Course 2022-2023 in Sustainable Finance
Lecture 2. ESG Scoring

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March 2023

[1]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

ESG data

Several issues:

- **E**: climate change mitigation, climate change adaptation, preservation of biodiversity, pollution prevention, circular economy
- **S**: inequality, inclusiveness, labor relations, investment in human capital and communities, human rights
- **G**: management structure, employee relations, executive remuneration

$\Rightarrow$ requires a lot of alternative data

Sovereign ESG data

Sovereign ESG framework

- World Bank
- Data may be download at the following webpage:
  `https://datatopics.worldbank.org/esg/framework.html`
- E: 27 variables
- S: 22 variables
- G: 18 variables

Sovereign ESG data

Table 1: The World Bank database of sovereign ESG indicators

Environmental

- Emissions & pollution (5)
- Natural capital endowment and management (6)
- Energy use & security (7)
- Environment/ climate risk & resilience (6)
- Food security (3)

Social

- Education & skills (3)
- Employment (3)
- Demography (3)
- Poverty & inequality (4)
- Health & nutrition (5)
- Access to services (4)

Governance

- Human rights (2)
- Government effectiveness (2)
- Stability & rule of law (4)
- Economic environment (3)
- Gender (4)
- Innovation (3)

Sovereign ESG data

Table 2: Indicators of the environmental pillar (World Bank database)

- **Emissions & pollution** (1) CO_2 emissions (metric tons per capita); (2) GHG net emissions/removals by LUCF (Mt of CO_2 equivalent); (3) Methane emissions (metric tons of CO_2 equivalent per capita); (4) Nitrous oxide emissions (metric tons of CO_2 equivalent per capita); (5) PM2.5 air pollution, mean annual exposure (micrograms per cubic meter);
- **Natural capital endowment & management**: (1) Adjusted savings: natural resources depletion (% of GNI); (2) Adjusted savings: net forest depletion (% of GNI); (3) Annual freshwater withdrawals, total (% of internal resources); (4) Forest area (% of land area); (5) Mammal species, threatened; (6) Terrestrial and marine protected areas (% of total territorial area);
- **Energy use & security**: (1) Electricity production from coal sources (% of total); (2) Energy imports, net (% of energy use); (3) Energy intensity level of primary energy (MJ/$2011 PPP GDP); (4) Energy use (kg of oil equivalent per capita); (5) Fossil fuel energy consumption (% of total); (6) Renewable electricity output (% of total electricity output); (7) Renewable energy consumption (% of total final energy consumption);
- **Environment/climate risk & resilience**: (1) Cooling degree days (projected change in number of degree Celsius); (2) Droughts, floods, extreme temperatures (% of population, average 1990-2009); (3) Heat Index 35 (projected change in days); (4) Maximum 5-day rainfall, 25-year return level (projected change in mm); (5) Mean drought index (projected change, unitless); (6) Population density (people per sq. km of land area)
- **Food security**: (1) Agricultural land (% of land area); (2) Agriculture, forestry, and fishing, value added (% of GDP); (3) Food production index (2004-2006 = 100);

Source: `https://datatopics.worldbank.org/esg/framework.html`.

Sovereign ESG data

Table 3: Indicators of the social pillar (World Bank database)

- **Education & skills**: (1) Government expenditure on education, total (% of government expenditure); (2) Literacy rate, adult total (% of people ages 15 and above); (3) School enrollment, primary (% gross);
- **Employment**: (1) Children in employment, total (% of children ages 7-14); (2) Labor force participation rate, total (% of total population ages 15-64) (modeled ILO estimate); (3) Unemployment, total (% of total labor force) (modeled ILO estimate);
- **Demography**: (1) Fertility rate, total (births per woman); (2) Life expectancy at birth, total (years); (3) Population ages 65 and above (% of total population);
- **Poverty & inequality**: (1) Annualized average growth rate in per capita real survey mean consumption or income, total population (%); (2) Gini index (World Bank estimate); (3) Income share held by lowest 20%; (4) Poverty headcount ratio at national poverty lines (% of population);
- **Health & nutrition**: (1) Cause of death, by communicable diseases and maternal, prenatal and nutrition conditions (% of total); (2) Hospital beds (per 1,000 people); (3) Mortality rate, under-5 (per 1,000 live births); (4) Prevalence of overweight (% of adults); (5) Prevalence of undernourishment (% of population);
- **Access to services**: (1) Access to clean fuels and technologies for cooking (% of population); (2) Access to electricity (% of population); (3) People using safely managed drinking water services (% of population); (4) People using safely managed sanitation services (% of population);

Source: `https://datatopics.worldbank.org/esg/framework.html`.

Sovereign ESG data

Table 4: Indicators of the governance pillar (World Bank database)

- **Human rights**: (1) Strength of legal rights index (0 = weak to 12 = strong); (2) Voice and accountability (estimate);
- **Government effectiveness**: (1) Government effectiveness (estimate); (2) Regulatory quality (estimate);
- **Stability & rule of law**: (1) Control of corruption (estimate); (2) Net migration; (3) Political stability and absence of violence/terrorism (estimate); (4) Rule of law (estimate)
- **Economic environment**: (1) Ease of doing business index (1 = most business-friendly regulations); (2) GDP growth (annual %); (3) Individuals using the internet (% of population);
- **Gender**: (1) Proportion of seats held by women in national parliaments (%); (2) Ratio of female to male labor force participation rate (%) (modeled ILO estimate); (3) School enrollment, primary and secondary (gross), gender parity index (GPI); (4) Unmet need for contraception (% of married women ages 15-49);
- **Innovation**: (1) Patent applications, residents; (2) Research and development expenditure (% of GDP); (3) Scientific and technical journal articles;

Source: `https://datatopics.worldbank.org/esg/framework.html`.

Where to find the data?

- National accounts statistics collected by OECD, United Nations Statistics Division (UNSD), etc.
- Internal departments and specialized databases of the World Bank: World Bank Open Data, Business Enabling Environment (BEE), Climate Change Knowledge Portal (CCKP), Global Electrification Database (GEP), etc.
- International organizations: Emission Database for Global Atmospheric Research (EDGAR), Food and Agriculture Organization FAO, International Energy Agency (IEA), International Labour Organization (ILO), World Health Organization (WHO), etc.
- NGOs: Climate Watch, etc.;
- Academic resources: International disasters database (EM-DAT) of the Centre for Research on the Epidemiology of Disasters (Université Catholique de Louvain), etc.

Other frameworks

The most known are FTSE (Beyond Ratings), Moody's (Vigeo-Eiris), MSCI, Sustainalytics, RepRisk and Verisk Mapplecroft.

⇒ The average cross-correlation between data providers is equal to 85% for the ESG score, 42% for the environmental score, 85% for the social score and 71% for the governance score

Bias towards richest countries

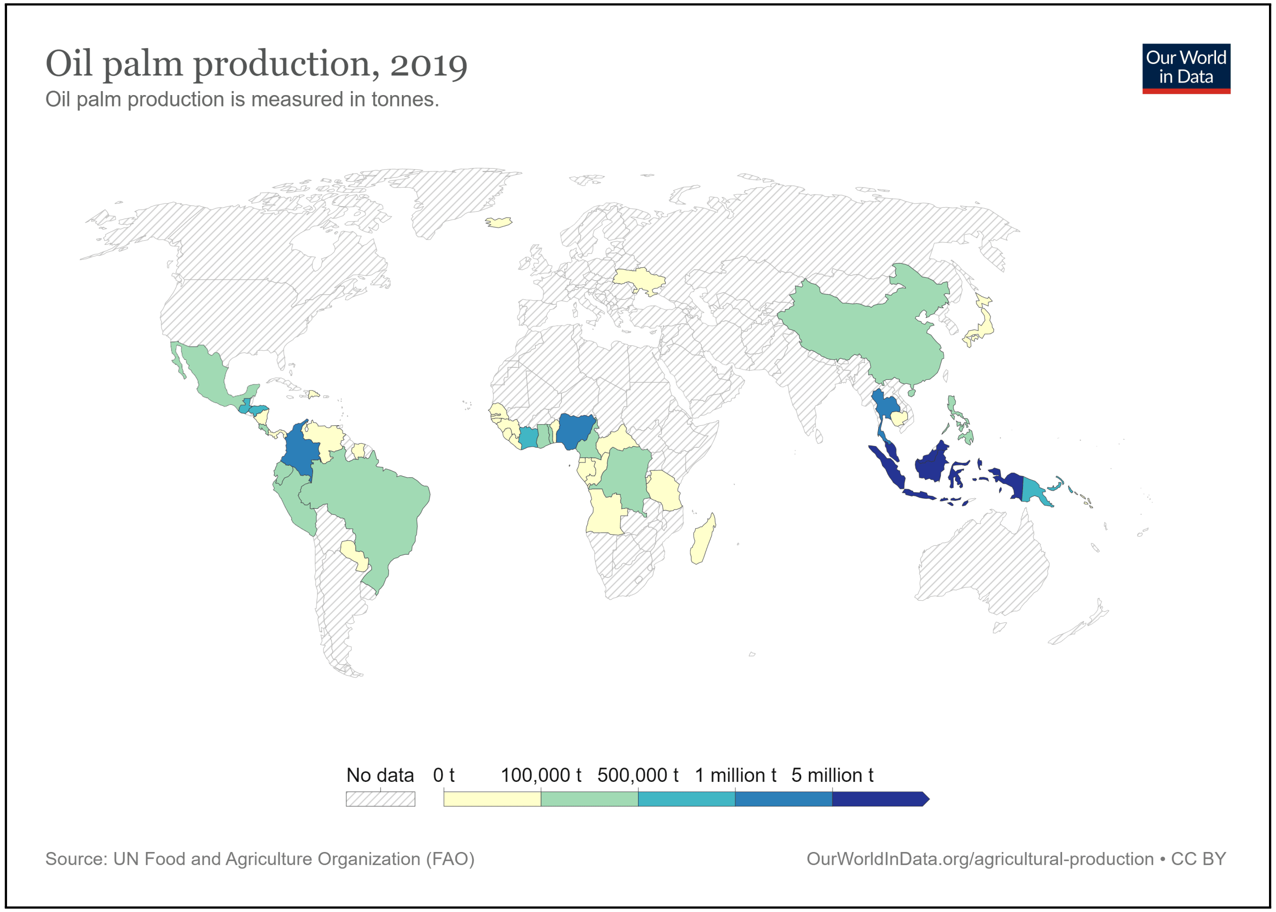
Table 5: Correlation of ESG scores with countrys national income (GNI per capita)

Factor	ESG	E	S	G
ISS	68%	7%	86%	77%
FTSE (Beyond Ratings)	91%	74%	88%	84%
MSCI	84%	10%	90%	77%
RepRisk	78%	79%	75%	37%
RobecoSAM	89%	82%	85%	85%
Sustainalytics	95%	83%	94%	93%
V.E	60%	23%	79%	39%
Total	81%	51%	85%	70%

Source: Gratcheva *et al.* (2020).

The mushrooming growth of data

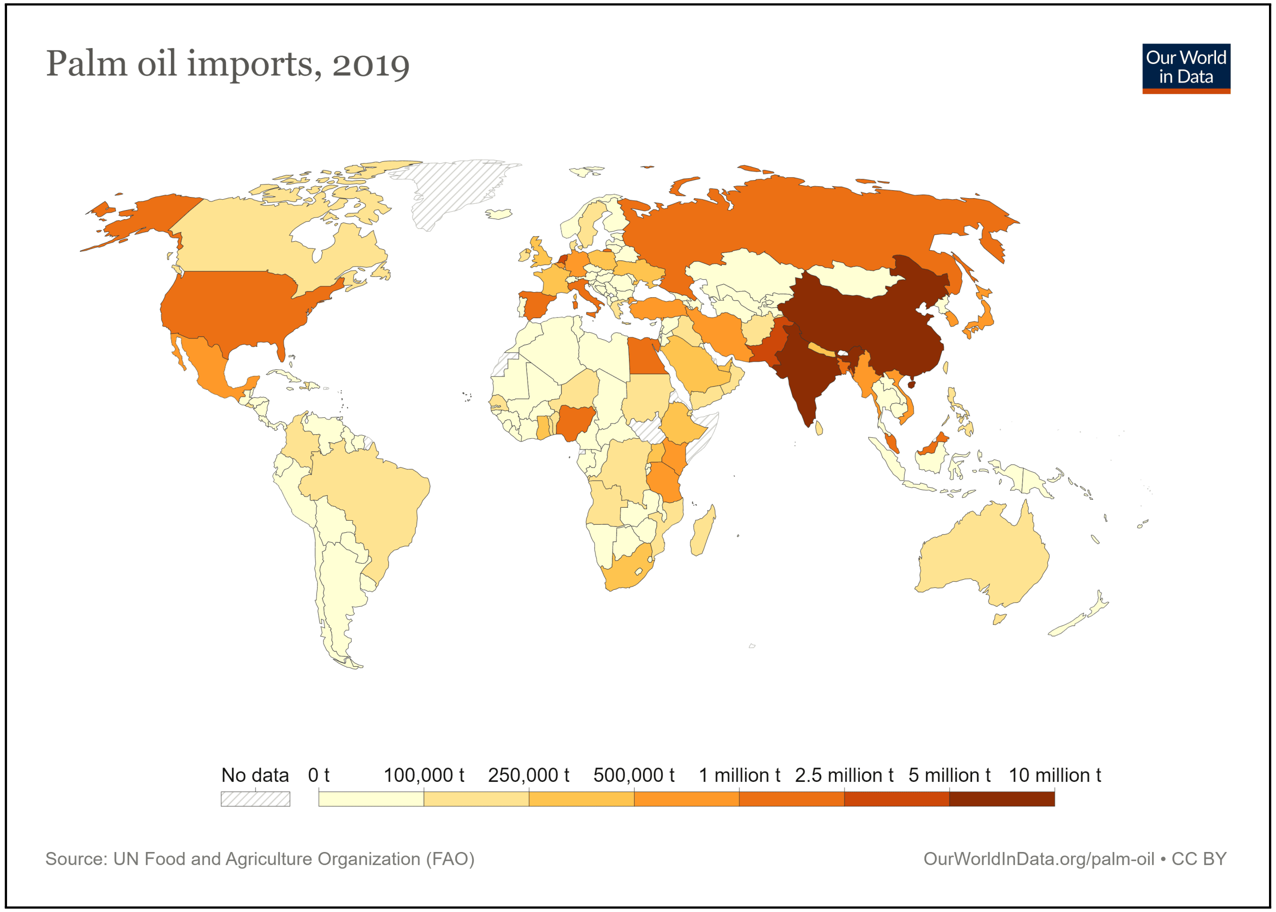
Figure 1: Palm oil production (2019)



Source: Our World in Data, `https://ourworldindata.org/palm-oil`.

The mushrooming growth of data

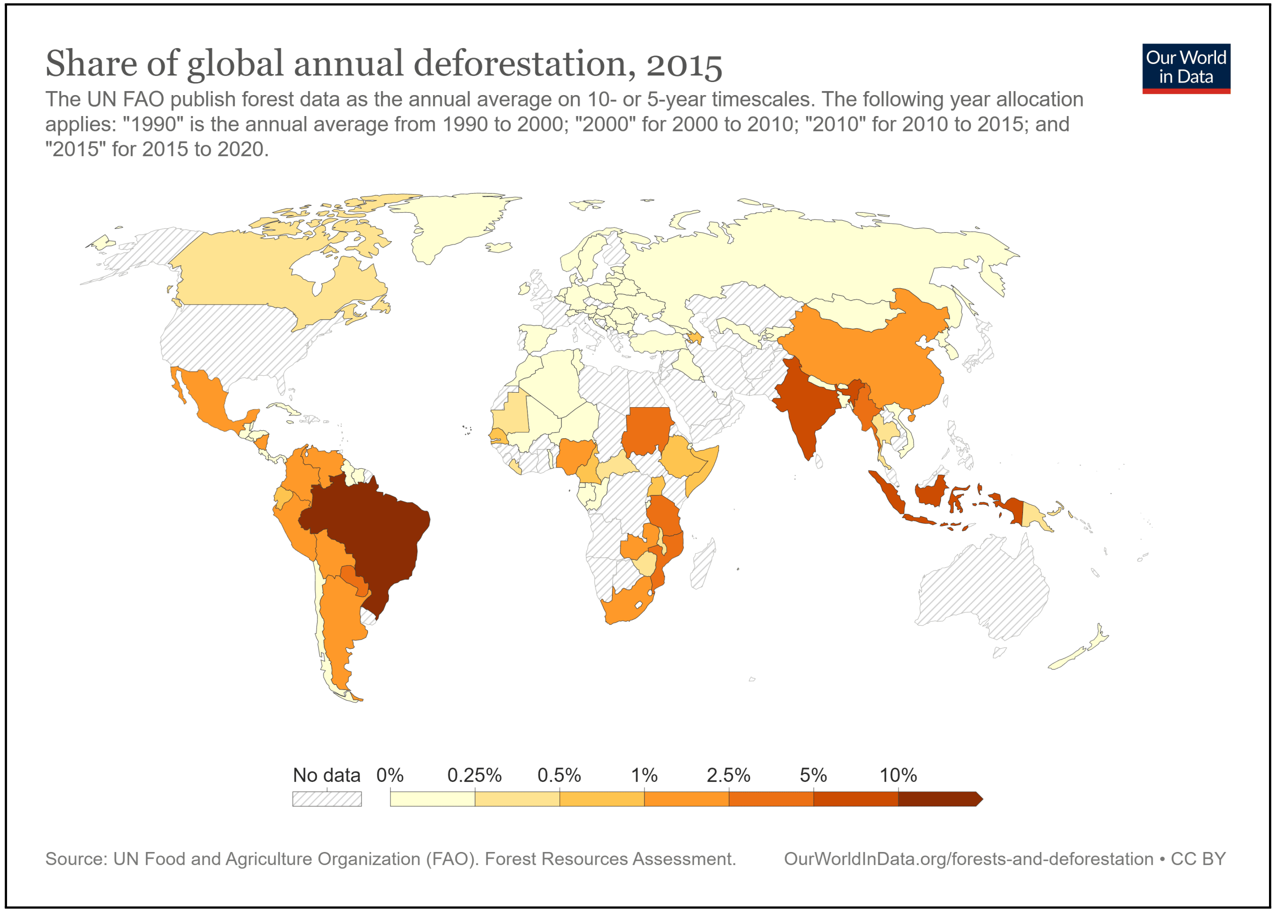
Figure 2: Palm oil imports (2019)



Source: Our World in Data, `https://ourworldindata.org/palm-oil`.

The mushrooming growth of data

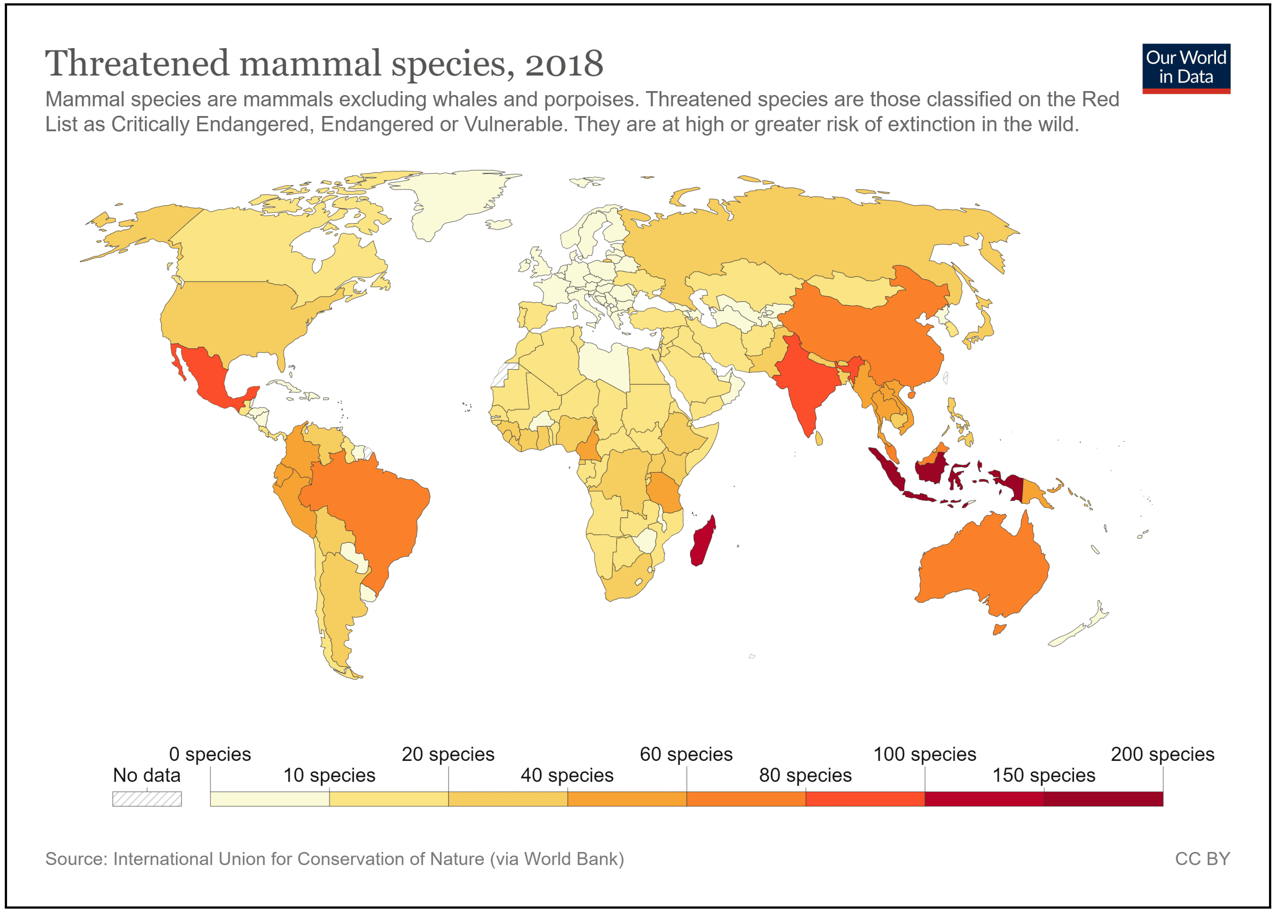
Figure 3: Share of global annual deforestation (2015)



Source: Our World in Data, `https://ourworldindata.org/deforestation`.

The mushrooming growth of data

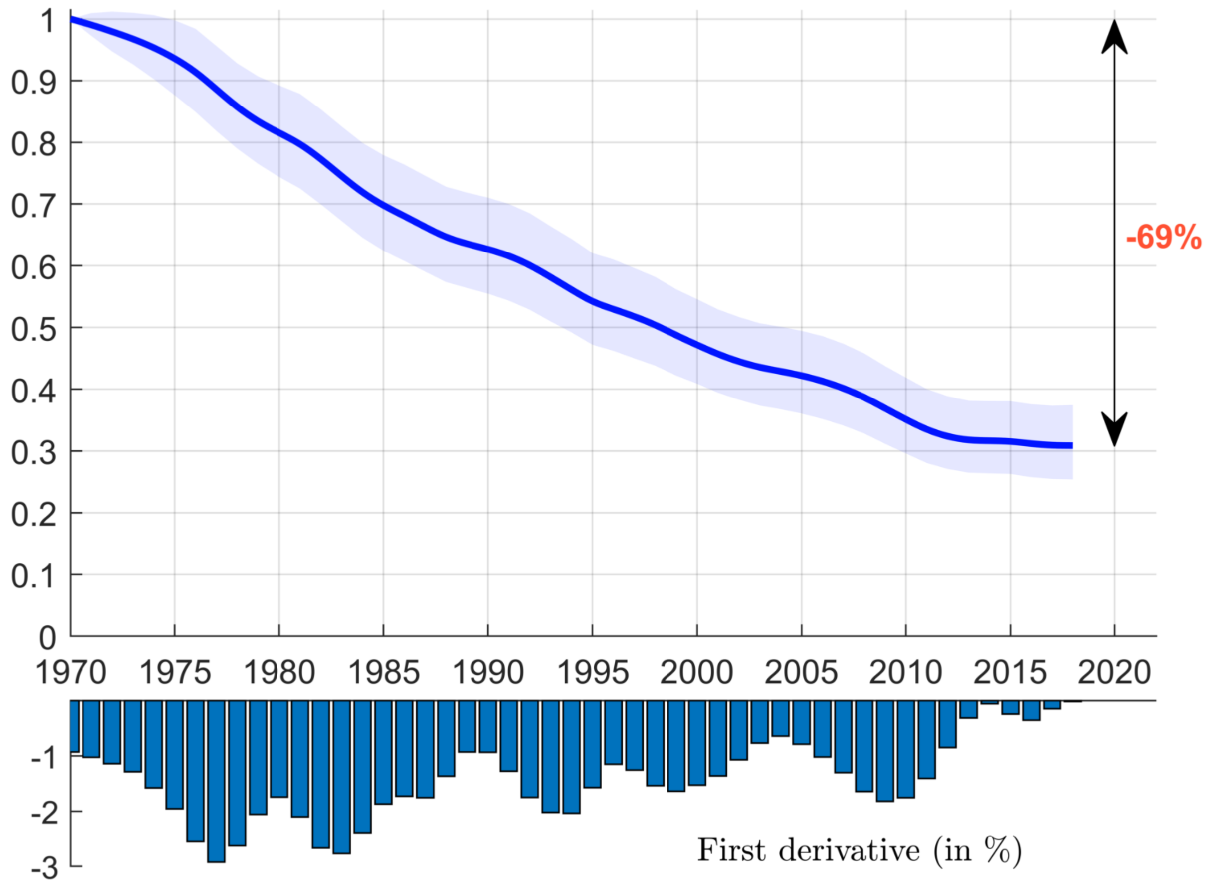
Figure 4: Threatened mammal species (2018)



Source: Our World in Data, `https://ourworldindata.org/biodiversity`.

An example with the biodiversity risk

Figure 5: Global living planet index



Source: `https://livingplanetindex.org/latest_results` & Author's calculation.

An example with the biodiversity risk

Some databases:

- the Red List Index (RLI)
- World Database on Protected Areas (WDPA)
- Integrated Biodiversity Assessment Tool (IBAT)
- Exploring Natural Capital Opportunities, Risks and Exposure (ENCORE)
- Etc.

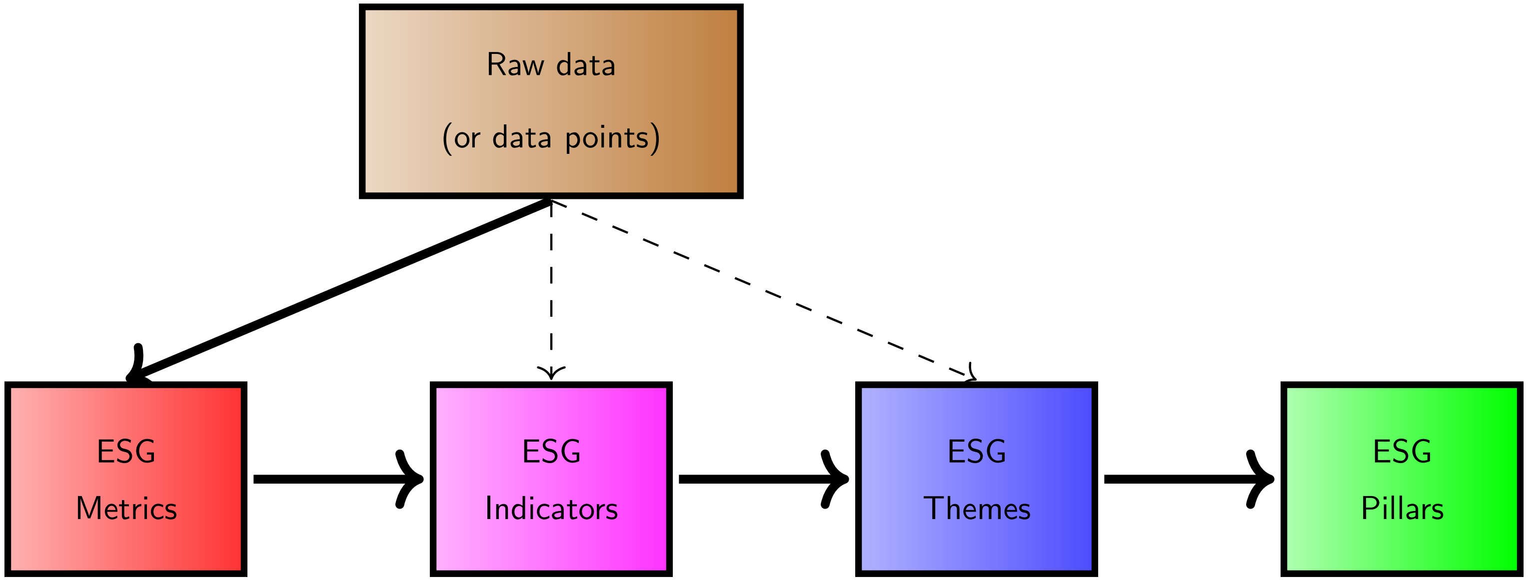
Corporate ESG data

Data sources:

1. Corporate publications (self-reporting)
   1. Annual reports
   2. Corporate sustainability reports
2. Financial and regulatory filings (standardized reporting)
   1. Mandatory reports (SFDR, CSRD, EUTR, etc.)
   2. Non-mandatory frameworks (PRI, TCFD, CDP, etc.)
3. News and other media
4. NGO reports and websites
5. Company assessment and due diligence questionnaire (DDQ)
6. Internal models

Corporate ESG data

Figure 6: From raw data to ESG pillars



Corporate ESG data

Table 6: An example of ESG criteria (corporate issuers)

Environmental	Social	Governance
• Carbon emissions	• Employment conditions	• Board independence
• Energy use	• Community involvement	• Corporate behaviour
• Pollution	• Gender equality	• Audit and control
• Waste disposal	• Diversity	• Executive compensation
• Water use	• Stakeholder opposition	• Shareholder' rights
• Renewable energy	• Access to medicine	• CSR strategy
• Green cars*		
• Green financing*		

$^{(\star)}$means a specific criterion related to one or several sectors (Green cars ⇒ Automobiles, Green financing ⇒ Financials)

Corporate ESG data

Some examples:

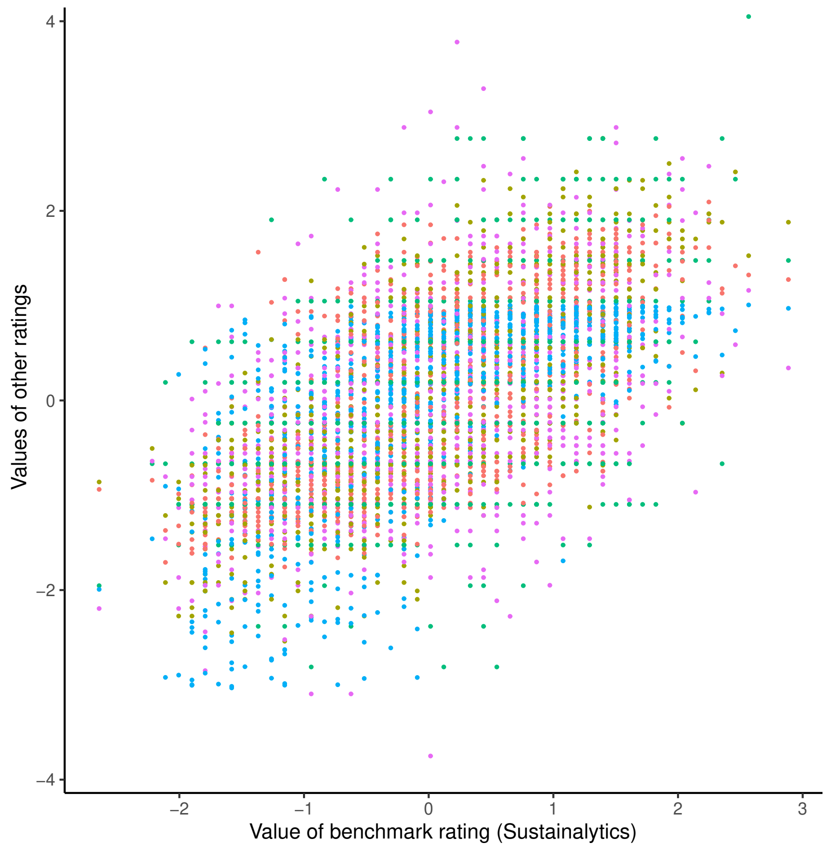
- Bloomberg rates 11 800 public companies. They use more than 120 ESG indicators and 2 000+ data points.
- ISS ESG rates about 10 000 issuers. They use more than 800 indicators and applies approximately 100 indicators per company.
- FTSE Russell rates about 7 200 securities. They use more than 300 indicators and 14 themes.
- MSCI rates 10 000 companies (14 000 issuers including subsidiaries) and 680 000 securities globally. They use 10 themes, 1000+ data points, 80 exposure metrics and 250+ management metrics.
- Refinitiv rates 12 000 public and private companies. They consider 10 themes. These themes are built using 186 metrics and 630+ data points.
- S&P Dow Jones Indices uses between 16 to 27 criteria scores, a questionnaire and 1 000 data points.
- Sustainalytics rates more than 16 300 companies. They consider 20 material ESG issues, based on 350+ indicators.

The race for alternative data

- Controversies ⇒ NLP (RepRisk, daily basis: 500 000+ documents, 100 000+ sources, 23 languages)
- Geospatial data ⇒ Physical risk

The divergence of corporate ESG ratings

Figure 7: ESG rating disagreement



Source: Berg *et al.* (2022).

The divergence of corporate ESG ratings

Berg *et al.* (2022) identify three sources of divergence:

1. "**Measurement** *divergence refers to situation where rating agencies measure the same indicator using different ESG metrics* (56%)
2. **Scope** *divergence refers to situation where ratings are based on different set of ESG indicators* (38%)
3. **Weight** *divergence emerges when rating agencies take different views on the relative importance of ESG indicators*" (6%)

The divergence of corporate ESG ratings

Table 7: Rank correlation among ESG ratings

	MSCI	Refinitiv	S&P Global	
MSCI	100%			
Refinitiv	43%	100%		
S&P Global	45%	69%	100%	
Sustainalytics	53%	64%	69%	100%

Source: Billio *et al.* (2021).

One-level tree structure

- $X_1, \ldots, X_m$ are m features
- The score is linear:

$$\mathcal{S} = \sum_{j=1}^{m} \omega_j X_j$$

- ω_j is the weight of the j^{th} metric

One-level tree structure

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One-level tree structure

The Altman Z score is equal to:

$$Z = 1.2 \cdot X_1 + 1.4 \cdot X_2 + 3.3 \cdot X_3 + 0.6 \cdot X_4 + 1.0 \cdot X_5$$

where the variables X_j represent the following financial ratios:

X_j	Ratio
X_1	Working capital / Total assets
X_2	Retained earnings / Total assets
X_3	Earnings before interest and tax / Total assets
X_4	Market value of equity / Total liabilities
X_5	Sales / Total assets

$$Z_i \Rightarrow Z_i^{\star} = \left(Z_i - m_z\right)/\sigma_z \Rightarrow \text{Decision rule}$$

Two-level tree structure

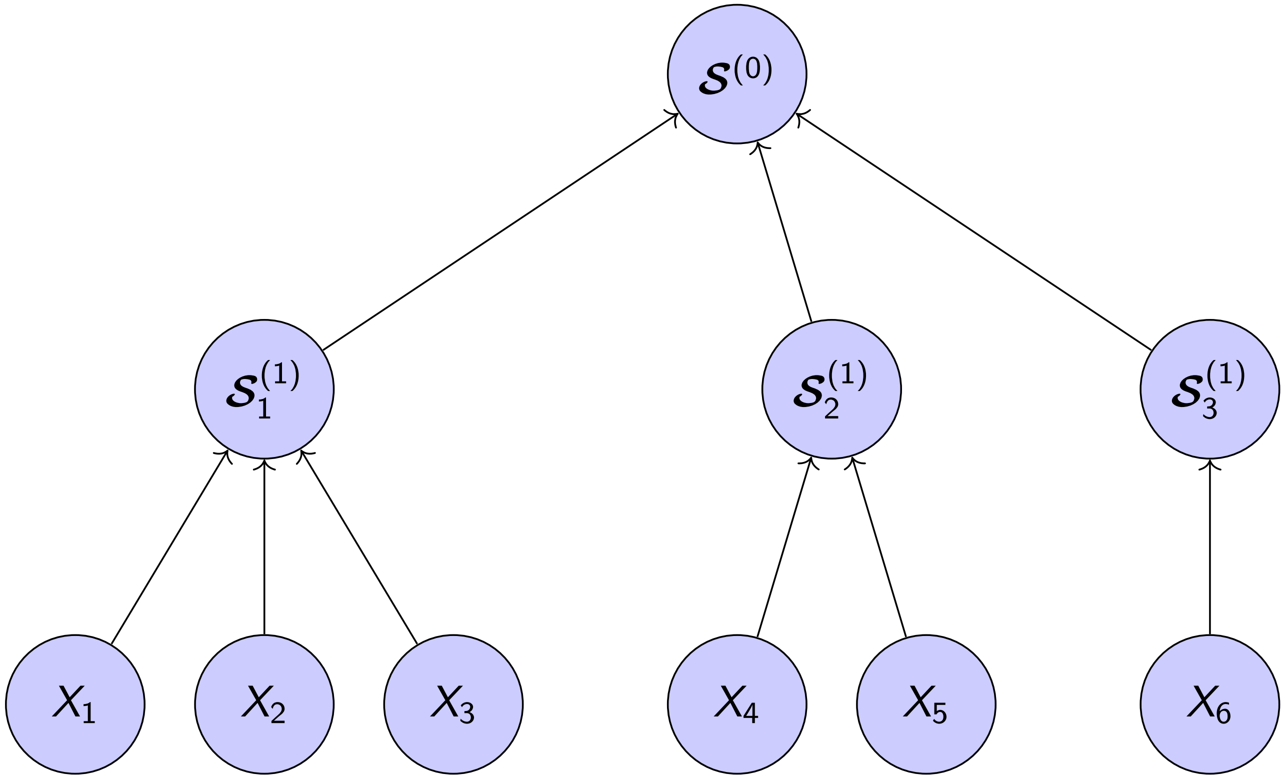
The intermediary scores are equal to:

$$\mathcal{S}_k^{(1)} = \sum_{j=1}^{m} \omega_{j,k}^{(1)} X_j$$

whereas the expression of the final score is:

$$\mathcal{S} := \mathcal{S}_1^{(0)} = \sum_{k=1}^{m_{(1)}} \omega_k^{(0)} \mathcal{S}_k^{(1)}$$

Figure 8: A two-level non-overlapping tree



- Level 1: $\omega_{1,1}^{(1)} = 50\%$; $\omega_{2,1}^{(1)} = 25\%$; $\omega_{3,1}^{(1)} = 25\%$; $\omega_{4,2}^{(1)} = 50\%$; $\omega_{5,2}^{(1)} = 50\%$; $\omega_{6,3}^{(1)} = 100\%$;
- Level 0: $\omega_1^{(0)} = \omega_2^{(0)} = \omega_3^{(0)} = 33.33\%$;

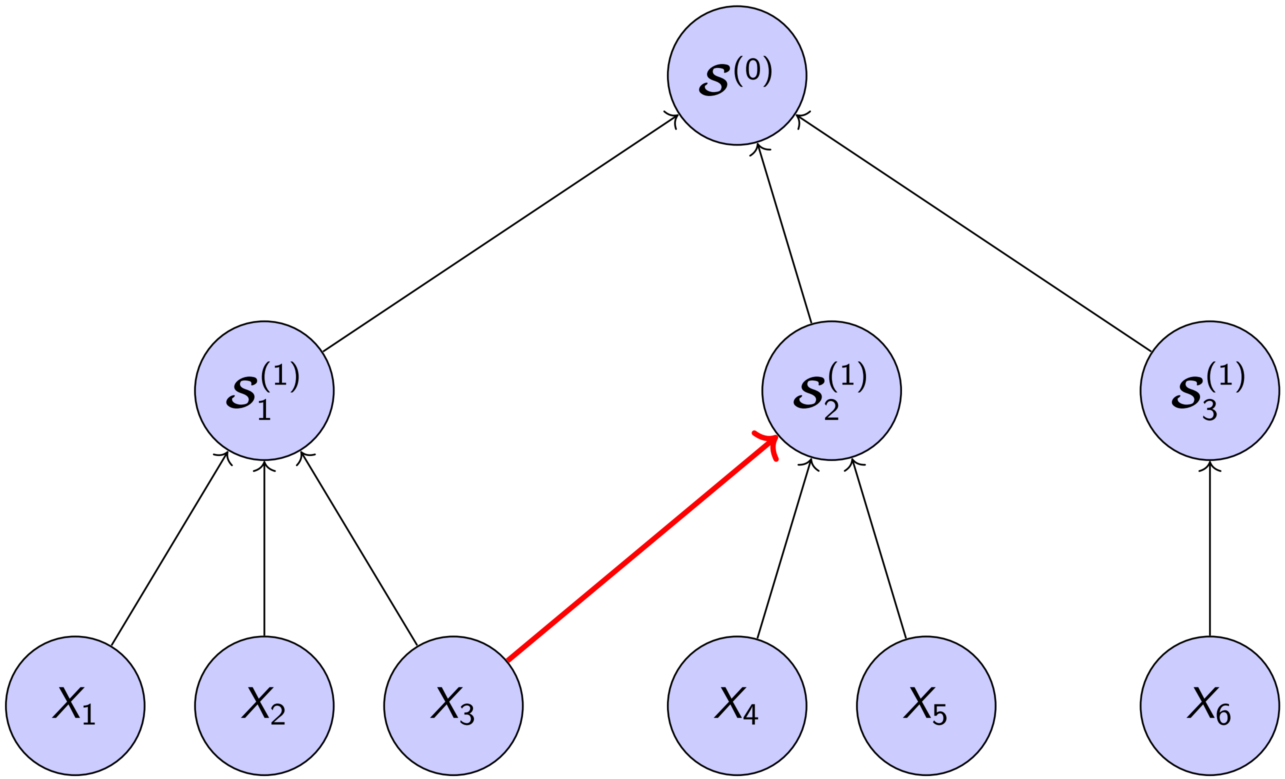
Two-level tree structure

$$\left\{\begin{array}{l} \mathcal{S}_1^{(1)} = 0.5 \cdot X_1 + 0.25 \cdot X_2 + 0.25 \cdot X_3 \\ \mathcal{S}_2^{(1)} = 0.5 \cdot X_4 + 0.5 \cdot X_5 \\ \mathcal{S}_3^{(1)} = X_6 \end{array}\right.$$

$$\mathcal{S} = \frac{\mathcal{S}_1^{(1)} + \mathcal{S}_2^{(1)} + \mathcal{S}_3^{(1)}}{3}$$

Two-level tree structure

Figure 9: A two-level overlapping ~~tree~~ graph



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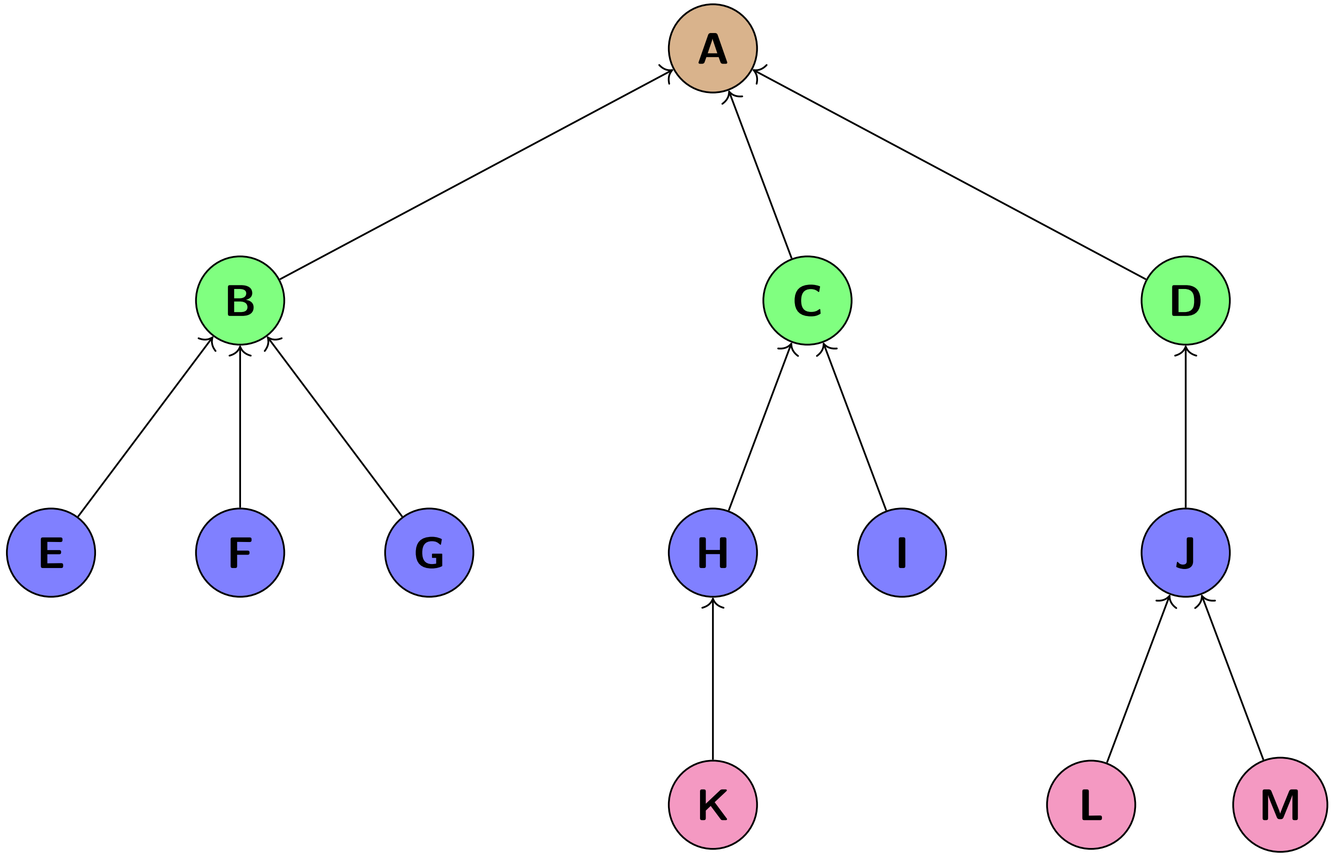
- Level 0: $\omega_1^{(0)} = \omega_2^{(0)} = \omega_3^{(0)} = 33.33\%$;

Tree and graph theory

Figure 10: Tree data structure



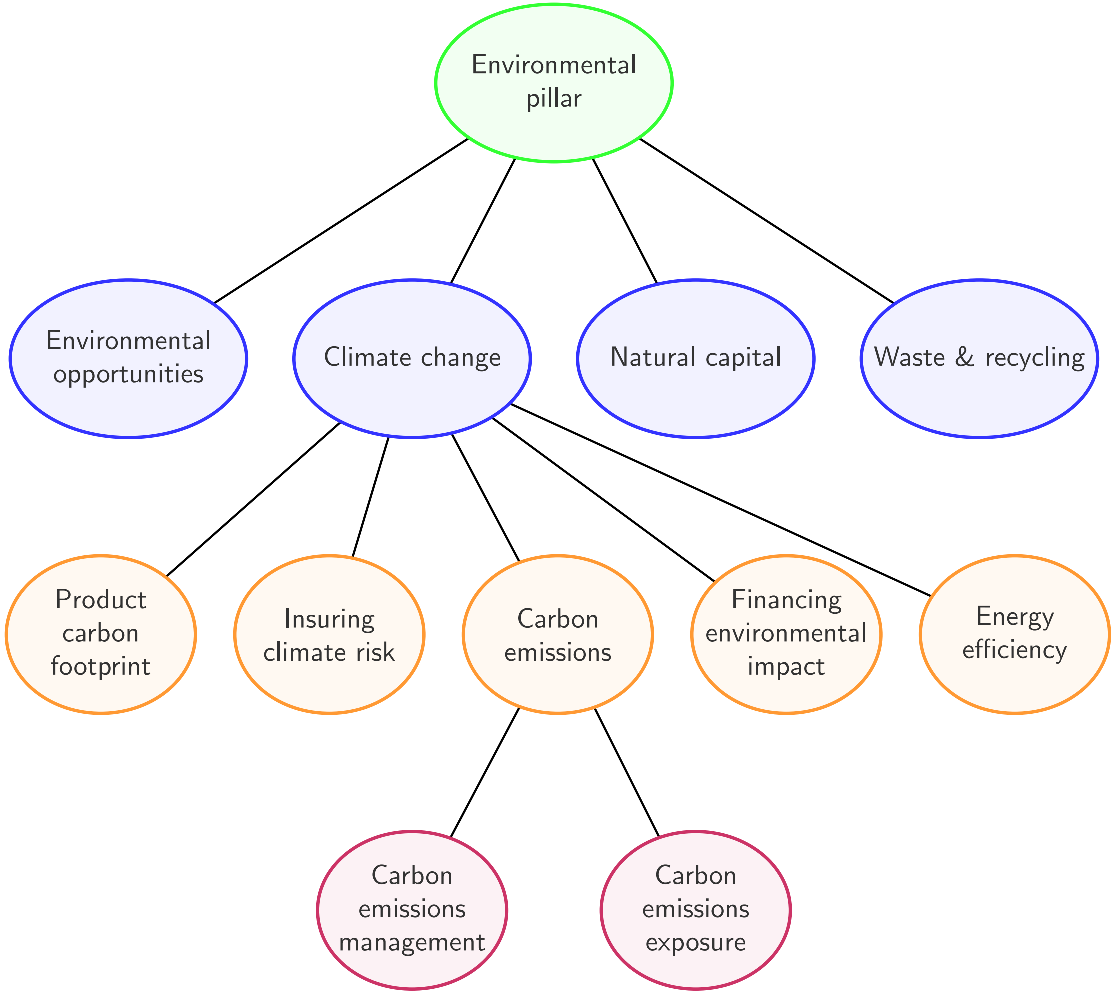
Tree and graph theory

- L is the number of levels
- We have $\boldsymbol{\mathcal{S}}_j^{(L)} = X_j$
- The value of the k^{th} node at level ℓ is given by:

$$\boldsymbol{\mathcal{S}}_k^{(\ell)} = \sum_{j=1}^{m_{(\ell+1)}} \omega_{j,k}^{(\ell)} \boldsymbol{\mathcal{S}}_j^{(\ell+1)}$$

An example of ESG scoring tree

Figure 11: An example of ESG scoring tree (MSCI methodology)



Source: MSCI (2020)

Score normalization

Let $\omega_{(\ell)}$ be the $m_{(\ell+1)} \times m_{(\ell)}$ matrix, whose elements are $\omega_{j,k}^{(\ell)}$ for $j = 1, \ldots, m_{(\ell+1)}$ and $k = 1, \ldots, m_{(\ell)}$

The final score is equal to:

$$\mathcal{S} = \omega^\top X$$

where:

$$\omega = \omega_{(L-1)} \cdots \omega_{(1)} \omega_{(0)}$$

Score normalization

If $X \sim \mathbf{F}$, we obtain:

$$\begin{aligned}
\mathbf{G}(s) &= \Pr\{\boldsymbol{\mathcal{S}} \leq s\} \\
&= \Pr\{\omega^\top X \leq s\} \\
&= \int \cdots \int \mathbb{1}\{\omega^\top x \leq s\}\, \mathrm{d}\mathbf{F}(x) \\
&= \int \cdots \int \mathbb{1}\left\{\sum_{j=1}^{m} \omega_j x_j \leq s\right\} \mathrm{d}\mathbf{F}(x_1, \ldots, x_m) \\
&= \int \cdots \int \mathbb{1}\left\{\sum_{j=1}^{m} \omega_j x_j \leq s\right\} \mathrm{d}\mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_m(x_m))
\end{aligned}$$

Therefore, the distribution $\mathbf{G}$ depends on the copula function $\mathbf{C}$ and the marginals $(\mathbf{F}_1, \ldots, \mathbf{F}_m)$ of $\mathbf{F}$

$$\mathbf{F}_1 \equiv \mathbf{F}_1 \equiv \ldots \equiv \mathbf{F}_m \Rightarrow \mathbf{G} \equiv \mathbf{F}_1?$$

Score normalization

In the independent case, we obtain a a convolution probability distribution:

$$\mathbf{G}(s) = \int \cdots \int \mathbb{1}\left\{\sum_{j=1}^{m} \omega_j x_j \leq s\right\} \prod_{j=1}^{m} \mathrm{d}\mathbf{F}_j(x_j)$$

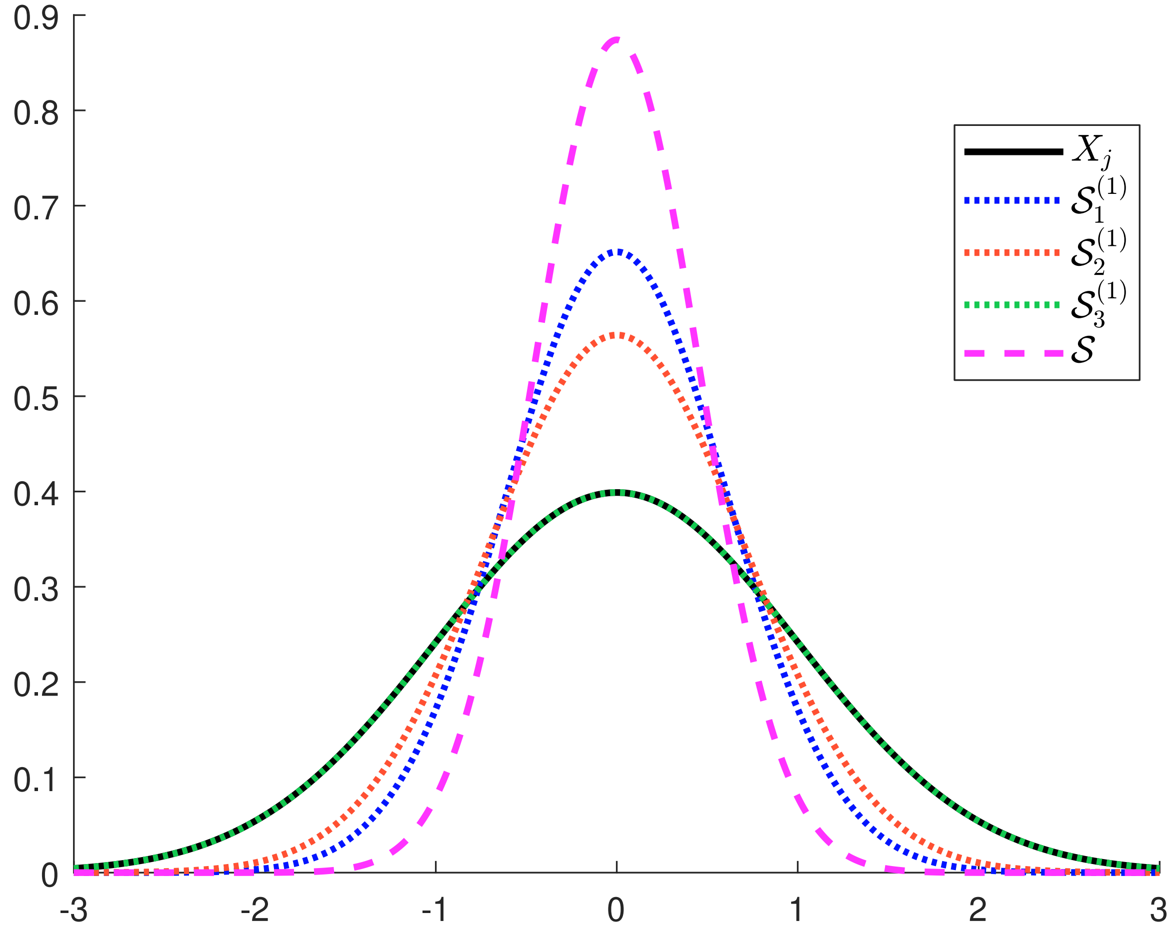
If $X_j \sim \mathcal{N}\left(\mu_j, \sigma_j^2\right)$, we have $\omega_j X_j \sim \mathcal{N}\left(\omega_j \mu_j, \omega_j^2 \sigma_j^2\right)$. We deduce that:

$$\mathcal{S} \sim \mathcal{N}\left(\sum_{j=1}^{m} \omega_j \mu_j, \sum_{j=1}^{m} \omega_j^2 \sigma_j^2\right) \equiv \mathcal{N}\left(\omega^\top \mu, \omega^\top \Sigma \omega\right)$$

where $\mu = (\mu_1, \ldots, \mu_m)$ and $\Sigma = \mathrm{diag}\left(\sigma_1^2, \ldots, \sigma_m^2\right)$.

Score normalization

Figure 12: Probability distribution of the scores based on the previous tree



Score normalization

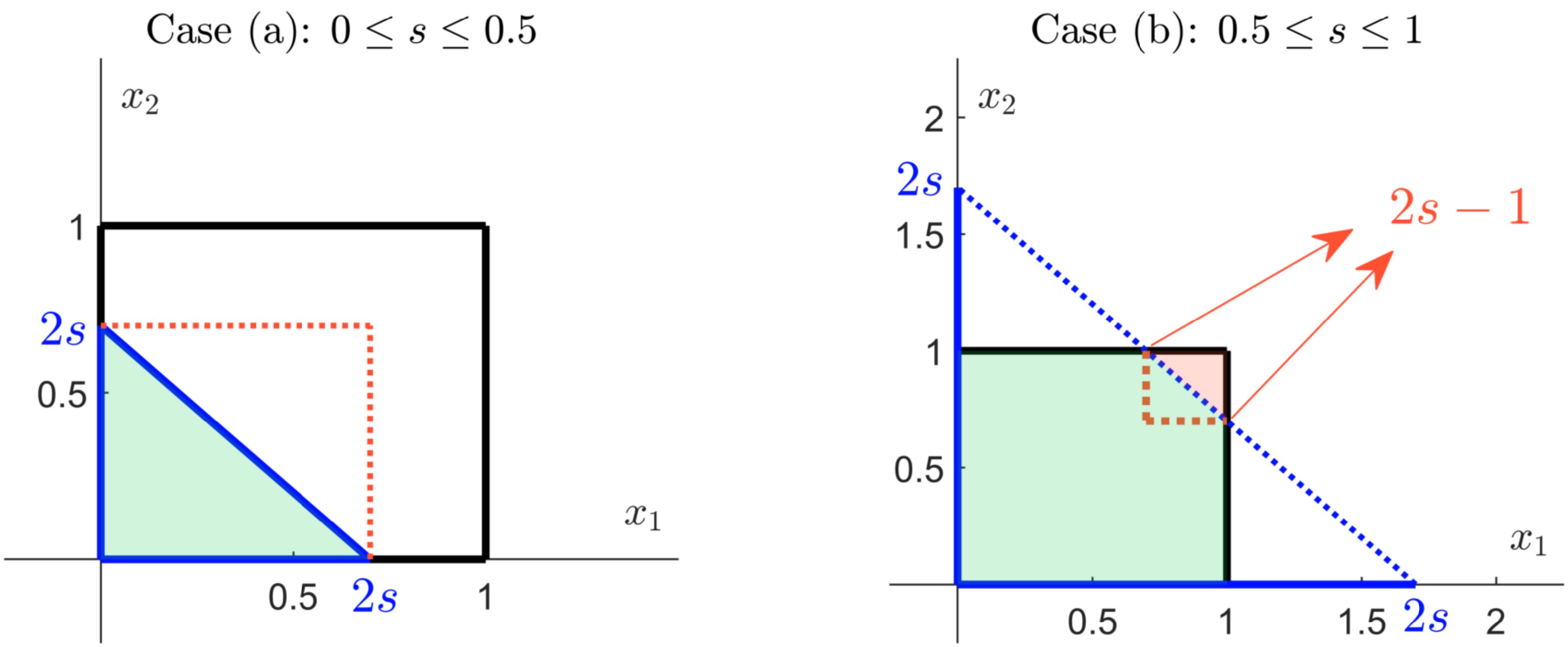
Exercise

We assume that $X_1 \sim \mathcal{U}_{[0,1]}$ and $X_2 \sim \mathcal{U}_{[0,1]}$ are two independent random variables. We consider the score $\boldsymbol{\mathcal{S}}$ defined as:

$$\boldsymbol{\mathcal{S}} = \frac{X_1 + X_2}{2}$$

Score normalization

Figure 13: Geometric interpretation of the probability mass function



Score normalization

We deduce that:

$$\Pr\left\{\mathcal{S} \leq s\right\} = \begin{cases} \dfrac{1}{2}(2s)^2 = 2s^2 & \text{if} \quad 0 \leq s \leq \dfrac{1}{2} \\ 1 - \dfrac{1}{2}(2-2s)^2 = -1 + 4s - 2s^2 & \text{if} \quad \dfrac{1}{2} \leq s \leq 1 \end{cases}$$

The density function is then:

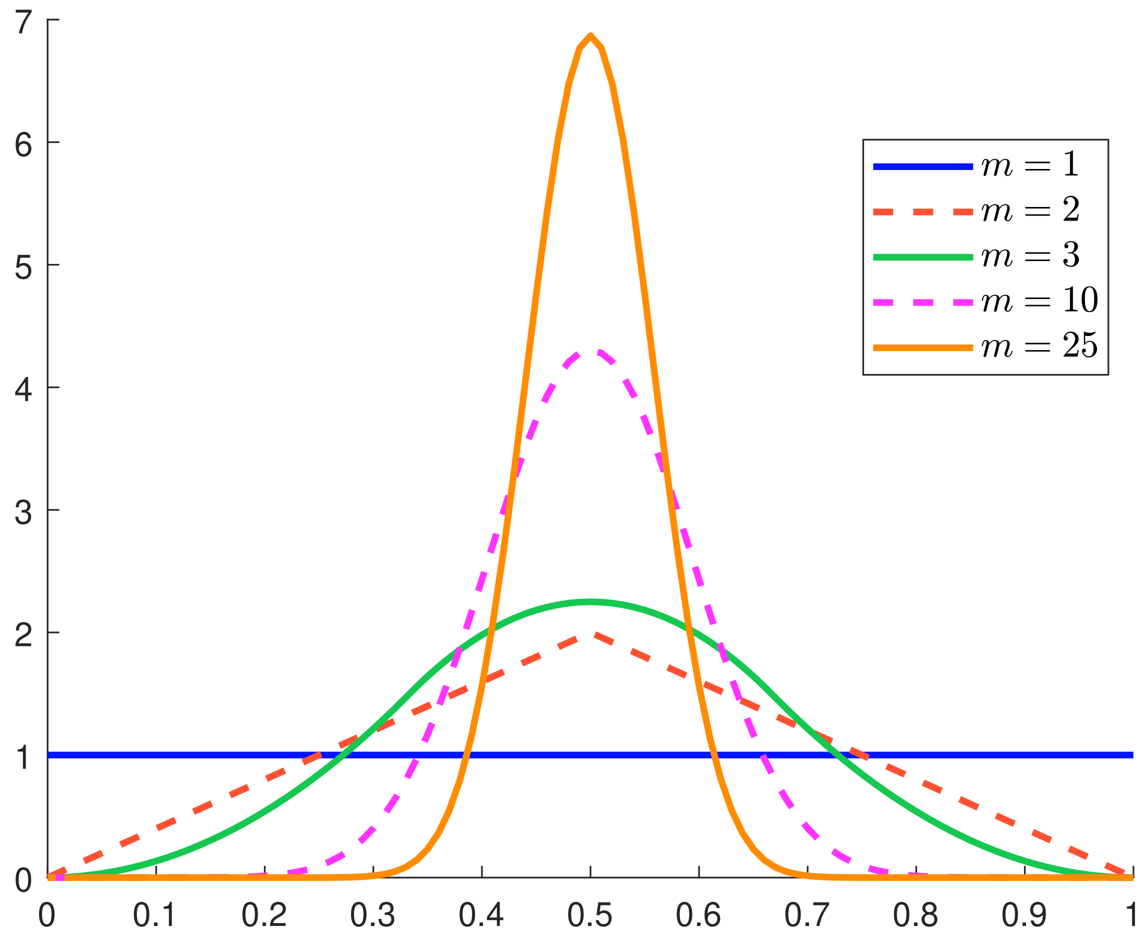
$$g(s) = \begin{cases} 4s & \text{if} \quad 0 \leq s \leq \dfrac{1}{2} \\ 4 - 4s & \text{if} \quad \dfrac{1}{2} \leq s \leq 1 \end{cases}$$

In the general case, we have:

$$\mathcal{S} = \frac{X_1 + X_2 + \cdots + X_m}{m} \sim \mathfrak{B}ates(m)$$

Score normalization

Figure 14: Probability density function of $\mathcal{S}$ (uniform distribution)



Score normalization

Exercise

We assume that $X \sim \mathcal{N}(\mu, \Sigma)$ with $\mu_j = 0$, $\sigma_j = 1$ and $\rho_{j,k} = \rho$ for $j \neq k$. Show that:

$$\mathbb{E}[\boldsymbol{\mathcal{S}}] = 0$$

and

$$\operatorname{var}(\boldsymbol{\mathcal{S}}) = \rho \mathcal{S}^2(w) + (1 - \rho)\mathcal{H}(\omega)$$

where $\mathcal{S}(w) = \sum_{j=1}^m \omega_j$ is the sum index and $\mathcal{H}(\omega) = \sum_{j=1}^m \omega_j^2$ is the Herfindahl index. Deduce that:

$$\sigma_{\boldsymbol{\mathcal{S}}} = \sqrt{\rho + (1 - \rho)\mathcal{H}(\omega)}$$

Score normalization

How to normalize?

$$\mathcal{S}_k^{(\ell)} = \varphi\left(\sum_{j=1}^{m_{(\ell+1)}} \omega_{j,k}^{(\ell)} \mathcal{S}_j^{(\ell+1)}\right)$$

Score normalization

1. m-score normalization:

$$m_i = \frac{x_i - x^-}{x^+ - x^-}$$

where $x^- = \min x_i$ and $x^+ = \max x_i$

2. q-score normalization:

$$q_i = \mathbf{H}(x_i)$$

where $\mathbf{H}$ is the distribution function of X

3. z-score normalization:

$$z_i = \frac{x_i - \mu}{\sigma}$$

where μ and σ are the mathematical expectation and standard deviation of X

4. b-score normalization:

$$b_i = \mathfrak{B}^{-1}(\mathbf{H}(x_i); \alpha, \beta)$$

where $\mathcal{B}(\alpha, \beta)$ is the beta distribution

Score normalization

Probability integral transform (PIT)

If $X \sim \mathbf{H}$ and is continuous, $Y = \mathbf{H}(X)$ is a uniform random variable.

We have $Y \in [0,1]$ and:

$$\begin{aligned} \Pr\{Y \leq y\} &= \Pr\{\mathbf{H}(X) \leq y\} \\ &= \Pr\left\{X \leq \mathbf{H}^{-1}(y)\right\} \\ &= \mathbf{H}\left(\mathbf{H}^{-1}(y)\right) \\ &= y \end{aligned}$$

Score normalization

Computing the empirical distribution $\hat{\mathbf{H}}$

- Let $\{x_1, x_2, \ldots, x_n\}$ be the sample
- We have:

$$q_i = \hat{\mathbf{H}}(x_i) = \Pr\{X \leq x_i\} = \frac{\#\{x_j \leq x_i\}}{n_q}$$

- $n_q = n$ or $n_q = n + 1$?

Score normalization

Exercise

What is the normalization shape of this transformation?

$$\boldsymbol{\mathcal{S}} = \frac{2}{1 + e^{-z}} - 1$$

Hint: compute the density function.

Score normalization

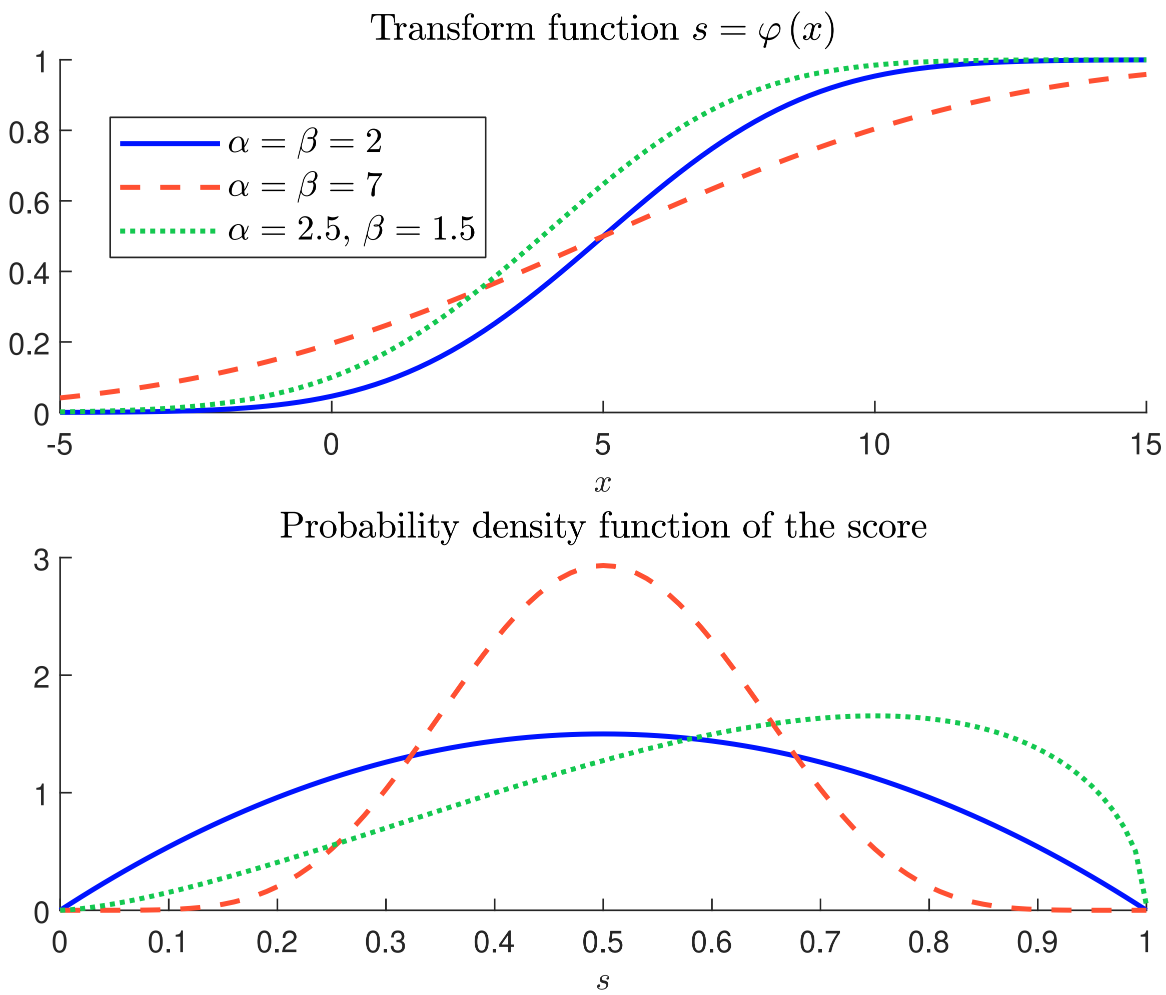
Example

The data are normally distributed with mean $\mu = 5$ and standard deviation $\sigma = 2$. To map these data into a 0/1 score, we consider the following transform:

$$s := \varphi(x) = \mathfrak{B}^{-1}\left(\Phi\left(\frac{x-5}{2}\right); \alpha, \beta\right)$$

Score normalization

Figure 15: Transforming data into $\mathcal{B}$-score



Score normalization

Example

We consider the raw data of 9 companies that belong to the same industry. The first variable measures the carbon intensity of the scope $1+2$ in 2020, while the second variable is the variation of carbon emissions between 2015 and 2020. We would like to create the score $\mathcal{S} \equiv 70\% \cdot X_1 + 30\% \cdot X_2$.

Firm	Carbon intensity in $\mathrm{tCO_2e}$/\$ mn)	Carbon momentum (in %)
1	94.0	−3.0
2	38.6	−5.5
3	30.6	5.6
4	74.4	−1.3
5	97.1	−16.8
6	57.1	−3.5
7	132.4	8.5
8	92.5	−9.1
9	64.9	−4.6

Score normalization

- q-score $0/100$
- z-score
- $qz = 100 \cdot \Phi(z)$
- $zq = \Phi^{-1}\left(\frac{q}{100}\right)$
- $bz = \mathfrak{B}^{-1}(\Phi(z); \alpha, \beta)$ where $\alpha = \beta = 2$
- $bz^{\star} = \mathfrak{B}^{-1}(\Phi(z); \alpha, \beta)$ where $\alpha = 2.5$ and $\beta = 1.5$.

Score normalization

Table 8: Computation of the score $\mathcal{S} \equiv 70\% \cdot X_1 + 30\% \cdot X_2$ (q-score 0/100 normalization)

#	X_1	q_1	X_2	q_2	s	$\mathcal{S}$	$\mathfrak{R}$
1	94.00	70.00	−3.00	60.00	67.00	80.00	8
2	38.60	20.00	−5.50	30.00	23.00	10.00	1
3	30.60	10.00	5.60	80.00	31.00	20.00	2
4	74.40	50.00	−1.30	70.00	56.00	60.00	6
5	97.10	80.00	−16.80	10.00	59.00	70.00	7
6	57.10	30.00	−3.50	50.00	36.00	30.00	3
7	132.40	90.00	8.50	90.00	90.00	90.00	9
8	92.50	60.00	−9.10	20.00	48.00	50.00	5
9	64.90	40.00	−4.60	40.00	40.00	40.00	4
Mean	75.73	50.00	−3.30	50.00	50.00	50.00	
Std-dev.	31.95	27.39	7.46	27.39	20.60	27.39	

Score normalization

Table 9: Computation of the score $\mathcal{S} \equiv 70\% \cdot X_1 + 30\% \cdot X_2$ (z-score normalization)

#	X_1	z_1	X_2	z_2	s	$\mathcal{S}$	$\mathfrak{R}$
1	94.00	0.572	−3.00	0.040	0.412	0.543	8
2	38.60	−1.162	−5.50	−0.295	−0.902	−1.188	1
3	30.60	−1.413	5.60	1.193	−0.631	−0.831	2
4	74.40	−0.042	−1.30	0.268	0.051	0.067	6
5	97.10	0.669	−16.80	−1.810	−0.075	−0.099	5
6	57.10	−0.583	−3.50	−0.027	−0.416	−0.548	3
7	132.40	1.774	8.50	1.582	1.716	2.261	9
8	92.50	0.525	−9.10	−0.778	0.134	0.177	7
9	64.90	−0.339	−4.60	−0.174	−0.290	−0.382	4
Mean	75.73	0.000	−3.30	0.000	0.000	0.000	
Std-dev.	31.95	1.000	7.46	1.000	0.759	1.000	

Score normalization

Table 10: Comparison of the different scoring methods

#	q		z		qz		zq		bz		$bz^{\star}$	
	$\mathcal{S}$	$\mathfrak{R}$	$\mathcal{S}$	$\mathfrak{R}$	$\mathcal{S}$	$\mathfrak{R}$	$\mathcal{S}$	$\mathfrak{R}$	$\mathcal{S}$	$\mathfrak{R}$	$\mathcal{S}$	$\mathfrak{R}$
1	80.00	8	0.54	8	76.27	8	0.84	8	0.66	8	0.81	8
2	10.00	1	−1.19	1	9.19	1	−1.28	1	0.20	1	0.30	1
3	20.00	2	−0.83	2	21.37	2	−0.84	2	0.29	2	0.38	2
4	60.00	6	0.07	6	54.13	5	0.25	6	0.52	6	0.70	6
5	70.00	7	−0.10	5	56.65	6	0.52	7	0.51	5	0.64	5
6	30.00	3	−0.55	3	24.42	3	−0.52	3	0.34	3	0.50	3
7	90.00	9	2.26	9	98.04	9	1.28	9	0.93	9	0.96	9
8	50.00	5	0.18	7	60.39	7	0.00	5	0.56	7	0.72	7
9	40.00	4	−0.38	4	30.96	4	−0.25	4	0.39	4	0.56	4
Mean	50.00		0.00		47.94		0.00		0.49		0.62	
Std-dev.	27.39		1.00		28.79		0.82		0.22		0.21	

An example with the CEO pay ratio

The CEO pay ratio is calculated by dividing the CEO's compensation by the pay of the median employee. It is one of the key metrics for the G pillar. It has been imposed by the Dodd-Frank Act, which requires that publicly traded companies disclose:

1. the median total annual compensation of all employees other than the CEO;
2. the ratio of the CEO's annual total compensation to that of the median employee;
3. the wage ratio of the CEO to the median employee.

$\Rightarrow$ the average S&P 500 company's CEO-to-worker pay ratio was 324-to-1 in 2021 (AFL-CIO)

An example with the CEO pay ratio

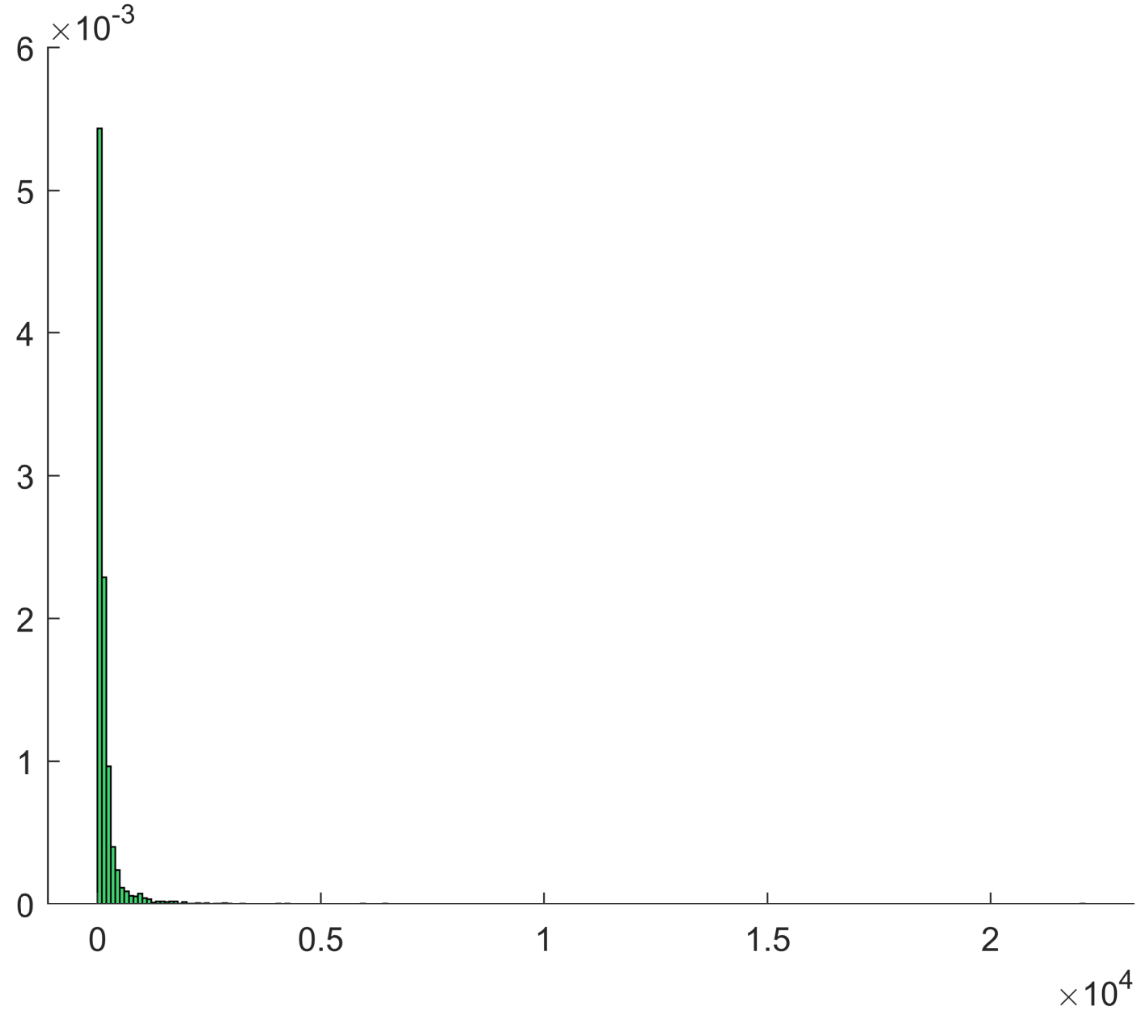
Table 11: Examples of CEO pay ratio (June 2021)

Company name	P	R	Company name	P	R
Abercrombie & Fitch	1 954	4,293	Netflix	202 931	190
McDonald's	9 291	1,939	BlackRock	133 644	182
Coca-Cola	11 285	1,657	Pfizer	98 972	181
Gap	6 177	1,558	Goldman Sachs	138 854	178
Alphabet	258 708	1,085	MSCI	55 857	165
Walmart	22 484	983	Verisk Analytics	77 055	117
Estee Lauder	30 733	697	Facebook	247 883	94
Ralph Lauren	21 358	570	Invesco	125 282	92
NIKE	25 386	550	Boeing	158 869	90
Citigroup	52 988	482	Citrix Systems	181 769	80
PepsiCo	45 896	368	Harley-Davidson	187 157	59
Microsoft	172 512	249	Amazon.com	28 848	58
Apple	57 596	201	Berkshire Hathaway	65 740	6

Source: `https://aflcio.org` (June 2021)

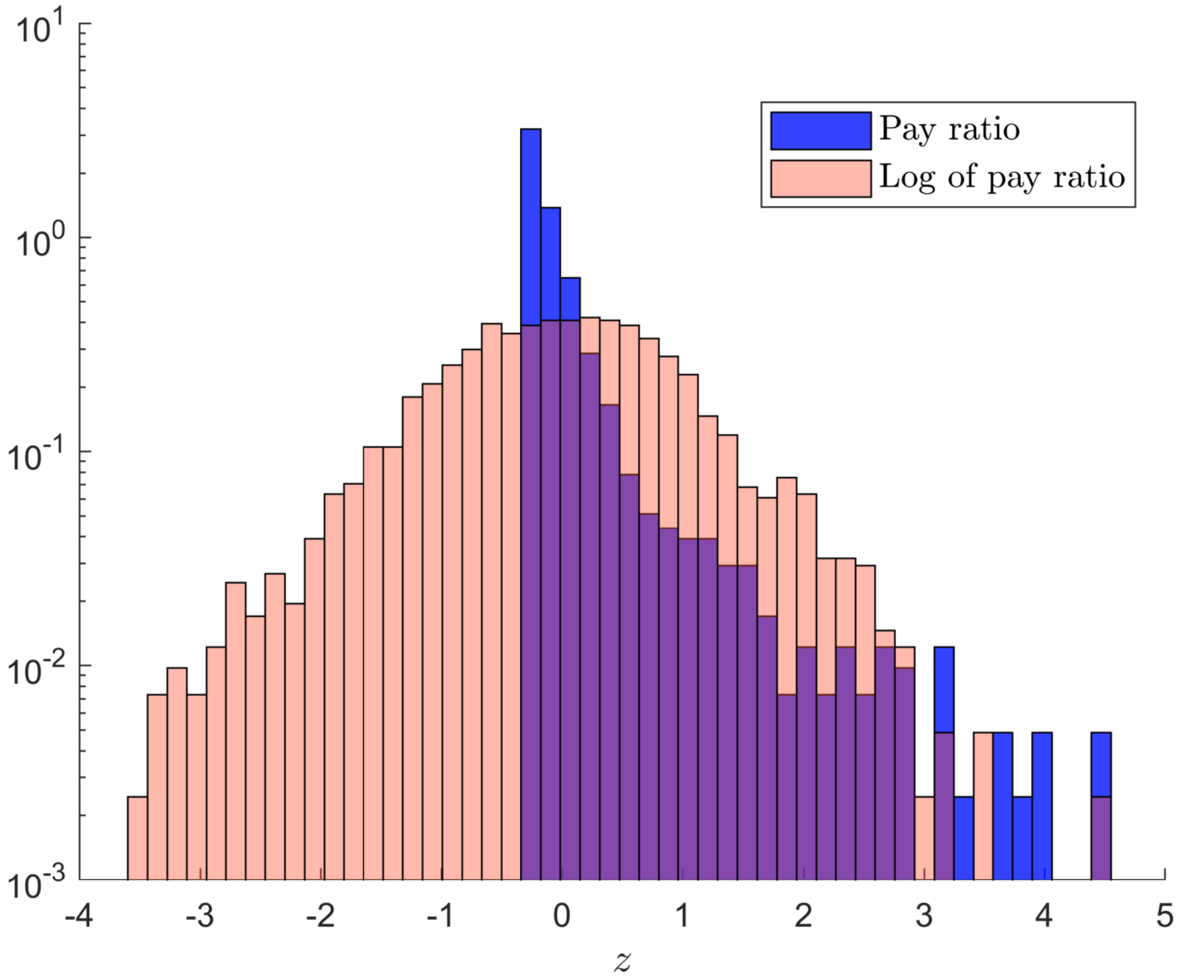
An example with the CEO pay ratio

Figure 16: Histogram of the CEO pay ratio



An example with the CEO pay ratio

Figure 17: Histogram of z-score applied to the CEO pay ratio



An example with the CEO pay ratio

What is the solution? Give the transform function $y = \varphi(x)$.

Hint: use the beta distribution.

Other statistical methods

Unsupervised learning

- Clustering (K-means, hierarchical clustering)
- Dimension reduction (PCA, NMF)

Other statistical methods

Supervised learning

- Discriminant analysis (LDA, QDA)
- Binary choice models (logistic regression, probit model)
- Regression models (OLS, lasso)

$\Rightarrow$ Advanced learning models (k-NN, neural networks and support vector machines) are not relevant in the case of ESG scoring

We need to define the response variable Y

Other statistical methods

Example with credit scoring models

- Let $\boldsymbol{\mathcal{S}}_i(t)$ be the credit score of individual i at time t
- We have:

$$Y_i(t) = \mathbb{1}\{\tau_i \leq t + \delta\} = \mathbb{1}\{D_i(t+\delta) = 1\}$$

  where τ_i and D_i are the default time and the default indicator function, and δ is the time horizon (e.g., one year)
- The calibration problem of the credit scoring model is:

$$\Pr\{Y_i(t) = 0\} = f(\boldsymbol{\mathcal{S}}_i(t))$$

  where f is an increasing function

Application to ESG scoring models

- Let $\mathcal{S}_i(t)$ be the ESG score of company i at time t
- Endogenous response variable:
  (a) Best-in-class oriented scoring system:
  $$Y_i(t) = \mathbb{1}\{\mathcal{S}_i(t+h) \geq s^\star\}$$
  where $s^\star$ is the best-in-class threshold
  (b) Worst-in-class oriented scoring system: $Y_i(t) = \mathbb{1}\{\mathcal{S}_i(t+h) \leq s^\star\}$
  where $s^\star$ is the worst-in-class threshold
- Exogenous response variable
  (c) Binary response:
  $$Y_i(t) = \mathbf{1}\{\mathcal{C}_i(t+h) \geq 0\}$$
  where $\mathcal{C}_i(t)$ is the controversy index
  d Continuous response:
  $$Y_i(t) = \mathcal{C}_i(t+h)$$
- The calibration problem of the ESG scoring model is $\Pr\{Y_i(t) = 0\} = f(\mathcal{S}_i(t))$ or $Y_i(t) = f(\mathcal{S}_i(t))$ where the function f is increasing for case (a) and decreasing for cases (b), (c) and (d)

Performance evaluation criteria

- ESG scoring and rating
  - Shannon entropy
  - Confusion matrix
  - Binary classification ratios (TPR, FNR, TNR, FPR, PPV, ACC, F_1)
- ESG scoring
  - Performance, selection and discriminant curves
  - ROC curve
  - Gini coefficient

Definition

Table 12: Credit rating system of S&P, Moody's and Fitch

	Prime Maximum Safety	High Grade High Quality			Upper Medium Grade		
S&P/Fitch	AAA	AA+	AA	AA−	A+	A	A−
Moody's	Aaa	Aa1	Aa2	Aa3	A1	A2	A3

	Lower Medium Grade			Non Investment Grade Speculative		
S&P/Fitch	BBB+	BBB	BBB−	BB+	BB	BB−
Moody's	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3

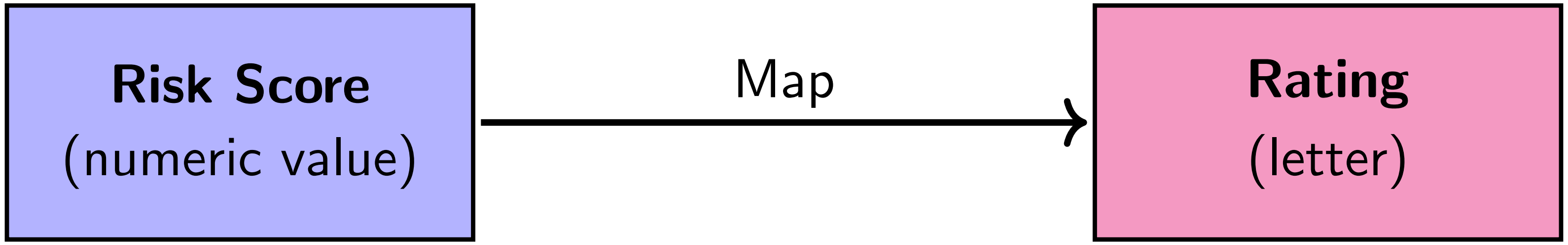
	Highly Speculative			Substantial Risk	In Poor Standing		Extremely Speculative
S&P/Fitch	B+	B	B−	CCC+	CCC	CCC−	CC
Moody's	B1	B2	B3	Caa1	Caa2	Caa3	Ca

Definition

- Amundi: A (high), B,... to G (low) — 7-grade scale
- FTSE Russell: 0 (low), 1,... to 5 (high) — 6-grade scale
- ISS ESG: 1 (high), 2,... to 10 (low) — 10-grade scale
- MSCI: AAA (high), AA,... to CCC (low) — 7-grade scale
- Refinitiv: A+ (high), A, A-, B+,... to D- (low) — 12-grade scale
- RepRisk: AAA (high), AA,... to D (low) — 8-grade scale
- Sustainanalytics: 1 (low), 2,... to 5 (high) — 5-grade scale

ESG rating process

Figure 18: From ESG score to ESG rating



Two-step approach:

1. Specification of the map function:

$$\begin{array}{rccc} \mathcal{M}ap: & \Omega_{\mathcal{S}} & \longrightarrow & \Omega_{\mathcal{R}} \\ & \mathcal{S} & \longmapsto & \mathcal{R} = \mathcal{M}ap(\mathcal{S}) \end{array}$$

   where $\Omega_{\mathcal{S}}$ is the support of ESG scores, $\Omega_{\mathcal{R}}$ is the ordered state space of ESG ratings and $\mathcal{R}$ is the ESG rating

2. Validation (and the possible *forcing*) of the rating by the analyst

ESG rating process

Example with the MSCI ESG rating system

- $\Omega_{\mathcal{S}} = [0, 10]$
- $\Omega_{\mathcal{R}} = \{\text{CCC}, \text{B}, \text{BB}, \text{BBB}, \text{A}, \text{AA}, \text{AAA}\}$
- The map function is defined as

$$\mathcal{M}ap\left(s\right) = \begin{cases} \text{CCC} & \text{if } \boldsymbol{\mathcal{S}} \in [0, {}^{10}/_{7}] & (0 - 1.429) \\ \text{B} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{10}/_{7}, {}^{20}/_{7}] & (1.429 - 2.857) \\ \text{BB} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{20}/_{7}, {}^{30}/_{7}] & (2.857 - 4.286) \\ \text{BBB} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{30}/_{7}, {}^{40}/_{7}] & (4.286 - 5.714) \\ \text{A} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{40}/_{7}, {}^{50}/_{7}] & (5.714 - 7.143) \\ \text{AA} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{50}/_{7}, {}^{60}/_{7}] & (7.143 - 8.571) \\ \text{AAA} & \text{if } \boldsymbol{\mathcal{S}} \in [{}^{60}/_{7}, 10] & (8.571 - 10) \end{cases}$$

ESG rating process

- The map function is an increasing piecewise function
- $\mathcal{S} \sim \mathbf{F}$ and $\mathcal{S} \in (s^-, s^+)$
- $\left\{s_0^\star = s^-, s_1^\star, \ldots, s_{K-1}^\star, s_K^\star = s^+\right\}$ are the knots of the piecewise function
- $\Omega_{\mathcal{R}} = \{R_1, \ldots, R_K\}$ is the set of grades

$\Rightarrow$ The frequency distribution of the ratings is given by:

$$\begin{aligned} p_k &= \Pr\{\mathcal{R} = R_k\} \\ &= \Pr\left\{s_{k-1}^\star \leq \mathcal{S} < s_k^\star\right\} \\ &= \mathbf{F}(s_k^\star) - \mathbf{F}\left(s_{k-1}^\star\right) \end{aligned}$$

ESG rating process

If we would like to build a rating system with pre-defined frequencies $(p_1, \ldots, p_K)$, we have to solve the following equation:

$$\mathbf{F}\left(s_k^\star\right) - \mathbf{F}\left(s_{k-1}^\star\right) = p_k$$

We deduce that:

$$\begin{aligned} \mathbf{F}\left(s_k^\star\right) &= p_k + \mathbf{F}\left(s_{k-1}^\star\right) \\ &= p_k + p_{k-1} + \mathbf{F}\left(s_{k-2}^\star\right) \\ &= \left(\sum_{j=1}^{k} p_j\right) + \mathbf{F}\left(s_0^\star\right) \end{aligned}$$

and:

$$s_k^\star = \mathbf{F}^{-1}\left(\sum_{j=1}^{k} p_j\right)$$

ESG rating process

Exercise

- We assume that $\mathcal{S} \sim \mathcal{U}_{[a,b]}$
- Show that $p_k = K^{-1}$ If the rating system consists in K equally-sized intervals
- Show that the knots of the map function are equal to:

$$s_k^\star = a + (b - a) \left(\sum_{j=1}^{k} p_j \right)$$

  when we impose pre-defined frequencies $(p_1, \ldots, p_K)$
- If we consider a 0/100 uniform score and $\Omega_{\mathcal{R}} \times \mathbb{P} = (\text{CCC}, 5\%), (\text{B}, 10\%), (\text{BB}, 20\%), (\text{BBB}, 30\%), (\text{A}, 20\%), (\text{AA}, 10\%), (\text{AAA}, 5\%)$, show that $s_{\text{CCC}}^\star = 5$, $s_{\text{B}}^\star = 15$, $s_{\text{BB}}^\star = 35$, $s_{\text{BBB}}^\star = 65$, $s_{\text{A}}^\star = 85$ and $s_{\text{AA}}^\star = 95$

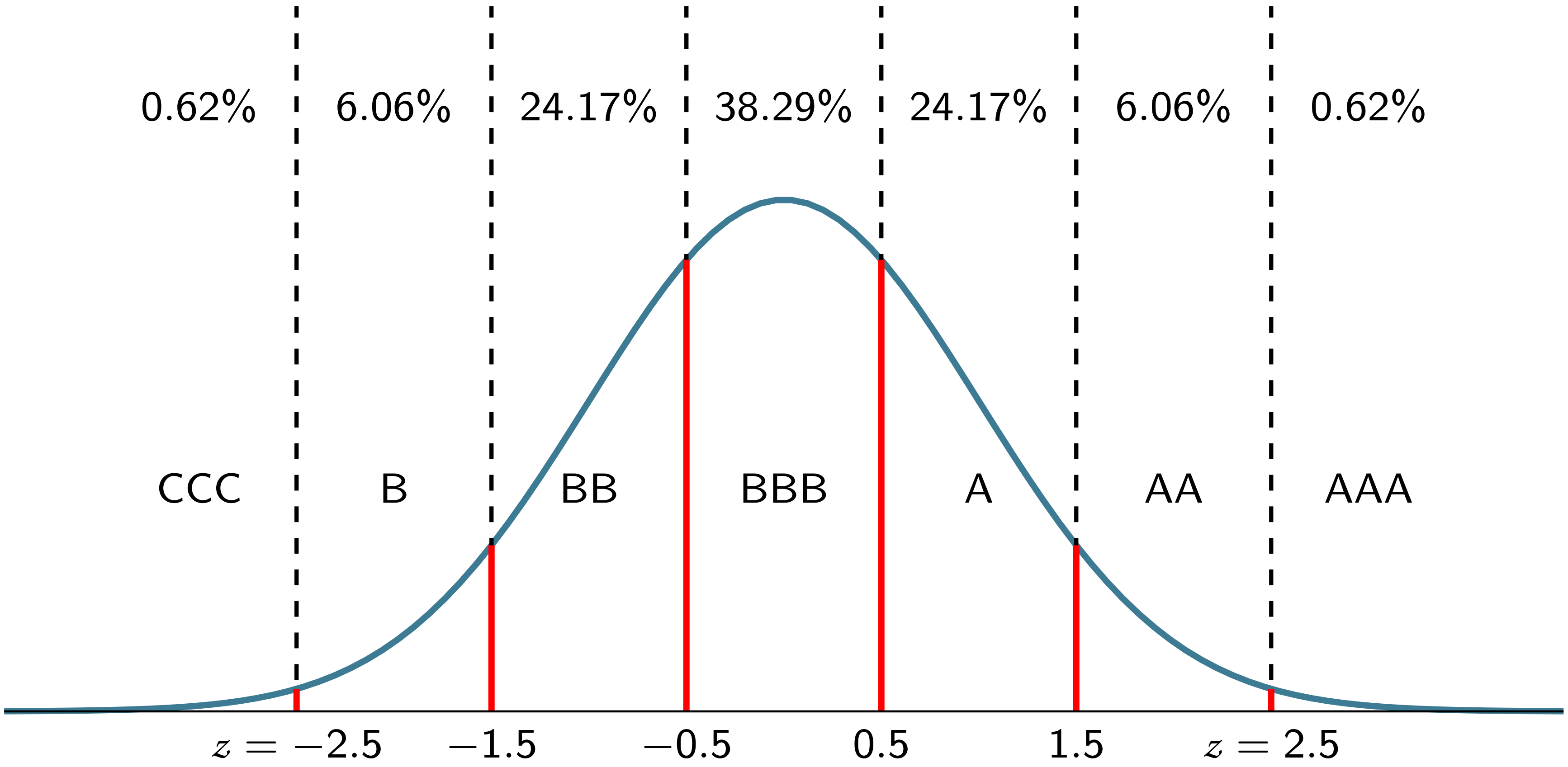
ESG rating process

For a z-score system ($\mathcal{S} \sim \mathcal{N}(0,1)$), we obtain:

$$p_k = \Phi\left(s_k^{\star}\right) - \Phi\left(s_{k-1}^{\star}\right)$$

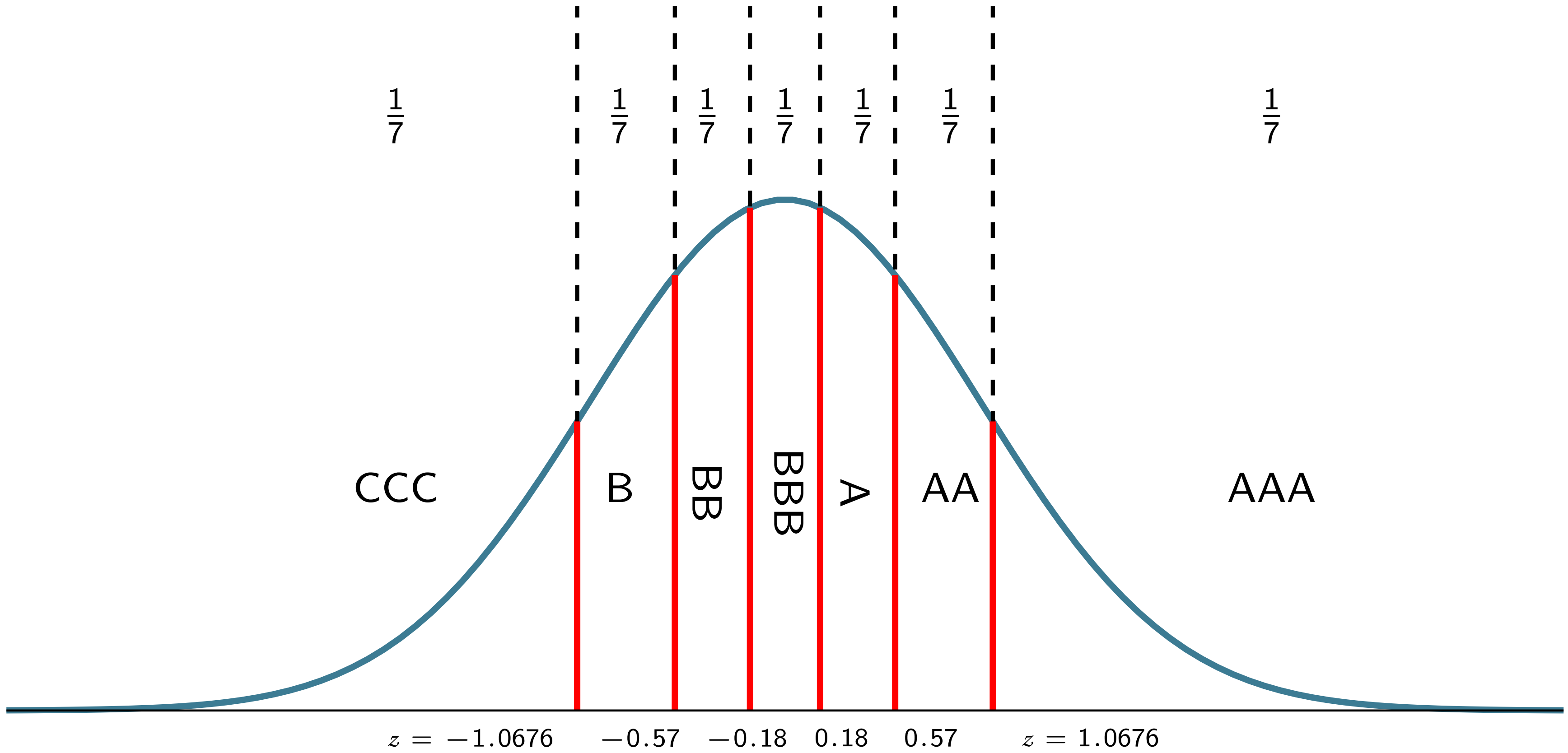
ESG rating process

Figure 19: Map function of a z-score (equal-space ratings)



ESG rating process

Figure 20: Map function of a z-score (equal-frequency ratings)



Rating migration matrix

Which rating model do you prefer? This one...

Table 13: ESG migration matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
AA	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
A	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
BBB	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
BB	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
B	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%
CCC	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%	14.3%

$$\Rightarrow \mathcal{I}\left(\mathcal{R}\left(t\right) \mid \mathcal{R}\left(s\right)\right) = \ln 7$$

Rating migration matrix

Which rating model do you prefer? Or this one...

Table 14: ESG migration matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	100%	0%	0%	0%	0%	0%	0%
AA	0%	100%	0%	0%	0%	0%	0%
A	0%	0%	100%	0%	0%	0%	0%
BBB	0%	0%	0%	100%	0%	0%	0%
BB	0%	0%	0%	0%	100%	0%	0%
B	0%	0%	0%	0%	0%	100%	0%
CCC	0%	0%	0%	0%	0%	0%	100%

$$\Rightarrow \mathcal{I}\left(\mathcal{R}\left(t\right) \mid \mathcal{R}\left(s\right)\right) = 0$$

Rating migration matrix

Which rating model do you prefer? Or this one?

Table 15: ESG migration matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	96%	4%	0%	0%	0%	0%	0%
AA	2%	96%	2%	0%	0%	0%	0%
A	0%	2%	96%	2%	0%	0%	0%
BBB	0%	0%	2%	96%	2%	0%	0%
BB	0%	0%	0%	2%	96%	2%	0%
B	0%	0%	0%	0%	2%	96%	2%
CCC	0%	0%	0%	0%	0%	4%	96%

$\Rightarrow 0 < \mathcal{I}\left(\mathcal{R}\left(t\right) \mid \mathcal{R}\left(s\right)\right) \ll \ln 7$

Rating migration matrix

A good reference on Markov chains is:

Norris, J. R. (1997).
Markov Chains.
Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press.

Rating migration matrix

Discrete time modeling

Definition

- $\mathcal{R}$ is a time-homogeneous Markov chain
- $\Omega_{\mathcal{R}} = \{R_1, \ldots, R_K\}$ is the state space of the chain
- $\mathbb{K} = \{1, \ldots, K\}$ is the corresponding index set
- The transition matrix is defined as $P = (p_{i,j})$
- $p_{i,j}$ is the probability that the entity migrates from rating R_i to rating R_j
- The matrix P satisfies the following properties:
  - $\forall i, j \in \mathbb{K}$, $p_{i,j} \geq 0$
  - $\forall i \in \mathbb{K}$, $\sum_{j=1}^{K} p_{i,j} = 1$

Rating migration matrix

Discrete time modeling

Table 16: ESG migration matrix #1 (one-year transition probability in %)

	AAA	AA	A	BBB	BB	B	CCC
AAA	92.76	5.66	0.90	0.45	0.23	0.00	0.00
AA	4.15	82.73	11.86	0.89	0.30	0.07	0.00
A	0.18	15.47	72.98	10.46	0.82	0.09	0.00
BBB	0.07	1.32	19.60	69.49	9.03	0.42	0.07
BB	0.04	0.19	1.55	19.36	70.88	7.75	0.23
B	0.00	0.05	0.24	1.43	21.54	74.36	2.38
CCC	0.00	0.00	0.22	0.44	2.21	13.24	83.89

Rating migration matrix

Discrete time modeling

The probability that the entity reaches the state R_j at time t given that it has reached the state R_i at time s is equal to:

$$p\left(s,i;t,j\right)=\Pr\left\{\mathcal{R}\left(t\right)=R_j\mid\mathcal{R}\left(s\right)=R_i\right\}=p_{i,j}^{(t-s)}$$

We note $p_{i,j}^{(n)}$ the n-step transition probability:

$$p_{i,j}^{(n)}=\Pr\left\{\mathcal{R}\left(t+n\right)=R_j\mid\mathcal{R}\left(t\right)=R_i\right\}$$

and the associated n-step transition matrix $P^{(n)}=\left(p_{i,j}^{(n)}\right)$

Rating migration matrix

Discrete time modeling

For $n = 2$, we obtain:

$$\begin{aligned} p_{i,j}^{(2)} &= \Pr\left\{\mathcal{R}\left(t+2\right) = R_j \mid \mathcal{R}\left(t\right) = R_i\right\} \\ &= \sum_{k=1}^{K} \Pr\left\{\mathcal{R}\left(t+2\right) = R_j, \mathfrak{R}\left(t+1\right) = R_k \mid \mathcal{R}\left(t\right) = R_i\right\} \\ &= \sum_{k=1}^{K} \Pr\left\{\mathcal{R}\left(t+2\right) = R_j \mid \mathcal{R}\left(t+1\right) = R_k\right\} \cdot \Pr\left\{\mathcal{R}\left(t+1\right) = R_k \mid \mathcal{R}\left(t\right) = R_i\right\} \\ &= \sum_{k=1}^{K} p_{i,k} \cdot p_{k,j} \end{aligned}$$

Rating migration matrix

Discrete time modeling

- The forward Chapman-Kolmogorov equation is :

$$p_{i,j}^{(n+m)} = \sum_{k=1}^{K} p_{i,k}^{(n)} \cdot p_{k,j}^{(m)} \qquad \forall n, m > 0$$

  or $P^{(n+m)} = P^{(n)} \cdot P^{(m)}$ with $P^{(0)} = I$

- We have:

$$\begin{aligned} P^{(n)} &= P^{(n-1)} \cdot P^{(1)} \\ &= P^{(n-2)} \cdot P^{(1)} \cdot P^{(1)} \\ &= \prod_{t=1}^{n} P^{(1)} \\ &= P^{n} \end{aligned}$$

- We deduce that:

$$p\left(t, i; t+n, j\right) = p_{i,j}^{(n)} = \mathbf{e}_i^\top P^n \mathbf{e}_j$$

Rating migration matrix

Discrete time modeling

Table 17: Two-year transition probability in % (migration matrix #1)

	AAA	AA	A	BBB	BB	B	CCC
AAA	86.28	10.08	2.25	0.92	0.44	0.02	0.00
AA	7.30	70.52	18.68	2.67	0.66	0.15	0.00
A	0.95	24.24	57.16	15.20	2.19	0.25	0.01
BBB	0.21	5.06	28.22	52.11	12.93	1.33	0.14
BB	0.09	0.79	6.07	27.45	53.68	11.37	0.55
B	0.01	0.18	0.98	6.26	31.47	57.28	3.82
CCC	0.00	0.05	0.50	1.32	6.31	21.13	70.70

Rating migration matrix

Discrete time modeling

We have:

$$
\begin{aligned}
p_{\mathrm{AAA},\mathrm{AAA}}^{(2)} &= p_{\mathrm{AAA},\mathrm{AAA}} \times p_{\mathrm{AAA},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{AA}} \times p_{\mathrm{AA},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{A}} \times p_{\mathrm{A},\mathrm{AAA}} + \\
&\quad p_{\mathrm{AAA},\mathrm{BBB}} \times p_{\mathrm{BBB},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{BB}} \times p_{\mathrm{BB},\mathrm{AAA}} + \\
&\quad p_{\mathrm{AAA},\mathrm{B}} \times p_{\mathrm{B},\mathrm{AAA}} + p_{\mathrm{AAA},\mathrm{CCC}} \times p_{\mathrm{CCC},\mathrm{AAA}} \\
&= 0.9276^2 + 0.0566 \times 0.0415 + 0.0090 \times 0.0018 + \\
&\quad 0.0045 \times 0.0007 + 0.0023 \times 0.0004 \\
&= 86.28\%
\end{aligned}
$$

Rating migration matrix

Discrete time modeling

Table 18: Five-year transition probability in % (migration matrix #1)

	AAA	AA	A	BBB	BB	B	CCC
AAA	70.45	18.69	6.97	2.61	1.08	0.18	0.01
AA	13.13	50.21	26.03	7.90	2.22	0.48	0.03
A	4.35	33.20	37.78	17.99	5.52	1.08	0.09
BBB	1.50	16.49	32.49	30.90	14.61	3.63	0.38
BB	0.50	5.98	17.83	30.10	31.35	12.85	1.39
B	0.15	1.90	7.40	18.95	35.11	31.26	5.23
CCC	0.05	0.64	2.55	6.93	17.96	38.54	43.33

Rating migration matrix

Discrete time modeling

Stationary distribution

- $\pi_k^{(n)} = \Pr\left\{\mathcal{R}(n) = R_k\right\}$ is the probability of the state R_k at time n:
- $\pi^{(n)} = \left(\pi_1^{(n)}, \ldots, \pi_K^{(n)}\right)$ satisfies $\pi^{(n+1)} = P^\top \pi^{(n)}$
- The Markov chain $\mathcal{R}$ has a stationary distribution $\pi^\star$ if $\pi^\star = P^\top \pi^\star$
- $\mathcal{T}_k = \inf\left\{n : \mathcal{R}(n) = R_k \mid \mathcal{R}(0) = R_k\right\}$ is the return period of state R_k
- The average return period is then equal to:

$$\boldsymbol{\tau}_k := \mathbb{E}\left[\mathcal{T}_k\right] = \frac{1}{\pi_k^\star}$$

Rating migration matrix

Discrete time modeling

- We obtain:

$$\pi^\star = (17.78\%, 29.59\%, 25.12\%, 15.20\%, 8.35\%, 3.29\%, 0.67\%)$$

- The average return periods are then equal to 5.6, 3.4, 4.0, 6.6, 12.0, 30.4 and 149.0 years

$\Rightarrow$ Best-in-class (or winning-) oriented system

Rating migration matrix

Discrete time modeling

Table 19: ESG migration matrix #2 (one-month transition probability in %)

	AAA	AA	A	BBB	BB	B	CCC
AAA	93.50	5.00	0.50	0.50	0.50	0.00	0.00
AA	2.00	93.00	4.00	0.50	0.50	0.00	0.00
A	0.00	3.00	93.00	3.90	0.10	0.00	0.00
BBB	0.00	0.10	2.80	94.00	3.00	0.10	0.00
BB	0.00	0.00	0.10	3.50	94.50	1.80	0.10
B	0.00	0.00	0.00	0.10	3.70	96.00	0.20
CCC	0.00	0.00	0.00	0.40	0.50	0.60	98.50

$\Rightarrow$ The stationary distribution is $\pi^{\star} = (3.11\%, 10.10\%, 17.46\%, 27.76\%, 25.50\%, 12.68\%, 3.39\%)$ and the average return periods are equal to 32.2, 9.9, 5.7, 3.6, 3.9, 7.9 and 29.5 years

$\Rightarrow$ balanced rating system

Rating migration matrix

Discrete time modeling

Table 20: One-year probability transition in % (migration matrix #2)

	AAA	AA	A	BBB	BB	B	CCC
AAA	48.06	29.71	10.34	6.42	4.95	0.49	0.03
AA	11.65	49.25	24.10	9.60	4.87	0.49	0.03
A	2.02	17.51	49.67	24.72	5.52	0.54	0.03
BBB	0.27	3.53	17.46	55.50	20.21	2.88	0.16
BB	0.03	0.60	4.21	23.43	57.45	13.27	1.01
B	0.00	0.08	0.74	5.94	27.10	64.18	1.96
CCC	0.00	0.07	0.57	4.22	5.77	5.85	83.51

Rating migration matrix

Discrete time modeling

Table 21: One-month probability transition in % (migration matrix #1)

	AAA	AA	A	BBB	BB	B	CCC
AAA	99.36	0.53	0.05	0.04	0.02	0.00	0.00
AA	0.39	98.31	1.26	0.01	0.03	0.01	0.00
A	**−0.02**	1.65	97.14	1.21	0.02	0.01	0.00
BBB	0.01	**−0.07**	2.28	96.72	1.06	**−0.01**	0.01
BB	0.00	0.02	**−0.12**	2.29	96.92	0.88	0.01
B	0.00	0.00	0.04	**−0.15**	2.45	97.42	0.25
CCC	0.00	0.00	0.02	0.04	0.05	1.37	98.53

$\Rightarrow$ Negative probabilities

The ESG rating system is not Markovian!

Rating migration matrix

Discrete time modeling

Mean hitting time

- Let $\mathcal{A} \subset \mathbb{K}$ be a given subset. The first hitting time of $\mathcal{A}$ is given by:

$$\mathcal{T}(\mathcal{A}) = \inf \{n : \mathcal{R}(n) \in \mathcal{A}\}$$

- The mean first hitting time to target $\mathcal{A}$ from state k is defined as:

$$\boldsymbol{\tau}_k(\mathcal{A}) = \mathbb{E}[\mathcal{T}(\mathcal{A}) \mid \mathcal{R}(0) = R_k]$$

- We can show that $\boldsymbol{\tau}_k(\mathcal{A}) = 1 + \sum_{j=1}^{K} p_{k,j} \boldsymbol{\tau}_j(\mathcal{A})$
- The solution is given by the LP problem:

$$\boldsymbol{\tau}(\mathcal{A}) = \arg\min \sum_{k=1}^{K} x_k \quad \text{s.t.} \quad \begin{cases} x_k = 0 & \text{if } k \in \mathcal{A} \\ x_k = 1 + \sum_{j=1}^{K} p_{k,j} x_j & \text{if } k \notin \mathcal{A} \\ x_k \geq 0 \end{cases}$$

Rating migration matrix

Discrete time modeling

- $\mathcal{B} = \{\text{AAA}, \text{AA}, \text{A}\}$
- $\mathcal{W} = \{\text{BB}, \text{B}, \text{CCC}\}$

Rating system	$\mathcal{W}$-target				$\mathcal{B}$-target			
	AAA	AA	A	BBB	BBB	BB	B	CCC
#1	79.21	70.04	62.34	46.54	7.50	13.28	17.58	22.68
#2	10.24	9.92	9.13	6.68	8.68	11.99	14.26	17.54

Rating migration matrix

Estimation of the transition matrix

Theoretical approach:

- Bayes theorem:

$$\begin{aligned} p_{i,j} &= \Pr\left\{\mathcal{R}(t+1) = R_j \mid \mathcal{R}(t) = R_i\right\} \\ &= \frac{\Pr\left\{\mathcal{R}(t+1) = R_j, \mathcal{R}(t) = R_i\right\}}{\Pr\left\{\mathcal{R}(t) = R_i\right\}} \end{aligned}$$

- We have seen that:

$$\Pr\left\{\mathcal{R}(t) = R_k\right\} = \mathbf{F}\left(s_k^\star\right) - \mathbf{F}\left(s_{k-1}^\star\right) = p_k$$

- We deduce that:

$$p_{i,j} = \frac{\mathbf{C}\left(\mathbf{F}\left(s_i^\star\right), \mathbf{F}\left(s_j^\star\right)\right) - \mathbf{C}\left(\mathbf{F}\left(s_{i-1}^\star\right), \mathbf{F}\left(s_j^\star\right)\right) - \mathbf{C}\left(\mathbf{F}\left(s_i^\star\right), \mathbf{F}\left(s_{j-1}^\star\right)\right) + \mathbf{C}\left(\mathbf{F}\left(s_{i-1}^\star\right), \mathbf{F}\left(s_{j-1}^\star\right)\right)}{\mathbf{F}\left(s_i^\star\right) - \mathbf{F}\left(s_{i-1}^\star\right)}$$

where $\mathbf{C}$ is the copula function of the random vector $\left(\boldsymbol{\mathcal{S}}(t), \boldsymbol{\mathcal{S}}(t+1)\right)$

Rating migration matrix

Estimation of the transition matrix

Non-parametric approach:

$$\hat{p}_{i,j}(t) = \frac{\#\{\mathcal{R}(t+1) = R_j, \mathcal{R}(t) = R_i\}}{\#\{\mathcal{R}(t) = R_i\}} = \frac{n_{i,j}(t)}{n_{i,\cdot}(t)}$$

$\Rightarrow$ cohort method vs. pooling method

Rating migration matrix

Estimation of the transition matrix

Table 22: Number of observations $n_{i,j}$ (migration matrix #1)

$n_{i,j}$	AAA	AA	A	BBB	BB	B	CCC	$n_{i,\cdot}(t)$	$\hat{p}_{i,\cdot}(t)$
AAA	2 050	125	20	10	5	0	0	2 210	3.683%
AA	280	5 580	800	60	20	5	0	6 745	11.242%
A	20	1 700	8 020	1 150	90	10	0	10 990	18.317%
BBB	10	190	2 820	10 000	1 300	60	10	14 390	23.983%
BB	5	25	200	2 500	9 150	1 000	30	12 910	21.517%
B	0	5	25	150	2 260	7 800	250	10 490	17.483%
CCC	0	0	5	10	50	300	1 900	2 265	3.775%
$n_{\cdot,j}(t)$	2 365	7 625	11 890	13 850	12 875	9 175	2 190	60 000	
$\hat{p}_{\cdot,j}(t)$	3.942%	12.708%	19.817%	23.133%	21.458%	15.292%	3.650%		100.00%

Rating migration matrix
Estimation of the transition matrix

- For the migration matrix #1, we have:

$$\pi^{\star} = (17.78\%, 29.59\%, 25.12\%, 15.20\%, 8.35\%, 3.29\%, 0.67\%)$$

- The initial empirical distribution of ratings is:

$$\hat{\pi}^{(0)} = (3.683\%, 11.242\%, 18.317\%, 23.983\%, 21.517\%, 17.483\%, 3.775\%)$$

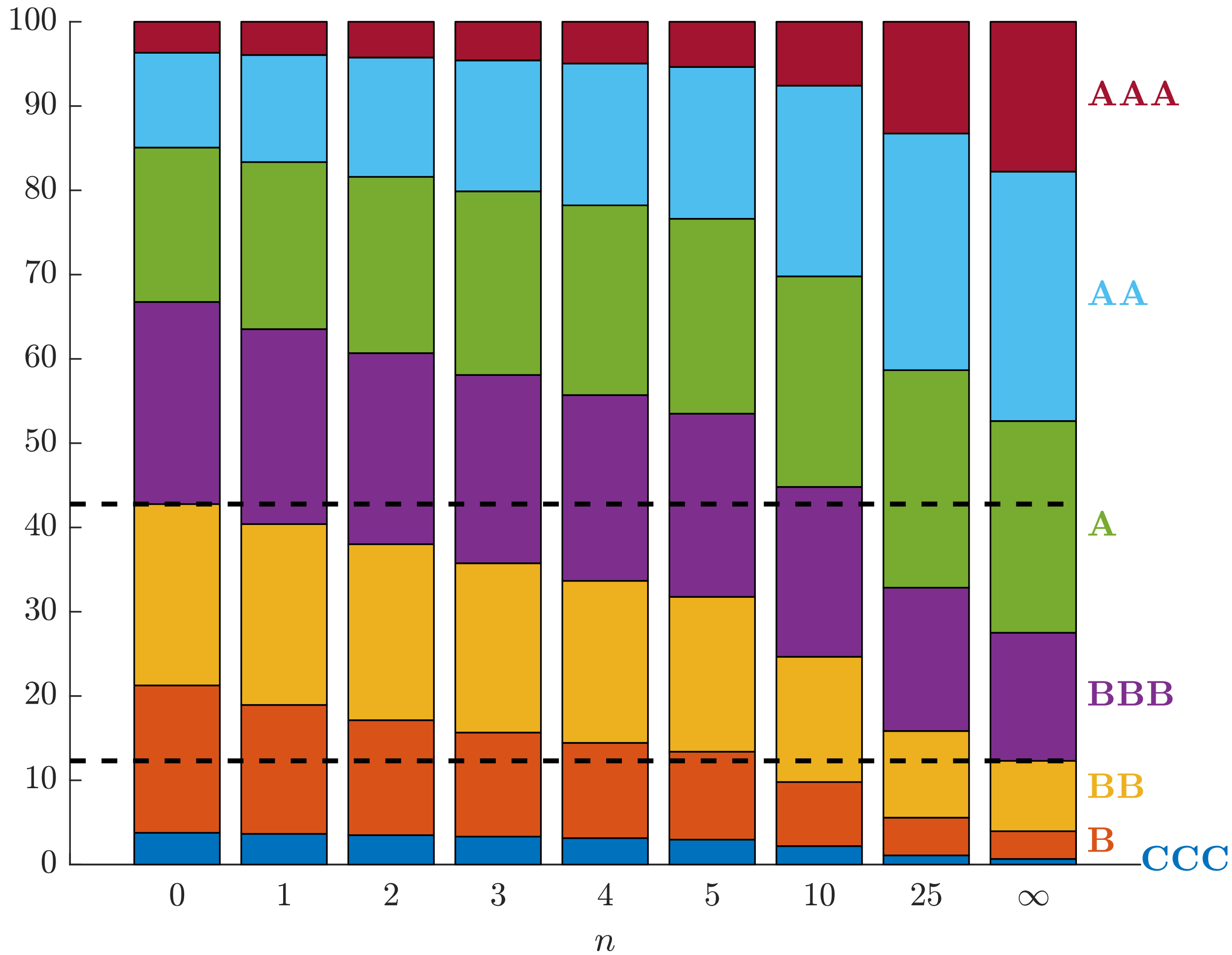
- We have:

$$\begin{aligned} \hat{\pi}^{(1)} &= \hat{P}^{\top}\hat{\pi}^{(0)} \\ &= (3.942\%, 12.708\%, 19.817\%, 23.133\%, 21.458\%, 15.290\%, 3.650\%) \end{aligned}$$

Rating migration matrix

Estimation of the transition matrix

Figure 21: Dynamics of the probability distribution $\pi^{(n)}$ (migration matrix #1)



Rating migration matrix

Continuous-time modeling

Markov generator

- $t \in \mathbb{R}_+$
- The transition matrix is defined as follows:

$$P_{i,j}(s;t) = p(s,i;t,j) = \Pr\{\mathcal{R}(t) = R_j \mid \mathcal{R}(s) = R_i\}$$

- If $\mathcal{R}$ is a time-homogenous Markov, we have:

$$P(t) = P(0;t) = \exp(t\Lambda)$$

- $\Lambda = (\lambda_{i,j})$ is the Markov generator matrix $\Lambda = (\lambda_{i,j})$ where $\lambda_{i,j} \geq 0$ for all $i \neq j$ and $\lambda_{i,i} = -\sum_{j\neq i}^{K} \lambda_{i,j}$

Rating migration matrix

Continuous-time modeling

An example

- Rating system with three states: A (good rating), B (average rating) and C (bad rating)
- The Markov generator is equal to:

$$\Lambda = \begin{pmatrix} -0.30 & 0.20 & 0.10 \\ 0.15 & -0.40 & 0.25 \\ 0.10 & 0.15 & -0.25 \end{pmatrix}$$

Rating migration matrix

Continuous-time modeling

- The one-year transition probability matrix is equal to:

$$P(1) = e^{\Lambda} = \begin{pmatrix} 75.63\% & 14.84\% & 9.53\% \\ 11.63\% & 69.50\% & 18.87\% \\ 8.52\% & 11.73\% & 79.75\% \end{pmatrix}$$

- For the two-year maturity, we get:

$$P(2) = e^{2\Lambda} = \begin{pmatrix} 59.74\% & 22.65\% & 17.61\% \\ 18.49\% & 52.24\% & 29.27\% \\ 14.60\% & 18.76\% & 66.63\% \end{pmatrix}$$

- We verify that $P(2) = P(1) \cdot P(1)$ because:

$$P(t) = e^{t\Lambda} = \left(e^{\Lambda}\right)^{t} = P(1)^{t}$$

- We have:

$$P\left(\frac{1}{12}\right) = e^{\frac{1}{12}\Lambda} = \begin{pmatrix} 97.54\% & 1.62\% & 0.83 \\ 1.22\% & 96.74\% & 2.03 \\ 0.82\% & 1.22\% & 97.95 \end{pmatrix}$$

Rating migration matrix

Matrix function

Matrix function

We consider the matrix function in the space $\mathbb{M}$ of square matrices:

$$\begin{aligned} f: \quad \mathbb{M} &\longrightarrow \mathbb{M} \\ A &\longmapsto B = f(A) \end{aligned}$$

For instance, if $f(x) = \sqrt{x}$ and A is positive, we can define the matrix B such that:

$$BB^* = B^*B = A$$

B is called the square root of A and we note $B = A^{1/2}$

Rating migration matrix

Matrix function

- We consider the following Taylor expansion:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots$$

- We can show that if the series converge for $|x - x_0| < \alpha$, then the matrix $f(A)$ defined by the following expression:

$$f(A) = f(x_0) + (A - x_0 I) f'(x_0) + \frac{(A - x_0 I)^2}{2!} f''(x_0) + \ldots$$

converges to the matrix B if $|A - x_0 I| < \alpha$ and we note $B = f(A)$

Rating migration matrix

Matrix function

- In the case of the exponential function, we have:

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- We deduce that the exponential of the matrix A is equal to:

$$B = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k"!}$$

- The logarithm of A is the matrix B such that $e^B = A$ and we note $B = \ln A$

Rating migration matrix

Matrix function

- Let A and B be two $n \times n$ square matrices. We have the properties:

$$
\begin{aligned}
f\left(A^\top\right) &= f(A)^\top \\
Af(A) &= f(A)A \\
f\left(B^{-1}AB\right) &= B^{-1}f(A)B
\end{aligned}
$$

- It follows that:

$$
\begin{cases}
e^{A^\top} = \left(e^A\right)^\top & \\
e^{B^{-1}AB} = B^{-1}e^A B & \\
Ae^B = e^B A & \text{if } AB = BA \\
e^{A+B} = e^A e^B = e^B e^A & \text{if } AB = BA
\end{cases}
$$

Rating migration matrix

Matrix function

Definition

The Schur decomposition of the $n \times n$ matrix A is equal to:

$$A = QTQ^*$$

where Q is a unitary matrix and T is an upper triangular matrix

For transcendental functions, we have:

$$f(A) = Qf(T)Q^*$$

where $A = QTQ^*$ is the Schur decomposition of A

Rating migration matrix

Continuous-time modeling

Estimation of the Markov generator

We have:

$$\hat{\Lambda} = \frac{1}{t} \ln \left(\hat{P}(t) \right)$$

$\Rightarrow$ $\hat{\Lambda}$ may not verify the Markov conditions: $\hat{\lambda}_{i,j} \geq 0$ for all $i \neq j$ and $\sum_{j=1}^{K} \lambda_{i,j} = 0$

Rating migration matrix

Continuous-time modeling

Table 23: Non-Markov generator $\Lambda' = \ln(P)$ of the migration matrix #1 (in %)

	AAA	AA	A	BBB	BB	B	CCC
AAA	−7.663	6.427	0.542	0.466	0.245	**−0.016**	**−0.000**
AA	4.770	−20.604	15.451	**−0.001**	0.318	0.066	**−0.001**
A	**−0.267**	20.259	−35.172	14.953	0.152	0.083	**−0.008**
BBB	0.102	**−1.051**	28.263	−40.366	13.100	**−0.128**	0.080
BB	0.032	0.307	**−1.762**	28.351	−37.889	10.832	0.129
B	**−0.005**	**−0.008**	0.503	**−2.240**	30.227	−31.482	3.006
CCC	0.000	**−0.024**	0.194	0.469	0.365	16.806	−17.810

Rating migration matrix

Continuous-time modeling

Israel-Rosenthal-Wei estimators

1. The first approach consists in adding the negative values back into the diagonal values:

$$\begin{cases} \bar{\lambda}_{i,j} = \max\left(\hat{\lambda}_{i,j}, 0\right) & i \neq j \\ \bar{\lambda}_{i,i} = \hat{\lambda}_{i,i} + \sum_{j \neq i} \min\left(\hat{\lambda}_{i,j}, 0\right) \end{cases}$$

2. The second estimator carries forward the negative values on the matrix entries which have the correct sign:

$$\begin{cases} G_i = \left|\hat{\lambda}_{i,i}\right| + \sum_{j \neq i} \max\left(\hat{\lambda}_{i,j}, 0\right), B_i = \sum_{j \neq i} \max\left(-\hat{\lambda}_{i,j}, 0\right) \\ \tilde{\lambda}_{i,j} = \begin{cases} 0 & \text{if } i \neq j \text{ and } \hat{\lambda}_{i,j} < 0 \\ \hat{\lambda}_{i,j} - B_i \left|\hat{\lambda}_{i,j}\right| / G_i & \text{if } G_i > 0 \\ \hat{\lambda}_{i,j} & \text{if } G_i = 0 \end{cases} \end{cases}$$

Rating migration matrix

Continuous-time modeling

Table 24: Markov generator of the migration matrix #1 (in %)

	AAA	AA	A	BBB	BB	B	CCC
AAA	−7.679	6.427	0.542	0.466	0.245	0.000	0.000
AA	4.770	−20.606	15.451	0.000	0.318	0.066	0.000
A	0.000	20.259	−35.447	14.953	0.152	0.083	0.000
BBB	0.102	0.000	28.263	−41.545	13.100	0.000	0.080
BB	0.032	0.307	0.000	38.351	−39.651	10.832	0.129
B	0.000	0.000	0.503	0.000	30.227	−33.735	3.006
CCC	0.000	0.000	0.194	0.469	0.365	16.806	−17.834

Rating migration matrix

Continuous-time modeling

Table 25: ESG migration Markov matrix #1 (one-year transition probability in %)

	AAA	AA	A	BBB	BB	B	CCC
AAA	92.75	5.66	0.90	0.45	0.23	0.01	0.00
AA	4.17	82.73	11.85	0.89	0.30	0.07	0.00
A	0.40	15.51	72.79	10.39	0.81	0.10	0.01
BBB	0.12	2.11	19.60	68.69	8.91	0.50	0.07
BB	0.04	0.43	2.79	19.25	69.65	7.61	0.23
B	0.01	0.09	0.65	2.98	21.21	72.71	2.35
CCC	0.00	0.02	0.25	0.58	2.19	13.09	83.87

Rating migration matrix

Continuous-time modeling

Table 26: **Original** migration matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	92.76	5.66	0.90	0.45	0.23	0.00	0.00
AA	4.15	82.73	11.86	0.89	0.30	0.07	0.00
A	**0.18**	15.47	72.98	10.46	0.82	0.09	0.00
BBB	0.07	**1.32**	19.60	69.49	9.03	0.42	0.07
BB	0.04	0.19	**1.55**	19.36	70.88	7.75	0.23
B	0.00	0.05	0.24	**1.43**	21.54	74.36	2.38
CCC	0.00	0.00	0.22	0.44	2.21	13.24	83.89

Table 27: **New** migration matrix

	AAA	AA	A	BBB	BB	B	CCC
AAA	92.75	5.66	0.90	0.45	0.23	0.01	0.00
AA	4.17	82.73	11.85	0.89	0.30	0.07	0.00
A	**0.40**	15.51	72.79	10.39	0.81	0.10	0.01
BBB	0.12	**2.11**	19.60	68.69	8.91	0.50	0.07
BB	0.04	0.43	**2.79**	19.25	69.65	7.61	0.23
B	0.01	0.09	0.65	**2.98**	21.21	72.71	2.35
CCC	0.00	0.02	0.25	0.58	2.19	13.09	83.87

Rating migration matrix

Continuous-time modeling

Why it is important that ESG ratings satisfy the Markov property

- Lack of memory:

$t-2$		$t-1$		t		$t+1$
AAA	$\longrightarrow$	BBB	$\longrightarrow$	BBB	$\longrightarrow$	?
BBB	$\longrightarrow$	BBB	$\longrightarrow$	BBB	$\longrightarrow$	?
BB	$\longrightarrow$	BB	$\longrightarrow$	BBB	$\longrightarrow$	?

- Non-Markov property:

$$\Pr\left\{\mathcal{R}_{c_1}(t+1) = R_j \mid \mathcal{R}_{c_1}(t) = R_i\right\} \neq \Pr\left\{\mathcal{R}_{c_2}(t+1) = R_j \mid \mathcal{R}_{c_2}(t) = R_i\right\}$$

for two different companies c_1 and c_2

Rating migration matrix

Continuous-time modeling

How to perform a dynamic analysis?

- We deduce that:

$$\pi_k(t,\mathcal{A}) = \Pr\{\mathcal{R}(t) \in \mathcal{A} \mid \mathcal{R}(0) = k\} = \sum_{j\in\mathcal{A}} \mathbf{e}_k^\top e^{t\Lambda} \mathbf{e}_j$$

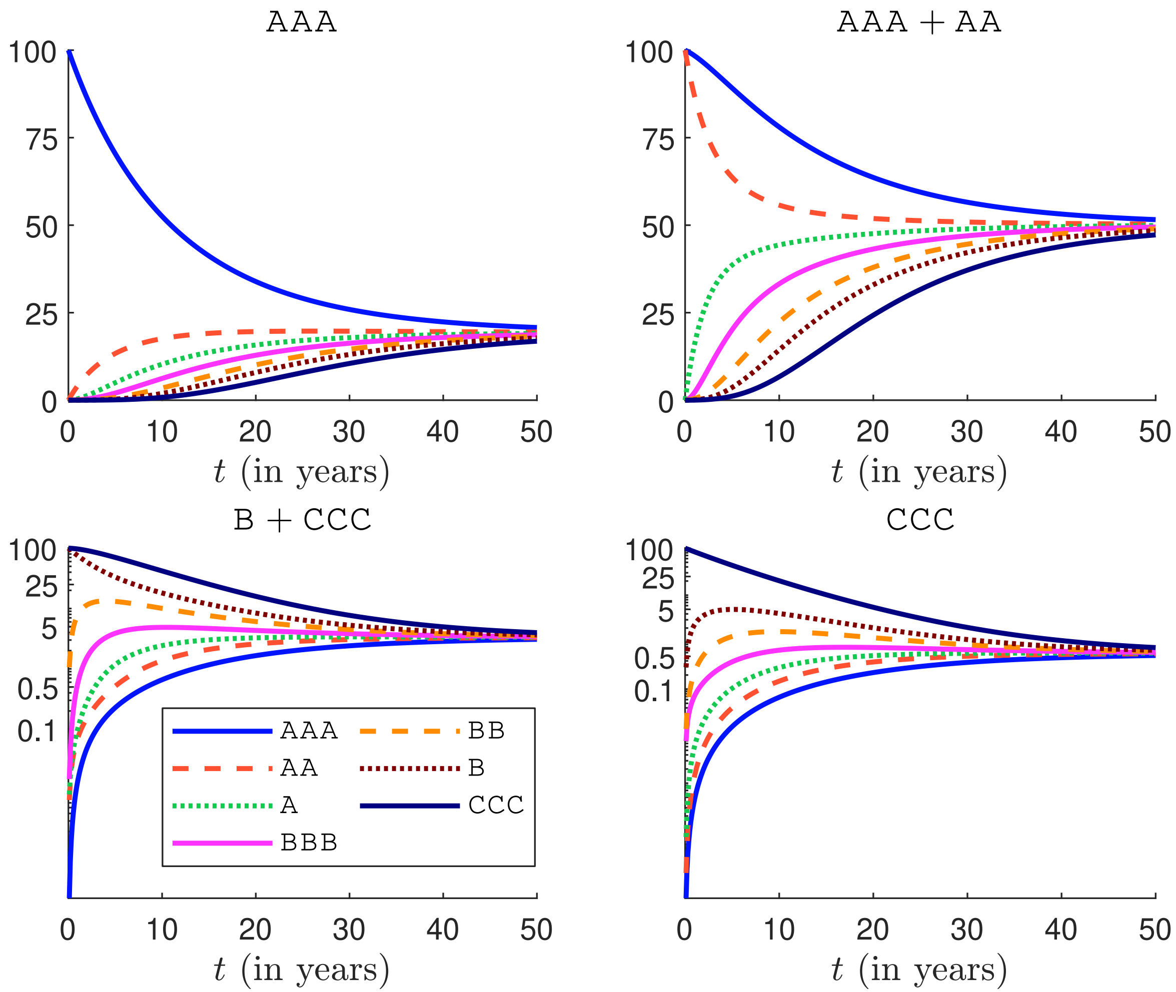
- Some properties
  - $\partial_t \exp(\Lambda t) = \Lambda \exp(\Lambda t)$
  - $\partial_t^m \exp(\Lambda t) = \Lambda^m \exp(\Lambda t)$
  - $\int_0^t e^{\Lambda s}\,\mathrm{d}s = \left(e^{\Lambda t} - I_K\right)\Lambda^{-1}$
- For example, the "*time density function*" is given by:

$$\pi_k^{(m)}(t,\mathcal{A}) := \frac{\partial\,\pi_k(t,\mathcal{A})}{\partial\,t^m} = \sum_{j\in\mathcal{A}} \mathbf{e}_k^\top \Lambda^m e^{t\Lambda} \mathbf{e}_j$$

Rating migration matrix

Continuous-time modeling

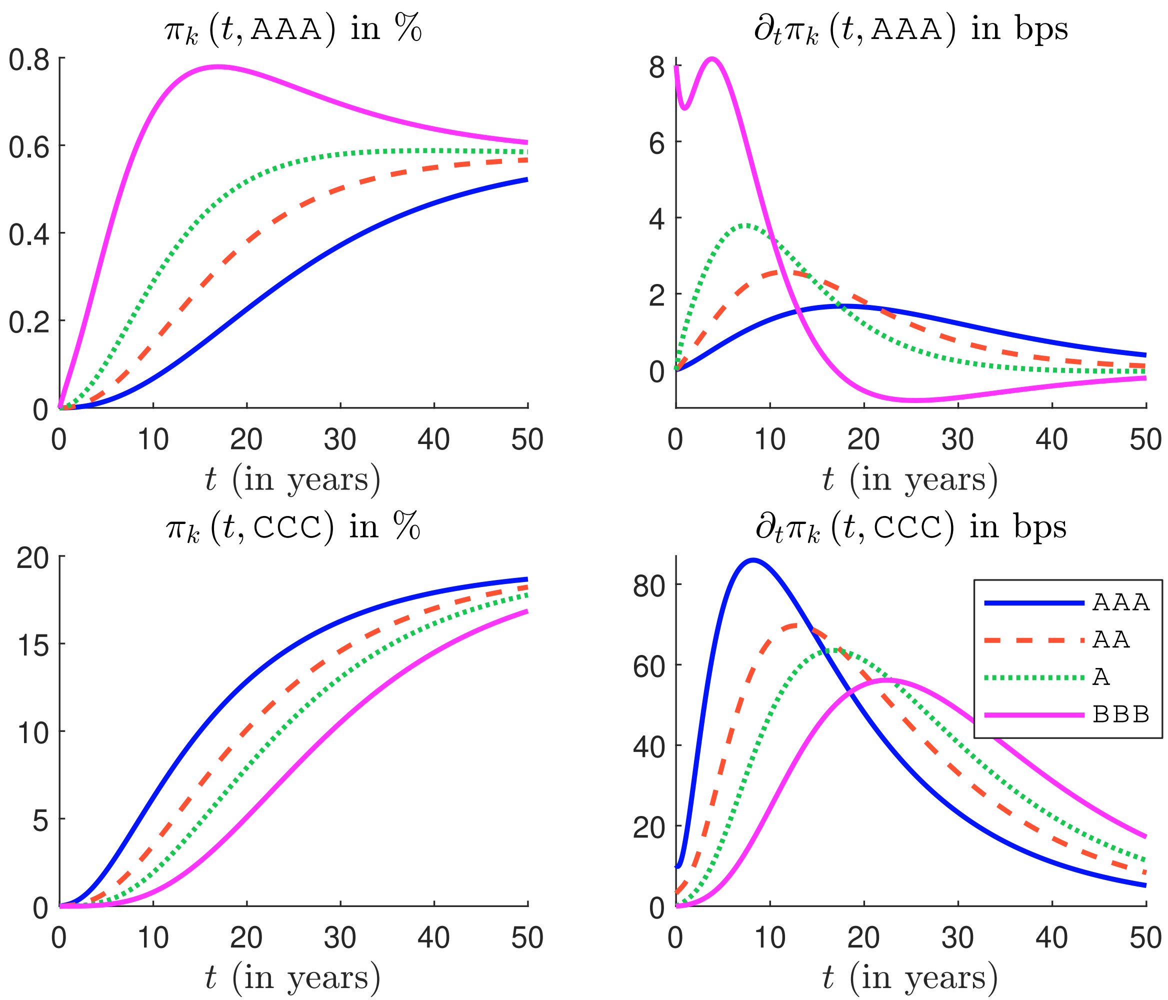
Figure 22: Probability $\pi_k(t, \mathcal{A})$ to reach $\mathcal{A}$ at time t (migration matrix #1)



Rating migration matrix

Continuous-time modeling

Figure 23: Dynamic analysis (migration matrix #1)



Rating migration matrix

Comparison with credit ratings

Table 28: Example of credit migration matrix (one-year probability transition in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.82	6.50	0.56	0.06	0.06	0.00	0.00	0.00
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.00
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.04
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.22
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.06
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Source: Kavvathas (2001).

Rating migration matrix

Comparison with credit ratings

The trace statistics is equal to:

$$\lambda(t) = \frac{\operatorname{trace}\left(e^{t\Lambda}\right)}{K}$$

Rating migration matrix

Comparison with credit ratings

Figure 24: Trace statistics of credit and ESG migration matrices