u)$ and therefore Δ NII. This last parameter is particularly important when we consider embedded options and customer behavior[49]. For instance, $\lambda = 0.1$ implies that the contract is repriced every ten years on average (top/left panel). It is obvious that the sensitivity is lower for this contract than for a contract that is repriced every 2 years (top/right panel).

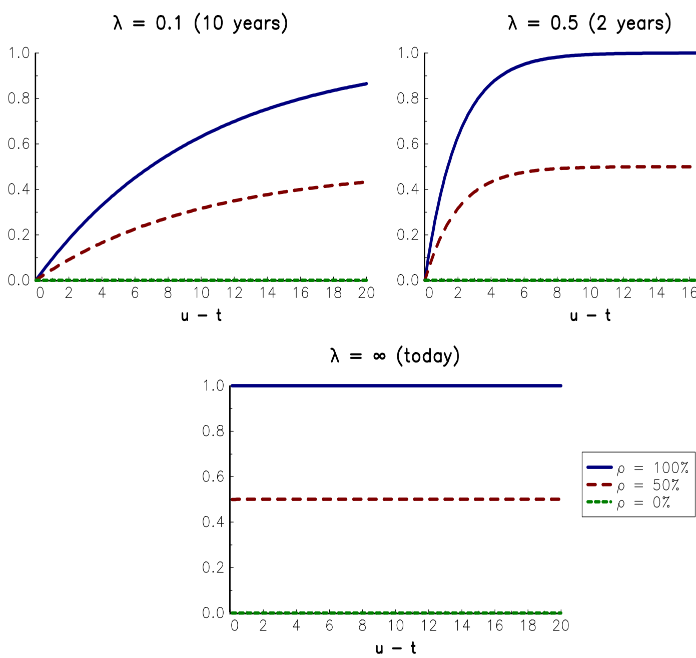

FIGURE 7.13: Sensitivity of the customer rate with respect to the market rate

[49]See Section 7.3 on page 427.

7.2.2.3 Hedging strategies

The question of hedging is not an easy task. There is no one optimal solution, but several answers. Moreover, this problem will be even more complicated when we will integrate the behavioral and embedded options.

To hedge or not to hedge Since the net interest income is sensitive to interest rate changes, it is important to define a hedging policy and to understand how it may impact the income statement of the bank. Let us define the hedged net interest income as the sum of the net interest income and the hedge P&L:

$$\mathrm{NII}_{\mathcal{H}}(t,u) = \mathrm{NII}(t,u) + \mathcal{H}(t,u)$$

In order to obtain a tractable formula of the hedge P&L $\mathcal{H}(t,u)$, we consider a forward rate agreement (FRA), which is an exchange contract between the future interest rate $r(u)$ at the pricing date u and the current forward rate $f(t,u)$ at the maturity date u. The hedge P&L is then:

$$\mathcal{H}(t,u) = N_{\mathcal{H}}(t,u) \cdot (f(t,u) - r(u))$$

where $N_{\mathcal{H}}(t,u)$ is the notional of the hedging strategy. We deduce that:

$$\begin{array}{rcl} \Delta\,\mathrm{NII}_{\mathcal{H}}(t,u) & = & \Delta\,\mathrm{NII}(t,u) + \Delta\mathcal{H}(t,u) \\ & = & \mathrm{GAP}(t,u)\cdot\Delta R(u) - N_{\mathcal{H}}(t,u)\cdot\Delta r(u) \\ & = & (\mathrm{GAP}(t,u)\cdot\rho(t,u) - N_{\mathcal{H}}(t,u))\cdot\Delta r(u) \end{array}$$

because we have $\Delta R(u) = \rho(t,u)\cdot\Delta r(u)$. The hedged NII is equal to zero if the notional of the hedge is equal to the product of the interest rate gap and the sensitivity $\rho(t,u)$:

$$\Delta\,\mathrm{NII}_{\mathcal{H}}(t,u) = 0 \Leftrightarrow N_{\mathcal{H}}(t,u) = \mathrm{GAP}(t,u)\cdot\rho(t,u)$$

In this case, we obtain:

$$\mathrm{NII}_{\mathcal{H}}(t,u) - \mathrm{NII}(t,u) = \mathrm{GAP}(t,u)\cdot\rho(t,u)\cdot(f(t,u) - r(u))$$

We can draw several conclusions from the above mathematical framework:

- When the correlation between the customer rate and the market rate is equal to one, the notional of the hedge is exactly equal to the interest rate gap. Otherwise, it is lower.

- When the interest rate gap is closed, the bank does not need to hedge the net interest income.

- If the bank hedges the net interest income, the difference $\mathrm{NII}_{\mathcal{H}}(t,u) - \mathrm{NII}(t,u)$ is positive if the gap and the difference between $f(t,u)$ and $r(u)$ have the same sign. For example, if the gap is positive, a decrease of interest rates is not favorable. This implies that the hedged NII is greater than the non-hedged NII only if the forward rate $f(t,u)$ is greater than the future market rate $r(u)$. This situation is equivalent to anticipate that the forward rate is overestimated.

We conclude that hedging the interest rate gap is not systematic and depends on the expectations of the bank. It is extremely rare that the bank fully hedges the net interest income. The other extreme situation where the NII is fully exposed to interest rate changes is also not very common. Generally, the bank prefers to consider a partial hedging. Moreover, we reiterate that the previous analysis is based on numerous assumptions[50]. Therefore, it is useless to compute a precise hedging strategy because of these approximations. This is why banks prefer to put in place macro hedging strategies with a limited number of instruments.

Hedging instruments In order to hedge the interest rate gap, the bank uses interest rate derivatives. They may be classified into two categories: those that hedge linear interest rate risks and those that hedge non-linear interest rate risks. The first category is made up of interest rate swaps (IRS) and forward rate agreements (FRA), while the second category concerns options such as caps, floors and swaptions. An IRS is a swap where two counterparties exchange a fixed rate against a floating rate or two floating rates. This is the hedging instrument which is certainly the most used in asset and liability management. The fixed rate is calibrated such that the initial value of the swap is equal to zero, meaning that the cost of entering into an IRS is low. This explains the popularity of IRS among ALM managers. However, these hedging instruments only concern linear changes in interest rates like the FRA instruments. In general, the ALM manager doesn't close fully all the interest rate gaps because this is not the purpose of a macro hedging strategy. In practice, two or three maturities are sufficient to highly reduce the risk.

Remark 82 *In order to hedge non-linear risks (slope of the yield curve, embedded options, etc.), the bank may use options. However, they are more expensive than IRS and are much less used by banks. One of the difficulties is the high degree of uncertainty around customer behavioral modeling.*

7.2.3 Simulation approach

We present here a general top-down econometric-based simulation framework in order to model the dynamics of the outstanding amount for the different items of the balance sheet. The underlying idea is that these items respond differently to key economic and market variables. The focus is then to model the earnings-at-risk profile of these items. The different profiles can also be aggregated in order to understand the income risk of each business line of the bank.

The framework is based on the cointegration theory and error correction models[51]. It is made up of two econometric models. We first begin by modeling the economic and market variables $x(t) = (x_1(t), \ldots, x_m(t))$ with a VECM:

$$\Phi_x(L)\,\Delta x(t) = \Pi_x x(t-1) + \varepsilon_x(t) \tag{7.21}$$

where $\Phi(L) = I_m - \Phi_1 L - \ldots - \Phi_p L^p$ is the lag polynomial and $\varepsilon_x(t) \sim \mathcal{N}(0, \Sigma_x)$. By definition, Equation (7.21) is valid if we have verified that each component of $x(t)$ is integrated of order one. The choice of the number p of lags is important. Generally, we consider a monthly econometric model, where the variables $x(t)$ are the economic growth $g(t)$, the inflation rate $\pi(t)$, the short-term market rate $r(t)$, the long-term interest rate $R(t)$, etc. In practice, $p = 3$ is used in order to have quarterly relationship between economic and market variables. The goal of this first econometric models is to simulate joint scenarios $\mathcal{S}_x$ of the economy and the market. Each scenario is represented by the current values of $x(t)$ and the future paths of $x(t+h)$:

$$\mathcal{S}_x = \{x(t+h) = (x_1(t+h), \ldots, x_m(t+h)), h = 0, 1, 2, \ldots\} \tag{7.22}$$

[50] They concern the sensitivity to markets rates, the behavior of customers, the new production, the interest rate shocks, etc.

[51] They are developed in Section 10.2.3 on page 655.

These scenarios do not necessarily correspond to extreme shocks, but they model the probability distribution of all future outcomes.

The second step consists in relating the growth of the outstanding amount $y_i(t)$ of item i to the variables $x(t)$. For instance, let us assume that:

$$y_i(t) = y_i(t-1) + 0.7 \times g(t) - 0.3 \times \pi(t)$$

This means that an economic growth of 1% implies that the outstanding amount of item i will increase by 70 bps, while the inflation has a negative impact on $y_i(t)$. The first idea is then to consider an ARX (q) model:

$$y_i(t) = \sum_{k=1}^{q} \phi_{i,k} y_i(t-k) + \sum_{j=1}^{m} \beta_{i,j} x_j(t) + \varepsilon_x(t)$$

However, this type of model has two drawbacks. It assumes that the current value of $y_i(t)$ is related to the current value of $x_j(t)$ and there are no substitution effects between the different items of the balance sheet. This is why it is better to consider again a VECM approach with exogenous variables:

$$\Phi_y(L)\,\Delta y(t) = \Pi_y y(t-1) + B_1 x(t) + B_2 \Delta x(t) + \varepsilon_y(t) \tag{7.23}$$

where $y(t) = (y_1(t), \ldots, y_n(t))$ and $\varepsilon_y(t) \sim \mathcal{N}(0, \Sigma_y)$. In this case, the current value of $y_i(t)$ is related to the current value of $x(t)$, the monthly variation $\Delta x(t)$ and the growth of the outstanding amount of the other items. Generally, the number q of lags is less than p. Indeed, the goal of the model (7.23) is to include short-term substitution effects between the different items whereas long-term substitution effects are more explained by the dynamics of economic and market variables.

Once the model (7.23) is estimated, we can simulate the future values of the outstanding amount for the different items with respect to the scenario $\mathcal{S}_x$ of the exogenous variables:

$$\mathcal{S}_y \mid \mathcal{S}_x = \{y(t+h) = (y_1(t+h), \ldots, y_n(t+h)), h = 0, 1, 2, \ldots\}$$

This framework allows going beyond the static gap analysis of interest rates, because the outstanding amounts are stochastic. For example, Figure 7.14 shows an earnings-at-risk analysis of the net interest income for the next six months. For each month, we report the median of NII and the 90% confidence interval.

Remark 83 *The previous framework can be used for assessing a given scenario, for example a parallel shock of interest rates. By construction, it will not give the same result than the income gap analysis, because this latter does not take into account the feedback effects of interest rates on the outstanding amount.*

7.2.4 Funds transfer pricing

According to Bessis (2015), the main objective of funds transfer pricing systems is to exchange funds and determine the profit allocation between business units. This means that all liquidity and interest rate risks are transferred to the ALM unit, which is in charge of managing them. Business units can then lend or borrow funding at a given internal price. This price is called the funds transfer price or the internal transfer rate, and is denoted by FTP. For example, the FTP charges interests to the business unit for client loans, whereas the FTP compensates the business unit for raising deposits. This implies that the balance sheet of the different business units is immunized to changes of market rates, and the internal transfer rates determine the net interest income of each business unit.

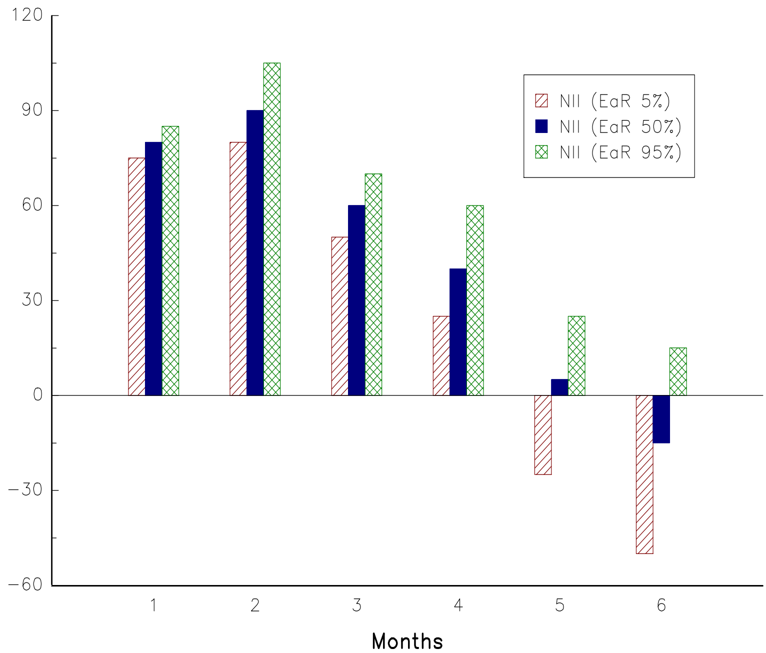

FIGURE 7.14: Earnings-at-risk analysis

7.2.4.1 Net interest and commercial margins

The net interest margin (NIM) is equal to the net interest income divided by the amount of assets:

$$\mathrm{NIM}(t,u) = \frac{\sum_{i\in\mathcal{A}ssets} N_i(t,u)\cdot R_i(t,u) - \sum_{j\in\mathcal{L}iabilities} N_j(t,u)\cdot R_j(t,u)}{\sum_{i\in\mathcal{A}ssets} N_i(t,u)}$$

Let $\mathrm{RA}(t,u)$ and $\mathrm{RL}(t,u)$ be the interest earning assets and interest bearing liabilities (or asset and liability amounts that are sensitive to interest rates). Another expression of the NIM is:

$$\mathrm{NIM}(t,u) = \frac{\mathrm{RA}(t,u)\cdot R_{\mathrm{RA}}(t,u) - \mathrm{RL}(t,u)\cdot R_{\mathrm{RL}}(t,u)}{\mathrm{RA}(t,u)}$$

where R_{RA} and R_{RL} represent the weighted average interest rate of interest earning assets and interest bearing liabilities. The net interest margin differs from the net interest spread (NIS), which is the difference between interest earning rates and interest bearing rates:

$$\begin{aligned}\mathrm{NIS}(t,u) &= \frac{\sum_{i\in\mathcal{A}ssets} N_i(t,u)\cdot R_i(t,u)}{\sum_{i\in\mathcal{A}ssets} N_i(t,u)} - \frac{\sum_{j\in\mathcal{L}iabilities} N_j(t,u)\cdot R_j(t,u)}{\sum_{j\in\mathcal{L}iabilities} N_j(t,u)} \\ &= R_{\mathrm{RA}}(t,u) - R_{\mathrm{RL}}(t,u)\end{aligned}$$

Example 73 *We consider the following interest earning and bearing items:*

Assets	$N_i(t,u)$	$R_i(t,u)$	*Liabilities*	$N_j(t,u)$	$R_j(t,u)$
Loans	*100*	*5%*	*Deposits*	*100*	*0.5%*
Mortgages	*100*	*4%*	*Debts*	*60*	*2.5%*

The interest income is equal to $100 \times 5\% + 100 \times 4\% = 9$ and the interest expense is $100 \times 0.5\% + 60 \times 2.5\% = 2$. We deduce that the net interest income is equal to $9 - 2 = 7$.

Moreover, we obtain[52] $\mathrm{RA}(t,u) = 200$, $R_{\mathrm{RA}}(t,u) = 4.5\%$, $\mathrm{RL}(t,u) = 160$ and $R_{\mathrm{RL}}(t,u) = 1.25\%$. We deduce that:

$$\mathrm{NIM}(t,u) = \frac{200 \times 4.5\% - 160 \times 1.25\%}{200} = \frac{7}{200} = 3.5\%$$

and:

$$\mathrm{NIS}(t,u) = 4.5\% - 1.25\% = 3.25\%$$

The net interest margin and spread are expressed in percent. NIM is the profitability ratio of the assets whereas NIS is the interest rate spread captured by the bank.

Remark 84 *In Figure 7.15, we have reported the average net interest margin in % for all US banks from 1984 to 2019. The average NIM was equal to* 3.36% *at the end of the first quarter of 2019. During the last 15 years, the average value is equal to* 3.78%*, the maximum* 4.91% *has been reached during Q1 1994 whereas the minimum* 2.95% *was observed in Q1 2015.*

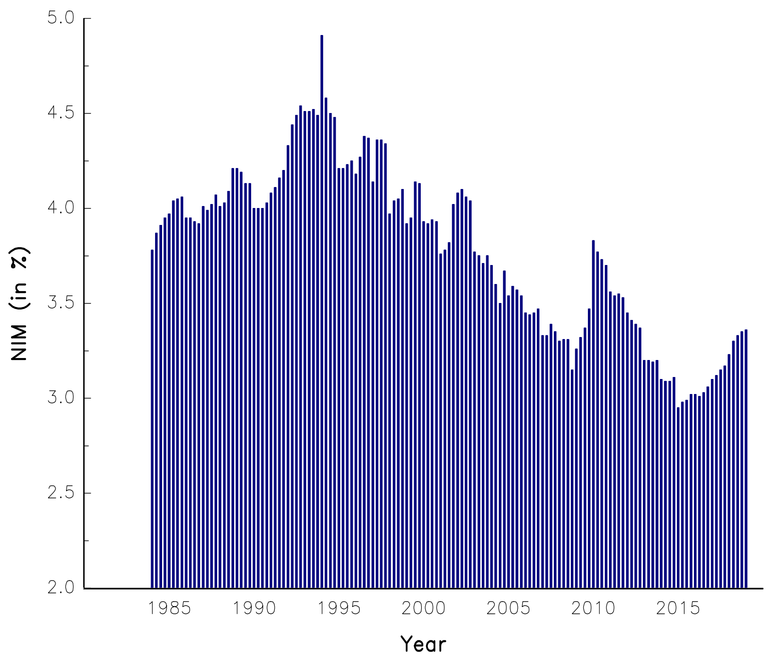

FIGURE 7.15: Evolution of the net interest margin in the US

Source: Federal Financial Institutions Examination Council (US), Net Interest Margin for all US Banks [USNIM], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/USNIM, July 9, 2019.

[52]We have:

$$R_{\mathrm{RA}}(t,u) = \frac{100 \times 5\% + 100 \times 4\%}{100 + 100} = 4.5\%$$

and:

$$R_{\mathrm{RL}}(t,u) = \frac{100 \times 0.5\% + 60 \times 2.5\%}{100 + 60} = 1.25\%$$

Let us now see how to calculate the commercial margin rate. A first idea is to approximate it by the net interest margin or the net interest spread. However, these quantities are calculated at the global level of the bank, not at the level of a business unit and even less at the level of a product. Let us consider an asset i. From a theoretical point of view, the commercial margin rate is the spread between the client rate of this asset $R_i(t,u)$ and the corresponding market rate $r(t,u)$:

$$m_i(t,u) = R_i(t,u) - r(t,u)$$

Here, we assume that $R_i(t,u)$ and $r(t,u)$ have the same maturity u. If we consider a liability j, we obtain a similar formula:

$$m_j(t,u) = r(t,u) - R_j(t,u)$$

In this framework, we assume that the business unit borrows at the market rate $r(t,u)$ in order to finance the asset i or lends to the market at the same rate $r(t,u)$. A positive commercial margin rate implies that $R_i(t,u) > r(t,u)$ and $r(t,u) > R_j(t,u)$. In the case where we can perfectly match the asset i with the liability j, the commercial margin rate is the net interest spread:

$$\begin{aligned} m(t,u) &= m_i(t,u) + m_j(t,u) \\ &= R_i(t,u) - R_j(t,u) \end{aligned}$$

As already said, a funds transfer pricing system is equivalent to interpose the ALM unit between the business unit and the market. In the case of assets, we decompose the commercial margin rate of the bank as follows:

$$\begin{aligned} m_i(t,u) &= R_i(t,u) - r(t,u) \\ &= \underbrace{(R_i(t,u) - \mathrm{FTP}_i(t,u))}_{m_i^{(c)}(t,u)} + \underbrace{(\mathrm{FTP}_i(t,u) - r(t,u))}_{m_i^{(t)}(t,u)} \end{aligned}$$

where $m_i^{(c)}(t,u)$ and $m_i^{(t)}(t,u)$ are the commercial margin rate of the business unit and the transformation margin rate of the ALM unit. For liabilities, we also have:

$$\begin{aligned} m_j(t,u) &= m_j^{(c)}(t,u) + m_j^{(t)}(t,u) \\ &= (\mathrm{FTP}_j(t,u) - R_j(t,u)) + (r(t,u) - \mathrm{FTP}_j(t,u)) \end{aligned}$$

The goal of FTP is then to lock the commercial margin rate $m_i^{(c)}(t,u)$ (or $m_j(t,u)$) over the lifetime of the product contract.

Let us consider Example 73. The FTP for the loans and the mortgages is equal to 3%, while the FTP for deposits is equal to 1.5% and the FTP for debts is equal to 2.5%. If we assume that the market rate is equal to 2.5%, we obtain the following results:

Assets	$m_i^{(c)}(t,u)$	$m_i^{(t)}(t,u)$	Liabilities	$m_j^{(c)}(t,u)$	$m_j^{(t)}(t,u)$
Loans	2%	0.5%	Deposits	1.0%	1.0%
Mortgages	1%	0.5%	Debts	0.0%	0.0%

It follows that the commercial margin of the bank is equal to:

$$\begin{aligned} M^{(c)} &= 100 \times 2\% + 100 \times 1\% + 100 \times 1\% + 60 \times 0\% \\ &= 4 \end{aligned}$$

For the transformation margin, we have:

$$\begin{aligned} M^{(t)} &= 100 \times 0.5\% + 100 \times 0.5\% + 100 \times 1.0\% + 60 \times 0\% \\ &= 2.0 \end{aligned}$$

We don't have $M^{(c)} + M^{(t)} = \text{NII}$ because assets and liabilities are not compensated:

$$\begin{aligned} \text{NII} - \left(M^{(c)} + M^{(t)}\right) &= (\text{RA}(t,u) - \text{RL}(t,u)) \cdot r(t,u) \\ &= 40 \times 2.5\% \\ &= 1 \end{aligned}$$

In fact, in a funds transfer pricing system, the balance sheet issue is the problem of the ALM unit. It is also interesting to notice that we can now calculate the commercial margin of each product: $M_{Loans}^{(c)} = 2$, $M_{Mortgages}^{(c)} = 1$ and $M_{Deposits}^{(c)} = 1$. We can then aggregate them by business units. For example, if the business unit is responsible for loans and deposits, its commercial margin is equal to 3.

7.2.4.2 Computing the internal transfer rates

Since the business unit knows the internal prices of funding, the commercial margin rates are locked and the commercial margin has a smooth profile. The business unit can then focus on its main objective, which is selling products and not losing time in managing interest rate and liquidity risks. However, in order to do correctly its job, the internal prices must be fair. The determination of FTPs is then crucial because it has a direct impact on the net income of the business unit. A system of arbitrary or wrong prices can lead to a false analysis of the income allocation, where some business units appear to be highly profitable when the exact opposite is true. The consequence is then a wrong allocation of resources and capital.

The reference rate If we consider the transformation margin rate, we have $m_i^{(t)}(t,u) = \text{FTP}_i(t,u) - r(t,u)$. The internal prices are fair if the corresponding mark-to-market is equal to zero on average, because the goal of FTP is to smooth the net interest income of each business unit and to allocate efficiently the net interest income between the different business units. For a contract with a bullet maturity, this implies that:

$$\text{FTP}_i(t,u) = \mathbb{E}[r(t,u)]$$

The transformation margin can then be interpreted as an interest rate swap[53] receiving a fixed leg $\text{FTP}_i(t,u)$ and paying a floating leg $r(t,u)$. It follows that the funds transfer price is equal to the market swap rate at the initial date t with the same maturity than the asset item i (Demey *et al.*, 2003).

In practice, it is impossible to have funds transfer prices that depend on the initial date and the maturity of each contract. Let us first assume that the bank uses the short market rate $r(u)$ for determining the funds transfer prices and considers globally the new production $\text{NP}(t)$ instead of the different individual contracts. The mark-to-market of the transformation margin satisfies then the following equation:

$$\mathbb{E}_t\left[\int_t^\infty B(t,u)\,\text{NP}(t)\,\mathbf{S}(t,u)\,(\text{FTP}(t,u) - r(u))\,\mathrm{d}u\right] = 0$$

[53]In the case of liabilities, the transformation margin is an interest rate swap paying the fixed leg $\text{FTP}_i(t,u)$ and receiving the floating leg $r(t,u)$.

As noticed by Demey *et al.* (2003), we need another constraint to determine explicitly the internal transfer rate, because the previous equation is not sufficient. For instance, if we assume that the internal transfer rate is constant over the lifetime of the new production — $\mathrm{FTP}(t,u) = \mathrm{FTP}(t)$, we obtain:

$$\mathrm{FTP}(t,u) = \frac{\mathbb{E}_t\left[\int_t^\infty B(t,u)\,\mathbf{S}(t,u)\,r(u)\;\mathrm{d}u\right]}{\mathbb{E}_t\left[\int_t^\infty B(t,u)\,\mathbf{S}(t,u)\;\mathrm{d}u\right]}$$

The drawback of this approach is that the commercial margin is not locked, and the business unit is exposed to the interest rate risk. On the contrary, we can assume that the commercial margin rate of the business unit is constant:

$$R(u) - \mathrm{FTP}(t,u) = m$$

Demey *et al.* (2003) show that[54]:

$$\mathrm{FTP}(t,u) = R(u) + \frac{\mathbb{E}_t\left[\int_t^\infty B(t,u)\,\mathbf{S}(t,u)\,(r(u) - R(u))\;\mathrm{d}u\right]}{\mathbb{E}_t\left[\int_t^\infty B(t,u)\,\mathbf{S}(t,u)\;\mathrm{d}u\right]}$$

The term structure of funds transfer prices According to Bessis (2015), there are two main approaches for designing a funds transfer pricing system: cash netting and central cash pool systems. In the first case, the business unit transfers to the ALM unit only the net cash balance, meaning that the internal transfer rates apply only to a fraction of asset and liability items. This system presents a major drawback, because business units are exposed to interest rate and liquidity risks. On the contrary, all funding and investment items are transferred into the ALM book in the second approach. In this case, all items have their own internal transfer rate. In order to reduce the complexity of the FTP system, assets and liabilities are generally classified into homogeneous pools in terms of maturity, credit, etc. In this approach, each pool has its own FTP. For example, the reference rate of long maturity pools is a long-term market rate while the reference rate of short maturity pools is a short-term market rate. In Figure 7.16, we have represented the term structure of the FTPs. Previously, we have seen that the reference rate is the market swap rate, meaning that the reference curve is the IRS curve. In practice, the FTP curve will differ from the IRS curve for several reasons. For instance, the reference curve can be adjusted by adding a credit spread in order to reflect the credit-worthiness of the bank, a bid-ask spread in order to distinguish assets and liabilities, a behavior-based spread because of prepayment and embedded options, and a liquidity spread. Therefore, we can decompose the funds transfer price as follows:

$$\mathrm{FTP}(t,u) = \mathrm{FTP}^{\mathrm{IR}}(t,u) + \mathrm{FTP}^{\mathrm{Liquidity}}(t,u) + \mathrm{FTP}^{\mathrm{Other}}(t,u)$$

where $\mathrm{FTP}^{\mathrm{IR}}(t,u)$ is the interest rate component, $\mathrm{FTP}^{\mathrm{IR}}(t,u)$ is the liquidity component and $\mathrm{FTP}^{\mathrm{Other}}(t,u)$ corresponds to the other components. The FTP curve can then be different than the IRS curve for the reasons presented above. But it can also be different because of business or ALM decisions. For instance, if the bank would like to increase its mortgage market share, it can reduce the client rate $R_i(t,u)$ meaning that the commercial

[54] Using this formulation, we can show the following results:

- for a loan with a fixed rate, the funds transfer price is exactly the swap rate with the same maturity than the loan and the same amortization scheme than the new production;
- if the client rate $R(u)$ is equal to the short-term market rate $r(u)$, the funds transfer price $\mathrm{FTP}(t,u)$ is also equal to $r(u)$.

margin $m_i^{(c)}(t,u)$ decreases, or it can maintain the commercial margin by reducing the internal transfer rate $\mathrm{FTP}_i(t,u)$. Another example concerns the investment maturity of retail deposits. Each time this maturity is revisited, it has a big impact on the retail business unit because a shorter maturity will reduce the internal transfer price and a longer maturity will increase the internal transfer price. Therefore, the FTP of deposits highly impacts the profitability of the retail business unit.

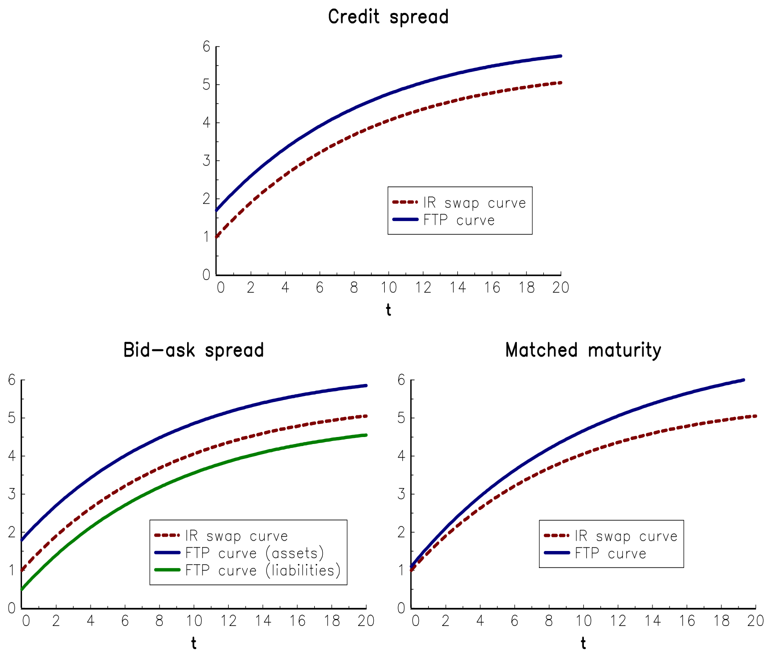

FIGURE 7.16: The term structure of FTP rates

7.3 Behavioral options

In this section, we focus on three behavioral options that make it difficult to calculate liquidity and interest rate risks. They have been clearly identified by the BCBS (2016d) and concern non-maturity deposits, prepayment risk and redemption (or early termination) issues. For NMDs, the challenge is to model the deposit volume and the associated implicit duration. For the two other risks, the goal is to calculate prepayment rates and redemption ratios on a yearly basis.

7.3.1 Non-maturity deposits

Let us assume that the deposit balance of the client A is equal to \$500. In this case, we can assume that the duration of this deposit is equal to zero day, because the client could withdraw her deposit volume today. Let us now consider 1 000 clients, whose deposit balance is equal to \$500. On average, we observe that the probability to withdraw \$500 at once is equal to 50%. The total amount that may be withdrawn today is then between \$0 and \$500 000. However, it is absurd to think that the duration of deposits is equal to zero, because the probability that \$500 000 are withdrawn is less than $10^{-300}\%$! Since we have $\Pr\{S > 275000\} < 0.1\%$, we can decide that 55% of the deposit balance has a duration of zero day, 24.75% has a duration of one day, 11.14% has a duration of two days, etc. It follows that the duration of deposits depends on the average behavior of customers and the number of account holders, but many other parameters may have an impact on non-maturity deposits. From a contractual point of view, deposits have a very short-term duration. From a statistical point of view, we notice that a part of these deposits are in fact very stable because of the law of large numbers.

NMDs are certainly the balance sheet item that is the most difficult to model. There are multiple reasons. The first reason is the non-specification of a maturity in the contract. The second reason is that NMDs are the most liquid instruments and their transaction costs are equal to zero, implying that subscriptions and redemptions are very frequent. This explains that the volume of deposits is the most volatile among the different banking products at the individual level. Another reason is the large number of embedded options that creates significant gamma and vega option risks (Blöchlinger, 2015). Finally, the volume of NMDs is very sensitive to the monetary policy (Bank of Japan, 2014), because NMDs are part of the M1 money supply, but also of the M2 money supply. Indeed, NMDs is made up of demand deposits (including overnight deposits and checkable accounts) and savings accounts. M1 captures demand deposits (and also currency in circulation) while M2 − M1 is an approximation of savings accounts. In what follows, we do not make a distinction between NMDs, but it is obvious that the bank must distinguish demand deposits and savings accounts in practice. Generally, academics model behavioral options related to NMDs by analyzing substitution effects between NMDs and term deposits. In the real life, demand-side substitution is more complex since it also concerns the cross-effects between demand deposits and savings accounts.

7.3.1.1 Static and dynamic modeling

In the case of non-maturity deposits, it is impossible to make the distinction between the entry dates. This means that the stock amortization function $\mathbf{S}^{\star}(t,u)$ must be equal to the amortization function $\mathbf{S}(t,u)$ of the new production. This implies that the hazard rate $\lambda(t,u)$ of the amortization function $\mathbf{S}(t,u)$ does not depend on the entry date t:

$$\lambda(t,u) = \lambda(u)$$

Indeed, we have by definition:

$$\mathbf{S}(t,u) = \exp\left(-\int_t^u \lambda(s)\,\mathrm{d}s\right)$$

and we verify that[55]:

$$\mathbf{S}^{\star}(t,u) = \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, S(s,u)\,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, S(s,t)\,\mathrm{d}s} = \mathbf{S}(t,u)$$

According to Demey *et al.* (2003), the concept of new production has no meaning. Then, we must focus on the modeling of the current volume of NMDs, which is given by Equation

[55]This result is based on the following computation:

$$\frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\int_s^u \lambda(v)\,\mathrm{d}v}\,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\int_s^t \lambda(v)\,\mathrm{d}v}\,\mathrm{d}s} = \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\left(\int_s^t \lambda(v)\,\mathrm{d}v + \int_t^u \lambda(v)\,\mathrm{d}v\right)}\,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\int_s^t \lambda(v)\,\mathrm{d}v}\,\mathrm{d}s} = e^{-\int_t^u \lambda(v)\,\mathrm{d}v}$$

(7.7) on page 386:

$$N(t) = \int_{-\infty}^{t} \mathrm{NP}(s)\, \mathbf{S}(s,t)\, \mathrm{d}s$$

It follows that:

$$\begin{aligned} \frac{\mathrm{d}N(t)}{\mathrm{d}t} &= \mathrm{NP}(t)\, \mathbf{S}(t,t) - \int_{-\infty}^{t} \mathrm{NP}(s)\, f(s,t)\, \mathrm{d}s \\ &= \mathrm{NP}(t) - \lambda(t) \int_{-\infty}^{t} \mathrm{NP}(s)\, \mathbf{S}(s,t)\, \mathrm{d}s \end{aligned}$$

or:

$$\mathrm{d}N(t) = (\mathrm{NP}(t) - \lambda(t)\, N(t))\, \mathrm{d}t \tag{7.24}$$

Therefore, the variation of $N(t)$ is the difference between deposit inflows $\mathrm{NP}(t)$ and deposit outflows $\lambda(t)\, N(t)$. In the case where the new production and the hazard rate are constant – $\mathrm{NP}(t) = \mathrm{NP}$ and $\lambda(t) = \lambda$, we obtain[56] $N(t) = N_\infty + (N_0 - N_\infty)\, e^{-\lambda(t-t_0)}$ where $N_0 = N(t_0)$ is the current value and $N_\infty = \lambda^{-1}\,\mathrm{NP}$ is the long-term value of $N(t)$. In this case, Equation (7.24) becomes:

$$\mathrm{d}N(t) = \lambda(N_\infty - N(t))\, \mathrm{d}t \tag{7.25}$$

We recognize the deterministic part of the Ornstein-Uhlenbeck process:

$$\mathrm{d}N(t) = \lambda(N_\infty - N(t))\, \mathrm{d}t + \sigma\, \mathrm{d}W(t) \tag{7.26}$$

where $W(t)$ is a Brownian motion. In this case, the solution is given by[57]:

$$N(t) = N_0 e^{-\lambda(t-t_0)} + N_\infty \left(1 - e^{-\lambda(t-t_0)}\right) + \sigma \int_{t_0}^{t} e^{-\lambda(t-s)}\, \mathrm{d}W(s) \tag{7.27}$$

The estimation of the parameters $(\lambda, N_\infty, \sigma)$ can be done using the generalized method of moments (GMM) or the method of maximum likelihood (ML). In this case, we can show that:

$$N(t) \mid N(s) = N_s \sim \mathcal{N}\left(\mu_{(s,t)}, \sigma^2_{(s,t)}\right)$$

where:

$$\mu_{(s,t)} = N_s e^{-\lambda(t-s)} + N_\infty \left(1 - e^{-\lambda(t-s)}\right)$$

and:

$$\sigma^2_{(s,t)} = \sigma^2 \left(\frac{1 - e^{-2\lambda(t-s)}}{2\lambda}\right)$$

Example 74 *We consider a deposit account with the following characteristics:* $N_\infty = \$1\,000$, $\lambda = 10$ *and* $\sigma = \$1\,000$.

The frequency λ means that the average duration of the deposit balance is equal to $1/\lambda$. In our case, we find $1/10 = 0.1$ years or 1.2 months. The new production is $\mathrm{NP} = \lambda N_\infty = \$10\,000$. This new production can be interpreted as the annual income of the client

[56] The solution of Equation (7.24) is given by:

$$N(t) - \frac{\mathrm{NP}}{\lambda} = \left(N_0 - \frac{\mathrm{NP}}{\lambda}\right) e^{-\lambda(t-t_0)}$$

[57] See Appendix A.3.8.2 on page 1075.

that is funded the deposit account. In Figure 7.17, the top panel represents the expected value $\mu_{(0,t)}$ of the deposit balance by considering different current values N_0, the top left panel corresponds to the density function[58] $f_{(s,t)}(x)$ of $N(t)$ given that $N(s) = N_s$ and the bottom panel shows three simulations of the stochastic process $N(t)$.

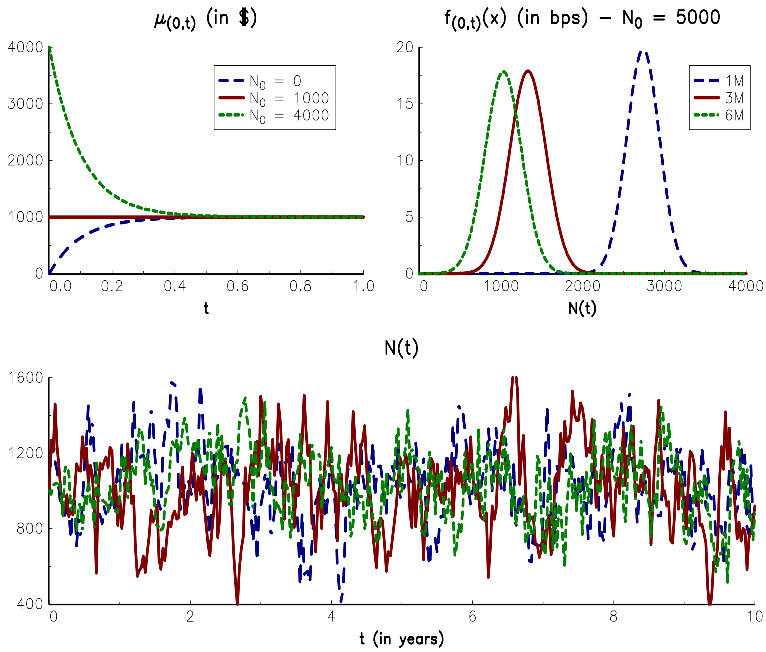

FIGURE 7.17: Statistics of the deposit amount $N(t)$

Another extension of Model (7.25) is to make the distinction between stable and non-stable deposits. Let g be the growth rate of deposits. The total amount of deposits $D(t)$ is given by:

$$D(t) = e^{g(t-s)} \sum_{i=1}^{n_t} N_i(t)$$

where n_t is the number of deposit accounts and $N_i(t)$ is the deposit balance of the i^{th} deposit account. It follows that:

$$\begin{aligned} D(t) &= e^{g(t-s)} \sum_{i=1}^{n_t} N_{\infty,i} + e^{g(t-s)} \sum_{i=1}^{n_t} (N_{s,i} - N_{\infty,i}) e^{-\lambda_i(t-s)} + \\ & \quad e^{g(t-s)} \sum_{i=1}^{n_t} \sigma_i \sqrt{\frac{1 - e^{-2\lambda_i(t-s)}}{2\lambda_i}} \varepsilon_i(t) \end{aligned}$$

where $\varepsilon_i(t) \sim \mathcal{N}(0,1)$. By considering a representative agent, we can replace the previous equation by the following expression:

$$D(t) = D_\infty e^{g(t-s)} + (D_s - D_\infty) e^{(g-\lambda)(t-s)} + \varepsilon(t) \tag{7.28}$$

[58] We have:

$$f_{(s,t)}(x) = \frac{1}{\sigma(s,t)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_{(s,t)}}{\sigma_{(s,t)}}\right)^2\right)$$

where $D_\infty = \sum_{i=1}^{n_t} N_{\infty,i}$, $D_s = \sum_{i=1}^{n_t} N_{s,i}$, λ^{-1} is the weighted average duration of deposits and $\varepsilon(t)$ is the stochastic part. Demey *et al.* (2003) notice that we can decompose $D(t)$ into two terms:

$$D(t) = D_{\text{long}}(s,t) + D_{\text{short}}(s,t)$$

where $D_{\text{long}}(s,t) = D_\infty e^{g(t-s)}$ and $D_{\text{short}}(s,t) = (D_s - D_\infty) e^{(g-\lambda)(t-s)} + \varepsilon(t)$. This breakdown seems appealing at first sight, but it presents a major drawback. Indeed, the short component $D_{\text{short}}(s,t)$ may be negative. In practice, it is better to consider the following equation:

$$D(t) = \underbrace{\varphi D_\infty e^{g(t-s)}}_{D_{\text{stable}}(s,t)} + \underbrace{(D_s - D_\infty) e^{(g-\lambda)(t-s)} + \varepsilon(t) + (1-\varphi) D_\infty e^{g(t-s)}}_{D_{\text{non-stable}}(s,t)}$$

where $D_{\text{stable}}(s,t)$ corresponds to the amount of stable deposits and $D_{\text{non-stable}}(s,t) = D(t) - D_{\text{stable}}(s,t)$ is the non-stable deposit amount. At time $t = s$, we verify that[59]:

$$D(t) = D_{\text{stable}} + D_{\text{non-stable}}(t)$$

The estimation of stable deposits is a two-step process. First, we estimate D_∞ by using the ML method. Second, we estimate the fraction $\varphi < 1$ of the long-run amount of deposits that can be considered as stable. Generally, we calibrate the parameter φ such that φN_∞ is the quantile of $D(t)$ at a given confidence level (e.g. 90% or 95%).

In Figure 7.18, we assume that the deposit amount $D(t)$ follows an Ornstein-Uhlenbeck process with parameters $D_\infty = \$1$ bn, $\lambda = 5$ and $\sigma = \$200$ mn. In the top/right panel, we have reported the $D_{\text{long}}/D_{\text{short}}$ breakdown. We verify that the short component may be negative, meaning the long component cannot be considered as a stable part. This is not the case with the $D_{\text{stable}}/D_{\text{non-stable}}$ breakdown given in the bottom panels. The big issue is of course the estimation of the parameter φ. One idea might be to calibrate φ such that $\Pr\{D(t) \le \varphi D_\infty\} = 1 - \alpha$ given the confidence level α. If we consider the Ornstein-Uhlenbeck dynamics, we obtain the following formula:

$$\varphi = 1 - \frac{\sigma \Phi^{-1}(1-\alpha)}{D_\infty \sqrt{2\lambda}}$$

In our example, this ratio is respectively equal to 85.3%, 89.6% and 91.9% when α takes the value 99%, 95% and 90%.

Remark 85 *We recall that the Basel Committee makes the distinction between stable and core deposits. It is assumed that the interest rate elasticity of NMDs is less than one. Core deposits are the proportion of stable deposits, whose pass through sensitivity is particularly low, meaning they are "unlikely to reprice even under significant changes in interest rate environment" (BCBS, 2016d, page 26).*

7.3.1.2 Behavioral modeling

If we assume that the growth rate g is equal to zero, the linearization of Equation (7.28) corresponds to the Euler approximation of the Ornstein-Uhlenbeck process:

$$D(t) \approx D(s) + \lambda (D_\infty - D(s)) + \varepsilon(t) \tag{7.29}$$

[59] The previous results are based on the dynamic analysis between time s and t. If we prefer to adopt a static analysis, the amount of non-stable deposits must be defined as follows:

$$D_{\text{non-stable}}(t) = D(t) - \varphi D_\infty$$

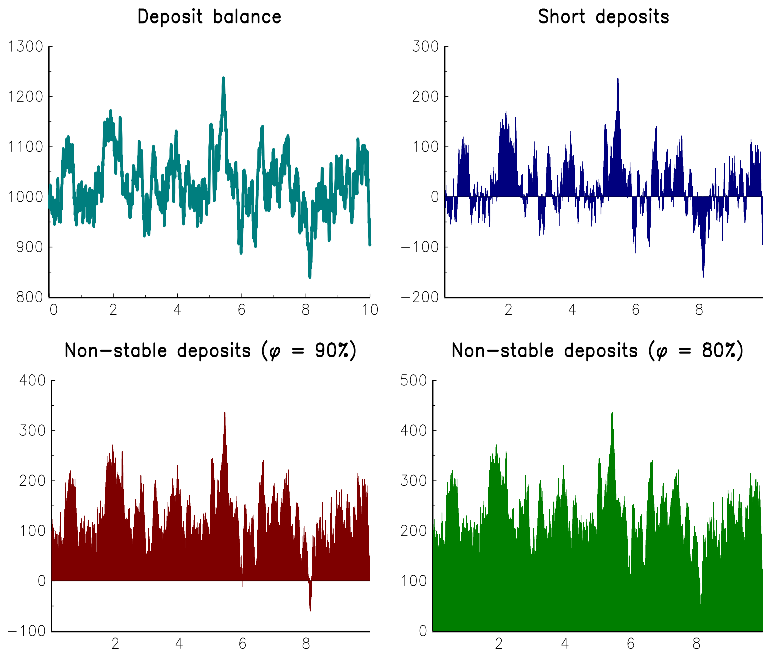

FIGURE 7.18: Stable and non-stable deposits

Here, $D(t)$ is the value of the non-maturity account balance or deposit volume. A similar expression is obtained by considering the individual deposit amount $N(t)$ instead of $D(t)$. In what follows, we use the same notation $D(t)$ for defining aggregated and individual deposit balances. Let us come back to the general case: $\mathrm{d}D(t) = (\mathrm{NP}(t) - \lambda(t) D(t))\,\mathrm{d}t$. By assuming that the new production is a function of the current balance, we have $\mathrm{NP}(t) = g(t, X(t)) D(t)$ where $g(t, X(t))$ depends on a set of explanatory variables $X(t)$. If follows that $\mathrm{d}\ln D(t) = (g(t, X(t)) - \lambda(t))\,\mathrm{d}t$ and:

$$\ln D(t) \approx \ln D(s) + g(s, X(s)) - \lambda(s) \tag{7.30}$$

Modeling the behavior of the client and introducing embedded options can be done by combining Equations (7.29) and (7.30):

$$\ln D(t) = \ln D(s) + \lambda(\ln D_\infty - \ln D(s)) + g(t, X(t)) + \varepsilon(t)$$

In this case, the main issue is to specify $g(t, X(t))$ and the explanatory variables that impact the dynamics of the deposit volume. Most of the time, $g(t, X(t))$ depends on two variables: the deposit rate $i(t)$ and the market rate $r(t)$. In what follows, we present several models that have been proposed for modeling either $D(t)$ or $i(t)$ or both. The two pioneer models are the deposit balance model of Selvaggio (1996) and the deposit rate model of Hutchison and Pennacchi (1996).

The Hutchison-Pennacchi-Selvaggio framework In Selvaggio (1996), the deposit rate $i(t)$ is exogenous and the bank account holder modifies his current deposit balance $D(t)$ to target a level $D^\star(t)$, which is defined as follows:

$$\ln D^\star(t) = \beta_0 + \beta_1 \ln i(t) + \beta_2 \ln Y(t)$$

where $Y(t)$ is the income of the account holder. The rational of this model is the following. In practice, the bank account holder targets a minimum positive balance in order to meet his current liquidity and consumption needs, which are a function of his income $Y(t)$. For example, we can assume that the client with a monthly income of \$10 000 targets a larger amount than the client with a monthly income of \$1 000. Moreover, we can assume that the target balance depends on the deposit rate $i(t)$. The elasticity coefficient must be positive, meaning that the client has a high incentive to transfer his money into a term deposit account if the deposit rate is low. At time t, the account holder can face two situations. If $D_{t-1} < D_t^\star$, he will certainly increase his deposit volume in order to increase his cash liquidity. If $D_{t-1} > D_t^\star$, he will certainly transfer a part of his deposit balance into his term account. Therefore, the behavior of the bank account holder can be represented by a mean-reverting AR(1) process:

$$\ln D(t) - \ln D(t-1) = (1-\phi)(\ln D^\star(t) - \ln D(t-1)) + \varepsilon(t) \tag{7.31}$$

where $\varepsilon(t) \sim \mathcal{N}(0, \sigma^2)$ is a white noise process and $\phi \leq 1$ is the mean-reverting parameter. It follows that:

$$\begin{aligned} \ln D(t) &= \phi \ln D(t-1) + (1-\phi)\ln D^\star(t) + \varepsilon(t) \\ &= \phi \ln D(t-1) + \beta_0' + \beta_1' \ln i(t) + \beta_2' \ln Y(t) + \varepsilon(t) \end{aligned} \tag{7.32}$$

where $\beta_k' = (1-\phi)\beta_k$. Let $d(t) = \ln D(t)$ be the logarithm of the deposit volume. The model of Selvaggio (1996) is then a ARX(1) process:

$$d(t) = \phi d(t-1) + (1-\phi) d^\star(t) + \varepsilon(t) \tag{7.33}$$

where $d^\star(t) = \ln D^\star(t)$ is the exogenous variable.

In practice, the bank does not know the value $\theta = (\phi, \beta_0, \beta_1, \beta_2, \sigma)$ of the parameters. Moreover, these parameters are customer-specific and are different from one customer to another. The bank can then estimate the vector θ for a given customer if it had a sufficient history. For instance, we consider that a two-year dataset of monthly observations or a ten-year dataset of quarterly observations is generally sufficient to estimate five parameters. However, the variables $i(t)$ and $Y(t)$ rarely change, meaning that it is impossible to estimate θ for a given customer. Instead of using a time-series analysis, banks prefer then to consider a cross-section/panel analysis. Because Model (7.33) is linear, we can aggregate the behavior of the different customers. The average behavior of a customer is given by Equation (7.32) where the parameters ϕ, β_0, β_1, β_2 and σ are equal to the mean of the customer parameters. This approach has the advantage to be more robust in terms of statistical inference. Indeed, the regression is performed using a large number of observations (number of customers $\times$ number of time periods).

In the previous model, the deposit interest rate is given and observed at each time period. Hutchison and Pennacchi (1996) propose a model for fixing the optimal value of $i(t)$. They assume that the bank maximizes its profit:

$$i^\star(t) = \arg\max \Pi(t)$$

where the profit $\Pi(t)$ is equal to the revenue minus the cost:

$$\Pi(t) = r(t) \cdot D(t) - (i(t) + c(t)) \cdot D(t)$$

In this expression, $r(t)$ is the market interest rate and $c(t)$ is the cost of issuing deposits. By assuming that $D(t)$ is an increasing function of $i(t)$, the first-order condition is:

$$(r(t) - (i(t) + c(t))) \cdot \frac{\partial D(t)}{\partial i(t)} - D(t) = 0$$

We deduce that:

$$
\begin{aligned}
i^{\star}(t) &= r(t) - c(t) - \left(\frac{\partial D(t)}{\partial i(t)}\right)^{-1} D(t) \\
&= r(t) - \left(c(t) + \left(\frac{\partial d(t)}{\partial i(t)}\right)^{-1}\right) \\
&= r(t) - s(t) \qquad (7.34)
\end{aligned}
$$

The deposit interest rate is then equal to the market interest rate $r(t)$ minus a spread[60] $s(t)$. Equations (7.32) and (7.34) are the backbone of various non-maturity deposit models.

The IRS framework Using arbitrage theory, Jarrow and van Deventer (1998) show that the deposit rate must be lower than the market rate[61] – $i(t) \leq r(t)$, and the current market value of deposits is the net present value of the cash flow stream $D(t)$:

$$
V(0) = \mathbb{E}\left[\sum_{t=0}^{\infty} B(0, t+1)(r(t) - i(t)) D(t)\right] \qquad (7.35)
$$

where $B(0,t)$ is the discount factor. Jarrow and van Deventer (1998) interpret $V(0)$ as an exotic interest rate swap, where the bank receives the market rate and pays the deposit rate. Since the present value of the deposit liability of the bank is equal to $L(0) = D(0) - V(0)$, the hedging strategy consists in "*investing $D(0)$ dollars in the shortest term bond $B(0,1)$ and shorting the exotic interest rate swap represented by $V(0)$*" (Jarrow and van Deventer, 1998, page 257). The complete computation of the hedging portfolio requires specifying $i(t)$ and $D(t)$. For example, Jarrow and van Deventer (1998) consider the following specification:

$$
\ln D(t) = \ln D(t-1) + \beta_0 + \beta_1 r(t) + \beta_2 (r(t) - r(t-1)) + \beta_3 t \qquad (7.36)
$$

and:

$$
i(t) = i(t) + \beta'_0 + \beta'_1 r(t) + \beta'_2 (r(t) - r(t-1)) \qquad (7.37)
$$

The deposit balance and the deposit rate are linear in the market rate $r(t)$ and the variation of the market rate $\Delta r(t)$. The authors also add a trend in Equation (7.36) in order to take into account macroeconomic variables that are not included in the model.

The previous model is fully tractable in continuous-time. Beyond these analytical formulas, the main interest of the Jarrow-van Deventer model is to show that the modeling of non-maturity deposits is related to the modeling of interest rate swaps. Another important contribution of this model is the introduction of the replicating portfolio. Indeed, it is common to break down deposits into stable and non-stable deposits, and stable deposits into core and non-core deposits. The idea is then to replicate the core deposits with a hedging portfolio with four maturities (3, 5, 7 and 10 years). In this case, the funds transfer pricing of non-maturity deposits is made up of four internal transfer rates corresponding to the maturity pillars of the replicating portfolio.

[60] We notice that the spread $s(t)$ is the sum of the cost $c(t)$ and the Lerner index $\eta(t)$, where $\eta(t) = 1/e(t)$ and $e(t)$ is the interest rate elasticity of the demand.

[61] This inequality is obtained by assuming no arbitrage opportunities for individuals and market segmentation. In particular, Jarrow and van Deventer (1998) consider that the competition among banks is imperfect because of entry and mobility barriers to the banking industry.

Asymmetric adjustment models O'Brien (2001) introduces an asymmetric adjustment of the deposit rate:

$$\Delta i(t) = \alpha(t) \cdot (\hat{\imath}(t) - i(t-1)) + \eta(t)$$

where $\hat{\imath}(t)$ is the conditional equilibrium deposit rate and:

$$\alpha(t) = \alpha^{+} \cdot \mathbb{1}\{\hat{\imath}(t) > i(t-1)\} + \alpha^{-} \cdot \mathbb{1}\{\hat{\imath}(t) < i(t-1)\}$$

If $\hat{\imath}_t > i(t-1)$, we obtain $\Delta i(t) = \alpha^{+} \cdot (\hat{\imath}(t) - i(t-1)) + \eta(t)$, otherwise we have $\Delta i(t) = \alpha^{-} \cdot (\hat{\imath}(t) - i(t-1)) + \eta(t)$. The distinction between α^{+} and α^{-} can be justified by the asymmetric behavior of banks and the rigidity of deposit rates. In particular, O'Brien (2001) suggests that $\alpha^{-} > \alpha^{+}$, implying that banks adjust more easily the deposit rate when the market rate decreases than when it increases. In this model, the deposit balance is a function of the spread $r(t) - i(t)$:

$$\ln D(t) = \beta_0 + \beta_1 \ln D(t-1) + \beta_2 (r(t) - i(t)) + \beta_3 \ln Y(t) + \varepsilon(t)$$

Moreover, O'Brien (2001) assumes that the conditional equilibrium deposit rate is a linear function of the market rate:

$$\hat{\imath}(t) = \gamma_0 + \gamma_1 \cdot r(t)$$

In the previous model, the asymmetric adjustment explicitly concerns the deposit interest rate $i(t)$ and implicitly impacts the deposit balance $D(t)$ because of the spread $r(t) - i(t)$. Frachot (2001) considers an extension of the Selvaggio model by adding a correction term that depends on the market interest rate $r(t)$ and a threshold:

$$\ln D(t) - \ln D(t-1) = (1-\phi)(\ln D^{\star}(t) - \ln D_{t-1}) + \delta_c(r(t), r^{\star}) \tag{7.38}$$

where $\delta_c(r(t), r^{\star}) = \delta \cdot \mathbb{1}\{r(t) \leq r^{\star}\}$ and $r^{\star}$ is the interest rate floor. When market interest rates are too low and below $r^{\star}$, the bank account holder does not make the distinction between deposit and term balances, and we have:

$$\delta_c(r(t), r^{\star}) = \begin{cases} \delta & \text{if } r(t) \leq r^{\star} \\ 0 & \text{otherwise} \end{cases}$$

Contrary to the Selvaggio model, the average behavior is not given by Equation (7.38) because of the non-linearity pattern. Let f be the probability density function of the threshold $r^{\star}$ among the different customers of the bank. On average, we have:

$$\begin{aligned} \mathbb{E}[\delta_c(r(t), r^{\star})] &= \int_0^{\infty} \delta \cdot \mathbb{1}\{r(t) \leq x\} \cdot f(x)\,\mathrm{d}x \\ &= \delta \cdot (1 - \mathbf{F}(r(t))) \end{aligned}$$

The average behavior is then given by the following equation:

$$d(t) - d(t-1) = (1-\phi)(d^{\star}(t) - d(t-1)) + \delta(1 - \mathbf{F}(r(t)))$$

where $d(t) = \ln D(t)$ and $d^{\star}(t) = \ln D^{\star}(t)$. For example, if we assume that the distribution of $r^{\star}$ is uniform on the range $[0; r^{\star}_{\max}]$, we obtain $f(x) = 1/r^{\star}_{\max}$ and $\mathbf{F}(x) = \min(x/r^{\star}_{\max}, 1)$. We deduce that:

$$\begin{aligned} d(t) - d(t-1) &= (1-\phi)(d^{\star}(t) - d(t-1)) + \delta\left(1 - \min\left(\frac{r(t)}{r^{\star}_{\max}}, 1\right)\right) \\ &= (1-\phi)(d^{\star}(t) - d(t-1)) + \delta\frac{\max(r^{\star}_{\max} - r(t), 0)}{r^{\star}_{\max}} \end{aligned}$$

In the case where $r^{\star} \sim \mathcal{N}\left(\mu_{\star}, \sigma_{\star}^{2}\right)$, we obtain:

$$d(t) - d(t-1) = (1-\phi)\left(d^{\star}(t) - d(t-1)\right) + \delta\Phi\left(\frac{\mu_{\star} - r(t)}{\sigma_{\star}}\right)$$

Another asymmetric model was proposed by OTS (2001):

$$d(t) = d(t-1) + \Delta \ln\left(\beta_0 + \beta_1 \arctan\left(\beta_2 + \beta_3 \frac{i(t)}{r(t)}\right) + \beta_4 i(t)\right) + \varepsilon(t)$$

where Δ corresponds to the frequency. The '*Net Portfolio Value Model*' published by the Office of Thrift Supervision[62] is a comprehensive report that contains dozens of models in order to implement risk management and ALM policies. For instance, Chapter 6 describes the methodologies for modeling liabilities and Section 6.D is dedicated to demand deposits. These models were very popular in the US in the 1990s. In 2011, the Office of the Comptroller of the Currency (OCC) provided the following parameters for the monthly model[63] of transaction accounts: $\beta_0 = 0.773$, $\beta_1 = -0.065$, $\beta_2 = -5.959$, $\beta_3 = 0.997$ and $\beta_4 = 1$ bp. In the case of money market accounts, the parameters were $\beta_0 = 0.643$, $\beta_1 = -0.069$, $\beta_2 = -6.284$, $\beta_3 = 2.011$ and $\beta_4 = 1$ bp.

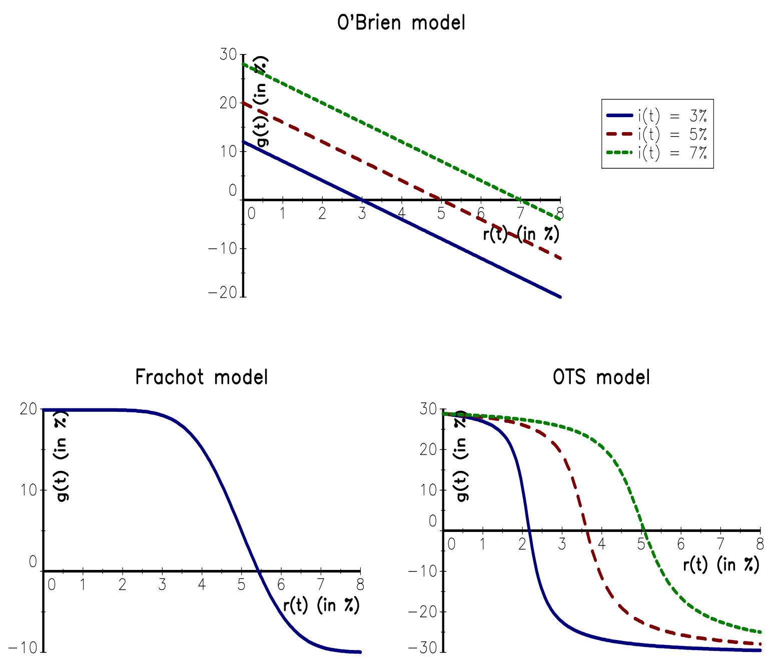

FIGURE 7.19: Impact of the market rate on the growth rate of deposits

In Figure 7.19, we compare the growth rate $g(t)$ of deposits for the different asymmetric models. For the O'Brien model, the growth rate is equal to $g(t) = \beta_2 (r(t) - i(t))$. In the case of the Frachot model, the market rate has only a positive impact because $\delta_c(r(t), r^{\star}) \geq 0$. This is why we consider an extended version where the correction term is equal to

[62] The mission of OTS is to "*supervise savings associations and their holding companies in order to maintain their safety and soundness and compliance with consumer laws and to encourage a competitive industry that meets America's financial services needs*".

[63] We have $\Delta = 1/12$.

$\delta_c(r(t), r^\star) - \delta^-$. The growth rate is then $g(t) = \delta(1 - \mathbf{F}(r(t))) - \delta^-$. Finally, the growth rate of the OTS model is equal to $g(t) = \ln\left(\beta_0 + \beta_1 \arctan\left(\beta_2 + \beta_3 \frac{i(t)}{r(t)}\right) + \beta_4 i(t)\right)$. Using several value of the deposit rate $i(t)$, we measure the impact of the market rate $r(t)$ on the growth rate $g(t)$ using the following parameters: $\beta_2 = -4$ (O'Brien model), $\delta = 30\%$, $\mu_\star = 5\%$, $\sigma_\star = 1\%$ and $\delta^- = 10\%$ (Frachot model), and $\beta_0 = 1.02$, $\beta_1 = 0.2$, $\beta_2 = -7$, $\beta_3 = 5$ and $\beta_4 = 0$ (OTS model). The O'Brien model is linear while the Frachot model is non-linear. However, the Frachot model does not depend on the level of the deposit rate $i(t)$. The OTS model combines non-linear effects and the dependence on the deposit rate.

Remark 86 *These different models have been extended in order to take into account other explanatory variables such that the CDS of the bank, the inflation rate, the deposit rate competition, lagged effects, etc. We can then use standard econometric and time-series tools for estimating the unknown parameters.*

7.3.2 Prepayment risk

A prepayment is the settlement of a debt or the partial repayment of its outstanding amount before its maturity date. It is an important risk for the ALM of a bank, because it highly impacts the net interest income and the efficiency of the hedging portfolio. For example, suppose that the bank has financed a 10-year mortgage paying 5% through a 10-year bond paying 4%. The margin on this mortgage is equal to 1%. Five years later, the borrower prepays the mortgage because of a fall in interest rates. In this case, the bank receives the cash of the mortgage refund whereas it continues to pay a coupon of 4%. Certainly, the cash will yield a lower return than previously, implying that the margin is reduced and may become negative.

Prepayment risk shares some common features with default risk. Indeed, the prepayment time can be seen as a stopping time exactly like the default time for credit risk. Prepayment and default are then the two actions that may terminate the loan contract. This is why they have been studied together in some research. However, they also present some strong differences. In the case of the default risk, the income of the bank is reduced because both interest and capital payments are shut down. In the case of the prepayment risk, the bank recovers the capital completely, but no longer receives the interest due. Moreover, while default risk increases when the economic environment is bad or interest rates are high, prepayment risk is more pronounced in a period of falling interest rates.

In the 1980s, prepayment has been extensively studied in the case of RMBS. The big issue was to develop a pricing model for GNMA[64] mortgage-backed pass-through securities (Dunn and McConnell, 1981; Brennan and Schwartz, 1985; Schwartz and Torous, 1989). In these approaches, the prepayment option is assimilated to an American call option and the objective of the borrower is to exercise the option when it has the largest value[65] (Schwartz and Torous, 1992). However, Deng *et al.* (2000) show that "*there exists significant heterogeneity among mortgage borrowers and ignoring this heterogeneity results in serious errors in estimating the prepayment behavior of homeowners*". Therefore, it is extremely difficult to model the prepayment behavior, because it is not always a rational decision and many factors affect prepayment decisions (Keys *et al.*, 2016; Chernov *et al.*, 2017). This microeconomic approach is challenged by a macroeconomic approach, whose goal is to model the prepayment rate at the portfolio level and not the prepayment time at the loan level.

[64]The Government National Mortgage Association (GNMA or Ginnie Mae) has already been presented on page 139.
[65]This implies that the call option is in the money.

In what follows, we focus on mortgage loans, because it is the main component of prepayment risk. However, the analysis can be extended to other loans, for example consumer credit, student loans and leasing contracts. The case of student loans is very interesting since students are looking forward to repay their loan as soon as possible once they have found a job and make enough money.

7.3.2.1 Factors of prepayment

Following Hayre *et al.* (2000), prepayments are caused by two main factors: refinancing and housing turnover. Let i_0 be the original interest rate of the mortgage or the loan. We note $i(t)$ the interest rate of the same mortgage if the household would finance it at time t. It is clear that the prepayment time $\boldsymbol{\tau}$ depends on the interest rate differential, and we can assume that the prepayment probability is an increasing function of the difference $\Delta i(t) = i_0 - i(t)$:

$$\mathbb{P}(t) = \Pr\{\boldsymbol{\tau} \leq t\} = \vartheta(i_0 - i(t))$$

where $\partial_x \vartheta(x) > 0$. For instance, if the original mortgage interest rate is equal to 10% and the current mortgage interest rate is equal to 0%, nobody benefits from keeping the original mortgage, and it is preferable to fully refinance the mortgage. This situation is particularly true in a period of falling interest rates. The real life example provided by Keys *et al.* (2016) demonstrates the strong implication that a prepayment may have on household budgeting:

> "*A household with a 30-year fixed-rate mortgage of* \$200 000 *at an interest rate of* 6.0% *that refinances when rates fall to* 4.5% *(approximately the average rate decrease between 2008 and 2010 in the US) saves more than* \$60 000 *in interest payments over the life of the loan, even after accounting for refinance transaction costs. Further, when mortgage rates reached all-time lows in late 2012, with rates of roughly* 3.35% *prevailing for three straight months, this household with a contract rate of* 6.5% *would save roughly* \$130 000 *over the life of the loan by refinancing*" (Keys *et al.*, 2016, pages 482-483).

As already said, the prepayment value is the premium of an American call option, meaning that we can derive the optimal option exercise. In this case, the prepayment strategy can be viewed as an arbitrage strategy between the market interest rate and the cost of refinancing. In practice, we observe that the prepayment probability $\mathbb{P}(t)$ depends on other factors: loan type, loan age, loan balance, monthly coupon (Elie *et al.*, 2002). For example, it is widely accepted that the prepayment probability is an increasing function of the monthly coupon.

The second factor for explaining prepayments is housing turnover. In this case, the prepayment decision is not motivated by refinancing, but it is explained by the home sale due to life events. For instance, marriage, divorce, death, children leaving home or changing jobs explain a large part of prepayment rates. Another reason is the housing market dynamics, in particular home prices that have an impact on housing turnover. These different factors explain that we also observe prepayments even when interest rates increase. For example, the upgrading housing decision (i.e. enhancing the capacity or improving the quality of housing) is generally explained by the birth of a new child, an inheritance or a salary increase.

Remark 87 *In addition to these two main factors, we also observe that some borrowers choose to reduce their debt even if it is not an optimal decision. When they have some financial saving, which may be explained by an inheritance for example, they proceed to partial prepayments.*

7.3.2.2 Structural models

As with the credit risk, there are two families of prepayment models. The objective of structural models is to explain the prepayment time $\boldsymbol{\tau}$ of a borrower while reduced-form models are interested in the prepayment rate of a loan portfolio.

Value of the American option The objective is to find the optimal value $\boldsymbol{\tau}$ such that the borrower minimizes the paid cash flows or maximizes the prepayment option. Let us consider a mortgage, whose maturity is equal to T. In continuous-time, the risk-neutral value of cash flows is equal to[66]:

$$V(t) = \inf_{\boldsymbol{\tau} \leq T} \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\boldsymbol{\tau}} m(u) \, e^{-\int_t^u r(s) \, \mathrm{d}s} \, \mathrm{d}u + e^{-\int_t^{\boldsymbol{\tau}} r(s) \, \mathrm{d}s} M(\boldsymbol{\tau}) \mid \mathcal{F}_t \right] \tag{7.39}$$

where $m(t)$ and $M(t)$ are the coupon and the mark-to-market value of the mortgage at time t. The first term that makes up $V(t)$ is the discounted value of the interest paid until the prepayment time $\boldsymbol{\tau}$ whereas the second term is the discounted value of the mortgage value at the prepayment time $\boldsymbol{\tau}$. Equation (7.39) is a generalization of the net present value of a mortgage in continuous-time[67]. The computation of the optimal stopping time can be done in a Hamilton-Jacobi-Bellman (HJB) framework. We introduce the state variable X_t, which follows a diffusion process:

$$\mathrm{d}X(t) = \mu(t, X(t)) \, \mathrm{d}t + \sigma(t, X(t)) \, \mathrm{d}W(t)$$

We note $V(t, X)$ the value of $V(t)$ when $X(t)$ is equal to X. In the absence of prepayment, we deduce that the value of $V(t, X)$ satisfies the following Cauchy problem[68]:

$$\begin{cases} -\partial_t V(t, X) + r(t) V(t, X) = \mathcal{A}_t V(t, X) + m(t) \\ V(T, X) = M(T) \end{cases}$$

where $\mathcal{A}_t$ is the infinitesimal generator of the diffusion process:

$$\mathcal{A}_t V(t, X) = \frac{1}{2} \sigma^2(t, X) \frac{\partial^2 V(t, X)}{\partial X^2} + \mu(t, X) \frac{\partial^2 V(t, X)}{\partial x^2}$$

The prepayment event changes the previous problem since we must verify that the value $V(t, X)$ is lower than the mortgage value $M(t)$ minus the refinancing cost $C(t)$. The option problem is then equivalent to solve the HJB equation or the variational inequality:

$$\min\left(\mathcal{L}_t V(t, X), V(t, X) + C(t) - M(t)\right) = 0$$

where:

$$\mathcal{L}_t V(t, X) = \mathcal{A}_t V(t, X) + m(t) + \partial_t V(t, X) - r(t) V(t, X)$$

This model can be extended to the case where there are several state variables or there is no maturity (perpetual mortgage).

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[66] $r(t)$ is the discount rate.

[67] The net present value is equal to:

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[\int_t^T m(u) \, e^{-\int_t^u r(s) \, \mathrm{d}s} \, \mathrm{d}u + e^{-\int_t^T r(s) \, \mathrm{d}s} N(T) \mid \mathcal{F}_t \right]$$

where $N(T)$ is the outstanding amount at the maturity.

[68] We use the Feynmac-Kac representation given on page 1070.

The Agarwal-Driscoll-Laibson model There are several possible specifications depending on the choice of the state variables, the dynamics of interest rates, etc. For example, using a similar framework than previously, Agarwal *et al.* (2013) propose the following optimal refinancing rule:

$$i_0 - i(t) \geq \delta^\star = \frac{1}{\psi}\left(\phi + W\left(-e^{-\phi}\right)\right) \tag{7.40}$$

where $W(x)$ is the Lambert W function[69], $\psi = \sigma^{-1}\sqrt{2(r+\lambda)}$ and $\phi = 1+\psi(r+\lambda)(C/M)$. The parameters are the real discount rate r, the rate λ of exogenous mortgage prepayment, the volatility σ of the mortgage rate $i(t)$, the refinancing cost C and the remaining mortgage value M. Equation (7.40) has been obtained by solving the HJB equation and assuming that $\mathrm{d}X(t) = \sigma\,\mathrm{d}W(t)$ and $X(t) = i(t) - i_0$.

Using the numerical values $r = 5\%$, $\lambda = 10\%$, $\sigma = 2\%$, and $C/M = 1\%$, $\delta^\star$ is equal to 110 bps. This means that the borrower has to prepay his mortgage if the mortgage rate falls by at least 110 bps. In Table 7.20, we consider the impact of one parameter by considering the other parameters unchanged. First, we assume that the cost function is $C = 2000 + 1\% \times M$, meaning that there is a fixed cost of \$2 000. It follows that $\delta^\star$ is a decreasing function of the mortgage value M, because fixed costs penalize low mortgage values. We also verify that $\delta^\star$ is an increasing function of r, σ and λ. In particular, the parameter σ has a big influence, because it indicates if the mortgage rate is volatile or not. In the case of a high volatility, it may be optimal that the borrower is waiting that $i(t)$ highly decreases. This is why the HJB equation finds a high value of $\delta^\star$.

TABLE 7.20: Optimal refinancing rule $\delta^\star$

M (in KUSD)	$\delta^\star$	r	$\delta^\star$	σ	$\delta^\star$	λ	$\delta^\star$
10	612	1%	101	1%	79	2%	89
100	198	2%	103	2%	110	5%	98
250	150	5%	110	3%	133	10%	110
500	131	8%	116	5%	171	15%	120
1 000	121	10%	120	10%	239	20%	128

7.3.2.3 Reduced-form models

Rate, coupon or maturity incentive? The previous approach can only be applied to the refinancing decision, but it cannot deal with all types of prepayment. Moreover, there is no guarantee that the right decision variable is the difference $i_0 - i(t)$ between the current mortgage rate and the initial mortgage rate. For instance, $i_0 - i(t) = 1\%$ implies a high impact for a 20-year remaining maturity, but has a small effect when the maturity is less than one year. A better decision variable is the coupon or annuity paid by the borrower. In the case of a constant payment mortgage, we recall that the annuity is equal to:

$$A(i,n) = \frac{i}{1-(1+i)^{-n}} N_0$$

where N_0 is the notional of the mortgage, i is the mortgage rate and n is the number of periods. If the mortgage rate drops from i_0 to $i(t)$, the absolute difference of the annuity is equal to $\mathfrak{D}_A(i_0, i(t)) = A(i_0, n) - A(i(t), n)$, whereas the relative difference of the annuity

[69] The Lambert W function is related to Shannon's entropy and satisfies $W(x)\,e^{W(x)} = x$.

is given by:

$$\begin{aligned}\mathfrak{D}_R(i_0, i(t)) &= \frac{\mathfrak{D}_A(i_0, i(t))}{A(i_0, n)} \\ &= 1 - \left(\frac{1-(1+i_0)^{-n}}{1-(1+i(t))^{-n}}\right)\frac{i(t)}{i_0}\end{aligned}$$

where n is the remaining number of periods. In a similar way, the relative cumulative difference $\mathfrak{C}(i_0, i(t))$ is equal to:

$$\begin{aligned}\mathfrak{C}(i_0, i(t)) &= \frac{\sum_{t=1}^{n}\mathfrak{D}_A(i_0, i(t))}{N_0} \\ &= n\left(\frac{i_0}{1-(1+i_0)^{-n}} - \frac{i(t)}{1-(1+i(t))^{-n}}\right)\end{aligned}$$

Finally, another interesting measure is the minimum number of periods $\mathfrak{N}(i_0, i(t))$ such that the new annuity is greater than or equal to the initial annuity:

$$\mathfrak{N}(i_0, i(t)) = \{x \in \mathbb{N} : A(i(t), x) \geq A(i(t), n), A(i(t), x+1) < A(i(t), n)\}$$

where $\mathfrak{N}(i_0, i(t))$ measures the maturity reduction of the loan by assuming that the borrower continues to pay the same annuity.

TABLE 7.21: Impact of a new mortgage rate (100 KUSD, 5%, 10-year)

i (in %)	A (in \$)	$\mathfrak{D}_A$ (in \$) Monthly	Annually	$\mathfrak{D}_R$ (in %)	$\mathfrak{C}$ (in %)	$\mathfrak{N}$ (in years)
5.0	1 061					
4.5	1 036	24	291	2.3	2.9	9.67
4.0	1 012	48	578	4.5	5.8	9.42
3.5	989	72	862	6.8	8.6	9.17
3.0	966	95	1 141	9.0	11.4	8.92
2.5	943	118	1 415	11.1	14.2	8.75
2.0	920	141	1 686	13.2	16.9	8.50
1.5	898	163	1 953	15.3	19.5	8.33
1.0	876	185	2 215	17.4	22.2	8.17
0.5	855	206	2 474	19.4	24.7	8.00

Let us illustrate the impact of a new rate $i(t)$ on an existing mortgage. We assume that the current outstanding amount is equal to \$100 000 and the amortization scheme is monthly. In Table 7.21, we show how the monthly annuity changes if the original rate is 5% and the remaining maturity is ten years. If the borrower refinances the mortgage at 2%, the monthly annuity is reduced by \$141, which represents 13.2% of the current monthly coupon. His total gain is then equal to 16.9% of the outstanding amount. If the borrower prefers to reduce the maturity and takes the annuity constant, he will gain 18 months. In Tables 7.22 and 7.23, we compute the same statistics when the remaining maturity is twenty years or the original rate is 10%. Banks have already experienced this kind of situation these last 30 years. For example, we report the average rate of 30-year and 15-year fixed rate mortgages in the US in Figure 7.20. We also calculate the differential rate between the 30-year mortgage rate lagged 15 years and the 15-year mortgage rate. We notice that this refinancing opportunity has reached 10% and more in the 1990s, and was above 3% most of the times these last 25 years. Of course, this situation is exceptional and explained by 30 years of falling interest rates.

TABLE 7.22: Impact of a new mortgage rate (100 KUSD, 5%, 20-year)

i (in %)	A (in \$)	$\mathfrak{D}_A$ (in \$) Monthly	Annually	$\mathfrak{D}_R$ (in %)	$\mathfrak{C}$ (in %)	$\mathfrak{N}$ (in years)
5.0	660					
4.5	633	27	328	4.1	6.6	18.67
4.0	606	54	648	8.2	13.0	17.58
3.5	580	80	960	12.1	19.2	16.67
3.0	555	105	1 264	16.0	25.3	15.83
2.5	530	130	1 561	19.7	31.2	15.17
2.0	506	154	1 849	23.3	37.0	14.50
1.5	483	177	2 129	26.9	42.6	14.00
1.0	460	200	2 401	30.3	48.0	13.50
0.5	438	222	2 664	33.6	53.3	13.00

TABLE 7.23: Impact of a new mortgage rate (100 KUSD, 10%, 10-year)

i (in %)	A (in \$)	$\mathfrak{D}_A$ (in \$) Monthly	Annually	$\mathfrak{D}_R$ (in %)	$\mathfrak{C}$ (in %)	$\mathfrak{N}$ (in years)
10.0	1 322					
9.0	1 267	55	657	4.1	6.6	9.33
8.0	1 213	108	1 299	8.2	13.0	8.75
7.0	1 161	160	1 925	12.1	19.3	8.33
6.0	1 110	211	2 536	16.0	25.4	7.92
5.0	1 061	261	3 130	19.7	31.3	7.58
4.0	1 012	309	3 709	23.3	37.1	7.25
3.0	966	356	4 271	26.9	42.7	6.92
2.0	920	401	4 816	30.4	48.2	6.67
1.0	876	445	5 346	33.7	53.5	6.50

Survival function with prepayment risk Previously, we have defined the amortization function $\mathbf{S}(t,u)$ as the fraction of the new production at time t that still remains in the balance sheet at time $u \geq t$: $\mathrm{NP}(t,u) = \mathrm{NP}(t)\,\mathbf{S}(t,u)$. We have seen that $\mathbf{S}(t,u)$ corresponds to a survival function. Therefore, we can use the property that the product of n_s survival functions is a survival function, meaning that we can decompose $\mathbf{S}(t,u)$ as follows:

$$\mathbf{S}(t,u) = \prod_{j=1}^{n_s} \mathbf{S}_j(t,u)$$

This implies that the hazard rate is an additive function:

$$\lambda(t,u) = \sum_{j=1}^{n_s} \lambda_j(t,u)$$

because we have:

$$e^{-\int_t^u \lambda(t,s)\,\mathrm{d}s} = \prod_{j=1}^{n_s} e^{-\int_t^u \lambda_j(t,s)\,\mathrm{d}s} = e^{-\int_t^u \left(\sum_{j=1}^{n_s} \lambda_j(t,s)\right)\mathrm{d}s}$$

If we apply this result to prepayment, we have:

$$\mathbf{S}(t,u) = \mathbf{S}_c(t,u) \cdot \mathbf{S}_p(t,u)$$

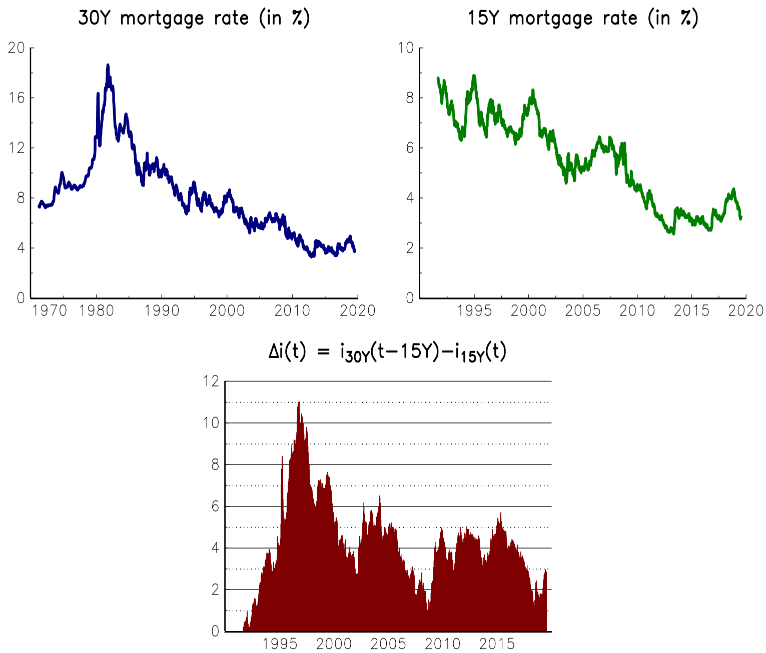

FIGURE 7.20: Evolution of 30-year and 15-year mortgage rates in the US

Source: Freddie Mac, 30Y/15Y Fixed Rate Mortgage Average in the United States [MORTGAGE30US/15US], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MORTGAGE30US, July 24, 2019.

where $\mathbf{S}_c(t,u)$ is the traditional amortization function (or the contract-based survival function) and $\mathbf{S}_p(t,u)$ is the prepayment-based survival function.

Example 75 *We consider a constant amortization mortgage (CAM) and assume that the prepayment-based hazard rate is constant and equal to λ_p.*

In Exercise 7.4.3 on page 450, we show that the survival function is equal to:

$$\mathbf{S}_c(t,u) = \mathbb{1}\{t \le u \le t+m\} \cdot \frac{1-e^{-i(t+m-u)}}{1-e^{-im}}$$

It follows that:

$$\begin{aligned}
\lambda_c(t,u) &= -\frac{\partial \ln \mathbf{S}_c(t,u)}{\partial u} \\
&= \frac{\partial \ln\left(1-e^{-im}\right)}{\partial u} - \frac{\partial \ln\left(1-e^{-i(t+m-u)}\right)}{\partial u} \\
&= \frac{ie^{-i(t+m-u)}}{1-e^{-i(t+m-u)}} \\
&= \frac{i}{e^{i(t+m-u)}-1}
\end{aligned}$$

Finally, we deduce that:

$$\lambda(t,u) = \mathbb{1}\{t \le u \le t+m\} \cdot \left(\frac{i}{e^{i(t+m-u)}-1} + \lambda_p\right)$$

In Figure 7.21, we report the survival function $\mathbf{S}(t,u)$ and the hazard rate $\lambda(t,u)$ of a 30-year mortgage at 5%. We also compare the amortization function $\mathbf{S}(t,u)$ obtained in continuous-time with the function calculated when we assume that the coupon is paid monthly. We notice that the continuous-time model is a good approximation of the discrete-time model.

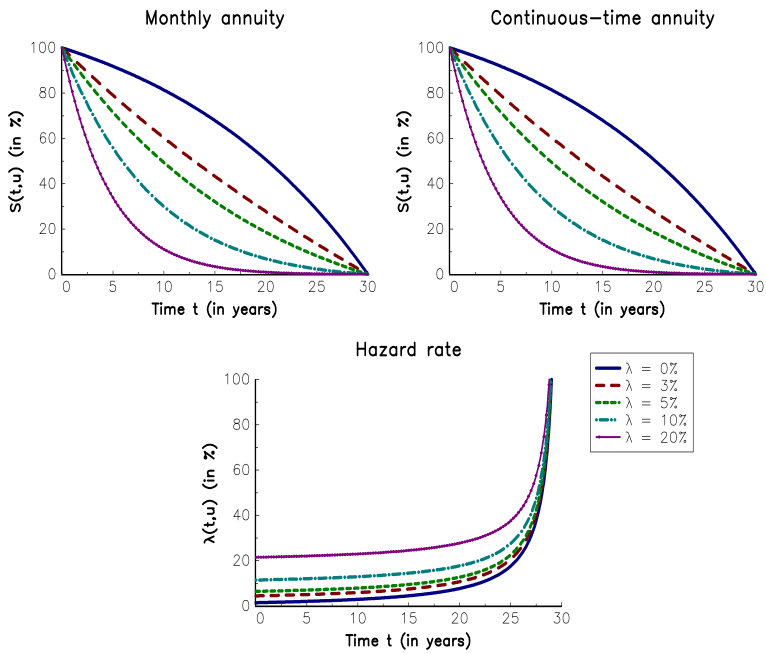

FIGURE 7.21: Survival function in the case of prepayment

Specification of the hazard function It is unrealistic to assume that the hazard function $\lambda_p(t,u)$ is constant because we do not make the distinction between economic and structural prepayments. In fact, it is better to decompose $\mathbf{S}_p(t,u)$ into the product of two survival functions:

$$\mathbf{S}_p(t,u) = \mathbf{S}_{\text{refinancing}}(t,u) \cdot \mathbf{S}_{\text{turnover}}(t,u)$$

where $\mathbf{S}_{\text{refinancing}}(t,u)$ corresponds to economic prepayments due to refinancing decisions and $\mathbf{S}_{\text{turnover}}(t,u)$ corresponds to structural prepayments because of housing turnover. In this case, we can assume that $\lambda_{\text{turnover}}(t,u)$ is constant and corresponds to the housing turnover rate. The specification of $\lambda_{\text{refinancing}}(t,u)$ is more complicated since it depends on several factors. For instance, Elie *et al.* (2002) show that $\lambda_{\text{refinancing}}(t,u)$ depends on the loan characteristics (type, age and balance), the cost of refinancing and the market rates. Moreover, they observe a seasonality in prepayment rates, which differs with respect to the loan type (monthly, quarterly or semi-annually).

As for deposit balances, the '*Net Portfolio Value Model*' published by the Office of Thrift Supervision (2001) gives very precise formulas for measuring prepayment. They assume that the prepayment rate is made up of three factors:

$$\lambda_p(t,u) = \lambda_{\text{age}}(u-t) \cdot \lambda_{\text{seasonality}}(u) \cdot \lambda_{\text{rate}}(u)$$

where λ_{age} measures the impact of the loan age, $\lambda_{\text{seasonality}}$ corresponds to the seasonality factor and λ_{rate} represents the influence of market rates. The first two components are specified as follows:

$$\lambda_{\text{age}}\left(\text{age}\right) = \begin{cases} 0.4 \cdot \text{age} & \text{if} \quad \text{age} \leq 2.5 \\ 1 & \text{if} \quad \text{age} \geq 2.5 \end{cases}$$

and:

$$\lambda_{\text{seasonality}}\left(u\right) = 1 + 0.20 \times \sin\left(1.571 \times \left(\frac{12 + \text{month}\left(u\right) - 3}{3}\right) - 1\right)$$

where $\text{age} = u - t$ is the loan age and $\text{month}\left(u\right)$ is the month of the date u. We notice that λ_{age} is equal to zero for a new mortgage – $u - t = 0$, increases linearly with mortgage age and remains constant after 30 months or 2.5 years. The refinancing factor of the OTS model has the following expression:

$$\lambda_{\text{rate}}\left(u\right) = \beta_0 + \beta_1 \arctan\left(\beta_2 \cdot \left(\beta_3 - \frac{i_0}{i\left(u - 0.25\right)}\right)\right)$$

where $i\left(u - 0.25\right)$ is the mortgage refinancing rate (lagged three months). In Figure 7.22, we represent the three components[70] while Figure 7.23 provides an example of the survival function $\mathbf{S}_p\left(t, u\right)$ where the mortgage rate drops from 5% to 1% after 6 years. The seasonality component has a small impact on the survival function because it is smoothed when computing the cumulative hazard function. On the contrary, the age and rate components change the prepayment speed.

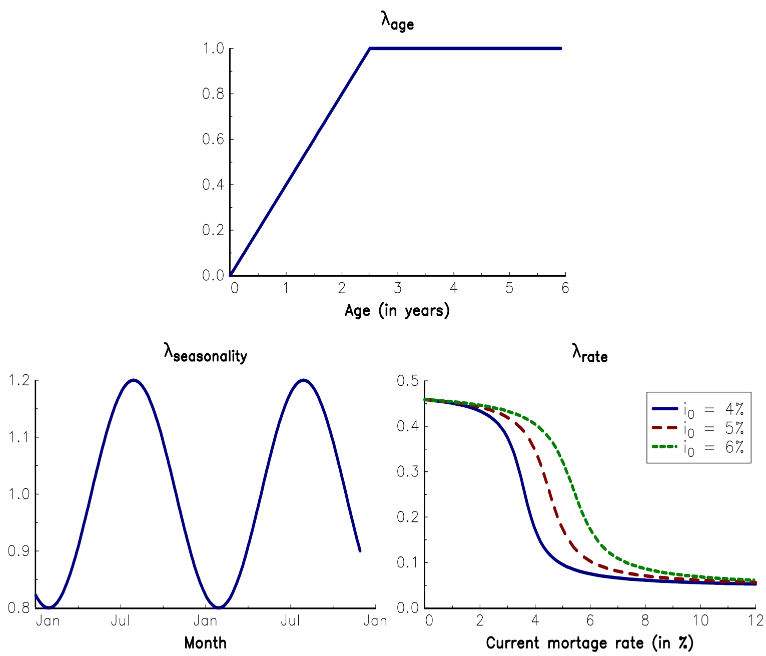

FIGURE 7.22: Components of the OTC model

[70]For the specification of λ_{rate}, we use the default values of OTS (2001, Equation 5.A.7): $\beta_0 = 0.2406$, $\beta_1 = -0.1389$, $\beta_2 = 5.952$, and $\beta_4 = 1.049$. We also assume that $i_0 = 5\%$.

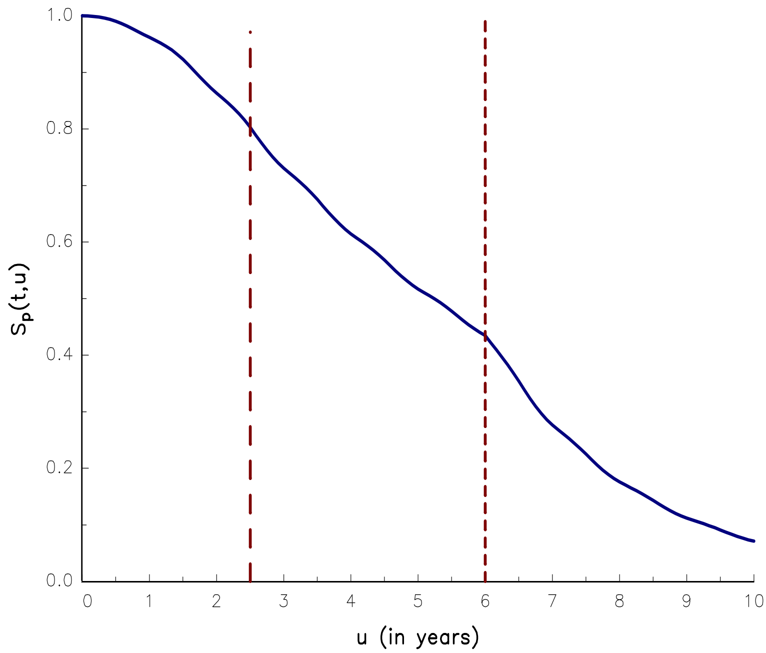

FIGURE 7.23: An example of survival function $\mathbf{S}_p(t,u)$ with a mortgage rate drop

7.3.2.4 Statistical measure of prepayment

In fact, the OTC model doesn't use the concept of hazard rate, but defines the constant prepayment rate CPR, which is the annualized rate of the single monthly mortality:

$$\text{SMM} = \frac{\text{prepayments during the month}}{\text{outstanding amount at the beginning of the month}}$$

The CPR and the SMM are then related by the following equation:

$$\text{CPR} = (1 - (1 - \text{SMM}))^{12}$$

In IRRBB, the CPR is also known as the conditional prepayment rate. It measures prepayments as a percentage of the current outstanding balance for the next year. By definition, it is related to the hazard function as follows:

$$\begin{aligned}
\text{CPR}(u,t) &= \Pr\{u < \boldsymbol{\tau} \leq u+1 \mid \boldsymbol{\tau} \geq u\} \\
&= \frac{\mathbf{S}_p(t,u) - \mathbf{S}_p(t,u+1)}{\mathbf{S}_p(t,u)} \\
&= 1 - \exp\left(-\int_u^{u+1} \lambda_p(t,s)\,\mathrm{d}s\right)
\end{aligned}$$

If $\lambda_p(t,s)$ is constant and equal to λ_p, we can approximate the CPR by the hazard rate λ_p because we have $\text{CPR}(u,t) \approx 1 - e^{-\lambda_p} \approx \lambda_p$.

We use the prepayment monitoring report published by the Federal Housing Finance Agency (FHFA). From 2008 to 2018, the CPR for 30-year mortgages varies between 5% to 35% in the US. The lowest value is reached at the end of 2008. This shows clearly that prepayments depend on the economic cycle. During a crisis, the number of defaults increases while the number of prepayments decreases. This implies that there is a negative correlation between default and prepayment rates. However, there is a high heterogeneity depending on the coupon rate and the issuance date as shown in Table 7.24. We generally observe that the CPR increases with the coupon rate. For example, in June 2018, the CPR is 7% greater for a 30-year mortgage issued between 2012 and 2016 with a 4.5% coupon than with a 3% coupon. We also verify the ramp effect because the prepayment rate is not of the same magnitude before and after January 2017, which corresponds to the 30-month age after which the prepayment rate can be assumed to be constant. This is why the CPR is only 5.3% and 12.8% for mortgages issued in 2018 and 2017 while it is equal to 17.4% for mortgages issued in 2016 when the coupon rate is 4.5%.

TABLE 7.24: Conditional prepayment rates in June 2018 by coupon rate and issuance date

Year	2012	2013	2014	2015	2016	2017	2018
Coupon = 3%	9.6%	10.2%	10.9%	10.0%	8.7%	5.3%	3.1%
Coupon = 4.5%	16.1%	15.8%	16.6%	17.9%	17.4%	12.8%	5.3%
Difference	6.5%	5.6%	5.7%	8.0%	8.7%	7.6%	2.2%

Source: RiskSpan dataset, FHFA (2018) and author's calculations.

7.3.3 Redemption risk

7.3.3.1 The funding risk of term deposits

A term deposit, also known as time deposit or certificate of deposit (CD), is a fixed-term cash investment. The client deposits a minimum sum of money into a banking account in exchange for a fixed rate over a specified period. A term deposit is then defined by three variables: the deposit or CD rate $i(t)$, the maturity period m and the minimum balance D^{-}. For example, the minimum deposit is generally \$1 000 in the US, and the typical maturities are 1M, 3M, 6M, 1Y, 2Y and 3Y. In some banks, the deposit rate may depends on the deposit amount[71]. Term deposits are an important source of bank funding with demand deposits and savings accounts. However, they differ from non-maturity deposits because they have a fixed maturity, their rates are higher and they may be redeemed with a penalty. When buying a term deposit, the investor can withdraw their funds only after the term ends. This is why CD rates are generally greater than NMD rates, because term deposits are a most stable funding resource for banks. Moreover, CD rates are generally more sensitive to market interest rates than NMD rates, because a term deposit is more an investment product while a demand deposit is more a transaction account. Under some conditions, the investor may withdraw his term deposit before the maturity date if he pays early redemption costs and fees, which generally correspond to a reduction of the deposit rate. For example, $i(t)$ may be reduced by 80% if the remaining maturity is greater than 50% of the CD maturity and 30% if the remaining maturity is less than 20% of the CD maturity.

According to Gilkeson *et al.* (1999), early time deposit withdrawals may be motivated by two reasons. As for prepayments, the first reason is economic. If market interest rates rise, the investor may have a financial incentive to close his old term deposit and reinvest his

[71] For example, Chase defines six CD rates for a given maturity and considers the following bands: below \$10K, \$10K – \$25K, \$25K – \$50K, \$50K – \$100K, \$100K – \$250K and \$250+ (source: https://www.chase.com/personal/savings/bank-cd).

money into a new term deposit. In this case, the investor is sensitive to the rate differential $i(t) - i_0$ where i_0 is the original CD rate and $i(t)$ is the current CD rate. In this case, early withdrawal risk can be viewed as the opposite of prepayment risk. Indeed, while the economic reason of prepayment risk is a fall of interest rates, the economic reason of redemption risk is a rise of interest rates. Since both risks imply a negative impact on the net interest income, the impact on the liquidity risk is different: the bank receives cash in case of a prepayment, while the funding of the bank is reduced in case of redemption. The second reason is related to negative liquidity shocks of depositors. For example, the client may need to get his money back because of life events: job loss, divorce, revenue decline, etc. In this case, redemption risk is explained by idiosyncratic liquidity shocks that are independent and can be measured by a structural constant rate. But redemption risk can also be explained by systemic liquidity shocks. For example, economic crises increase the likelihood of early withdrawals. In this case, we cannot assume that the redemption rate is constant because it depends on the economic cycle.

7.3.3.2 Modeling the early withdrawal risk

Redemption risk can be measured using the same approach we have used for prepayment risk. This is particularly true for the economic component and the idiosyncratic liquidity component. The systemic component of negative liquidity shocks requires a more appropriate analysis and makes the modeling more challenging. Another difficulty with the early withdrawal risk is the scarcity of academic models, professional publications and data. To our knowledge, there are only five academic publications on this topic and only three articles that give empirical results[72]: Cline and Brooks (2004), Gilkeson *et al.* (1999) and Gilkeson *et al.* (2000).

The redemption-based survival function of time deposits can be decomposed as:

$$\mathbf{S}_r(t,u) = \mathbf{S}_{\text{economic}}(t,u) \cdot \mathbf{S}_{\text{liquidity}}(t,u)$$

where $\mathbf{S}_{\text{economic}}(t,u)$ is the amortization function related to reinvestment financial incentives and $\mathbf{S}_{\text{liquidity}}(t,u)$ is the amortization function due to negative liquidity shocks.

Let us first focus on economic withdrawals. We note t the current date, m the maturity of the time deposit and N_0 the initial investment at time 0. In the absence of redemption, the value of the time deposit at the maturity is equal to $V_0 = N_0 (1 + i_0)^m$. If we assume that $\boldsymbol{\tau}$ is the withdrawal time, the value of the investment for $\boldsymbol{\tau} = t$ becomes:

$$V_r(t) = N_0 \cdot (1 + (1 - \varphi(t))\, i_0)^t \cdot (1 + i(t))^{m-t} - C(t)$$

where $\varphi(t)$ is the penalty parameter applied to interest paid and $C(t)$ is the break fee. For example, if we specify $\varphi(t) = 1 - t/m$, $\varphi(t)$ is a linear decreasing function between[73] $\varphi(0) = 100\%$ and $\varphi(m) = 0\%$. $C(t)$ may be a flat fee (e.g. $C(t) = \$1\,000$) or $C(t)$ may be a proportional fee: $C(t) = c(t) \cdot N_0$. The rational investor redeems the term deposit if the refinancing incentive is positive:

$$\mathrm{RI}(t) = \frac{V_r(t) - V_0}{N_0} > 0$$

In the case where $C(t) = c(t)\, N_0$, we obtain the following equivalent condition:

$$i(t) > i^\star(t) = \left(\frac{(1+i_0)^m + c(t)}{(1 + (1 - \varphi(t))\, i_0)^t} \right)^{1/(m-t)} - 1$$

[72] The two other theoretical publications are Stanhouse and Stock (2004), and Gao *et al.* (2018).

[73] $\varphi(t) = 100\%$ if the redemption occurs at the beginning of the contract and $\varphi(m) = 0\%$ when the term deposit matures.

An example of this refinancing incentive rule is given in Figure 7.24. This corresponds to a three-year term deposit whose rate is equal to 2%. The penalty applied to interest paid is given by $\varphi(t) = 1 - t/m$. We show the impact of the fee $c(t)$ on $i^{\star}(t)$. We observe that the investor has no interest to wait if the interest rate rise is sufficient. Therefore, there is an arbitrage between the current rate $i(t)$ and the original rate i_0. We deduce that the hazard function takes the following form: $\lambda_{\text{economic}}(t,u) = g(i(u) - i_0)$ or $\lambda_{\text{economic}}(t,u) = g(r(u) - i_0)$ where g is a function to estimate. For instance, Gilkeson *et al.* (1999) consider a logistic regression model and explain withdrawal rates by the refinancing incentive variable.

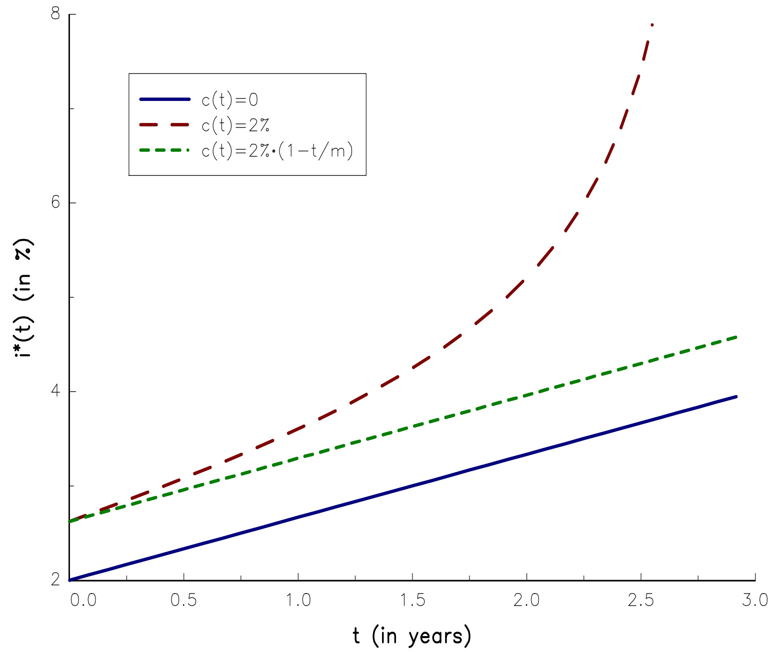

FIGURE 7.24: Refinancing incentive rule of term deposits

For early withdrawals due to negative liquidity shocks, we can decompose the hazard function into two effects:

$$\lambda_{\text{liquidity}}(t,u) = \lambda_{\text{structural}} + \lambda_{\text{cyclical}}(u)$$

where $\lambda_{\text{structural}}$ is the structural rate of redemption and $\lambda_{\text{cyclical}}(u)$ is the liquidity component due to the economic cycle. A simple way to model $\lambda_{\text{cyclical}}(u)$ is to consider a linear function of the GDP growth.

7.4 Exercises

7.4.1 Constant amortization of a loan

We consider a loan that is repaid by annual payments. We assume that the notional of the loan is equal to N_0, the maturity of the loan is n and i is the annual interest rate. We note $N(t)$ the outstanding amount, $I(t)$ the interest payment, $P(t)$ the principal payment at time t and $C(t)$ the present value.

1. Let C_0 be the present value of an annuity A that is paid annually during n years. Calculate C_0 as a function of A, n and i.

2. Determine the constant annuity A of the loan and the corresponding annuity rate $a_{(n)}$.

3. Calculate $I(1)$ and $P(1)$. Show that the outstanding amount $N(1)$ is equal to the present value $C(1)$ of the constant annuity A for the last $n-1$ years.

4. Calculate the general formula of $N(t)$, $I(t)$ and $P(t)$.

7.4.2 Computation of the amortization functions $\mathbf{S}(t,u)$ and $\mathbf{S}^\star(t,u)$

In what follows, we consider a debt instrument, whose remaining maturity is equal to m. We note t the current date and $T = t + m$ the maturity date.

1. We consider a bullet repayment debt. Define its amortization function $\mathbf{S}(t,u)$. Calculate the survival function $\mathbf{S}^\star(t,u)$ of the stock. Show that:

$$\mathbf{S}^\star(t,u) = \mathbb{1}\{t \leq u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)$$

   in the case where the new production is constant. Comment on this result.

2. Same question if we consider a debt instrument, whose amortization rate is constant.

3. Same question if we assume[74] that the amortization function is exponential with parameter λ.

4. Find the expression of $\mathcal{D}^\star(t)$ when the new production is constant.

5. Calculate the durations $\mathcal{D}(t)$ and $\mathcal{D}^\star(t)$ for the three previous cases.

6. Calculate the corresponding dynamics $\mathrm{d}N(t)$.

7.4.3 Continuous-time analysis of the constant amortization mortgage (CAM)

We consider a constant amortization mortgage, whose maturity is equal to m. We note i the interest rate and A the constant annuity.

1. Let N_0 be the amount of the mortgage at time $t = 0$. Write the equation of $\mathrm{d}N(t)$. Show that the annuity is equal to:

$$A = \frac{i \cdot N_0}{1 - e^{-im}}$$

   Deduce that the outstanding balance at time t is given by:

$$N(t) = \mathbb{1}\{t < m\} \cdot N_0 \cdot \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}$$

2. Find the expression of $\mathbf{S}(t,u)$ and $\mathbf{S}^\star(t,u)$.

3. Calculate the liquidity duration $\mathcal{D}(t)$.

[74] By definition of the exponential amortization, we have $m = +\infty$.

7.4.4 Valuation of non-maturity deposits

This exercise is based on the model of De Jong and Wielhouwer (2003), which is an application of the continuous-time framework of Jarrow and van Deventer (1998). The framework below has been used by de Jong and Wielhouwer to model variable rate savings accounts. However, it is valid for all types of non-maturity deposits (demand deposits and savings accounts). For instance, Jarrow and van Deventer originally develop the approach for all types of demand deposits[75].

1. Let $D(t)$ be the amount of savings accounts. We note $r(t)$ and $i(t)$ the market rate and the interest rate paid to account holders. We define the current market value of liabilities as follows:

$$L_0 = \mathbb{E}\left[\int_0^\infty e^{-r(t)t}\left(i(t)\,D(t) - \partial_t D(t)\right)\,\mathrm{d}t\right]$$

Explain the expression of L_0, in particular the two components $i(t)\,D(t)$ and $\partial_t D(t)$.

2. By considering that the short rate $r(t)$ is constant, demonstrate that:

$$L_0 = D_0 + \mathbb{E}\left[\int_0^\infty e^{-r(t)t}\left(i(t) - r(t)\right)D(t)\,\mathrm{d}t\right]$$

3. Calculate the current mark-to-market V_0 of savings accounts. How do you interpret V_0?

4. Let us assume that the margin $m(t) = r(t) - i(t)$ is constant and equal to m_0, and $D(t)$ is at the steady state D_∞. Show that:

$$V_0 = m_0 \cdot r_\infty^{-1} \cdot D_\infty$$

where r_∞ is a parameter to determine.

5. For the specification of the deposit rate $i(t)$ and the deposit balance $D(t)$, De Jong and Wielhouwer (2003) propose the following dynamics:

$$\mathrm{d}i(t) = \left(\alpha + \beta\left(r(t) - i(t)\right)\right)\mathrm{d}t$$

and:

$$\mathrm{d}D(t) = \gamma\left(D_\infty - D(t)\right)\mathrm{d}t - \delta\left(r(t) - i(t)\right)\mathrm{d}t$$

where α, $\beta \geq 0$, $\gamma \geq 0$ and $\delta \geq 0$ are four parameters. What is the rationale of these equations? Find the general expression of $i(t)$ and $D(t)$.

6. In the sequel, the market rate $r(t)$ is assumed to be constant and equal to r_0. Deduce the value of $i(t)$ and $D(t)$.

7. Calculate the net asset value V_0 and deduce its sensitivity with respect to the market rate r_0 when $\alpha = 0$.

8. Find the general expression of the sensitivity of V_0 with respect to the market rate r_0 when $\alpha \neq 0$. Deduce the duration $\mathcal{D}_D$ of the deposits.

[75] Janosi *et al.* (1999) provide an empirical analysis of the Jarrow-van Deventer model for negotiable orders of withdrawal accounts (NOW), passbook accounts, statement accounts and demand deposit accounts (DDAs), whereas Kalkbrener and Willing (2004) consider an application to savings accounts. Generally, these different accounts differ with respect to the specification of interest paid $i(t)$ and the dynamics of the deposit amount $D(t)$.

9. We consider a numerical application of the De Jong-Wielhouwer model with the following parameters: $r_0 = 10\%$, $i_0 = 5\%$, $D_0 = 100$, $D_\infty = 150$, $\beta = 0.5$, $\gamma = 0.7$ and $\delta = 0.5$. Make a graph to represent the relationship between the time t and the deposit rate $i(t)$ when α is equal to -1%, 0 and 1%. Why is it natural to consider that $\alpha < 0$? We now assume that $\alpha = -1\%$. Draw the dynamics of $D(t)$. What are the most important parameters that impact $D(t)$? What is the issue if we calculate the duration of the deposits with respect to r_0 when α is equal to zero? Make a graph to represent the relationship between the market rate r_0 and the duration when α is equal to -50 bps, -1% and -2%.

7.4.5 Impact of prepayment on the amortization scheme of the CAM

This is a continuation of Exercise 7.4.3 on page 450. We recall that the outstanding balance at time t is given by:

$$N(t) = \mathbb{1}\{t < m\} \cdot N_0 \cdot \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}$$

1. Find the dynamics $\mathrm{d}N(t)$.

2. We note $\tilde{N}(t)$ the modified outstanding balance that takes into account the prepayment risk. Let $\lambda_p(t)$ be the prepayment rate at time t. Write the dynamics of $\tilde{N}(t)$.

3. Show that $\tilde{N}(t) = N(t) \cdot \mathbf{S}_p(t)$ where $\mathbf{S}_p(t)$ is the prepayment-based survival function.

4. Calculate the liquidity duration $\tilde{\mathcal{D}}(t)$ associated to the outstanding balance $\tilde{N}(t)$ when the hazard rate of prepayments is constant and equal to λ_p.