Course 2023-2024 in Sustainable Finance
Lecture 13. Exercise — Equity and Bond Portfolio Optimization with Green Preferences

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

We consider an investment universe of 8 issuers. In the table below, we report the carbon emissions $\mathcal{CE}_{i,j}$ (in $\mathrm{ktCO_2e}$) of these companies and their revenues Y_i (in \$ bn), and we indicate in the last row whether the company belongs to sector $\mathcal{S}ector_1$ or $\mathcal{S}ector_2$:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25
$\mathcal{S}ector$	1	2	1	1	2	1	2	2

The benchmark b of this investment universe is defined as:

$$b = (22\%, 19\%, 17\%, 13\%, 11\%, 8\%, 6\%, 4\%)$$

In what follows, we consider long-only portfolios.

Question 1

We want to compute the carbon intensity of the benchmark.

Question (a)

Compute the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes 1, 2 and 3.

We have:

$$\mathcal{CI}_{i,j} = \frac{\mathcal{CE}_{i,j}}{Y_i}$$

For instance, if we consider the 8^{th} issuer, we have[2]:

$$\begin{aligned}
\mathcal{CI}_{8,1} &= \frac{\mathcal{CE}_{8,1}}{Y_8} = \frac{5}{25} = 0.20 \text{ tCO}_2\text{e}/\$ \text{ mn} \\
\mathcal{CI}_{8,2} &= \frac{\mathcal{CE}_{8,2}}{Y_8} = \frac{64}{25} = 2.56 \text{ tCO}_2\text{e}/\$ \text{ mn} \\
\mathcal{CI}_{8,3} &= \frac{\mathcal{CE}_{8,3}}{Y_8} = \frac{199}{25} = 7.96 \text{ tCO}_2\text{e}/\$ \text{ mn}
\end{aligned}$$

[2] Because $1 \text{ ktCO}_2\text{e}/\$$ bn $= 1 \text{ tCO}_2\text{e}/\$ \text{ mn}$.

Since we have:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25

we obtain:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,2}$	0.25	15.24	8.24	3.50	22.50	0.05	18.69	2.56
$\mathcal{CI}_{i,3}$	80.00	45.73	9.68	5.50	2.50	1.83	280.37	7.96

Question (b)

Deduce the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes $1+2$ and $1+2+3$.

We have:

$$\mathcal{CI}_{i,1-2} = \frac{\mathcal{CE}_{i,1} + \mathcal{CE}_{i,2}}{Y_i} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2}$$

and:

$$\mathcal{CI}_{i,1-3} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2} + \mathcal{CI}_{i,3}$$

We deduce that:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,1-2}$	0.50	30.49	14.00	4.00	35.00	0.29	299.07	2.76
$\mathcal{CI}_{i,1-3}$	80.50	76.22	23.68	9.50	37.50	2.12	579.44	10.72

Question (c)

Deduce the weighted average carbon intensity (WACI) of the benchmark if we consider the scope $1 + 2 + 3$.

We have:

$$\begin{aligned}\mathcal{CI}(b) &= \sum_{i=1}^{8} b_i \mathcal{CI}_i \\ &= 0.22 \times 80.50 + 0.19 \times 76.2195 + 0.17 \times 23.68 + 0.13 \times 9.50 + \\ &\quad 0.11 \times 37.50 + 0.08 \times 2.1275 + 0.06 \times 579.4393 + 0.04 \times 10.72 \\ &= 76.9427 \text{ tCO}_2\text{e}/\$ \text{ mn}\end{aligned}$$

Question (d)

We assume that the market capitalization of the benchmark portfolio is equal to \$10 tn and we invest \$1 bn.

Question (d).i

Deduce the market capitalization of each company (expressed in $ bn).

We have:

$$b_i = \frac{MC_i}{\sum_{k=1}^{8} \mathrm{MC}_k}$$

and $\sum_{k=1}^{8} \mathrm{MC}_k = \$10$ tn. We deduce that:

$$\mathrm{MC}_i = 10 \times b_i$$

We obtain the following values of market capitalization expressed in \$ bn:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
MC_i	2 200	1 900	1 700	1 300	1 100	800	600	400

Question (d).ii

Compute the ownership ratio for each asset (expressed in bps).

Let W be the wealth invested in the benchmark portfolio b. The wealth invested in asset i is equal to $b_i W$. We deduce that the ownership ratio is equal to:

$$\varpi_i = \frac{b_i W}{\mathrm{MC}_i} = \frac{b_i W}{b_i \sum_{k=1}^{n} \mathrm{MC}_k} = \frac{W}{\sum_{k=1}^{n} \mathrm{MC}_k}$$

When we invest in a capitalization-weighted portfolio, the ownership ratio is the same for all the assets. In our case, we have:

$$\varpi_i = \frac{1}{10 \times 1000} = 0.01\%$$

The ownership ratio is equal to 1 basis point.

Question (d).iii

Compute the carbon emissions of the benchmark portfolio[a] if we invest \$1 bn and we consider the scope $1 + 2 + 3$.

[a] We assume that the float percentage is equal to 100% for all the 8 companies.

Using the financed emissions approach, the carbon emissions of our investment is equal to:

$$\begin{aligned}\mathcal{CE}\,(\$1\text{ bn}) &= 0.01\% \times (75 + 75 + 24\,000) + \\ &\quad 0.01\% \times (5\,000 + 5\,000 + 15\,000) + \\ &\quad \ldots + \\ &\quad 0.01\% \times (5 + 64 + 199) \\ &= 12.3045 \text{ ktCO}_2\text{e}\end{aligned}$$

Question (d).iv

Compare the (exact) carbon intensity of the benchmark portfolio with the WACI value obtained in Question 1.(c).

We compute the revenues of our investment:

$$Y(\$1 \text{ bn}) = 0.01\% \sum_{i=1}^{8} Y_i = \$0.1287 \text{ bn}$$

We deduce that the exact carbon intensity is equal to:

$$\mathcal{CI}(\$1 \text{ bn}) = \frac{\mathcal{CE}(\$1 \text{ bn})}{Y(\$1 \text{ bn})} = \frac{12.3045}{0.1287} = 95.6061 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

We notice that the WACI of the benchmark underestimates the exact carbon intensity of our investment by 19.5%:

$$76.9427 < 95.6061$$

Question 2

We want to manage an equity portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We assume that the volatility of the stocks is respectively equal to 22%, 20%, 25%, 18%, 40%, 23%, 13% and 29%. The correlation matrix between these stocks is given by:

$$\rho = \begin{pmatrix}
100\% & & & & & & & \\
80\% & 100\% & & & & & & \\
70\% & 75\% & 100\% & & & & & \\
60\% & 65\% & 80\% & 100\% & & & & \\
70\% & 50\% & 70\% & 85\% & 100\% & & & \\
50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\
70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\
60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 60\% & 100\%
\end{pmatrix}$$

Question (a)

Compute the covariance matrix Σ.

The covariance matrix $\Sigma = (\Sigma_{i,j})$ is defined by:

$$\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$$

We obtain the following numerical values (expressed in bps):

$$\Sigma = \begin{pmatrix} 484.0 & 352.0 & 385.0 & 237.6 & 616.0 & 253.0 & 200.2 & 382.8 \\ 352.0 & 400.0 & 375.0 & 234.0 & 400.0 & 276.0 & 130.0 & 377.0 \\ 385.0 & 375.0 & 625.0 & 360.0 & 700.0 & 402.5 & 227.5 & 507.5 \\ 237.6 & 234.0 & 360.0 & 324.0 & 612.0 & 331.2 & 175.5 & 391.5 \\ 616.0 & 400.0 & 700.0 & 612.0 & 1600.0 & 552.0 & 416.0 & 754.0 \\ 253.0 & 276.0 & 402.5 & 331.2 & 552.0 & 529.0 & 149.5 & 466.9 \\ 200.2 & 130.0 & 227.5 & 175.5 & 416.0 & 149.5 & 169.0 & 226.2 \\ 382.8 & 377.0 & 507.5 & 391.5 & 754.0 & 466.9 & 226.2 & 841.0 \end{pmatrix}$$

Question (b)

Write the optimization problem if the objective function is to minimize the tracking error risk under the constraint of carbon intensity reduction.

The tracking error variance of portfolio w with respect to benchmark b is equal to:

$$\sigma^2 (w \mid b) = (w - b)^\top \Sigma (w - b)$$

The carbon intensity constraint has the following expression:

$$\sum_{i=1}^{8} w_i \mathcal{CI}_i \leq (1 - \mathcal{R}) \mathcal{CI}(b)$$

where $\mathcal{R}$ is the reduction rate and $\mathcal{CI}(b)$ is the carbon intensity of the benchmark. Let $\mathcal{CI}^\star = (1 - \mathcal{R}) \mathcal{CI}(b)$ be the target value of the carbon footprint. The optimization problem is then:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} \sigma^2 (w \mid b) \\ \text{s.t.} & \left\{ \begin{array}{l} \sum_{i=1}^{8} w_i \mathcal{CI}_i \leq \mathcal{CI}^\star \\ \sum_{i=1}^{8} w_i = 1 \\ 0 \leq w_i \leq 1 \end{array} \right. \end{aligned}$$

We add the second and third constraints in order to obtain a long-only portfolio.

Question (c)

Give the QP formulation of the optimization problem.

The objective function is equal to:

$$f(w) = \frac{1}{2}\sigma^2(w \mid b) = \frac{1}{2}(w-b)^\top \Sigma (w-b) = \frac{1}{2}w^\top \Sigma w - w^\top \Sigma b + \frac{1}{2}b^\top \Sigma b$$

while the matrix form of the carbon intensity constraint is:

$$\boldsymbol{\mathcal{CI}}^\top w \leq \boldsymbol{\mathcal{CI}}^\star$$

where $\boldsymbol{\mathcal{CI}} = (\boldsymbol{\mathcal{CI}}_1, \ldots, \boldsymbol{\mathcal{CI}}_8)$ is the column vector of carbon intensities. Since $b^\top \Sigma b$ is a constant and does not depend on w, we can cast the previous optimization problem into a QP problem:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2}w^\top Q w - w^\top R \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^- \leq w \leq w^+ \end{array} \right. \end{aligned}$$

We have $Q = \Sigma$, $R = \Sigma b$, $A = \mathbf{1}_8^\top$, $B = 1$, $C = \boldsymbol{\mathcal{CI}}^\top$, $D = \boldsymbol{\mathcal{CI}}^\star$, $w^- = \mathbf{0}_8$ and $w^+ = \mathbf{1}_8$.

Question (d)

$\mathcal{R}$ is equal to 20%. Find the optimal portfolio if we target scope $1+2$. What is the value of the tracking error volatility?

We have:

$$\begin{aligned}\mathcal{CI}(b) &= 0.22 \times 0.50 + 0.19 \times 30.4878 + \ldots + 0.04 \times 2.76 \\ &= 30.7305 \text{ tCO}_2\text{e}/\$ \text{ mn}\end{aligned}$$

We deduce that:

$$\mathcal{CI}^{\star} = (1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 30.7305 = 24.5844 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

Therefore, the inequality constraint of the QP problem is:

$$\begin{pmatrix} 0.50 & 30.49 & 14.00 & 4.00 & 35.00 & 0.29 & 299.07 & 2.76 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_7 \\ w_8 \end{pmatrix} \leq 24.5844$$

We obtain the following optimal solution:

$$w^{\star} = \begin{pmatrix} 23.4961\% \\ 17.8129\% \\ 17.1278\% \\ 15.4643\% \\ 10.4037\% \\ 7.5903\% \\ 4.0946\% \\ 4.0104\% \end{pmatrix}$$

The minimum tracking error volatility $\sigma\left(w^{\star} \mid b\right)$ is equal to 15.37 bps.

Question (e)

Same question if $\mathcal{R}$ is equal to 30%, 50%, and 70%.

Table 1: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-2}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.4961	24.2441	25.7402	30.4117
w_2	19.0000	17.8129	17.2194	16.0323	9.8310
w_3	17.0000	17.1278	17.1917	17.3194	17.8348
w_4	13.0000	15.4643	16.6964	19.1606	23.3934
w_5	11.0000	10.4037	10.1055	9.5091	7.1088
w_6	8.0000	7.5903	7.3854	6.9757	6.7329
w_7	6.0000	4.0946	3.1418	1.2364	0.0000
w_8	4.0000	4.0104	4.0157	4.0261	4.6874
$\mathcal{CI}(w)$	30.7305	24.5844	21.5114	15.3653	9.2192
$\sigma(w \mid b)$	0.00	15.37	23.05	38.42	72.45

In Table 1, we report the optimal solution $w^{\star}$ (expressed in %) of the optimization problem for different values of $\mathcal{R}$. We also indicate the carbon intensity of the portfolio (in $\mathrm{tCO_2e/\$\ mn}$) and the tracking error volatility (in bps). For instance, if $\mathcal{R}$ is set to 50%, the weights of assets #1, #3, #4 and #8 increase whereas the weights of assets #2, #5, #6 and #7 decrease. The carbon intensity of this portfolio is equal to 15.3653 $\mathrm{tCO_2e/\$\ mn}$. The tracking error volatility is below 40 bps, which is relatively low.

Question (f)

We target scope $1+2+3$. Find the optimal portfolio if $\mathcal{R}$ is equal to $20\%, 30\%$, 50% and 70%. Give the value of the tracking error volatility for each optimized portfolio.

In this case, the inequality constraint $Cw \leq D$ is defined by:

$$C = \mathcal{CI}_{1-3}^{\top} = \begin{pmatrix} 80.5000 \\ 76.2195 \\ 23.6800 \\ 9.5000 \\ 37.5000 \\ 2.1275 \\ 579.4393 \\ 10.7200 \end{pmatrix}^{\top}$$

and:

$$D = (1 - \mathcal{R}) \times 76.9427$$

We obtain the results given in Table 2.

Table 2: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-3}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.9666	24.9499	26.4870	13.6749
w_2	19.0000	17.4410	16.6615	8.8001	0.0000
w_3	17.0000	17.1988	17.2981	19.4253	24.1464
w_4	13.0000	16.5034	18.2552	25.8926	41.0535
w_5	11.0000	10.2049	9.8073	7.1330	3.5676
w_6	8.0000	7.4169	7.1254	7.0659	8.8851
w_7	6.0000	3.2641	1.8961	0.0000	0.0000
w_8	4.0000	4.0043	4.0065	5.1961	8.6725
$\mathcal{CI}(w)$	76.9427	61.5541	53.8599	38.4713	23.0828
$\sigma(w \mid b)$	0.00	21.99	32.99	104.81	414.48

Question (g)

Compare the optimal solutions obtained in Questions 2.(e) and 2.(f).

Figure 1: Impact of the scope on the tracking error volatility

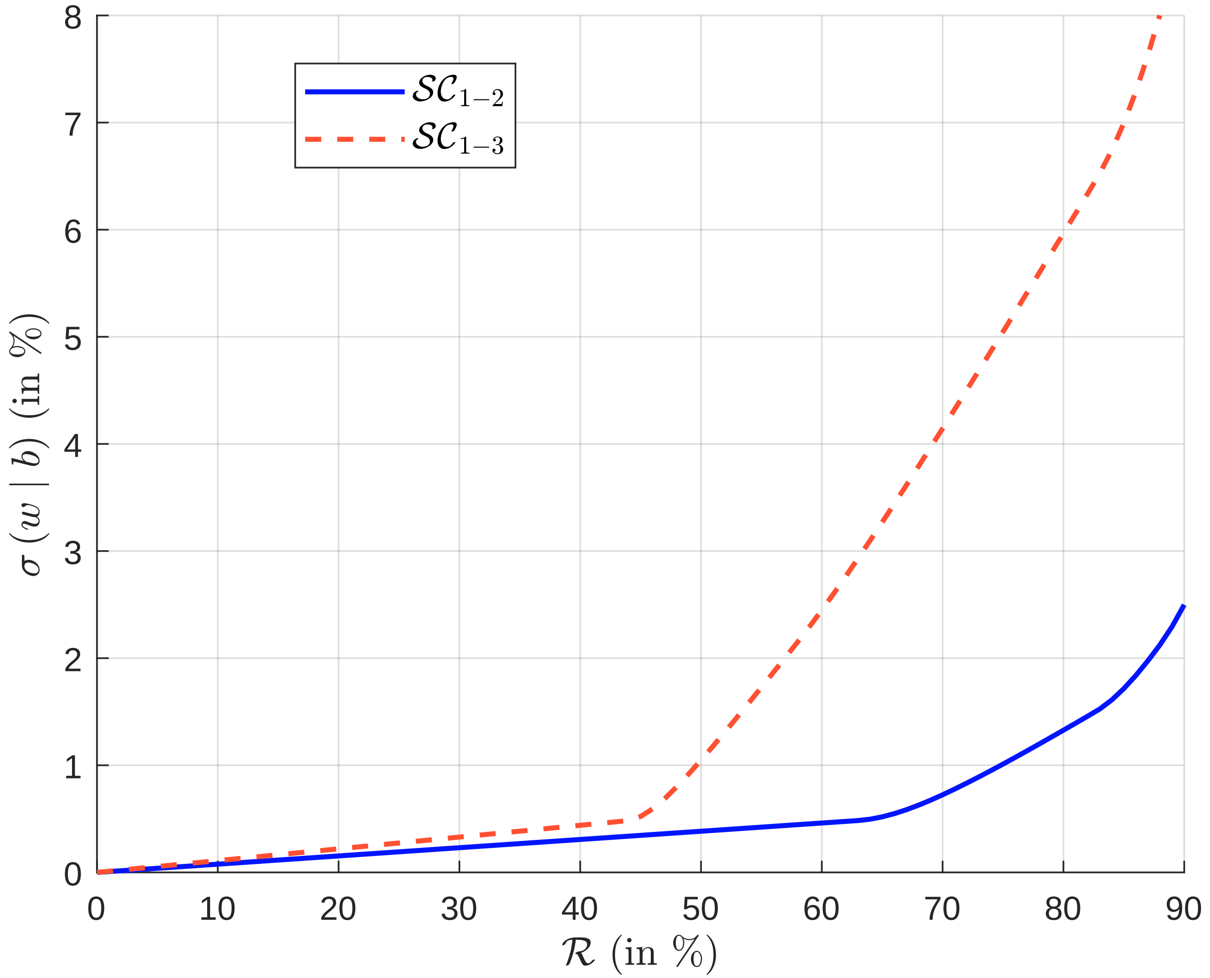
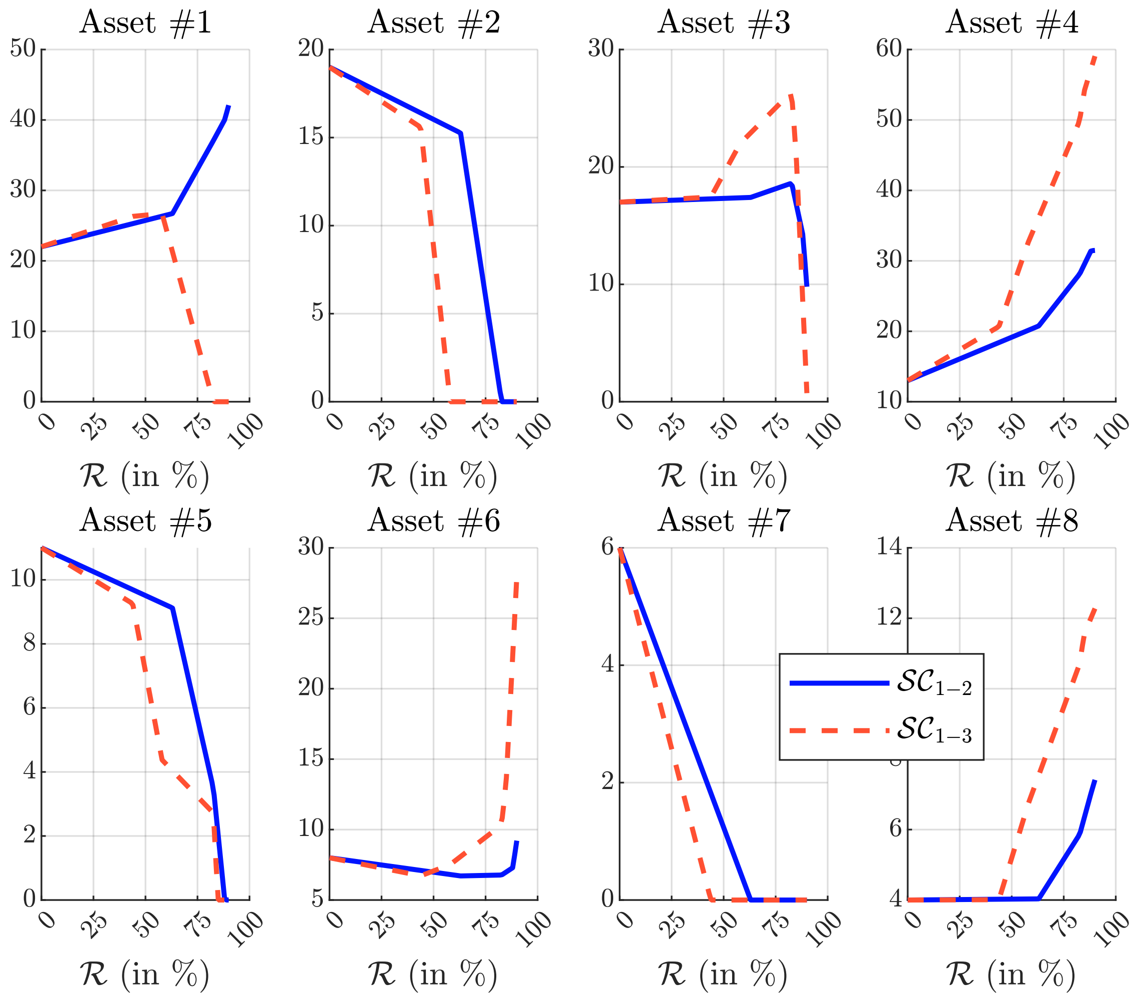


Figure 2: Impact of the scope on the portfolio allocation (in %)



In Figure 1, we report the relationship between the reduction rate $\mathcal{R}$ and the tracking error volatility $\sigma(w \mid b)$. The choice of the scope has little impact when $\mathcal{R} \leq 45\%$. Then, we notice a high increase when we consider the scope $1+2+3$. The portfolio's weights are given in Figure 2. For assets $\#1$ and $\#3$, the behavior is divergent when we compare scopes $1+2$ and $1+2+3$.

Question 3

We want to manage a bond portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We use the scope $1+2+3$. In the table below, we report the modified duration MD_i and the duration-times-spread factor DTS_i of each corporate bond i:

Asset	#1	#2	#3	#4	#5	#6	#7	#8
MD_i (in years)	3.56	7.48	6.54	10.23	2.40	2.30	9.12	7.96
DTS_i (in bps)	103	155	75	796	89	45	320	245
$\mathcal{S}$ector	1	2	1	1	2	1	2	2

Question 3 (Cont'd)

We remind that the active risk can be calculated using three functions. For the active share, we have:

$$\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) = \sigma^2_{\mathrm{AS}}\left(w \mid b\right) = \sum_{i=1}^{n} \left(w_i - b_i\right)^2$$

We also consider the MD-based tracking error risk:

$$\mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) = \sigma^2_{\mathrm{MD}}\left(w \mid b\right) = \sum_{j=1}^{n_{\boldsymbol{\mathcal{S}}ector}} \left(\sum_{i \in \boldsymbol{\mathcal{S}}ector_j} \left(w_i - b_i\right) \mathrm{MD}_i\right)^2$$

and the DTS-based tracking error risk:

$$\mathcal{R}_{\mathrm{DTS}}\left(w \mid b\right) = \sigma^2_{\mathrm{DTS}}\left(w \mid b\right) = \sum_{j=1}^{n_{\boldsymbol{\mathcal{S}}ector}} \left(\sum_{i \in \boldsymbol{\mathcal{S}}ector_j} \left(w_i - b_i\right) \mathrm{DTS}_i\right)^2$$

Question 3 (Cont'd)

Finally, we define the synthetic risk measure as a combination of AS, MD and DTS active risks:

$$\mathcal{R}(w \mid b) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

where $\varphi_{\mathrm{AS}} \geq 0$, $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the weight of each risk. In what follows, we use the following numerical values: $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 1$. The reduction rate $\boldsymbol{\mathcal{R}}$ of the weighted average carbon intensity is set to 50% for the scope $1 + 2 + 3$.

Question (a)

Compute the modified duration $\mathrm{MD}(b)$ and the duration-times-spread factor $\mathrm{DTS}(b)$ of the benchmark.

We have:

$$\begin{aligned}
\mathrm{MD}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{MD}_i \\
&= 0.22 \times 3.56 + 0.19 \times 7.48 + \ldots + 0.04 \times 7.96 \\
&= 5.96 \text{ years}
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{DTS}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{DTS}_i \\
&= 0.22 \times 103 + 0.19 \times 155 + \ldots + 0.04 \times 155 \\
&= 210.73 \text{ bps}
\end{aligned}$$

Question (b)

Let w_{ew} be the equally-weighted portfolio. Compute[a] $\mathrm{MD}\left(w_{\mathrm{ew}}\right)$, $\mathrm{DTS}\left(w_{\mathrm{ew}}\right)$, $\sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right)$.

[a] Precise the corresponding unit (years, bps or %) for each metric.

We have:

$$\begin{cases} \mathrm{MD}\left(w_{\mathrm{ew}}\right) = 6.20 \text{ years} \\ \mathrm{DTS}\left(w_{\mathrm{ew}}\right) = 228.50 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right) = 17.03\% \\ \sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right) = 1.00 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right) = 36.19 \text{ bps} \end{cases}$$

Question (c)

We consider the following optimization problem:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) \\ \text{s.t.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} w_i = 1 \\ \mathrm{MD}\left(w\right) = \mathrm{MD}\left(b\right) \\ \mathrm{DTS}\left(w\right) = \mathrm{DTS}\left(b\right) \\ \boldsymbol{\mathcal{CI}}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{CI}}\left(b\right) \\ 0 \leq w_i \leq 1 \end{array} \right. \end{aligned}$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have:

$$\begin{aligned}\mathcal{R}_{\mathrm{AS}}(w \mid b) &= (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\ & (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2\end{aligned}$$

The objective function is then:

$$f(w) = \frac{1}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b)$$

The optimal solution is equal to:

$$w^\star = (17.30\%, 17.41\%, 20.95\%, 14.41\%, 10.02\%, 11.09\%, 0\%, 8.81\%)$$

The risk metrics are:

$$\left\{\begin{array}{l}\mathrm{MD}(w^\star) = 5.96 \text{ years} \\ \mathrm{DTS}(w^\star) = 210.73 \text{ bps} \\ \sigma_{\mathrm{AS}}(w^\star \mid b) = 10.57\% \\ \sigma_{\mathrm{MD}}(w^\star \mid b) = 0.43 \text{ years} \\ \sigma_{\mathrm{DTS}}(w^\star \mid b) = 15.21 \text{ bps}\end{array}\right.$$

Question (d)

We consider the following optimization problem:

$$
\begin{aligned}
w^\star &= \arg\min \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\sum_{i=1}^{n} w_i = 1 \\
\mathrm{DTS}\left(w\right) = \mathrm{DTS}\left(b\right) \\
\boldsymbol{\mathcal{CI}}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{CI}}\left(b\right) \\
0 \leq w_i \leq 1
\end{array}
\right.
\end{aligned}
$$

Give the analytical value of the objective function. Find the optimal portfolio $w^\star$. Compute $\mathrm{MD}\left(w^\star\right)$, $\mathrm{DTS}\left(w^\star\right)$, $\sigma_{\mathrm{AS}}\left(w^\star \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^\star \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^\star \mid b\right)$.

We have[3]:

$$\begin{aligned}
\mathcal{R}_{\mathrm{MD}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{MD}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{MD}_i\right)^2 \\
&= \left(\sum_{i=1,3,4,6} w_i\,\mathrm{MD}_i - \sum_{i=1,3,4,6} b_i\,\mathrm{MD}_i\right)^2 + \\
&\quad \left(\sum_{i=2,5,7,8} w_i\,\mathrm{MD}_i - \sum_{i=2,5,7,8} b_i\,\mathrm{MD}_i\right)^2 \\
&= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + \\
&\quad (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2
\end{aligned}$$

The objective function is then:

$$f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b)$$

[3] We verify that $3.4089 + 2.5508 = 5.9597$ years.

The optimal solution is equal to:

$$w^{\star} = (16.31\%, 18.44\%, 17.70\%, 13.82\%, 11.67\%, 11.18\%, 0\%, 10.88\%)$$

The risk metrics are:

$$\left\{ \begin{array}{l} \mathrm{MD}\left(w^{\star}\right) = 5.93 \text{ years} \\ \mathrm{DTS}\left(w^{\star}\right) = 210.73 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right) = 11.30\% \\ \sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right) = 0.03 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right) = 3.70 \text{ bps} \end{array} \right.$$

Question (e)

We consider the following optimization problem:

$$w^{\star} = \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right)$$
$$\text{s.t.} \quad \left\{\begin{array}{l} \sum_{i=1}^{n} w_i = 1 \\ \mathcal{CI}\left(w\right) \leq \left(1 - \mathcal{R}\right)\mathcal{CI}\left(b\right) \\ 0 \leq w_i \leq 1 \end{array}\right.$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have[4]:

$$\begin{aligned}\mathcal{R}_{\mathrm{DTS}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 \\ &= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\ &\quad (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2\end{aligned}$$

The objective function is then:

$$f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

[4] We verify that $142.49 + 68.24 = 210.73$ bps.

The optimal solution is equal to:

$$w^{\star} = (16.98\%, 17.21\%, 18.26\%, 13.45\%, 12.10\%, 9.46\%, 0\%, 12.55\%)$$

The risk metrics are:

$$\left\{ \begin{array}{l} \mathrm{MD}\left(w^{\star}\right) = 5.97 \text{ years} \\ \mathrm{DTS}\left(w^{\star}\right) = 210.68 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right) = 11.94\% \\ \sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right) = 0.03 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right) = 0.06 \text{ bps} \end{array} \right.$$

Question (f)

Comment on the results obtained in Questions 3.(c), 3.(d) and 3.(e).

Table 3: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(c)	3.(d)	3.(e)
w_1	22.0000	17.3049	16.3102	16.9797
w_2	19.0000	17.4119	18.4420	17.2101
w_3	17.0000	20.9523	17.6993	18.2582
w_4	13.0000	14.4113	13.8195	13.4494
w_5	11.0000	10.0239	11.6729	12.1008
w_6	8.0000	11.0881	11.1792	9.4553
w_7	6.0000	0.0000	0.0000	0.0000
w_8	4.0000	8.8075	10.8769	12.5464
$\mathrm{MD}(w)$	5.9597	5.9597	5.9344	5.9683
$\mathrm{DTS}(w)$	210.7300	210.7300	210.7300	210.6791
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	10.5726	11.3004	11.9400
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.4338	0.0254	0.0308
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	15.2056	3.7018	0.0561
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713	38.4713

Question (g)

How to find the previous solution of Question 3.(e) using a QP solver?

The goal is to write the objective function into a quadratic function:

$$\begin{aligned} f(w) &= \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b) \\ &= \frac{1}{2}w^\top Q(b)\, w - w^\top R(b) + c(b) \end{aligned}$$

where:

$$\begin{aligned} \mathcal{R}_{\mathrm{AS}}(w \mid b) &= (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\ & \quad (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2 \\ \mathcal{R}_{\mathrm{MD}}(w \mid b) &= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + \\ & \quad (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2 \\ \mathcal{R}_{\mathrm{DTS}}(w \mid b) &= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\ & \quad (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2 \end{aligned}$$

We use the analytical approach which is described in Section 11.1.2 on pages 332-339. Moreover, we rearrange the universe such that the first fourth assets belong to the first sector and the last fourth assets belong to the second sector. In this case, we have:

$$w = \left(\underbrace{w_1, w_3, w_4, w_6}_{\mathcal{S}ector_1}, \underbrace{w_2, w_5, w_7, w_8}_{\mathcal{S}ector_2} \right)$$

The matrix $Q(b)$ is block-diagonal:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{4,4} \\ \mathbf{0}_{4,4} & Q_2 \end{pmatrix}$$

where the matrices Q_1 and Q_2 are equal to:

$$Q_1 = \begin{pmatrix} 11\,025.8400 & 8\,307.0600 & 82\,898.4700 & 4\,839.7000 \\ 8\,307.0600 & 6\,794.2900 & 61\,372.6050 & 3\,751.0500 \\ 82\,898.4700 & 61\,372.6050 & 636\,332.3225 & 36\,408.2250 \\ 4\,839.7000 & 3\,751.0500 & 36\,408.2250 & 2\,257.2500 \end{pmatrix}$$

and:

$$Q_2 = \begin{pmatrix} 25\,523.7600 & 14\,243.8000 & 51\,305.4400 & 39\,463.5200 \\ 14\,243.8000 & 8\,165.0000 & 29\,027.2000 & 22\,282.6000 \\ 51\,305.4400 & 29\,027.2000 & 104\,579.3600 & 80\,214.8800 \\ 39\,463.5200 & 22\,282.6000 & 80\,214.8800 & 61\,709.0400 \end{pmatrix}$$

The vector $R(b)$ is defined as follows:

$$R(b) = \begin{pmatrix} 15\,001.8621 \\ 11\,261.1051 \\ 114\,306.8662 \\ 6\,616.0617 \\ 11\,073.1996 \\ 6\,237.4080 \\ 22\,424.3824 \\ 17\,230.4092 \end{pmatrix}$$

Finally, the value of $c(b)$ is equal to:

$$c(b) = 12\,714.3386$$

Using a QP solver, we obtain the following numerical solution:

$$\begin{pmatrix} w_1 \\ w_3 \\ w_4 \\ w_6 \\ w_2 \\ w_5 \\ w_7 \\ w_8 \end{pmatrix} = \begin{pmatrix} 16.9796 \\ 18.2582 \\ 13.4494 \\ 9.4553 \\ 17.2102 \\ 12.1009 \\ 0.0000 \\ 12.5464 \end{pmatrix} \times 10^{-2}$$

We observe some small differences (after the fifth digit) because the QP solver is more efficient than a traditional nonlinear solver.

Question 4

We consider a variant of Question 3 and assume that the synthetic risk measure is:

$$\mathcal{D}(w \mid b) = \varphi_{\mathrm{AS}}\mathcal{D}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}}\mathcal{D}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}}\mathcal{D}_{\mathrm{DTS}}(w \mid b)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}(w \mid b) &= \frac{1}{2}\sum_{i=1}^{n} |w_i - b_i| \\
\mathcal{D}_{\mathrm{MD}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{MD}_i \right| \\
\mathcal{D}_{\mathrm{DTS}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{DTS}_i \right|
\end{aligned}$$

Question (a)

Define the corresponding optimization problem when the objective is to minimize the active risk and reduce the carbon intensity of the benchmark by $\mathcal{R}$.

The optimization problem is:

$$
\begin{aligned}
w^\star & = \arg\min \mathcal{D}\left(w \mid b\right) \\
\text{s.t.} & \quad \left\{
\begin{array}{l}
\mathbf{1}_8^\top w = 1 \\
\boldsymbol{\mathcal{CI}}^\top w \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \boldsymbol{\mathcal{CI}}\left(b\right) \\
\mathbf{0}_8 \leq w \leq \mathbf{1}_8
\end{array}
\right.
\end{aligned}
$$

Question (b)

Give the LP formulation of the optimization problem.

We use the absolute value trick and obtain the following optimization problem:

$$w^\star = \arg\min \frac{1}{2}\varphi_{\mathrm{AS}}\sum_{i=1}^{8}\tau_{i,w} + \varphi_{\mathrm{MD}}\sum_{j=1}^{2}\tau_{j,\mathrm{MD}} + \varphi_{\mathrm{DTS}}\sum_{j=1}^{2}\tau_{j,\mathrm{DTS}}$$

$$\text{s.t.} \quad \begin{cases} \mathbf{1}_8^\top w = 1 \\ \mathbf{0}_8 \leq w \leq \mathbf{1}_8 \\ \boldsymbol{\mathcal{CI}}^\top w \leq (1-\boldsymbol{\mathcal{R}})\,\boldsymbol{\mathcal{CI}}(b) \\ |w_i - b_i| \leq \tau_{i,w} \\ \left|\sum_{i\in\boldsymbol{\mathcal{S}}ector_j}(w_i - b_i)\,\mathrm{MD}_i\right| \leq \tau_{j,\mathrm{MD}} \\ \left|\sum_{i\in\boldsymbol{\mathcal{S}}ector_j}(w_i - b_i)\,\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \\ \tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0 \end{cases}$$

We can now formulate this problem as a standard LP problem:

$$\begin{aligned} x^{\star} & = & \arg\min c^{\top} x \\ & \text{s.t.} & \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where x is the 20×1 vector defined as follows:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

The 20×1 vector c is equal to:

$$c = \begin{pmatrix} \mathbf{0}_8 \\ \dfrac{1}{2}\varphi_{\mathrm{AS}}\mathbf{1}_8 \\ \varphi_{\mathrm{MD}}\mathbf{1}_2 \\ \varphi_{\mathrm{DTS}}\mathbf{1}_2 \end{pmatrix}$$

The equality constraint is defined by $A = \begin{pmatrix} \mathbf{1}_8^\top & \mathbf{0}_8^\top & \mathbf{0}_2^\top & \mathbf{0}_2^\top \end{pmatrix}$ and $B = 1$. The bounds are $x^- = \mathbf{0}_{20}$ and $x^+ = \infty \cdot \mathbf{1}_{20}$.

For the inequality constraint, we have[5]:

$$Cx \leq D \Leftrightarrow \left(\begin{array}{cccc} I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ -I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ -C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ -C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ \mathcal{CI}^{\top} & \mathbf{0}_{1,8} & 0 & 0 \end{array} \right) x \leq \left(\begin{array}{c} b \\ -b \\ \mathrm{MD}^{\star} \\ -\mathrm{MD}^{\star} \\ \mathrm{DTS}^{\star} \\ -\mathrm{DTS}^{\star} \\ (1-\mathcal{R})\,\mathcal{CI}(b) \end{array} \right)$$

where:

$$C_{\mathrm{MD}} = \left(\begin{array}{cccccccc} 3.56 & 0.00 & 6.54 & 10.23 & 0.00 & 2.30 & 0.00 & 0.00 \\ 0.00 & 7.48 & 0.00 & 0.00 & 2.40 & 0.00 & 9.12 & 7.96 \end{array} \right)$$

and:

$$C_{\mathrm{DTS}} = \left(\begin{array}{cccccccc} 103 & 0 & 75 & 796 & 0 & 45 & 0 & 0 \\ 0 & 155 & 0 & 0 & 89 & 0 & 320 & 245 \end{array} \right)$$

The 2×1 vectors $\mathrm{MD}^{\star}$ and $\mathrm{DTS}^{\star}$ are respectively equal to $(3.4089, 2.5508)$ and $(142.49, 68.24)$.

[5] C is a 25×8 matrix and D is a 25×1 vector.

Question (c)

Find the optimal portfolio when $\mathcal{R}$ is set to 50%. Compare the solution with this obtained in Question 3.(e).

We obtain the following solution:

$$
\begin{aligned}
w^\star &= (18.7360, 15.8657, 17.8575, 13.2589, 11, 9.4622, 0, 13.8196) \times 10^{-2} \\
\tau_w^\star &= (3.2640, 3.1343, 0.8575, 0.2589, 0, 1.4622, 6, 9.8196) \times 10^{-2} \\
\tau_{\mathrm{MD}} &= (0, 0) \\
\tau_{\mathrm{DTS}} &= (0, 0)
\end{aligned}
$$

Table 4: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(e)	4.(c)
w_1	22.0000	16.9796	18.7360
w_2	19.0000	17.2102	15.8657
w_3	17.0000	18.2582	17.8575
w_4	13.0000	13.4494	13.2589
w_5	11.0000	12.1009	11.0000
w_6	8.0000	9.4553	9.4622
w_7	6.0000	0.0000	0.0000
w_8	4.0000	12.5464	13.8196
$\mathrm{MD}(w)$	5.9597	5.9683	5.9597
$\mathrm{DTS}(w)$	210.7300	210.6791	210.7300
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	11.9400	12.4837
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.0308	0.0000
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0561	0.0000
$\mathcal{D}_{\mathrm{AS}}(w \mid b)$	0.0000	25.6203	24.7964
$\mathcal{D}_{\mathrm{MD}}(w \mid b)$	0.0000	0.0426	0.0000
$\mathcal{D}_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0608	0.0000
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713

In Table 4, we compare the two solutions[6]. They are very close. In fact, we notice that the LP solution matches perfectly the MD and DTS constraints, but has a higher AS risk $\sigma_{\mathrm{AS}}(w \mid b)$. If we note the two solutions $w^\star(\mathcal{L}_1)$ and $w^\star(\mathcal{L}_2)$, we have:

$$\left\{\begin{array}{l} \mathcal{R}(w^\star(\mathcal{L}_2) \mid b) = 1.4524 < \mathcal{R}(w^\star(\mathcal{L}_1) \mid b) = 1.5584 \\ \mathcal{D}(w^\star(\mathcal{L}_2) \mid b) = 13.9366 > \mathcal{D}(w^\star(\mathcal{L}_1) \mid b) = 12.3982 \end{array}\right.$$

There is a trade-off between the $\mathcal{L}_1$- and $\mathcal{L}_2$-norm risk measures. This is why we cannot say that one solution dominates the other.

[6]The units are the following: % for the weights w_i, and the active share metrics $\sigma_{\mathrm{AS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{AS}}(w \mid b)$; years for the modified duration metrics $\mathrm{MD}(w)$, $\sigma_{\mathrm{MD}}(w \mid b)$ and $\mathcal{D}_{\mathrm{MD}}(w \mid b)$; bps for the duration-times-spread metrics $\mathrm{DTS}(w)$, $\sigma_{\mathrm{DTS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{DTS}}(w \mid b)$; $\mathrm{tCO_2e/\$\ mn}$ for the carbon intensity $\mathrm{DTS}(w)$.