Course 2022-2023 in Sustainable Finance
Lecture 10. Climate Portfolio Construction

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[1]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Quadratic programming

Definition

We have:

$$
\begin{aligned}
x^\star &= \arg\min \frac{1}{2} x^\top Q x - x^\top R \\
&\text{s.t.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^- \leq x \leq x^+ \end{array} \right.
\end{aligned}
$$

where x is a $n \times 1$ vector, Q is a $n \times n$ matrix, R is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^- and x^+ are two $n \times 1$ vectors

Quadratic form

A quadratic form is a polynomial with terms all of degree two

$$\mathcal{QF}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j = x^\top A x$$

Canonical form

$$\mathcal{QF}(x_1, \ldots, x_n) = \frac{1}{2}\left(x^\top A x + x^\top A^\top x\right) = \frac{1}{2} x^\top \left(A + A^\top\right) x = \frac{1}{2} x^\top Q x$$

Generalized quadratic form

$$\mathcal{QF}(x; Q, R, c) = \frac{1}{2} x^\top Q x - x^\top R + c$$

Quadratic form

Main properties

1. $\varphi \cdot \mathcal{QF}(w; Q, R, c) = \mathcal{QF}(w; \varphi Q, \varphi R, \varphi c)$
2. $\mathcal{QF}(x; Q_1, R_1, c_1) + \mathcal{QF}(x; Q_2, R_2, c_2) = \mathcal{QF}(x; Q_1 + Q_2, R_1 + R_2, c_1 + c_2)$
3. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(x; Q, R + Qy, \frac{1}{2}y^\top Qy + y^\top R + c\right)$
4. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(y; Q, Qx - R, \frac{1}{2}x^\top Qx - x^\top R + c\right)$
5. $\dfrac{1}{2}\sum_{i=1}^n q_i x_i^2 = \mathcal{QF}(x; \mathcal{D}(q), \mathbf{0}_n, 0)$ where $q = (q_1, \ldots, q_n)$ is a $n \times 1$ vector and $\mathcal{D}(q) = \operatorname{diag}(q)$
6. $\dfrac{1}{2}\sum_{i=1}^n q_i (x_i - y_i)^2 = \mathcal{QF}\left(x; \mathcal{D}(q), \mathcal{D}(q)y, \frac{1}{2}y^\top \mathcal{D}(q)y\right)$
7. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i x_i\right)^2 = \mathcal{QF}(x; \mathcal{T}(q), \mathbf{0}_n, 0)$ where $\mathcal{T}(q) = qq^\top$
8. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i (x_i - y_i)\right)^2 = \mathcal{QF}\left(x; \mathcal{T}(q), \mathcal{T}(q)y, \frac{1}{2}y^\top \mathcal{T}(q)y\right)$

Quadratic form

Main properties

We note $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i = \mathbb{1}\{i \in \Omega\}$

1. $\frac{1}{2}\sum_{i\in\Omega} q_i x_i^2 = \mathcal{QF}\left(x; \mathcal{D}\left(\omega \circ q\right), \mathbf{0}_n, 0\right)$
2. $\frac{1}{2}\sum_{i\in\Omega} q_i \left(x_i - y_i\right)^2 =$
   $\mathcal{QF}\left(x; \mathcal{D}\left(\omega \circ q\right), \mathcal{D}\left(\omega \circ q\right) y, \frac{1}{2} y^\top \mathcal{D}\left(\omega \circ q\right) y\right)$
3. $\frac{1}{2}\left(\sum_{i\in\Omega} q_i x_i\right)^2 = \mathcal{QF}\left(x; \mathcal{T}\left(\omega \circ q\right), \mathbf{0}_n, 0\right)$
4. $\frac{1}{2}\left(\sum_{i\in\Omega} q_i \left(x_i - y_i\right)\right)^2 =$
   $\mathcal{QF}\left(x; \mathcal{T}\left(\omega \circ q\right), \mathcal{T}\left(\omega \circ q\right) y, \frac{1}{2} y^\top \mathcal{T}\left(\omega \circ q\right) y\right)$
5. $\mathcal{D}\left(\omega \circ q\right) = \operatorname{diag}\left(\omega \circ q\right) = \mathcal{D}\left(\omega\right) \mathcal{D}\left(q\right)$
6. $\mathcal{T}\left(\omega \circ q\right) = \left(\omega \circ q\right)\left(\omega \circ q\right)^\top = \left(\omega\omega^\top\right) \circ qq^\top = \mathcal{T}\left(\omega\right) \circ \mathcal{T}\left(q\right)$

Equity portfolio

Basic optimization problems

Mean-variance optimization

The long-only mean-variance optimization problem is given by:

$$
\begin{aligned}
w^\star &= \arg\min \frac{1}{2} w^\top \Sigma w - \gamma w^\top \mu \\
&\text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right.
\end{aligned}
$$

where:

- γ is the risk-tolerance coefficient
- the equality constraint is the budget constraint $\left(\sum_{i=1}^n w_i = 1\right)$
- the bounds correspond to the no short-selling restriction $(w_i \geq 0)$

QP form

$Q = \Sigma$, $R = \gamma \mu$, $A = \mathbf{1}_n^\top$, $B = 1$, $w^- = \mathbf{0}_n$ and $w^+ = \mathbf{1}$

Equity portfolio

Basic optimization problems

Tracking error optimization

The tracking error optimization problem is defined as:

$$w^{\star} = \arg\min \frac{1}{2} w^{\top} \Sigma w - w^{\top} \left(\gamma \mu + \Sigma b\right)$$
$$\text{s.t.} \quad \begin{cases} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{cases}$$

QP form

$Q = \Sigma$, $\boxed{R = \gamma \mu + \Sigma b}$, $A = \mathbf{1}_n^{\top}$, $B = 1$, $w^{-} = \mathbf{0}_n$ and $w^{+} = \mathbf{1}$

$\Rightarrow$ Portfolio replication: $\boxed{R = \Sigma b}$

Specification of the constraints

Sector weight constraint

- We have

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+$$

- $\boldsymbol{s}_j$ is the $n \times 1$ sector-mapping vector: $\boldsymbol{s}_{i,j} = \mathbb{1}\{i \in \mathcal{S}ector_j\}$
- We notice that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \boldsymbol{s}_j^\top w$$

- We deduce that:

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+ \Leftrightarrow \left\{ \begin{array}{l} s_j^- \leq \boldsymbol{s}_j^\top w \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} -\boldsymbol{s}_j^\top w \leq -s_j^- \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right.$$

QP form

$$\underbrace{\begin{pmatrix} -\boldsymbol{s}_j^\top \\ \boldsymbol{s}_j^\top \end{pmatrix}}_{C} w \leq \underbrace{\begin{pmatrix} -s_j^- \\ s_j^+ \end{pmatrix}}_{D}$$

Specification of the constraints

Score constraint

- General constraint:

$$\sum_{i=1}^{n} w_i \mathcal{S}_i \geq \mathcal{S}^{\star} \Leftrightarrow -\mathcal{S}^{\top} w \leq -\mathcal{S}^{\star}$$

QP form

- $C = -\mathcal{S}^{\top}$
- $D = -\mathcal{S}^{\star}$

Specification of the constraints

Score constraint

- Sector-specific constraint:

$$\begin{aligned}\sum_{i \in \mathcal{S}ector_j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star & \Leftrightarrow \sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} \cdot w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\ & \Leftrightarrow \sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\ & \Leftrightarrow \sum_{i=1}^{n} w_i \cdot \left(\boldsymbol{s}_{i,j} \mathcal{S}_i\right) \geq \mathcal{S}_j^\star \\ & \Leftrightarrow \left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top w \geq \mathcal{S}_j^\star\end{aligned}$$

QP form

- $C = -\left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top$
- $D = -\mathcal{S}_j^\star$

Equity portfolios

Example #1

- The capitalization-weighted equity index is composed of 8 stocks
- The weights are equal to 23%, 19%, 17%, 13%, 9%, 8%, 6% and 5%
- The ESG score, carbon intensity and sector of the eight stocks are the following:

Stock	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{S}$	−1.20	0.80	2.75	1.60	−2.75	−1.30	0.90	−1.70
$\mathcal{CI}$	125	75	254	822	109	17	341	741
$\mathcal{S}ector$	1	1	2	2	1	2	1	2

Equity portfolios

Example #1 (Cont'd)

- The stock volatilities are equal to 22%, 20%, 25%, 18%, 35%, 23%, 13% and 29%
- The correlation matrix is given by:

$$\mathbb{C} = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

Equity portfolios

QP problem

- We have:

$$w^{\star} = \arg\min \frac{1}{2} w^{\top} Q w - w^{\top} R$$
$$\text{s.t.} \quad \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^{-} \leq w \leq w^{+} \end{array} \right.$$

Equity portfolios

Objective function

- Using $\Sigma_{i,j} = \mathbb{C}_{i,j}\sigma_i\sigma_j$, we obtain:

$$Q = \Sigma = 10^{-4} \times$$

$$\begin{pmatrix}
484.00 & 352.00 & 385.00 & 237.60 & 539.00 & 253.00 & 200.20 & 382.80 \\
352.00 & 400.00 & 375.00 & 234.00 & 350.00 & 276.00 & 130.00 & 377.00 \\
385.00 & 375.00 & 625.00 & 360.00 & 612.50 & 402.50 & 227.50 & 507.50 \\
237.60 & 234.00 & 360.00 & 324.00 & 535.50 & 331.20 & 175.50 & 391.50 \\
539.00 & 350.00 & 612.50 & 535.50 & 1225.00 & 483.00 & 364.00 & 659.75 \\
253.00 & 276.00 & 402.50 & 331.20 & 483.00 & 529.00 & 149.50 & 466.90 \\
200.20 & 130.00 & 227.50 & 175.50 & 364.00 & 149.50 & 169.00 & 301.60 \\
382.80 & 377.00 & 507.50 & 391.50 & 659.75 & 466.90 & 301.60 & 841.00
\end{pmatrix}$$

Equity portfolios
Objective function

- We have:

$$R = \Sigma b = \begin{pmatrix} 3.74 \\ 3.31 \\ 4.39 \\ 3.07 \\ 5.68 \\ 3.40 \\ 2.02 \\ 4.54 \end{pmatrix} \times 10^{-2}$$

Equity portfolios

Constraint specification (bounds)

- The portfolio is long-only

QP form

  - $w^{-} = \mathbf{0}_8$
  - $w^{+} = \mathbf{1}_8$

Equity portfolios

Constraint specification (equality)

- The budget constraint $\sum_{i=1}^{8} w_i = 1 \Rightarrow$ a first linear equation $A_0 w = B_0$

  QP form

  - $A_0 = \mathbf{1}_8^\top$
  - $B_0 = 1$

Equity portfolios

Constraint specification (equality)

- We can impose the sector neutrality of the portfolio meaning that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \sum_{i \in \mathcal{S}ector_j} b_i$$

The sector neutrality constraint can be written as:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} w = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

QP form

- $A_1 = \boldsymbol{s}_1^\top = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
- $A_2 = \boldsymbol{s}_2^\top = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- $B_1 = \boldsymbol{s}_1^\top b = \sum_{i \in \mathcal{S}ector_1} b_i$
- $B_2 = \boldsymbol{s}_2^\top b = \sum_{i \in \mathcal{S}ector_2} b_i$

Equity portfolios

Constraint specification (inequality)

- We can impose a relative reduction of the benchmark carbon intensity:

$$\mathcal{CI}(w) \leq (1-\mathcal{R})\,\mathcal{CI}(b) \Leftrightarrow C_1 w \leq D_1$$

QP form

  - $C_1 = \mathcal{CI}^\top$ (because $\mathcal{CI}(w) = \mathcal{CI}^\top w$)
  - $D_1 = (1-\mathcal{R})\,\mathcal{CI}(b)$

- We can impose an absolute increase of the benchmark ESG score:

$$\mathcal{S}(w) \geq \mathcal{S}(b) + \Delta\mathcal{S}^\star$$

Since $\mathcal{S}(w) = \mathcal{S}^\top w$, we deduce that $C_2 w \leq D_2$

QP form

  - $C_2 = -\mathcal{S}^\top$
  - $D_2 = -\left(\mathcal{S}(b) + \Delta\mathcal{S}^\star\right)$

Equity portfolios

Combination of constraints

Set of constraints	Carbon intensity	ESG score	Sector neutrality	A	B	C	D
#1	✓			A_0	B_0	C_1	D_1
#2		✓		A_0	B_0	C_2	D_2
#3	✓	✓		A_0	B_0	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$
#4	✓	✓	✓	$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}$	$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}$	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

Equity portfolios

Results

Table 1: $\mathcal{R} = 30\%$ and $\Delta\mathcal{S}^\star = 0.50$ (Example #1)

		Benchmark	Set #1	Set #2	Set #3	Set #4
Weights (in %)	$w_1^\star$	23.00	18.17	25.03	8.64	12.04
	$w_2^\star$	19.00	24.25	14.25	29.27	23.76
	$w_3^\star$	17.00	16.92	21.95	26.80	30.55
	$w_4^\star$	13.00	2.70	27.30	1.48	2.25
	$w_5^\star$	9.00	12.31	3.72	10.63	8.51
	$w_6^\star$	8.00	11.23	1.34	6.30	10.20
	$w_7^\star$	6.00	11.28	1.68	16.87	12.69
	$w_8^\star$	5.00	3.15	4.74	0.00	0.00
Statistics	$\sigma(w^\star \mid b)$ (in %)	0.00	0.50	1.18	1.90	2.12
	$\mathcal{CI}(w^\star)$	261.72	183.20	367.25	183.20	183.20
	$\mathcal{R}(w^\star \mid b)$ (in %)		30.00	−40.32	30.00	30.00
	$\mathcal{S}(w^\star)$	0.17	0.05	0.67	0.67	0.67
	$\mathcal{S}(w^\star) - \mathcal{S}(b)$		−0.12	0.50	0.50	0.50
	$w^\star(\mathcal{S}ector_1)$ (in %)	57.00	66.00	44.67	65.41	57.00
	$w^\star(\mathcal{S}ector_2)$ (in %)	43.00	34.00	55.33	34.59	43.00

Equity portfolios

Dealing with constraints on relative weights

- The carbon intensity of the j^{th} sector within the portfolio w is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right)=\sum_{i\in\mathcal{S}ector_j}\tilde{w}_i\mathcal{CI}_i$$

where $\tilde{w}_i$ is the normalized weight in the sector bucket:

$$\tilde{w}_i=\frac{w_i}{\sum_{k\in\mathcal{S}ector_j}w_k}$$

- Another expression of $\mathcal{CI}\left(w;\mathcal{S}ector_j\right)$ is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right)=\frac{\sum_{i\in\mathcal{S}ector_j}w_i\mathcal{CI}_i}{\sum_{i\in\mathcal{S}ector_j}w_i}=\frac{\left(\boldsymbol{s}_j\circ\mathcal{CI}\right)^\top w}{\boldsymbol{s}_j^\top w}$$

Equity portfolios

Dealing with constraints on relative weights

- If we consider the constraint $\mathcal{CI}\left(w;\mathcal{S}ector_j\right) \leq \mathcal{CI}_j^{\star}$, we obtain:

$$\begin{aligned}(*) \quad & \Leftrightarrow \quad \mathcal{CI}\left(w;\mathcal{S}ector_j\right) \leq \mathcal{CI}_j^{\star} \\ & \Leftrightarrow \quad \left(\boldsymbol{s}_j \circ \mathcal{CI}\right)^{\top} w \leq \mathcal{CI}_j^{\star}\left(\boldsymbol{s}_j^{\top} w\right) \\ & \Leftrightarrow \quad \left(\left(\boldsymbol{s}_j \circ \mathcal{CI}\right) - \mathcal{CI}_j^{\star}\boldsymbol{s}_j\right)^{\top} w \leq 0 \\ & \Leftrightarrow \quad \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^{\star}\right)\right)^{\top} w \leq 0\end{aligned}$$

QP form

- $C = \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^{\star}\right)\right)^{\top}$
- $D = 0$

Equity portfolios

Dealing with constraints on relative weights

Example #2

- Example #1
- We would like to reduce the carbon footprint of the benchmark by 30%
- We impose the sector neutrality

Equity portfolios

Dealing with constraints on relative weights

QP form

- $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- $B = \begin{pmatrix} 100\% \\ 57\% \\ 43\% \end{pmatrix}$
- $C = \begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \end{pmatrix}$
- $D = 183.2040$

Equity portfolios

Dealing with constraints on relative weights

- The optimal solution is:

$$w^{\star} = (21.54\%, 18.50\%, 21.15\%, 3.31\%, 10.02\%, 15.26\%, 6.94\%, 3.27\%)$$

- $\sigma(w^{\star} \mid b) = 112$ bps
- $\mathcal{CI}(w^{\star}) = 183.20$ vs. $\mathcal{CI}(b) = 261.72$

BUT

$$\left\{\begin{array}{l} \mathcal{CI}(w^{\star}; \mathcal{S}ector_1) = 132.25 \\ \mathcal{CI}(w^{\star}; \mathcal{S}ector_2) = 250.74 \end{array}\right. \quad \text{versus} \quad \left\{\begin{array}{l} \mathcal{CI}(b; \mathcal{S}ector_1) = 128.54 \\ \mathcal{CI}(b; \mathcal{S}ector_2) = 438.26 \end{array}\right.$$

The global reduction of 30% is explained by:

- an increase of 2.89% of the carbon footprint for the first sector
- a decrease of 42.79% of the carbon footprint for the second sector

Equity portfolios

Dealing with constraints on relative weights

- We impose $\mathcal{R}_1 = 20\%$

QP form

- $C = \begin{pmatrix} \mathcal{CI}^\top \\ \left(\boldsymbol{s}_1 \circ \left(\mathcal{CI} - (1 - \mathcal{R}_1)\,\mathcal{CI}\left(b; \mathcal{S}ector_1\right)\right)\right)^\top \end{pmatrix} =$
$\begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \\ 22.1649 & -27.8351 & 0 & 0 & 6.1649 & 0 & 238.1649 & 0 \end{pmatrix}$
- $D = \begin{pmatrix} 183.2040 \\ 0 \end{pmatrix}$

Equity portfolios

Dealing with constraints on relative weights

- Solving the new QP problem gives the following optimal portfolio:

$$w^{\star} = (22.70\%, 22.67\%, 19.23\%, 5.67\%, 11.39\%, 14.50\%, 0.24\%, 3.61\%)$$

- $\sigma(w^{\star} \mid b) = 144$ bps
- $\mathcal{CI}(w^{\star}) = 183.20$
  - $\mathcal{CI}(w^{\star}; \mathcal{S}ector_1) = 102.84$ (reduction of 20%)
  - $\mathcal{CI}(w^{\star}; \mathcal{S}ector_2) = 289.74$ (reduction of 33.89%)

Risk measure of a bond portfolio

- We consider a zero-coupon bond, whose price and maturity date are $B(t,T)$ and T:

$$B_t(t,T) = e^{-(r(t)+s(t))(T-t)+L(t)}$$

where $r(t)$, $s(t)$ and $L(t)$ are the interest rate, the credit spread and the liquidity premium

- We deduce that:

$$\begin{aligned} \mathrm{d}\ln B(t,T) &= -(T-t)\,\mathrm{d}r(t) - (T-t)\,\mathrm{d}s(t) + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - (D\,s(t))\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - \mathrm{DTS}(t)\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \end{aligned}$$

where:

  - $D = T - t$ is the remaining maturity (or duration)
  - $\mathrm{DTS}(t)$ is the duration-times-spread factor

Risk measure of a bond portfolio

- If we assume that $r(t)$, $s(t)$ and $L(t)$ are independent, the risk of the defaultable bond is equal to:

$$\sigma^2 (\mathrm{d} \ln B(t,T)) = D^2 \sigma^2 (\mathrm{d} r(t)) + \mathrm{DTS}(t)^2 \sigma^2 \left(\frac{\mathrm{d} s(t)}{s(t)} \right) + \sigma^2 (\mathrm{d} L(t))$$

- Three risk components

$$\sigma^2 (\mathrm{d} \ln B(t,T)) = D^2 \sigma_r^2 + \mathrm{DTS}(t)^2 \sigma_s^2 + \sigma_L^2$$

$\Longrightarrow$ **The historical volatility of a bond price is not a relevant risk measure**

Bond portfolio optimization

Without a benchmark

- Duration risk:

$$\mathrm{MD}(w) = \sum_{i=1}^{n} w_i \, \mathrm{MD}_i$$

- DTS risk:

$$\mathrm{DTS}(w) = \sum_{i=1}^{n} w_i \, \mathrm{DTS}_i$$

- Clustering approach = generalization of the sector approach, e.g. (EUR, Financials, AAA to A−, 1Y-3Y)
- We have:

$$\mathrm{MD}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{MD}_i$$

and:

$$\mathrm{DTS}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{DTS}_i$$

Bond portfolio optimization

Without a benchmark

Objective function without a benchmark

We have:

$$w^{\star} = \arg\min \frac{\varphi_{\mathrm{MD}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{MD}_j(w) - \mathrm{MD}_j^{\star}\right)^2 + \frac{\varphi_{\mathrm{DTS}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{DTS}_j(w) - \mathrm{DTS}_j^{\star}\right)^2 - \gamma \sum_{i=1}^{n} w_i \mathcal{C}_i$$

where:

- $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the relative weight of each risk component
- $\mathcal{C}_i$ is the expected carry of bond i and γ is the risk-tolerance coefficient

Bond portfolio optimization

Without a benchmark

QP form

$$
\begin{aligned}
w^{\star} & = & \arg\min \mathcal{QF}\left(w; Q, R, c\right) \\
& \text{s.t.} & \left\{\begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array}\right.
\end{aligned}
$$

where $\mathcal{QF}\left(w; Q, R, c\right)$ is the quadratic form of the objective function

Bond portfolio optimization

Without a benchmark

We have:

$$
\begin{aligned}
\frac{1}{2}\left(\mathrm{MD}_j(w) - \mathrm{MD}_j^\star\right)^2 &= \frac{1}{2}\left(\sum_{i \in \mathcal{S}ector_j} w_i \,\mathrm{MD}_i - \mathrm{MD}_j^\star\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i \,\mathrm{MD}_i - \mathrm{MD}_j^\star\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j} \,\mathrm{MD}_i \, w_i\right)^2 - w^\top \left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \mathrm{MD}_j^\star + \frac{1}{2} \mathrm{MD}_j^{\star 2} \\
&= \mathcal{QF}\left(w; \mathcal{T}\left(\boldsymbol{s}_j \circ \mathrm{MD}\right), \left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \mathrm{MD}_j^\star, \frac{1}{2} \mathrm{MD}_j^{\star 2}\right)
\end{aligned}
$$

where $\mathrm{MD} = (\mathrm{MD}_1, \ldots, \mathrm{MD}_n)$ is the vector of modified durations and $\mathcal{T}(u) = uu^\top$

Bond portfolio optimization

Without a benchmark

We deduce that:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\mathrm{MD}_j(w)-\mathrm{MD}_j^{\star}\right)^2=\mathcal{QF}\left(w;Q_{\mathrm{MD}},R_{\mathrm{MD}},c_{\mathrm{MD}}\right)$$

where:

$$\left\{\begin{array}{l}Q_{\mathrm{MD}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\mathcal{T}\left(\boldsymbol{s}_j\circ\mathrm{MD}\right)\\R_{\mathrm{MD}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\boldsymbol{s}_j\circ\mathrm{MD}\right)\mathrm{MD}_j^{\star}\\c_{\mathrm{MD}}=\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\mathrm{MD}_j^{\star 2}\end{array}\right.$$

Bond portfolio optimization

Without a benchmark

In a similar way, we have:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\mathrm{DTS}_j\left(w\right)-\mathrm{DTS}_j^{\star}\right)^2=\mathcal{QF}\left(w;Q_{\mathrm{DTS}},R_{\mathrm{DTS}},c_{\mathrm{DTS}}\right)$$

where:

$$\left\{\begin{array}{l}Q_{\mathrm{DTS}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\mathcal{T}\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)\\ R_{\mathrm{MD}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)\mathrm{DTS}_j^{\star}\\ c_{\mathrm{DTS}}=\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\mathrm{DTS}_j^{\star^2}\end{array}\right.$$

Bond portfolio optimization

Without a benchmark

We have:

$$-\gamma \sum_{i=1}^{n} w_i \mathcal{C}_i = \gamma \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \mathcal{C}, 0\right) = \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \gamma \mathcal{C}, 0\right)$$

where $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ is the vector of expected carry values

Bond portfolio optimization

Without a benchmark

Quadratic form of the objective function

The function to optimize is:

$$\begin{aligned} \mathcal{QF}(w; Q, R, c) &= \varphi_{\mathrm{MD}} \mathcal{QF}(w; Q_{\mathrm{MD}}, R_{\mathrm{MD}}, c_{\mathrm{MD}}) + \\ & \quad \varphi_{\mathrm{DTS}} \mathcal{QF}(w; Q_{\mathrm{DTS}}, R_{\mathrm{DTS}}, c_{\mathrm{DTS}}) + \\ & \quad \mathcal{QF}(w; \mathbf{0}_{n,n}, \gamma \mathcal{C}, 0) \end{aligned}$$

where:

$$\left\{ \begin{array}{l} Q = \varphi_{\mathrm{MD}} Q_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}} \\ R = \gamma \mathcal{C} + \varphi_{\mathrm{MD}} R_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}} \\ c = \varphi_{\mathrm{MD}} c_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}} \end{array} \right.$$

Bond portfolio optimization

With a benchmark

- The MD- and DTS-based tracking error variances are equal to:

$$\mathcal{R}_{\mathrm{MD}}(w \mid b) = \sigma^2_{\mathrm{MD}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{MD}_i \right)^2$$

  and:

$$\mathcal{R}_{\mathrm{DTS}}(w \mid b) = \sigma^2_{\mathrm{DTS}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{DTS}_i \right)^2$$

  This means that $\mathrm{MD}_j^\star = \sum_{i \in \mathcal{S}ector_j} b_i\,\mathrm{MD}_i$ and $\mathrm{DTS}_j^\star = \sum_{i \in \mathcal{S}ector_j} b_i\,\mathrm{DTS}_i$.

- The active share risk is defined as:

$$\mathcal{R}_{\mathrm{AS}}(w \mid b) = \sigma^2_{\mathrm{AS}}(w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2$$

Bond portfolio optimization

With a benchmark

Objective function with a benchmark

The optimization problem becomes:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\
\text{s.t.} &\quad \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right.
\end{aligned}
$$

where the synthetic risk measure is equal to:

$$
\mathcal{R}\left(w \mid b\right) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}\left(w \mid b\right)
$$

Bond portfolio optimization

With a benchmark

We can show that

$$
\begin{array}{rcl}
w^{\star} & = & \arg\min \mathcal{QF}\left(w; Q\left(b\right), R\left(b\right), c\left(b\right)\right) \\
& \text{s.t.} & \left\{\begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array}\right.
\end{array}
$$

where:

$$
\left\{\begin{array}{l}
Q\left(b\right) = \varphi_{\mathrm{AS}} Q_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} Q_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}}\left(b\right) \\
R\left(b\right) = \gamma \mathcal{C} + \varphi_{\mathrm{AS}} R_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} R_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}}\left(b\right) \\
c\left(b\right) = \gamma b^\top \mathcal{C} + \varphi_{\mathrm{AS}} c_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} c_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}}\left(b\right)
\end{array}\right.
$$

$Q_{\mathrm{AS}}\left(b\right) = I_n$, $R_{\mathrm{AS}}\left(b\right) = b$, $c_{\mathrm{AS}}\left(b\right) = \frac{1}{2} b^\top b$, $Q_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}}$, $R_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}} b = R_{\mathrm{MD}}$, $c_{\mathrm{MD}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{MD}} b = c_{\mathrm{MD}}$, $Q_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}}$, $R_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}} b = R_{\mathrm{DTS}}$, and $c_{\mathrm{DTS}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{DTS}} b = c_{\mathrm{DTS}}$

Bond portfolio optimization

With a benchmark

Example #3

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$. We have $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 0.001$.

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e}/\$ \mathrm{mn}$, MD_i in years and DTS_i in bps

Bond portfolio optimization

With a benchmark

The optimization problem is defined as:

$$\begin{aligned} w^{\star}\left(\mathcal{R}\right) & = & \arg\min \frac{1}{2} w^{\top} Q\left(b\right) w - w^{\top} R\left(b\right) \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_{9}^{\top} w = 1 \\ \mathcal{CI}^{\top} w \leq \left(1 - \mathcal{R}\right) \mathcal{CI}\left(b\right) \\ \dfrac{b}{4} \leq w \leq 4b \end{array} \right. \end{aligned}$$

where $\mathcal{R}$ is the reduction rate

Bond portfolio optimization

With a benchmark

Since the bonds are ordering by sectors, $Q(b)$ is a block diagonal matrix:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & Q_2 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & Q_3 \end{pmatrix} \times 10^3$$

where:

$$Q_1 = \begin{pmatrix} 0.3611 & 0.5392 & 0.2877 \\ 0.5392 & 1.2148 & 0.5926 \\ 0.2877 & 0.5926 & 0.4189 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3.2218 & 1.7646 & 0.5755 \\ 1.7646 & 1.2075 & 0.3810 \\ 0.5755 & 0.3810 & 0.2343 \end{pmatrix}$$

and:

$$Q_3 = \begin{pmatrix} 2.1328 & 1.7926 & 1.1899 \\ 1.7926 & 1.7653 & 1.1261 \\ 1.1899 & 1.1261 & 0.8664 \end{pmatrix}$$

$R(b) = (2.243, 4.389, 2.400, 6.268, 3.751, 1.297, 2.354, 2.120, 1.424)$
$\times 10^2$

Bond portfolio optimization

With a benchmark

Table 2: Weights in % of optimized bond portfolios (Example #3)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^{\star}$ (10%)	21.92	19.01	15.53	11.72	11.68	7.82	6.68	4.71	0.94
$w^{\star}$ (30%)	26.29	20.24	10.90	10.24	16.13	3.74	9.21	2.50	0.75
$w^{\star}$ (50%)	27.48	23.97	4.00	6.94	22.70	2.00	11.15	1.00	0.75

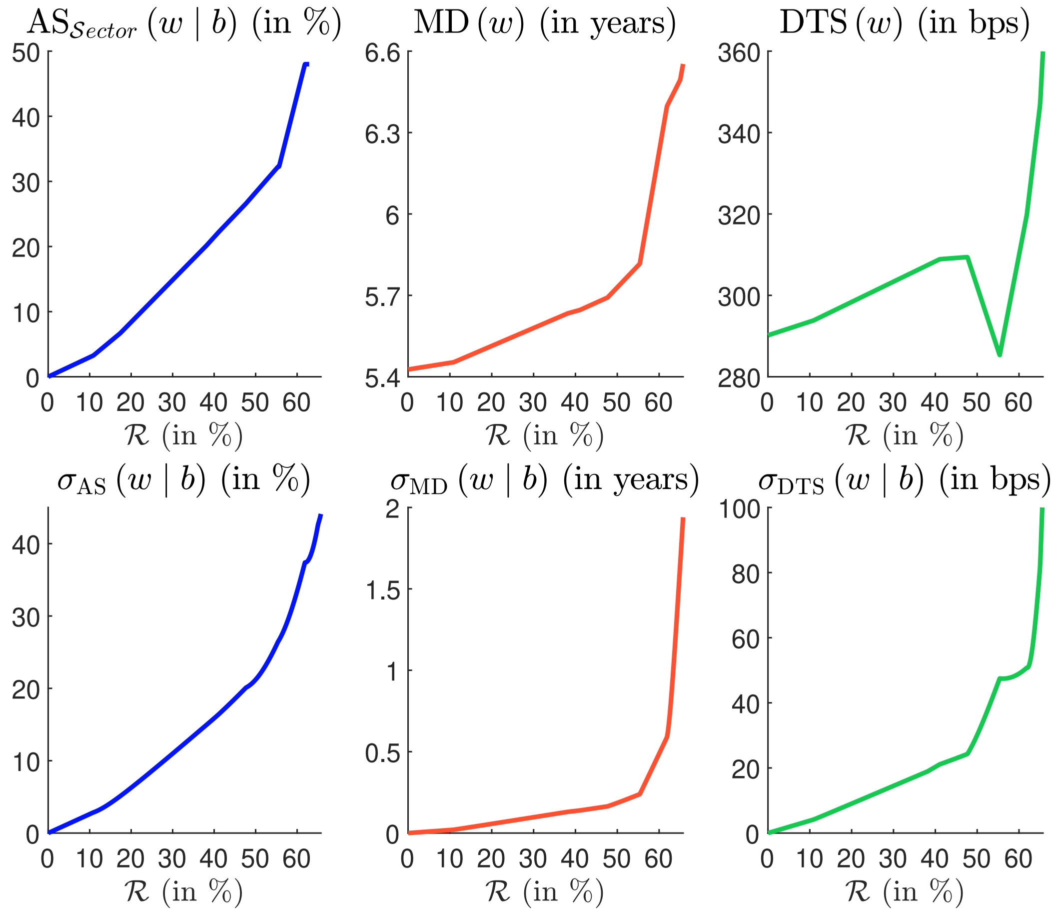
Table 3: Risk statistics of optimized bond portfolios (Example #3)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e/\$}$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^{\star}$ (10%)	3.00	5.45	293.53	2.62	0.02	3.80	165.95
$w^{\star}$ (30%)	14.87	5.58	303.36	10.98	0.10	14.49	129.07
$w^{\star}$ (50%)	28.31	5.73	302.14	21.21	0.19	30.11	92.19

Bond portfolio optimization

With a benchmark

Figure 1: Relationship between the reduction rate and the tracking risk (Example #3)



Advanced optimization problems
Large bond universe

- QP: $\mathbf{n \leq 5\,000}$ (the dimension of Q is $n \times n$)
- LP: $\mathbf{n \gg 10^6}$

- Some figures as of 31/01/2023
  - MSCI World Index (DM): $n = 1\,508$ stocks
  - MSCI World IMI (DM): $n = 5\,942$ stocks
  - MSCI World AC (DM + EM): $n = 2\,882$ stocks
  - MSCI World AC IMI (DM + EM): $n = 7\,928$ stocks
  - Bloomberg Global Aggregate Total Return Index: $n = 28\,799$ securities
  - ICE BOFA Global Broad Market Index: $n = 33\,575$ securities

- Trick: $\mathcal{L}_2$-norm risk measures $\Rightarrow$ $\mathcal{L}_1$-norm risk measures

Advanced optimization problems

Large bond universe

We replace the synthetic risk measure by:

$$\mathcal{D}\left(w \mid b\right) = \varphi'_{\mathrm{AS}}\mathcal{D}_{\mathrm{AS}}\left(w \mid b\right) + \varphi'_{\mathrm{MD}}\mathcal{D}_{\mathrm{MD}}\left(w \mid b\right) + \varphi'_{\mathrm{DTS}}\mathcal{D}_{\mathrm{DTS}}\left(w \mid b\right)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}\left(w \mid b\right) &= \frac{1}{2}\sum_{i=1}^{n}\left|w_i - b_i\right| \\
\mathcal{D}_{\mathrm{MD}}\left(w \mid b\right) &= \sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum\nolimits_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{MD}_i\right| \\
\mathcal{D}_{\mathrm{DTS}}\left(w \mid b\right) &= \sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum\nolimits_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{DTS}_i\right|
\end{aligned}$$

Advanced optimization problems
Large bond universe

The optimization problem becomes:

$$
\begin{aligned}
w^{\star} & = \arg\min \mathcal{D}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\
& \text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_n^\top w = 1 \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n
\end{array}
\right.
\end{aligned}
$$

Advanced optimization problems

Large bond universe

Absolute value trick

If $c_i \geq 0$, then:

$$\min \sum_{i=1}^{n} c_i \left| f_i(x) \right| + g(x) \Leftrightarrow \left\{ \begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} c_i \tau_i + g(x) \\ \text{s.t.} & \left\{ \begin{array}{l} \left| f_i(x) \right| \leq \tau_i \\ \tau_i \geq 0 \end{array} \right. \end{array} \right.$$

The problem becomes linear:

$$\left| f_i(x) \right| \leq \tau_i \Leftrightarrow -\tau_i \leq f_i(x) \wedge f_i(x) \leq \tau_i$$

Advanced optimization problems

Large bond universe

Linear programming

The standard formulation of a linear programming problem is:

$$
\begin{aligned}
x^{\star} & = \arg\min c^{\top} x \\
\text{s.t.} & \left\{ \begin{array}{l} Ax = b \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right.
\end{aligned}
$$

where x is a $n \times 1$ vector, c is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^{-} and x^{+} are two $n \times 1$ vectors.

Advanced optimization problems

Large bond universe

We have:

$$w^{\star} = \arg\min \frac{1}{2}\varphi'_{\mathrm{AS}} \sum_{i=1}^{n} \tau_{i,w} + \varphi'_{\mathrm{MD}} \sum_{j=1}^{n_{\mathcal{S}ector}} \tau_{j,\mathrm{MD}} + \varphi'_{\mathrm{DTS}} \sum_{j=1}^{n_{\mathcal{S}ector}} \tau_{j,\mathrm{DTS}} - \gamma \sum_{i=1}^{n} (w_i - b_i)\, \mathcal{C}_i$$

$$\text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \\ \left| w_i - b_i \right| \leq \tau_{i,w} \\ \left| \sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\, \mathrm{MD}_i \right| \leq \tau_{j,\mathrm{MD}} \\ \left| \sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\, \mathrm{DTS}_i \right| \leq \tau_{j,\mathrm{DTS}} \\ \tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0 \end{array} \right.$$

Advanced optimization problems

Large bond universe

$$\left| w_i - b_i \right| \leq \tau_{i,w} \Leftrightarrow \left\{ \begin{array}{l} w_i - \tau_{i,w} \leq b_i \\ -w_i - \tau_{i,w} \leq -b_i \end{array} \right.$$

Advanced optimization problems

Large bond universe

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \left| \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right| \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} \leq \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{MD}_i \leq \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} + \\
& \qquad \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{MD}_i \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \mathrm{MD}_j^\star \leq (\boldsymbol{s}_j \circ \mathrm{MD})^\top w \leq \tau_{j,\mathrm{MD}} + \mathrm{MD}_j^\star \\
& \Leftrightarrow \quad \left\{ \begin{array}{l} (\boldsymbol{s}_j \circ \mathrm{MD})^\top w - \tau_{j,\mathrm{MD}} \leq \mathrm{MD}_j^\star \\ -(\boldsymbol{s}_j \circ \mathrm{MD})^\top w - \tau_{j,\mathrm{MD}} \leq -\mathrm{MD}_j^\star \end{array} \right.
\end{aligned}
$$

Advanced optimization problems

Large bond universe

$$\left|\sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \Leftrightarrow \left\{ \begin{array}{l} (\boldsymbol{s}_j \circ \mathrm{DTS})^\top w - \tau_{j,\mathrm{DTS}} \leq \mathrm{DTS}_j^\star \\ -(\boldsymbol{s}_j \circ \mathrm{DTS})^\top w - \tau_{j,\mathrm{DTS}} \leq -\mathrm{DTS}_j^\star \end{array} \right.$$

Advanced optimization problems

LP formulation

- x is a vector of dimension $n_x = 2 \times (n + n_{\mathcal{S}ector})$:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The vector c is equal to:

$$c = \begin{pmatrix} -\gamma \mathcal{C} \\ \frac{1}{2} \varphi'_{\mathrm{AS}} \mathbf{1}_n \\ \varphi'_{\mathrm{MD}} \mathbf{1}_{n_{\mathcal{S}ector}} \\ \varphi'_{\mathrm{DTS}} \mathbf{1}_{n_{\mathcal{S}ector}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The linear equality constraint $Ax = B$ is defined by:

$$A = \begin{pmatrix} \mathbf{1}_n^\top & \mathbf{0}_n^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top \end{pmatrix}$$

and:

$$B = 1$$

Advanced optimization problems

LP formulation

- The linear inequality constraint $Cx \leq D$ is defined by:

$$C = \begin{pmatrix} I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ -I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ -C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \\ -C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \end{pmatrix}$$

end:

$$D = \begin{pmatrix} b \\ -b \\ \mathrm{MD}^{\star} \\ -\mathrm{MD}^{\star} \\ \mathrm{DTS}^{\star} \\ -\mathrm{DTS}^{\star} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- C_{MD} and C_{DTS} are two $n_{\mathcal{S}ector} \times n$ matrices, whose elements are:

$$\left(C_{\mathrm{MD}}\right)_{j,i} = \boldsymbol{s}_{i,j}\,\mathrm{MD}_i$$

  and:

$$\left(C_{\mathrm{DTS}}\right)_{j,i} = \boldsymbol{s}_{i,j}\,\mathrm{DTS}_i$$

- We have:

$$\mathrm{MD}^{\star} = \left(\mathrm{MD}_1^{\star}, \ldots, \mathrm{MD}_{n_{\mathcal{S}ector}}^{\star}\right)$$

  and

$$\mathrm{DTS}^{\star} = \left(\mathrm{DTS}_1^{\star}, \ldots, \mathrm{DTS}_{n_{\mathcal{S}ector}}^{\star}\right)$$

Advanced optimization problems

LP formulation

- The bounds are:

$$x^- = \mathbf{0}_{n_x}$$

and:

$$x^+ = \infty \cdot \mathbf{1}_{n_x}$$

Advanced optimization problems

LP formulation

- Additional constraints:

$$\left\{ \begin{array}{l} A'w = B' \\ C'w \leq D' \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \left(\begin{array}{cc} A' & \mathbf{0}_{n_A, n_x - n} \end{array} \right) x = B' \\ \left(\begin{array}{cc} C' & \mathbf{0}_{n_A, n_x - n} \end{array} \right) x \leq D' \end{array} \right.$$

Advanced optimization problems

Large bond universe

Toy example

We consider a toy example with four corporate bonds:

Issuer	#1	#2	#3	#4
b_i (in %)	35	15	20	30
$\mathcal{CI}_i$ (in $\text{tCO}_2\text{e}/\$\text{ mn}$)	117	284	162.5	359
MD_i (in years)	3.0	5.0	2.0	6.0
DTS_i (in bps)	100	150	200	250
$\mathcal{S}ector$	1	1	2	2

We would like to reduce the carbon footprint by 20%, and we set $\varphi'_{\text{AS}} = 100$, $\varphi'_{\text{MD}} = 25$ and $\varphi'_{\text{DTS}} = 1$

Advanced optimization problems

Large bond universe

We have $n = 4$, $n_{\mathcal{S}ector} = 2$ and:

$$x = (\underbrace{w_1, w_2, w_3, w_4}_{w}, \underbrace{\tau_{w_1}, \tau_{w_2}, \tau_{w_3}, \tau_{w_4}}_{\tau_w}, \underbrace{\tau_{\mathrm{MD}_1}, \tau_{\mathrm{MD}_2}}_{\tau_{\mathrm{MD}}}, \underbrace{\tau_{\mathrm{DTS}_1}, \tau_{\mathrm{DTS}_2}}_{\tau_{\mathrm{DTS}}})$$

Since the vector $\mathcal{C}$ is equal to $\mathbf{0}_4$, we obtain:

$$c = (0, 0, 0, 0, 50, 50, 50, 50, 25, 25, 1, 1)$$

Advanced optimization problems

Large bond universe

The equality system $Ax = B$ is defined by:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and:

$$B = 1$$

Advanced optimization problems

Large bond universe

The inequality system $Cx \leq D$ is given by:

$$C = \left(\begin{array}{cccc|c|cccc}
\multicolumn{4}{c|}{I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\
\hline
\multicolumn{4}{c|}{-I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\
\hline
3 & 5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & 2 & 6 & \mathbf{0}_{4,4} & 0 & -1 & 0 & 0 \\
-3 & -5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & -2 & -6 & & 0 & -1 & 0 & 0 \\
\hline
100 & 150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & 200 & 250 & \mathbf{0}_{4,4} & 0 & 0 & 0 & -1 \\
-100 & -150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & -200 & -250 & & 0 & 0 & 0 & -1 \\
\hline
117 & 284 & 162.5 & 359 & \mathbf{0}_{1,4} & 0 & 0 & 0 & 0
\end{array}\right)$$

and:

$$\begin{array}{rcl} D & = & (0.35, 0.15, 0.2, 0.3, \quad -0.35, -0.15, -0.2, -0.3, \ldots \\ & & 1.8, 2.2, -1.8, -2.2, \quad 57.5, 115, -57.5, -115, \quad 179) \end{array}$$

Advanced optimization problems

Large bond universe

- The last row of $Cx \leq D$ corresponds to the carbon footprint constraint
- We have:

$$\mathcal{CI}(b) = 223.75 \text{ tCO}_2\text{e/\$ mn}$$

and:

$$(1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 223.75 = 179.00 \text{ tCO}_2\text{e/\$ mn}$$

Advanced optimization problems

Large bond universe

We solve the LP program, and we obtain the following solution:

$$\begin{aligned} w^{\star} &= (47.34\%, 0\%, 33.3\%, 19.36\%) \\ \tau_{w}^{\star} &= (12.34\%, 15\%, 13.3\%, 10.64\%) \\ \tau_{\mathrm{MD}}^{\star} &= (0.3798, 0.3725) \\ \tau_{\mathrm{DTS}}^{\star} &= (10.1604, 0) \end{aligned}$$

Advanced optimization problems

Large bond universe

- Interpretation of $\tau_w^\star$:

$$w^\star \pm \tau_w^\star = \begin{pmatrix} 47.34\% \\ 0.00\% \\ 33.30\% \\ 19.36\% \end{pmatrix} \begin{pmatrix} - \\ + \\ - \\ + \end{pmatrix} \begin{pmatrix} 12.34\% \\ 15.00\% \\ 13.30\% \\ 10.64\% \end{pmatrix} = \begin{pmatrix} 35\% \\ 15\% \\ 20\% \\ 30\% \end{pmatrix} = b$$

- Interpretation of $\tau_{\mathrm{MD}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \,\mathrm{MD}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \,\mathrm{MD}_i \end{pmatrix} \pm \tau_{\mathrm{MD}}^\star = \begin{pmatrix} 1.42 \\ 1.83 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 0.38 \\ 0.37 \end{pmatrix} = \begin{pmatrix} 1.80 \\ 2.20 \end{pmatrix} = \begin{pmatrix} \mathrm{MD}_1^\star \\ \mathrm{MD}_2^\star \end{pmatrix}$$

- Interpretation of $\tau_{\mathrm{DTS}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \,\mathrm{DTS}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \,\mathrm{DTS}_i \end{pmatrix} \pm \tau_{\mathrm{DTS}}^\star = \begin{pmatrix} 47.34 \\ 115.00 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 10.16 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 57.50 \\ 115.00 \end{pmatrix} = \begin{pmatrix} \mathrm{DTS}_1^\star \\ \mathrm{DTS}_2^\star \end{pmatrix}$$

Advanced optimization problems

Large bond universe

Example #4 (Example #3 again)

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$ and assume that $\varphi'_{\mathrm{AS}} = \varphi_{\mathrm{AS}} = 100$, $\varphi'_{\mathrm{MD}} = \varphi_{\mathrm{MD}} = 25$ and $\varphi'_{\mathrm{DTS}} = \varphi'_{\mathrm{DTS}} = 0.001$

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$, MD_i in years and DTS_i in bps

Advanced optimization problems

Large bond universe

Table 4: Weights in % of optimized bond portfolios (Example #4)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^{\star}$ (10%)	21.70	19.00	16.00	12.00	11.00	8.00	7.46	4.00	0.84
$w^{\star}$ (30%)	34.44	19.00	4.00	11.65	11.98	6.65	7.52	4.00	0.75
$w^{\star}$ (50%)	33.69	19.37	4.00	3.91	24.82	2.00	10.46	1.00	0.75

Table 5: Risk statistics of optimized bond portfolios (Example #4)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e}/\$$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^{\star}$ (10%)	2.16	5.45	297.28	2.16	0.02	7.10	165.95
$w^{\star}$ (30%)	15.95	5.43	300.96	15.95	0.00	13.20	129.07
$w^{\star}$ (50%)	31.34	5.43	268.66	31.34	0.00	65.12	92.19

Equity portfolios

Threshold approach

The optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) \\
\text{s.t.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} w = 1 \\
w \in \Omega \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n \\
\mathcal{CI}(w) \leq (1 - \mathcal{R}) \, \mathcal{CI}(b)
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

Order-statistic approach

- $\mathcal{CI}_{i:n}$ is the order statistics of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$:

$$\min \mathcal{CI}_i = \mathcal{CI}_{1:n} \leq \mathcal{CI}_{2:n} \leq \cdots \leq \mathcal{CI}_{i:n} \leq \cdots \leq \mathcal{CI}_{n:n} = \max \mathcal{CI}_i$$

- The carbon intensity bound $\mathcal{CI}^{(m,n)}$ is defined as:

$$\mathcal{CI}^{(m,n)} = \mathcal{CI}_{n-m+1:n}$$

where $\mathcal{CI}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$

- Exclusion process:

$$\mathcal{CI}_i \geq \mathcal{CI}^{(m,n)} \Rightarrow w_i = 0$$

Equity portfolios

Order-statistic approach (Cont'd)

The optimization problem is:

$$\begin{aligned} w^{\star} & = & \arg\min \frac{1}{2}(w-b)^{\top}\Sigma(w-b) \\ & & \text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ w \in \Omega \\ \mathbf{0}_n \leq w \leq \mathbb{1}\left\{\mathcal{CI} < \mathcal{CI}^{(m,n)}\right\} \end{array} \right. \end{aligned}$$

Equity portfolios

Naive approach

We re-weight the remaining assets:

$$w_i^{\star} = \frac{\mathbb{1}\left\{\mathcal{CI}_i < \mathcal{CI}^{(m,n)}\right\} \cdot b_i}{\sum_{k=1}^{n} \mathbb{1}\left\{\mathcal{CI}_k < \mathcal{CI}^{(m,n)}\right\} \cdot b_k}$$

Equity portfolios

Example #5

We consider a capitalization-weighted equity index, which is composed of eight stocks. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e/\$\ mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the market one-factor model: the beta β_i of each stock is equal to 0.30, 1.80, 0.85, 0.83, 1.47, 0.94, 1.67 and 1.08, the idiosyncratic volatilities $\tilde{\sigma}_i$ are respectively equal to 10%, 5%, 6%, 12%, 15%, 4%, 8% and 7%, and the estimated market volatility σ_m is 18%.

Equity portfolios

The covariance matrix is:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where:

1. β is the vector of beta coefficients
2. σ_m^2 is the variance of the market portfolio
3. $D = diag\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of idiosyncratic variances

Equity portfolios

Table 6: Optimal decarbonization portfolios (Example #5, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.54	21.14	21.86	22.58	22.96	100.5
$w_2^\star$	19.00	19.33	19.29	18.70	18.11	17.23	97.2
$w_3^\star$	17.00	15.67	12.91	8.06	3.22	0.00	250.4
$w_4^\star$	13.00	12.28	10.95	8.74	6.53	3.36	352.3
$w_5^\star$	12.00	12.26	12.60	13.07	13.53	14.08	27.1
$w_6^\star$	8.00	11.71	16.42	22.57	28.73	34.77	54.2
$w_7^\star$	6.00	6.36	6.69	7.00	7.30	7.59	78.6
$w_8^\star$	5.00	1.86	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	30.01	61.90	104.10	149.65	196.87	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

The reduction rate and the weights are expressed in % whereas the tracking error volatility is measured in bps

Equity portfolios

Table 7: Optimal decarbonization portfolios (Example #5, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.40	22.35	26.46	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	19.90	20.07	20.83	7.57	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.94	21.41	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.24	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.12	12.32	12.79	13.04	14.26	18.78	100.00	27.1
$w_6^\star$	8.00	10.04	17.14	32.38	74.66	75.12	81.22	0.00	54.2
$w_7^\star$	6.00	6.37	6.70	7.53	4.73	10.62	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.37	1.68	2.25	3.98	4.04	4.30	15.41	
$\mathcal{CI}(w)$	160.57	145.12	113.48	73.78	55.08	52.93	49.11	27.10	
$\mathcal{R}(w \mid b)$	0.00	9.62	29.33	54.05	65.70	67.04	69.42	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %

Equity portfolios

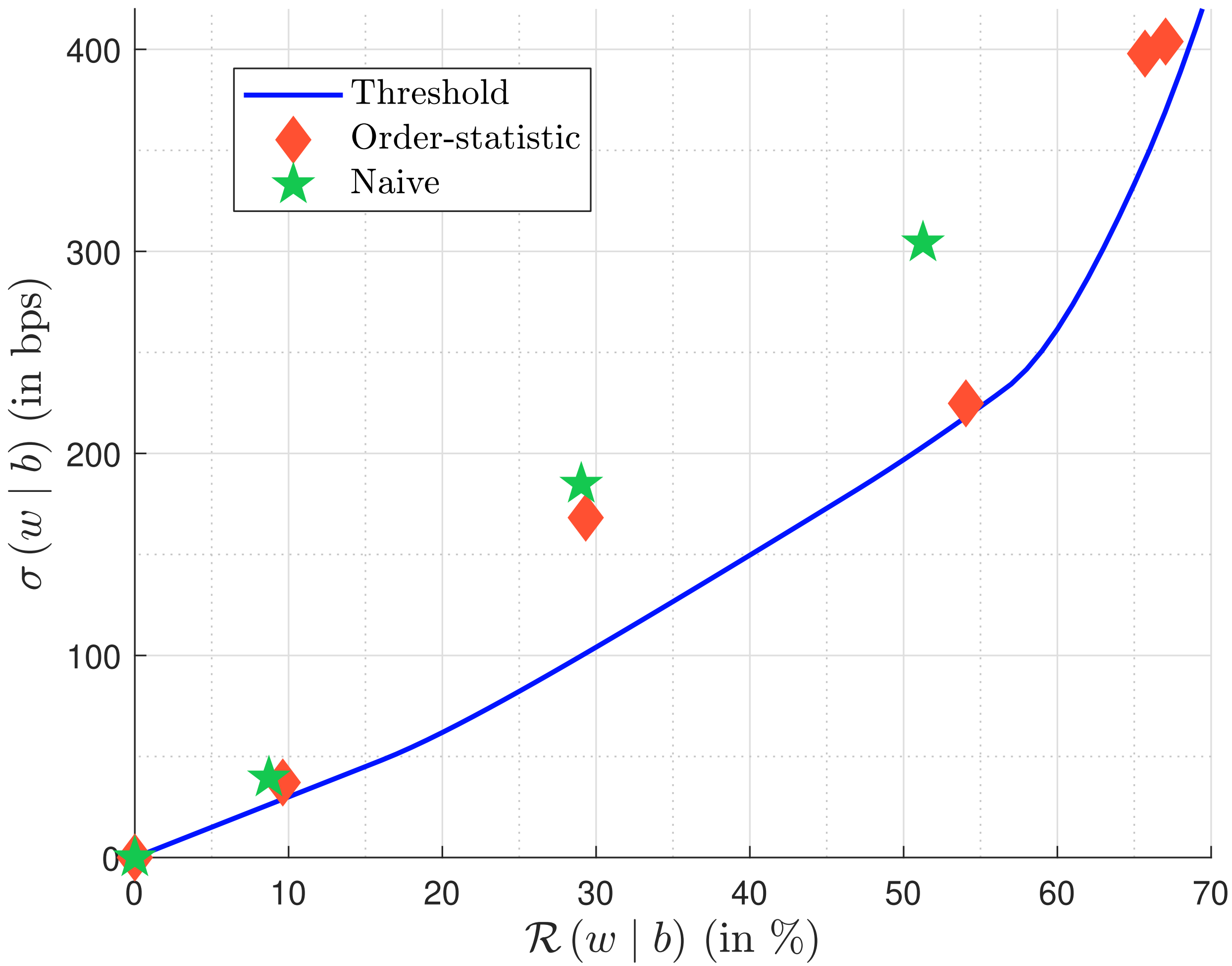
Table 8: Optimal decarbonization portfolios (Example #5, naive approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.05	24.39	30.77	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	20.00	23.17	29.23	42.22	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.89	20.73	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.68	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.63	14.63	18.46	26.67	46.15	60.00	100.00	27.1
$w_6^\star$	8.00	8.42	9.76	12.31	17.78	30.77	40.00	0.00	54.2
$w_7^\star$	6.00	6.32	7.32	9.23	13.33	23.08	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.39	1.85	3.04	9.46	8.08	8.65	15.41	
$\mathcal{CI}(w)$	160.57	146.57	113.95	78.26	68.38	47.32	37.94	27.10	
$\mathcal{R}(w \mid b)$	0.00	8.72	29.04	51.26	57.41	70.53	76.37	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %.

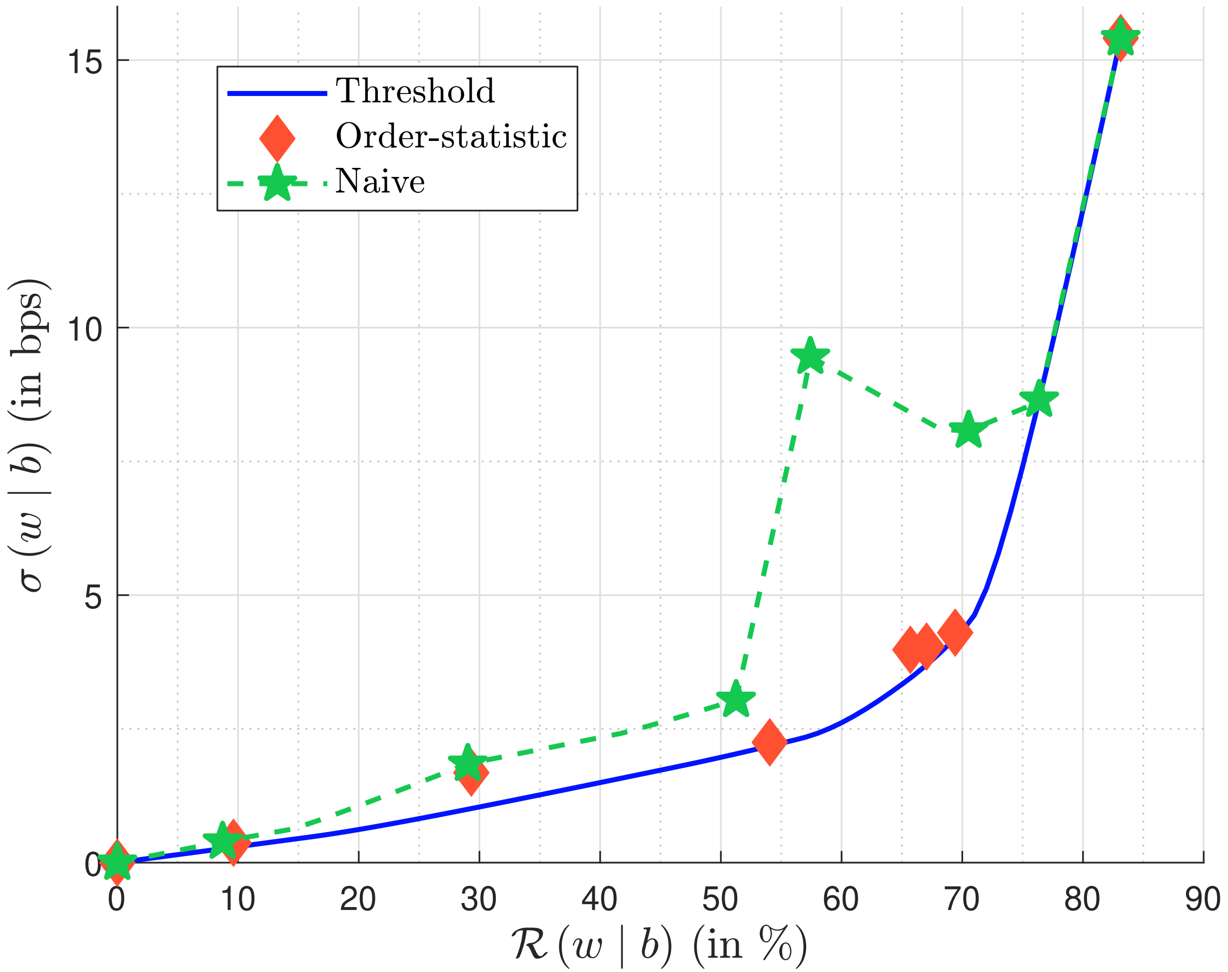
Equity portfolios

Figure 2: Efficient decarbonization frontier (Example #5)



Equity portfolios

Figure 3: Efficient decarbonization frontier of the interpolated naive approach (Example #5)



Bond portfolios

Example #6

We consider a debt-weighted bond index, which is composed of eight bonds. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the modified duration which is respectively equal to 3.1, 6.6, 7.2, 5, 4.7, 2.1, 8.1 and 2.6 years, and the duration-times-spread factor, which is respectively equal to 100, 155, 575, 436, 159, 145, 804 and 365 bps. There are two sectors. Bonds #1, #3, #4 and #8 belong to $\mathcal{S}ector_1$ while Bonds #2, #5, #6 and #7 belong to $\mathcal{S}ector_2$

Bond portfolios

Table 9: Optimal decarbonization portfolios (Example #6, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.62	23.93	26.72	30.08	33.44	100.5
$w_2^\star$	19.00	18.18	16.98	14.18	7.88	1.58	97.2
$w_3^\star$	17.00	18.92	21.94	22.65	16.82	11.00	250.4
$w_4^\star$	13.00	11.34	5.35	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	13.72	16.14	21.63	33.89	46.14	27.1
$w_6^\star$	8.00	9.60	10.47	10.06	7.21	4.36	54.2
$w_7^\star$	6.00	5.56	5.19	4.75	4.11	3.48	78.6
$w_8^\star$	5.00	1.05	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	6.87	15.49	24.07	31.97	47.58	
$\mathrm{MD}(w)$	5.48	5.49	5.45	5.29	4.90	4.51	
$\mathrm{DTS}(w)$	301.05	292.34	282.28	266.12	236.45	206.78	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.57	12.31	19.82	30.04	43.58	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.01	0.04	0.17	0.49	0.81	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	8.99	19.29	35.74	65.88	96.01	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

Bond portfolios

Table 10: Optimal decarbonization portfolios (Example #6, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.83	24.62	64.64	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	18.60	18.13	21.32	3.32	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.79	26.30	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	14.53	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.89	13.96	6.00	36.57	41.27	41.27	100.00	27.1
$w_6^\star$	8.00	9.74	11.85	0.00	60.11	58.73	58.73	0.00	54.2
$w_7^\star$	6.00	5.62	5.15	8.03	0.00	0.00	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	5.78	19.72	49.00	76.68	80.00	80.00	88.00	
$\mathrm{MD}(w)$	5.48	5.52	5.54	4.77	3.27	3.17	3.17	4.70	
$\mathrm{DTS}(w)$	301.05	295.08	284.71	171.82	150.45	150.78	150.78	159.00	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.73	17.94	50.85	66.96	68.63	68.63	95.33	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.03	0.04	0.63	2.66	2.64	2.64	3.21	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	6.21	16.87	128.04	197.22	197.29	197.29	199.22	
$\mathcal{CI}(w)$	160.57	147.94	122.46	93.63	45.72	43.02	43.02	27.10	
$\mathcal{R}(w \mid b)$	0.00	7.87	23.74	41.69	71.53	73.21	73.21	83.12	

Sector-specific constraints

Sector scenario

- Decarbonization scenario per sector:

$$\mathcal{CI}\left(w; \mathcal{S}ector_j\right) \leq \left(1 - \mathcal{R}_j\right) \mathcal{CI}\left(b; \mathcal{S}ector_j\right)$$

- We have:

$$\left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^\star\right)\right)^\top w \leq 0$$

where $\mathcal{CI}_j^\star = \left(1 - \mathcal{R}_j\right) \mathcal{CI}\left(b; \mathcal{S}ector_j\right)$

Sector-specific constraints

Sector scenario

QP form

$$C = \begin{pmatrix} \left(s_1 \circ \left(\mathcal{CI} - \mathcal{CI}_1^\star\right)\right)^\top \\ \vdots \\ \left(s_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^\star\right)\right)^\top \\ \vdots \\ \left(s_{n_{\mathcal{S}ector}} \circ \left(\mathcal{CI} - \mathcal{CI}_{n_{\mathcal{S}ector}}^\star\right)\right)^\top \end{pmatrix}$$

$$D = \begin{pmatrix} \left(1 - \mathcal{R}_1\right) \mathcal{CI}\left(b; \mathcal{S}ector_1\right) \\ \vdots \\ \left(1 - \mathcal{R}_j\right) \mathcal{CI}\left(b; \mathcal{S}ector_j\right) \\ \vdots \\ \left(1 - \mathcal{R}_{n_{\mathcal{S}ector}}\right) \mathcal{CI}\left(b; \mathcal{S}ector_{n_{\mathcal{S}ector}}\right) \end{pmatrix}$$

Sector-specific constraints

Sector scenario

Table 11: Carbon intensity and threshold in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$ per GICS sector (MSCI World, 2030)

Sector	$\mathcal{CI}(b;\mathcal{S}ector_j)$				$\mathcal{R}_j$	$\mathcal{CI}_j^\star$			
	$\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$	(in %)	$\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$
Communication Services	2	28	134	172	52.4	1	13	64	82
Consumer Discretionary	23	65	206	590	52.4	11	31	98	281
Consumer Staples	28	55	401	929	52.4	13	26	191	442
Energy	632	698	1 006	6 823	56.9	272	301	434	2 941
Financials	13	19	52	244	52.4	6	9	25	116
Health Care	10	22	120	146	52.4	5	10	57	70
Industrials	111	130	298	1 662	18.8	90	106	242	1 350
Information Technology	7	23	112	239	52.4	3	11	53	114
Materials	478	702	1 113	2 957	36.7	303	445	704	1 872
Real Estate	22	101	167	571	36.7	14	64	106	361
Utilities	1 744	1 794	2 053	2 840	56.9	752	773	885	1 224
MSCI World	130	163	310	992	36.6	82	103	196	629

Sector-specific constraints

Sector and weight deviation constraints (equity portfolio)

1. Asset weight deviation constraint:

$$\Omega := \mathcal{C}_1\left(m_w^-, m_w^+\right) = \left\{w : m_w^- b \leq w \leq m_w^+ b\right\}$$

2. Sector weight deviation constraint:

$$\Omega := \mathcal{C}_2\left(m_s^-, m_s^+\right) == \left\{\forall j : m_s^- \sum_{i \in \mathcal{S}ector_j} b_i \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq m_s^+ \sum_{i \in \mathcal{S}ector_j} b_i\right\}$$

3. $\mathcal{C}_2\left(m_s\right) = \mathcal{C}_2\left(1/m_s, m_s\right)$
4. $\mathcal{C}_3\left(m_w^-, m_w^+, m_s\right) = \mathcal{C}_1\left(m_w^-, m_w^+\right) \cap \mathcal{C}_2\left(m_s\right)$

Sector-specific constraints

Sector and weight deviation constraints (bond portfolio)

1. Modified duration constraint:

$$\Omega := \mathcal{C}'_1 = \{w : \mathrm{MD}(w) = \mathrm{MD}(b)\} = \left\{w : \sum_{i=1}^{n} (x_i - b_i)\,\mathrm{MD}_i = 0\right\}$$

2. DTS constraint

$$\Omega := \mathcal{C}'_2 = \{w : \mathrm{DTS}(w) = \mathrm{DTS}(b)\} = \left\{w : \sum_{i=1}^{n} (x_i - b_i)\,\mathrm{DTS}_i = 0\right\}$$

3. Maturity/rating buckets:

$$\Omega := \left\{w : \sum_{i \in \mathcal{B}ucket_j} (x_i - b_i) = 0\right\}$$

   1. $\mathcal{C}'_3$: $\mathcal{B}ucket_j$ is the j^{th} maturity bucket, e.g., 0–1, 1–3, 3–5, 5–7, 7–10 and 10+
   2. $\mathcal{C}'_4$: $\mathcal{B}ucket_j$ is the j^{th} rating category, e.g., AAA–AA (AAA, AA+, AA and AA−), A (A+, A and A−) and BBB (BBB+, BBB, BBB−)

Sector-specific constraints

HCIS constraint

Two types of sectors:

1. High climate impact sectors (HCIS):
   "**sectors that are key to the low-carbon transition**" (TEG, 2019)
2. Low climate impact sectors (LCIS)

Let $\mathcal{HCIS}(w) = \sum_{i \in \mathrm{HCIS}} w_i$ be the HCIS weight of portfolio w:

$$\mathcal{HCIS}(w) \geq \mathcal{HCIS}(b)$$

Sector-specific constraints

HCIS constraint

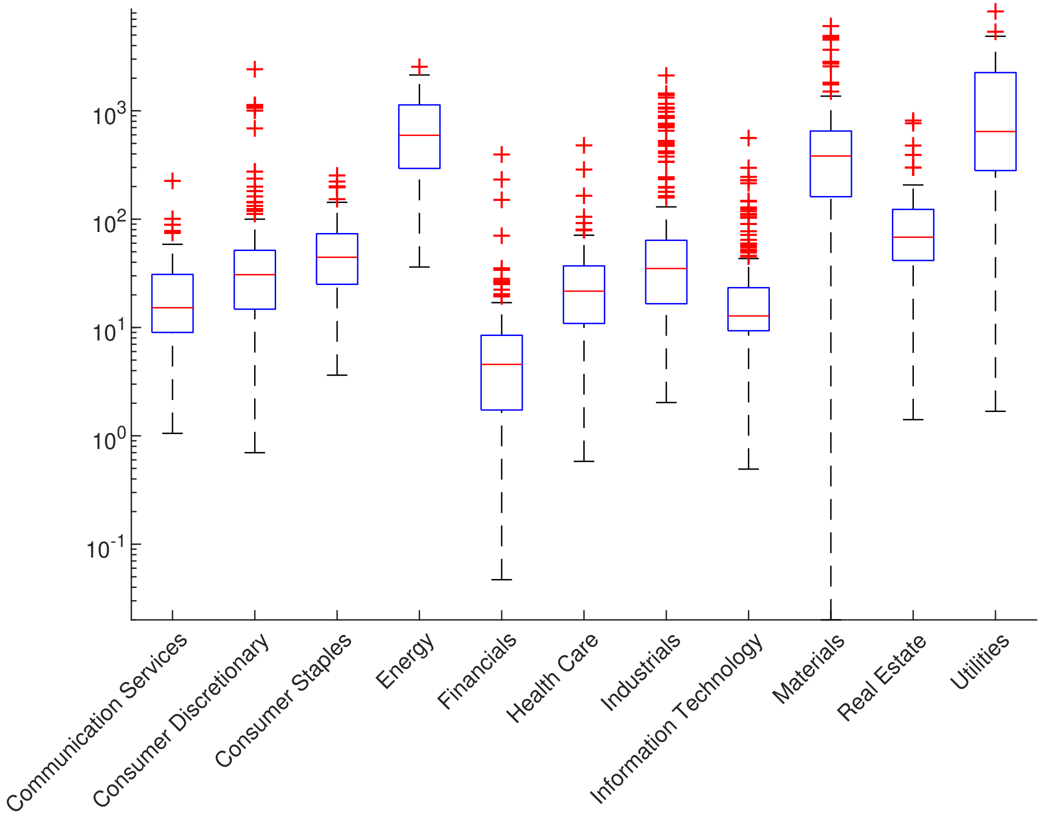
Table 12: Weight and carbon intensity when applying the HCIS filter (MSCI World, June 2022)

Sector	Index b_j	HCIS b'_j	$\mathcal{SC}_1$ $\mathcal{CI}$	$\mathcal{SC}_1$ $\mathcal{CI}'$	$\mathcal{SC}_{1-2}$ $\mathcal{CI}$	$\mathcal{SC}_{1-2}$ $\mathcal{CI}'$	$\mathcal{SC}_{1-3}^{\text{up}}$ $\mathcal{CI}$	$\mathcal{SC}_{1-3}^{\text{up}}$ $\mathcal{CI}'$	$\mathcal{SC}_{1-3}$ $\mathcal{CI}$	$\mathcal{SC}_{1-3}$ $\mathcal{CI}'$
Communication Services	7.58	0.00	2		28		134		172	
Consumer Discretionary	10.56	8.01	23	14	65	31	206	189	590	462
Consumer Staples	7.80	7.80	28	28	55	55	401	401	929	929
Energy	4.99	4.99	632	632	698	698	1 006	1 006	6 823	6 823
Financials	13.56	0.00	13		19		52		244	
Health Care	14.15	9.98	10	13	22	26	120	141	146	177
Industrials	9.90	7.96	111	132	130	151	298	332	1 662	1 921
Information Technology	21.08	10.67	7	12	23	30	112	165	239	390
Materials	4.28	4.28	478	478	702	702	1 113	1 113	2 957	2 957
Real Estate	2.90	2.90	22	22	101	101	167	167	571	571
Utilities	3.21	3.21	1 744	1 744	1 794	1 794	2 053	2 053	2 840	2 840
MSCI World	100.00	59.79	130	210	163	252	310	458	992	1 498

Source: MSCI (2022), Trucost (2022) & Author's calculations

Empirical results (equity portfolios)

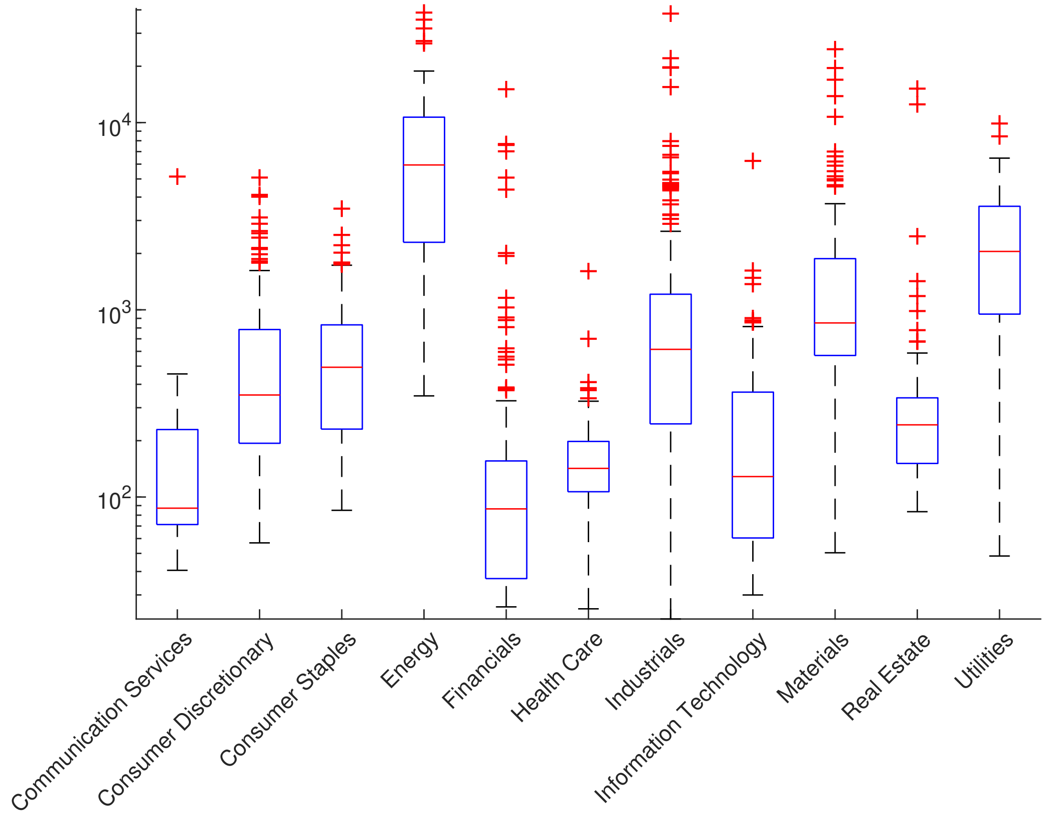
Figure 4: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-2}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results (equity portfolios)

Figure 5: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

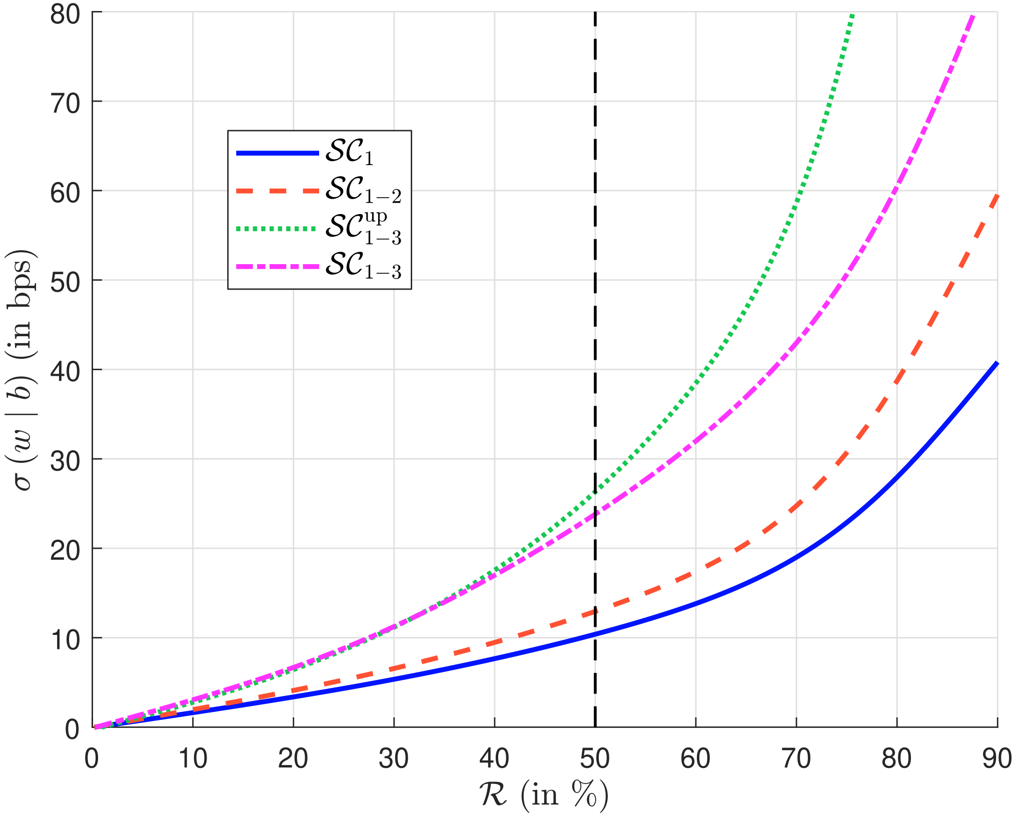
Barahhou *et al.* (2022) consider the basic optimization problem:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) \\ \text{s.t.} & \quad \begin{cases} \mathcal{CI}(w) \leq (1 - \mathcal{R}) \mathcal{CI}(b) \\ w \in \Omega_0 \cap \Omega \end{cases} \end{aligned}$$

What is the impact of constraints $\Omega_0 \cap \Omega$?

Equity portfolios

Figure 6: Impact of the carbon scope on the tracking error volatility (MSCI World, June 2022, $\mathcal{C}_0$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Table 13: Sector allocation in % (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)

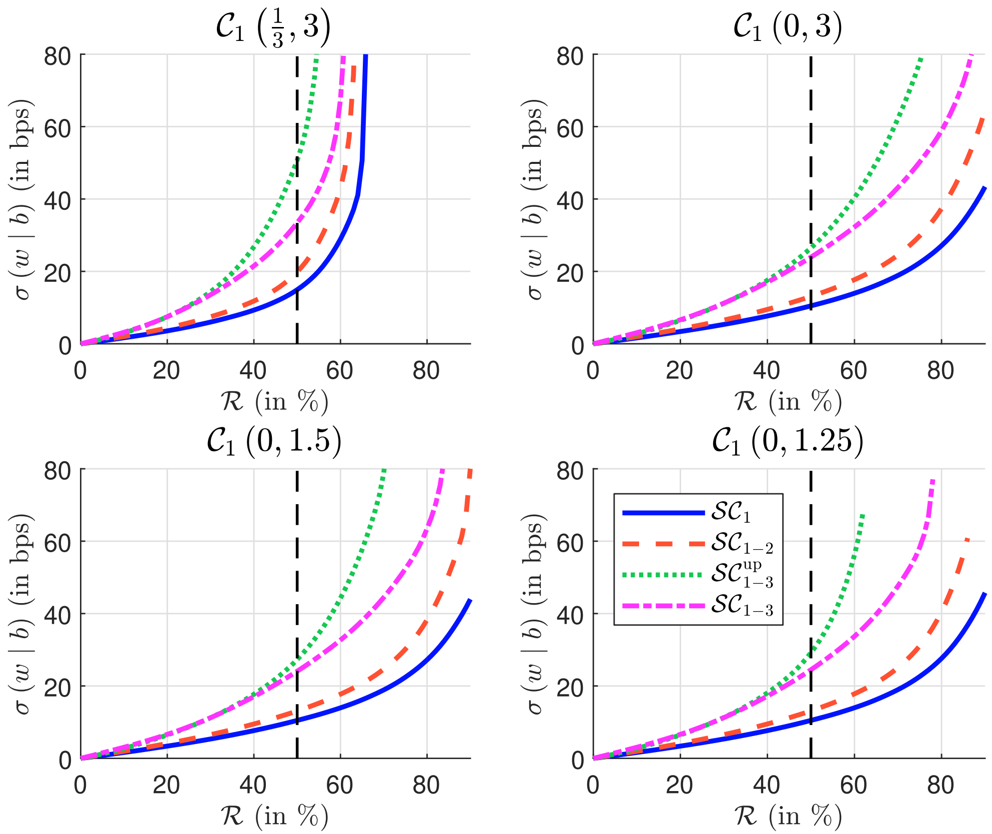
Sector	Index	Reduction rate $\mathcal{R}$						
		30%	40%	50%	60%	70%	80%	90%
Communication Services	7.58	7.95	8.15	8.42	8.78	9.34	10.13	12.27
Consumer Discretionary	10.56	10.69	10.69	10.65	10.52	10.23	9.62	6.74
Consumer Staples	7.80	7.80	7.69	7.48	7.11	6.35	5.03	1.77
Energy	**4.99**	4.14	3.65	3.10	2.45	**1.50**	**0.49**	**0.00**
Financials	**13.56**	14.53	15.17	15.94	16.90	**18.39**	**20.55**	**28.62**
Health Care	14.15	14.74	15.09	15.50	16.00	16.78	17.77	17.69
Industrials	9.90	9.28	9.01	8.71	8.36	7.79	7.21	6.03
Information Technology	21.08	21.68	22.03	22.39	22.88	23.51	24.12	24.02
Materials	**4.28**	3.78	3.46	3.06	2.56	**1.85**	**1.14**	**0.24**
Real Estate	2.90	3.12	3.27	3.41	3.57	3.72	3.71	2.51
Utilities	**3.21**	2.28	1.79	1.36	0.90	**0.54**	**0.24**	**0.12**

Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Portfolio decarbonization = strategy **long on Financials** and **short on Energy, Materials and Utilities**

Equity portfolios

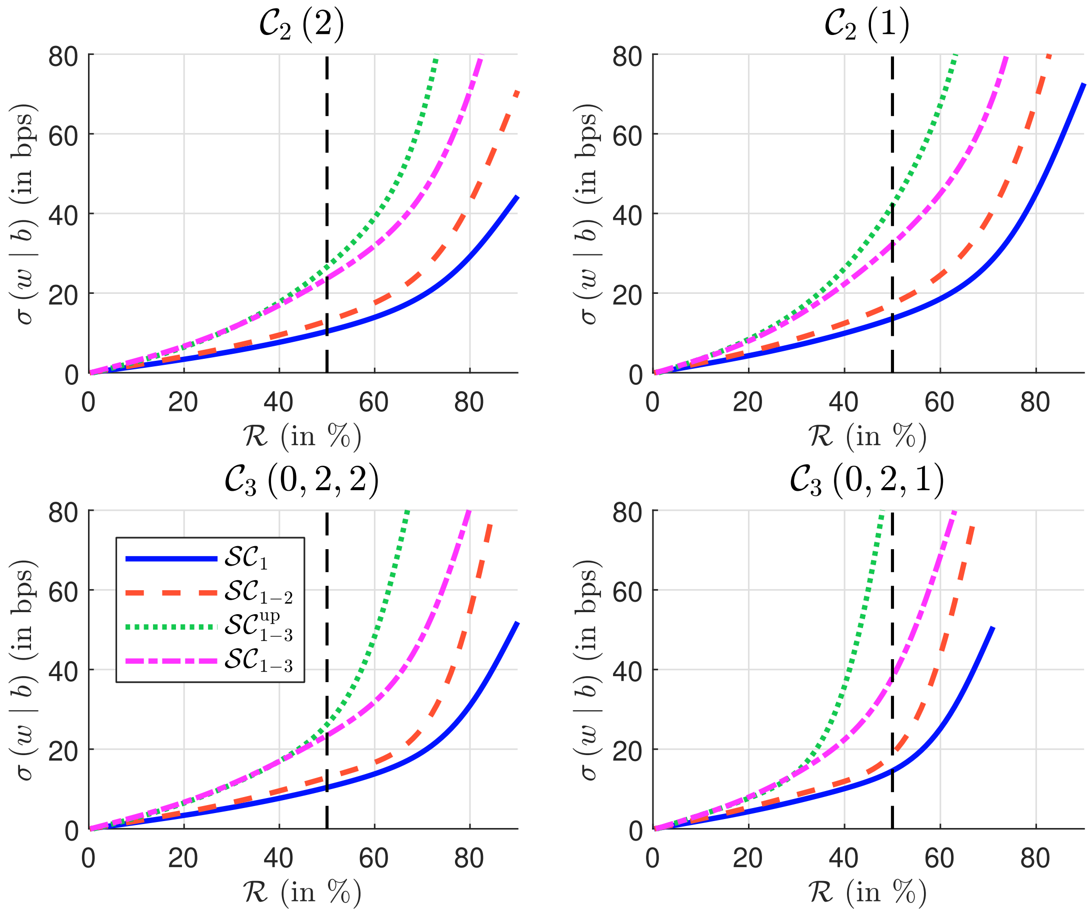
Figure 7: Impact of $\mathcal{C}_1$ constraint on the tracking error volatility (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

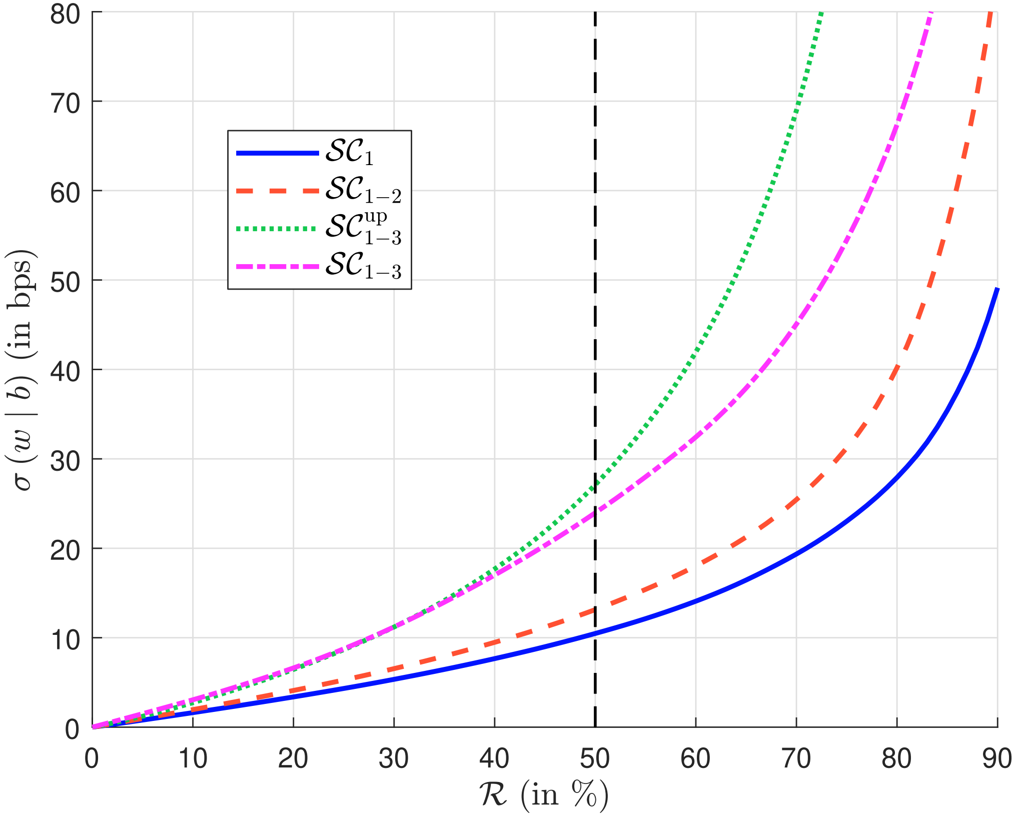
Figure 8: Impact of $\mathcal{C}_2$ and $\mathcal{C}_3$ constraints (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Figure 9: Tracking error volatility with $\mathcal{C}_3(0,10,2)$ constraint (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

First approach

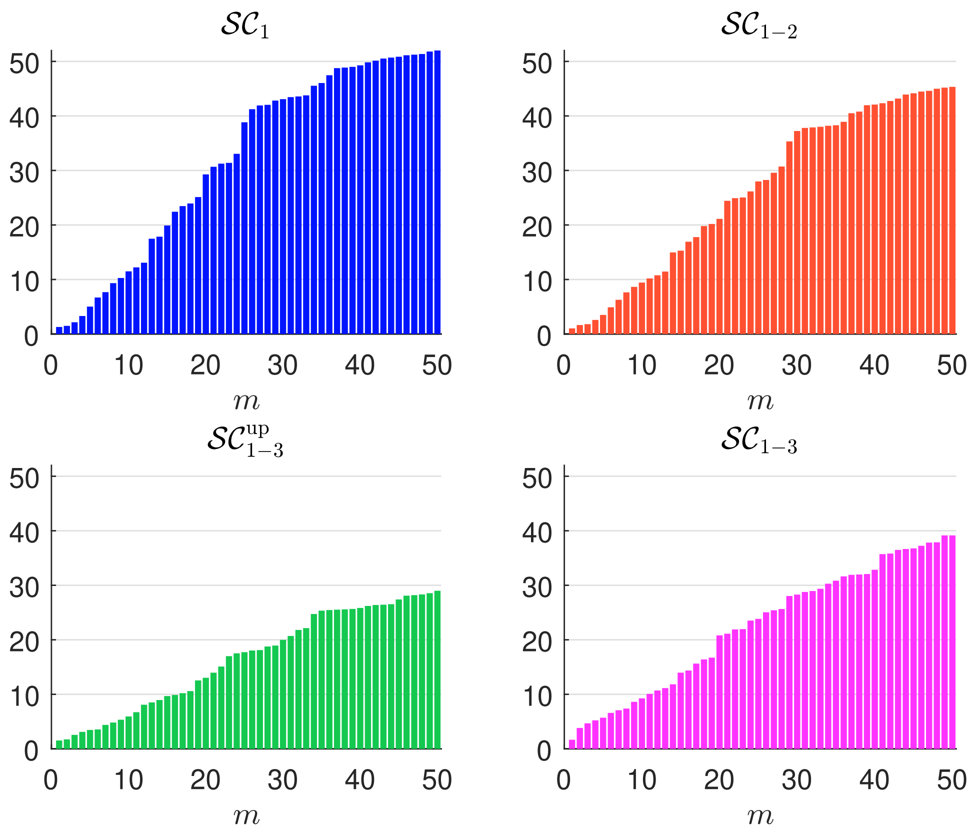
- The carbon footprint contribution of the m worst performing assets is:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CI}_i \geq \mathcal{CI}^{(m,n)}\right\} \cdot b_i \mathcal{CI}_i}{CI(b)}$$

where $\mathcal{CI}^{(m,n)} = \mathcal{CI}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic

Equity portfolios

Figure 10: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, first approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Second approach

- Another definition:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i \boldsymbol{\mathcal{CI}}_i}{CI\left(b\right)}$$

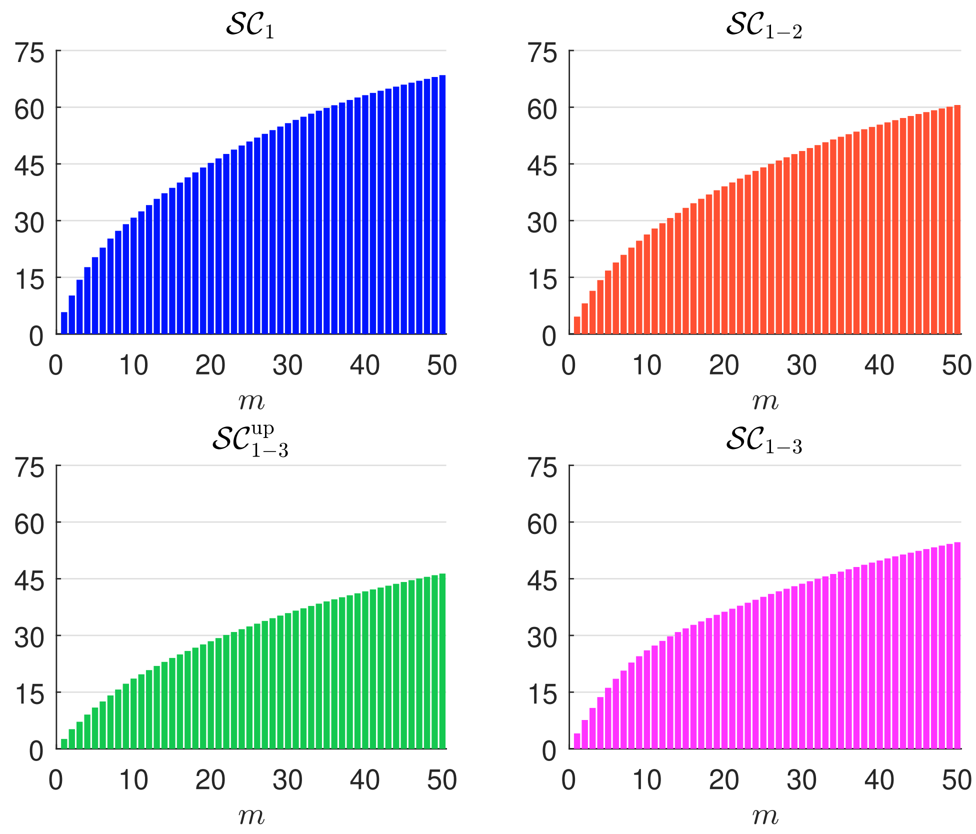
  where $\mathcal{CIC}_i = b_i \boldsymbol{\mathcal{CI}}_i$ and $\mathcal{CIC}^{(m,n)} = \mathcal{CIC}_{n-m+1:n}$

- Weight contribution:

$$\mathcal{WC}^{(m,n)} = \sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i$$

Equity portfolios

Figure 11: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 14: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	m							
	1	5	10	25	50	75	100	200
Communication Services						0.44	0.44	0.73
Consumer Discretionary				0.78	1.37	2.44	2.93	4.28
Consumer Staples		2.46	2.46	2.46	3.75	4.44	4.92	5.62
Energy		9.61	17.35	23.78	29.56	31.78	33.02	33.89
Financials						0.72	1.53	1.88
Health Care							0.21	0.37
Industrials			2.16	5.59	7.13	8.70	9.48	13.05
Information Technology				0.98	1.58	1.94	2.15	3.30
Materials	4.08	4.08	4.08	5.81	7.31	8.81	9.59	10.75
Real Estate					0.77	0.77	0.77	0.85
Utilities				0.81	3.20	3.89	5.24	7.98
Total	4.08	16.15	26.06	40.21	54.66	63.94	70.29	82.70

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 15: Weight contribution $\mathcal{WC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	b_j (in %)	m = 1	5	10	25	50	75	100	200
Communication Services	7.58						0.08	0.08	3.03
Consumer Discretionary	10.56				0.58	1.79	2.44	4.51	5.89
Consumer Staples	7.80		0.70	0.70	0.70	1.90	2.50	2.84	3.84
Energy	4.99		1.71	2.25	2.96	3.62	3.99	4.33	4.65
Financials	13.56						0.74	1.17	2.33
Health Care	14.15							0.95	1.34
Industrials	9.90			0.06	0.32	0.70	0.96	1.20	4.12
Information Technology	21.08				0.16	4.70	8.42	8.78	11.62
Materials	4.28	0.29	0.29	0.29	0.47	0.88	1.10	1.40	1.87
Real Estate	2.90					0.05	0.05	0.05	0.23
Utilities	3.21				0.31	0.86	1.04	1.31	2.33
Total		0.29	2.71	3.30	5.49	14.50	21.32	26.63	41.24

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

- The order-statistic optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} (w-b)^{\top} \Sigma (w-b) \\
&\text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq w^{(m,n)} \end{array} \right.
\end{aligned}
$$

where the upper bound $w^{(m,n)}$ is equal to $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}} < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC} < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

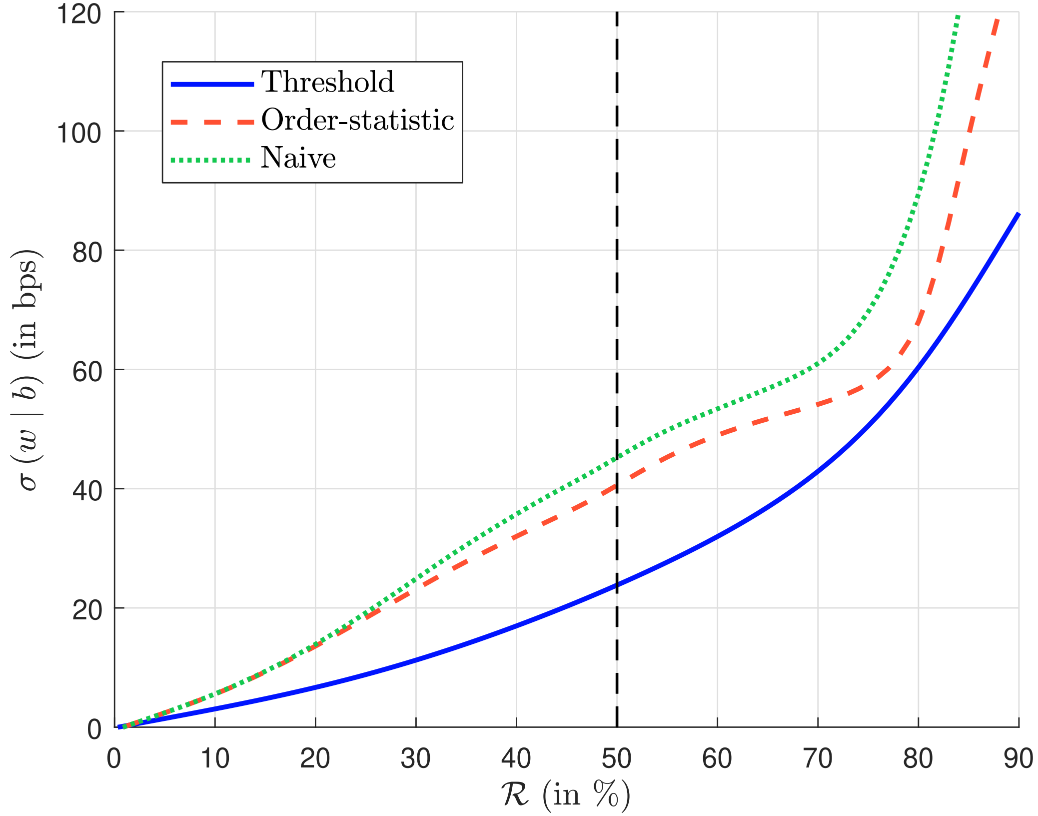
- The naive method is:

$$w_i^\star = \frac{e_i b_i}{\sum_{k=1}^{n} e_k b_k}$$

where e_i is defined as $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}}_i < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC}_i < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

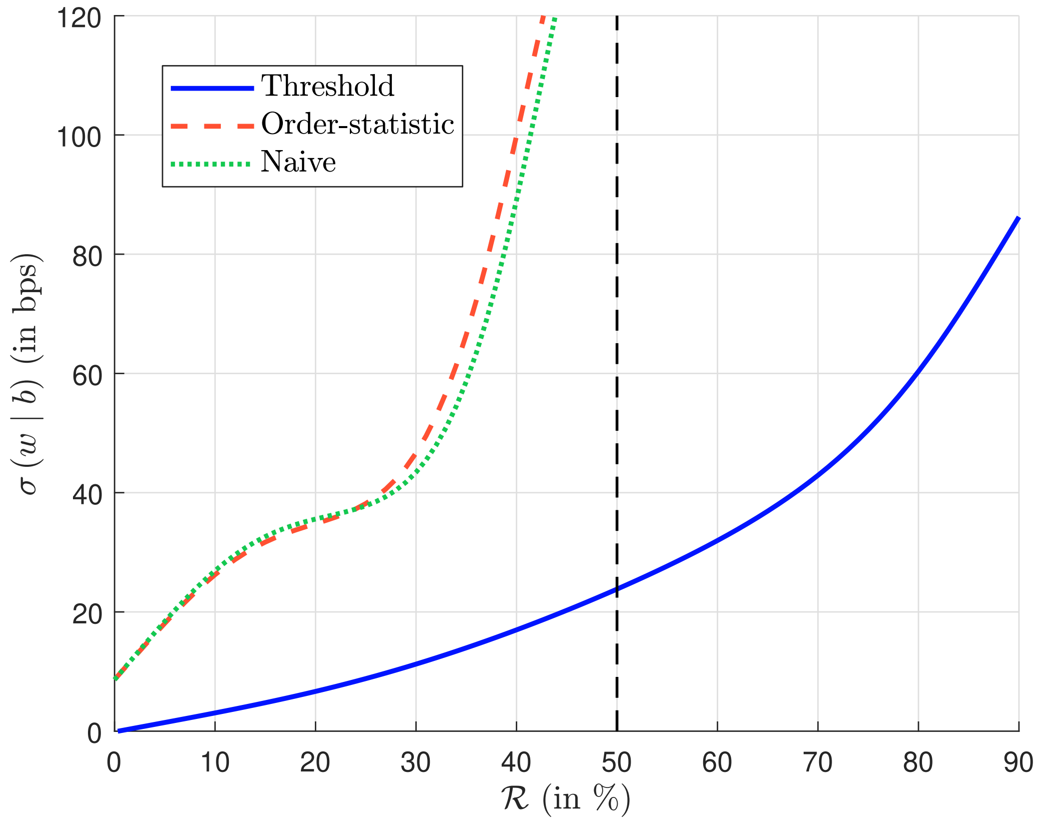
Figure 12: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, first ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Figure 13: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, second ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

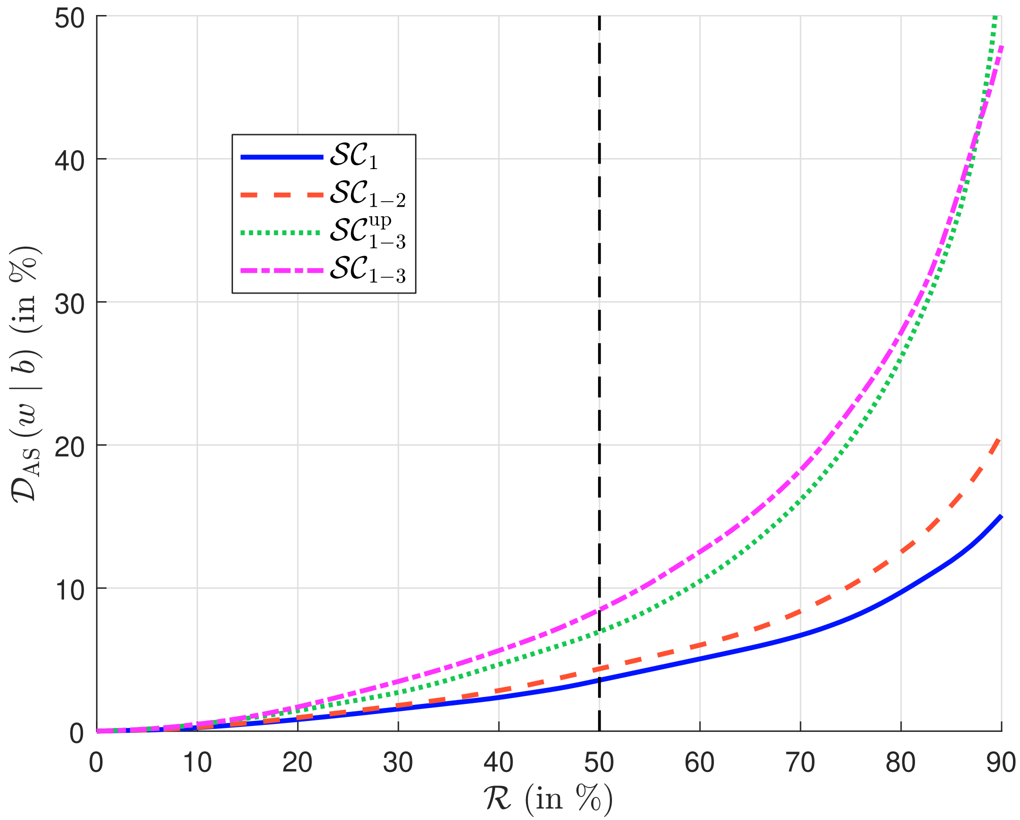
Bond portfolios

The optimization problem is:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\sum_{i=1}^{n}\left|w_i - b_i\right| + 50 \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum_{i \in \mathcal{S}ector_j} \left(w_i - b_i\right) \mathrm{DTS}_i \right| \\ \text{s.t.} & \left\{ \begin{array}{l} \mathcal{CI}\left(w\right) \leq \left(1 - \mathcal{R}\right) \mathcal{CI}\left(b\right) \\ w \in \mathcal{C}_0 \cap \mathcal{C}'_1 \cap \mathcal{C}'_3 \cap \mathcal{C}'_4 \end{array} \right. \end{aligned}$$

Bond portfolios

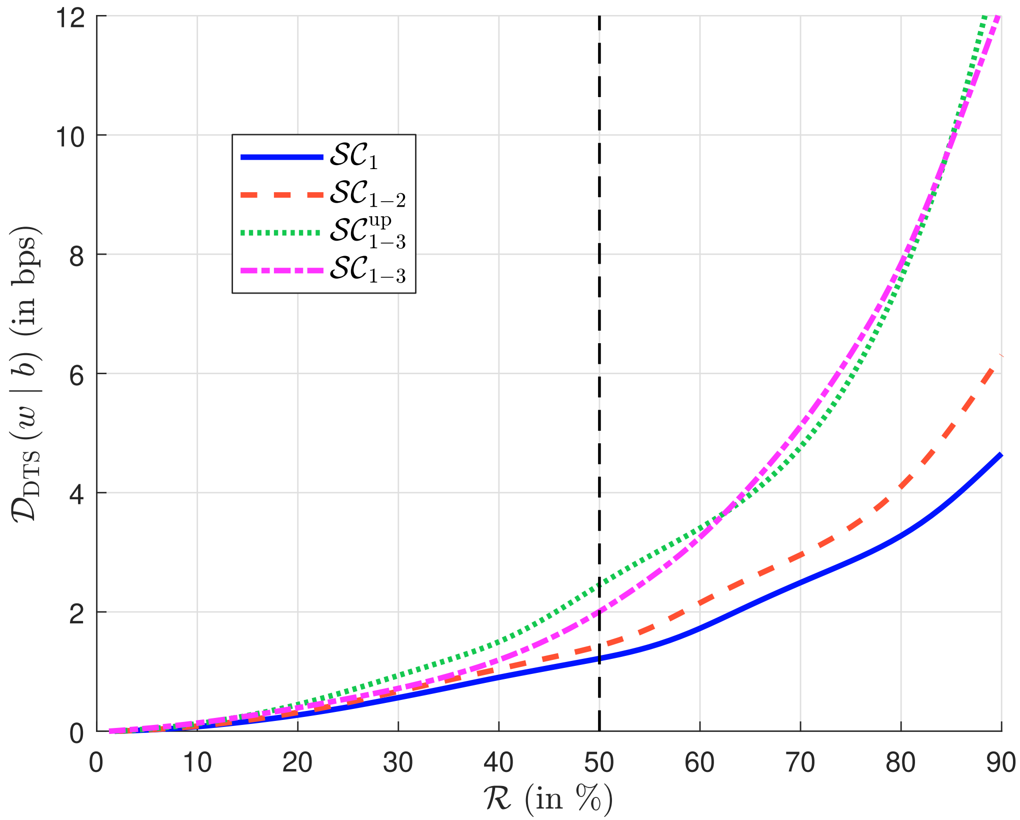
Figure 14: Impact of the carbon scope on the active share in % (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

Figure 15: Impact of the carbon scope on the DTS risk in bps (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

Table 16: Sector allocation in % (ICE Global Corp., June 2022, scope $\mathcal{SC}_{1-3}$)

Sector	Index	Reduction rate $\mathcal{R}$						
		30%	40%	50%	60%	70%	80%	90%
Communication Services	7.34	7.35	7.34	7.37	7.43	7.43	7.31	7.30
Consumer Discretionary	5.97	5.97	5.96	5.94	5.93	5.46	4.48	3.55
Consumer Staples	6.04	6.04	6.04	6.04	6.04	6.02	5.39	4.06
Energy	6.49	5.49	4.42	3.84	3.69	3.23	2.58	2.52
Financials	33.91	34.64	35.66	35.96	36.09	37.36	38.86	39.00
Health Care	7.50	7.50	7.50	7.50	7.50	7.50	7.52	7.48
Industrials	8.92	9.38	9.62	10.19	11.34	12.07	13.55	18.13
Information Technology	5.57	5.57	5.59	5.59	5.60	5.60	5.52	5.27
Materials	3.44	3.43	3.31	3.18	3.12	2.64	2.25	1.86
Real Estate	4.76	4.74	4.74	4.74	4.74	4.66	4.61	3.93
Utilities	10.06	9.89	9.82	9.64	8.52	8.04	7.92	6.88

Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)