Course 2024–2025 in Sustainable Finance
Lecture 3. Exercise
Equity Portfolio Optimization with ESG Scores

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[1]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

We consider the CAPM model:

$$R_i - r = \beta_i (R_m - r) + \varepsilon_i$$

where R_i is the return of asset i, R_m is the return of the market portfolio w_m, r is the risk free asset, β_i is the beta of asset i with respect to the market portfolio and ε_i is the idiosyncratic risk of asset i. We have $R_m \perp \varepsilon_i$ and $\varepsilon_i \perp \varepsilon_j$. We note σ_m the volatility of the market portfolio. Let $\tilde{\sigma}_i$, μ_i and $\boldsymbol{\mathcal{S}}_i$ be the idiosyncratic volatility, the expected return and the ESG score of asset i. We use a universe of 6 assets with the following parameter values:

Asset i	1	2	3	4	5	6
β_i	0.10	0.30	0.50	0.90	1.30	2.00
$\tilde{\sigma}_i$ (in %)	17.00	17.00	16.00	10.00	11.00	12.00
μ_i (in %)	1.50	2.50	3.50	5.50	7.50	11.00
$\boldsymbol{\mathcal{S}}_i$	1.10	1.50	2.50	-1.82	-2.35	-2.91

and $\sigma_m = 20\%$. The risk-free return r is set to 1% and the expected return of the market portfolio w_m is equal to $\mu_m = 6\%$.

Question 1

We assume that the CAPM is valid.

Question (a)

Calculate the vector μ of expected returns.

- Using the CAPM, we have:

$$\mu_i = r + \beta_i (\mu_m - r)$$

- For instance, we have:

$$\mu_1 = 1\% + 0.10 \times (6\% - 1\%) = 1.5\%$$

and:

$$\mu_2 = 1\% + 0.30 \times 5\% = 2.5\%$$

- Finally, we obtain $\mu = (1.5\%, 2.5\%, 3.5\%, 5.5\%, 7.5\%, 11\%)$

Question (b)

Compute the covariance matrix Σ. Deduce the volatility σ_i of the asset i and find the correlation matrix $\mathbb{C} = (\rho_{i,j})$ between asset returns.

- We have:

$$\Sigma = \sigma_m^2 \beta \beta^\top + D$$

where:

$$D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_6^2\right)$$

- The numerical value of Σ is:

$$\Sigma = \begin{pmatrix} 293 & & & & & \\ 12 & 325 & & & & \\ 20 & 60 & 356 & & & \\ 36 & 108 & 180 & 424 & & \\ 52 & 156 & 260 & 468 & 797 & \\ 80 & 240 & 400 & 720 & 1\,040 & 1\,744 \end{pmatrix} \times 10^{-4}$$

- We have:

$$\sigma_i = \sqrt{\Sigma_{i,i}}$$

- We obtain:

$$\sigma = (17.12\%, 18.03\%, 18.87\%, 20.59\%, 28.23\%, 41.76\%)$$

- We have:

$$\rho_{i,j} = \frac{\Sigma_{i,j}}{\sigma_i \sigma_j}$$

- We obtain the following correlation matrix expressed in %:

$$\mathbb{C} = \begin{pmatrix} 100.00 & & & & & \\ 3.89 & 100.00 & & & & \\ 6.19 & 17.64 & 100.00 & & & \\ 10.21 & 29.09 & 46.33 & 100.00 & & \\ 10.76 & 30.65 & 48.81 & 80.51 & 100.00 & \\ 11.19 & 31.88 & 50.76 & 83.73 & 88.21 & 100.00 \end{pmatrix}$$

Question (c)

Compute the tangency portfolio w^*. Calculate $\mu(w^*)$ and $\sigma(w^*)$. Deduce the Sharpe ratio and the ESG score of the tangency portfolio.

- We have:

$$w^* = \frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})} = \begin{pmatrix} 0.94\% \\ 2.81\% \\ 5.28\% \\ 24.34\% \\ 29.06\% \\ 37.57\% \end{pmatrix}$$

- We deduce:

$$\begin{aligned} \mu(w^*) &= w^{*\top}\mu = 7.9201\% \\ \sigma(w^*) &= \sqrt{w^{*\top}\Sigma w^*} = 28.3487\% \\ \mathrm{SR}(w^* \mid r) &= \frac{7.9201\% - 1\%}{28.3487\%} = 0.2441 \\ \mathcal{S}(w^*) &= \sum_{i=1}^{6} w_i^* \mathcal{S}_i = -2.0347 \end{aligned}$$

Question (d)

Compute the beta coefficient $\beta_i\left(w^*\right)$ of the six assets with respect to the tangency portfolio w^*, and the implied expected return $\tilde{\mu}_i$:

$$\tilde{\mu}_i = r + \beta_i\left(w^*\right)\left(\mu\left(w^*\right) - r\right)$$

- We have:

$$\beta_i\left(w^*\right) = \frac{\mathbf{e}_i^\top \Sigma w^*}{\sigma^2\left(w^*\right)}$$

- We obtain:

$$\beta\left(w^*\right) = \begin{pmatrix} 0.0723 \\ 0.2168 \\ 0.3613 \\ 0.6503 \\ 0.9393 \\ 1.4451 \end{pmatrix}$$

- The computation of $\tilde{\mu}_i = r + \beta_i\left(w^*\right)\left(\mu\left(w^*\right) - r\right)$ gives:

$$\tilde{\mu} = \begin{pmatrix} 1.50\% \\ 2.50\% \\ 3.50\% \\ 5.50\% \\ 7.50\% \\ 11.00\% \end{pmatrix}$$

Question (e)

Deduce the market portfolio w_m. Comment on these results.

- $\beta_i(w^*) \neq \beta_i(w_m)$ but risk premia are exact
- Let us assume that the allocation of w_m is equal to α of the tangency portfolio w^* and $1-\alpha$ of the risk-free asset. We deduce that:
$$\beta(w_m) = \frac{\Sigma w_m}{\sigma^2(w_m)} = \frac{\alpha \Sigma w^*}{\alpha^2 \sigma^2(w^*)} = \frac{1}{\alpha}\beta(w^*)$$
- We have:
$$\alpha = \frac{\beta_i(w^*)}{\beta_i(w_m)} = 72.25\%$$
- The market portfolio w_m is equal to 72.25% of the tangency portfolio w^* and 27.75% of the risk-free asset

- We have:

$$\mu(w_m) = r + \alpha(\mu(w^*) - r) = 1\% + 72.25\% \times (7.9201\% - 1\%) = 6\%$$

and:

$$\sigma(w_m) = \alpha\,\sigma(w^*) = 72.25\% \times 28.3487\% = 20.48\%$$

- We deduce that:

$$\mathrm{SR}(w_m \mid r) = \frac{6\% - 1\%}{20.48\%} = 0.2441$$

- We do not obtain the true value of the Sharpe ratio:

$$\mathrm{SR}(w_m \mid r) = \frac{6\% - 1\%}{20\%} = 0.25$$

- The tangency portfolio has an idiosyncratic risk:

$$\sqrt{w_m^\top \left(\sigma_m^2 \beta\beta^\top\right) w^\top} = 20\% < \sigma(w_m) = 20.48\%$$

Question 2

We consider long-only portfolios and we also impose a minimum threshold $\mathcal{S}^{\star}$ for the portfolio ESG score:

$$\mathcal{S}(w) = w^{\top}\mathcal{S} \geq \mathcal{S}^{\star}$$

Question (a)

Let γ be the risk tolerance. Write the mean-variance optimization problem.

- We have:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} w^{\top} \Sigma w - \gamma w^{\top} \mu \\
\text{s.t.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_6^{\top} w = 1 \\
w^{\top} \boldsymbol{\mathcal{S}} \geq \boldsymbol{\mathcal{S}}^{\star} \\
\mathbf{0}_6 \leq w \leq \mathbf{1}_6
\end{array}
\right.
\end{aligned}
$$

Question (b)

Find the QP form of the MVO problem.

- The matrix form of the QP problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} w^{\top} Q w - w^{\top} R \\
&\text{s.t.} \quad \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^{-} \leq w \leq w^{+} \end{array} \right.
\end{aligned}
$$

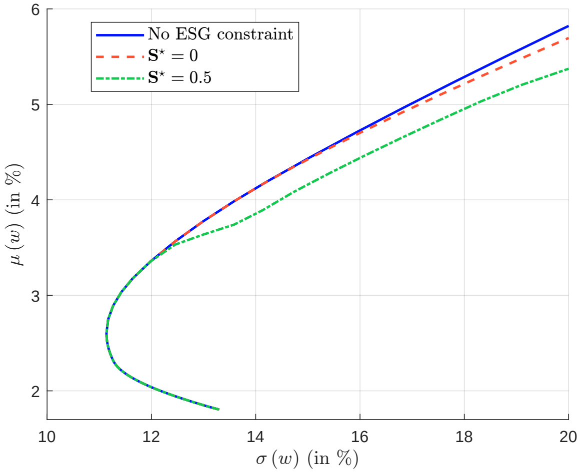
- We deduce that $Q = \Sigma$, $R = \gamma\mu$, $A = \mathbf{1}_6^{\top}$, $B = 1$, $C = -\boldsymbol{\mathcal{S}}^{\top}$, $D = -\boldsymbol{\mathcal{S}}^{\star}$, $w^{-} = \mathbf{0}_6$ and $w^{+} = \mathbf{1}_6$

Question (c)

Compare the efficient frontier when (1) there is no ESG constraint ($\mathcal{S}^{\star} = -\infty$), (2) we impose a positive ESG score ($\mathcal{S}^{\star} = 0$) and (3) the minimum threshold is set to 0.5 ($\mathcal{S}^{\star} = 0.5$). Comment on these results.

- To compute the efficient frontier, we consider several value of $\gamma \in [-1, 2]$
- For each value of γ, we compute the optimal portfolio $w^\star$ and deduce its expected return $\mu(w^\star)$ and its volatility $\sigma(w^\star)$

Figure 1: Impact of the minimum ESG score on the efficient frontier



Question (d)

For each previous cases, find the tangency portfolio w^* and the corresponding risk tolerance γ^*. Compute then $\mu(w^*)$, $\sigma(w^*)$, $\mathrm{SR}(w^* \mid r)$ and $\boldsymbol{\mathcal{S}}(w^*)$. Comment on these results.

- Let $w^\star(\gamma)$ be the MVO portfolio when the risk tolerance is equal to γ
- By using a fine grid of γ values, we can find the optimal value γ^* by solving numerically the following optimization problem with the brute force algorithm:

$$\gamma^* = \arg\max \frac{\mu(w^\star(\gamma)) - r}{\sigma(w^\star(\gamma))} \qquad \text{for } \gamma \in [0, 2]$$

- We deduce the tangency portfolio $w^* = w^\star(\gamma^*)$

Table 1: Impact of the minimum ESG score on the efficient frontier

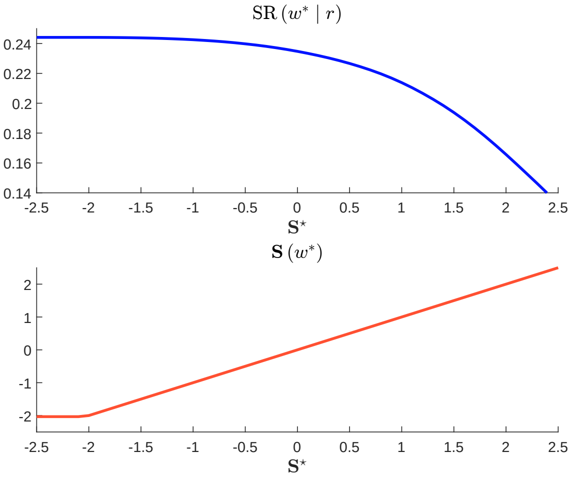
$\mathcal{S}^{\star}$	$-\infty$	0	0.5
γ^{*}	1.1613	0.8500	0.8500
w^{*} (in %)	0.9360	9.7432	9.1481
	2.8079	16.3317	19.0206
	5.2830	31.0176	40.3500
	24.3441	5.1414	0.0000
	29.0609	11.6028	3.8248
	37.5681	26.1633	27.6565
$\mu(w^{*})$ (in %)	7.9201	5.6710	5.3541
$\sigma(w^{*})$ (in%)	28.3487	19.8979	19.2112
$\mathrm{SR}(w^{*} \mid r)$	0.2441	0.2347	0.2266
$\mathcal{S}(w^{*})$	-2.0347	0.0000	0.5000

Question (e)

Draw the relationship between the minimum ESG score $\mathcal{S}^{\star}$ and the Sharpe ratio $\mathrm{SR}(w^{*} \mid r)$ of the tangency portfolio.

- We perform the same analysis as previously for several values $\mathcal{S}^{\star} \in [-2.5, 2.5]$
- We verify that the Sharpe ratio is a decreasing function of $\mathcal{S}^{\star}$

Figure 2: Relationship between the minimum ESG score $\mathcal{S}^{\star}$ and the Sharpe ratio $\mathrm{SR}\left(w^{*} \mid r\right)$ of the tangency portfolio



Question (f)

We assume that the market portfolio w_m corresponds to the tangency portfolio when $\mathcal{S}^{\star} = 0.5$.

- The market portfolio w_m is then equal to:

$$w_m = \begin{pmatrix} 9.15\% \\ 19.02\% \\ 40.35\% \\ 0.00\% \\ 3.82\% \\ 27.66\% \end{pmatrix}$$

- We deduce that:

$$\begin{aligned} \mu(w_m) &= 5.3541\% \\ \sigma(w_m) &= 19.2112\% \\ \mathrm{SR}(w_m \mid r) &= 0.2266 \\ \boldsymbol{\mathcal{S}}(w_m) &= 0.5 \end{aligned}$$

Question (f).i

Compute the beta coefficient $\beta_i\left(w_m\right)$ and the implied expected return $\tilde{\mu}_i\left(w_m\right)$ for each asset. Deduce then the alpha return α_i of asset i. Comment on these results.

- We have:

$$\beta_i(w_m) = \frac{\mathbf{e}_i^\top \Sigma w_m}{\sigma^2(w_m)}$$

and:

$$\tilde{\mu}_i(w_m) = r + \beta_i(w_m)(\mu(w_m) - r)$$

- We deduce that the alpha return is equal to:

$$\begin{aligned} \alpha_i &= \mu_i - \tilde{\mu}_i(w_m) \\ &= (\mu_i - r) - \beta_i(w_m)(\mu(w_m) - r) \end{aligned}$$

- We notice that $\alpha_i < 0$ for the first three assets and $\alpha_i > 0$ for the last three assets, implying that:

$$\begin{cases} \boldsymbol{\mathcal{S}}_i > 0 \Rightarrow \alpha_i < 0 \\ \boldsymbol{\mathcal{S}}_i < 0 \Rightarrow \alpha_i > 0 \end{cases}$$

Table 2: Computation of the alpha return due to the ESG constraint

Asset	$\beta_i(w_m)$	$\tilde{\mu}_i(w_m)$ (in %)	$\tilde{\mu}_i(w_m) - r$ (in %)	α_i (in bps)
1	0.1660	1.7228	0.7228	-22.28
2	0.4321	2.8813	1.8813	-38.13
3	0.7518	4.2733	3.2733	-77.33
4	0.8494	4.6984	3.6984	80.16
5	1.2395	6.3967	5.3967	110.33
6	1.9955	9.6885	8.6885	131.15

Question (f).ii

We consider the equally-weighted portfolio w_{ew}. Compute its beta coefficient $\beta\left(w_{\mathrm{ew}} \mid w_m\right)$, its implied expected return $\tilde{\mu}\left(w_{\mathrm{ew}}\right)$ and its alpha return $\alpha\left(w_{\mathrm{ew}}\right)$. Comment on these results.

- We have:

$$\beta\left(w_{\mathrm{ew}} \mid w_m\right) = \frac{w_{\mathrm{ew}}^\top \Sigma w_m}{\sigma^2\left(w_m\right)} = 0.9057$$

and:

$$\tilde{\mu}\left(w_{\mathrm{ew}}\right) = 1\% + 0.9057 \times (5.3541\% - 1\%) = 4.9435\%$$

- We deduce that:

$$\alpha\left(w_{\mathrm{ew}}\right) = \mu\left(w_{\mathrm{ew}}\right) - \tilde{\mu}\left(w_{\mathrm{ew}}\right) = 5.25\% - 4.9435\% = 30.65 \text{ bps}$$

- We verify that:

$$\alpha\left(w_{\mathrm{ew}}\right) = \sum_{i=1}^{6} w_{\mathrm{ew},i}\alpha_i = \frac{\sum_{i=1}^{6}\alpha_i}{6} = 30.65 \text{ bps}$$

- The equally-weighted portfolio has a positive alpha because:

$$\boldsymbol{\mathcal{S}}\left(w_{\mathrm{ew}}\right) = -0.33 \ll \boldsymbol{\mathcal{S}}\left(w_m\right) = 0.50$$

Question 3

The objective of the investor is twice. He would like to manage the tracking error risk of his portfolio with respect to the benchmark $b = (15\%, 20\%, 19\%, 14\%, 15\%, 17\%)$ and have a better ESG score than the benchmark. Nevertheless, this investor faces a long-only constraint because he cannot leverage his portfolio and he cannot also be short on the assets.

Question (a)

What is the ESG score of the benchmark?

- We have:

$$\mathcal{S}(b) = \sum_{i=1}^{6} b_i \mathcal{S}_i = -0.1620$$

Question (b)

We assume that the investor's portfolio is $w = (10\%, 10\%, 30\%, 20\%, 20\%, 10\%)$. Compute the excess score $\boldsymbol{\mathcal{S}}(w \mid b)$, the expected excess return $\mu(w \mid b)$, the tracking error volatility $\sigma(w \mid b)$ and the information ratio $\mathrm{IR}(w \mid b)$. Comment on these results.

- We have:

$$\left\{ \begin{array}{l} \boldsymbol{\mathcal{S}}\left(w \mid b\right) = \left(w - b\right)^{\top} \boldsymbol{\mathcal{S}} = 0.0470 \\ \mu\left(w \mid b\right) = \left(w - b\right)^{\top} \mu = -0.5 \text{ bps} \\ \sigma\left(w \mid b\right) = \sqrt{\left(w - b\right)^{\top} \Sigma \left(w - b\right)} = 2.8423\% \\ \mathrm{IR}\left(w \mid b\right) = \dfrac{\mu\left(w \mid b\right)}{\sigma\left(w \mid b\right)} = -0.0018 \end{array} \right.$$

- The portfolio w is not optimal since it improves the ESG score of the benchmark, but its information ratio is negative. Nevertheless, the expected excess return is close to zero (less than -1 bps).

Question (c)

Same question with the portfolio $w = (10\%, 15\%, 30\%, 10\%, 15\%, 20\%)$.

- We have:

$$\left\{\begin{array}{l}\boldsymbol{\mathcal{S}}\left(w \mid b\right) = \left(w-b\right)^{\top}\boldsymbol{\mathcal{S}} = 0.1305 \\ \mu\left(w \mid b\right) = \left(w-b\right)^{\top}\mu = 29.5 \text{ bps} \\ \sigma\left(w \mid b\right) = \sqrt{\left(w-b\right)^{\top}\Sigma\left(w-b\right)} = 2.4949\% \\ \mathrm{IR}\left(w \mid b\right) = \dfrac{\mu\left(w \mid b\right)}{\sigma\left(w \mid b\right)} = 0.1182\end{array}\right.$$

Question (d)

In the sequel, we assume that the investor has no return target. In fact, the objective of the investor is to improve the ESG score of the benchmark and control the tracking error volatility. We note γ the risk tolerance. Give the corresponding esg-variance optimization problem.

- The optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) - \gamma \boldsymbol{\mathcal{S}}\left(w \mid b\right) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_6^{\top} w = 1 \\
\mathbf{0}_6 \leq w \leq \mathbf{1}_6
\end{array}
\right.
\end{aligned}
$$

Question (e)

Find the matrix form of the corresponding QP problem.

- The objective function is equal to:

$$\begin{aligned}(*) &= \frac{1}{2}\sigma^2(w \mid b) - \gamma\boldsymbol{\mathcal{S}}(w \mid b) \\ &= \frac{1}{2}(w-b)^\top \Sigma (w-b) - \gamma (w-b)^\top \boldsymbol{\mathcal{S}} \\ &= \frac{1}{2} w^\top \Sigma w - w^\top (\Sigma b + \gamma \boldsymbol{\mathcal{S}}) + \underbrace{\left(\gamma b^\top \boldsymbol{\mathcal{S}} + \frac{1}{2} b^\top \Sigma b\right)}_{\text{does not depend on } w}\end{aligned}$$

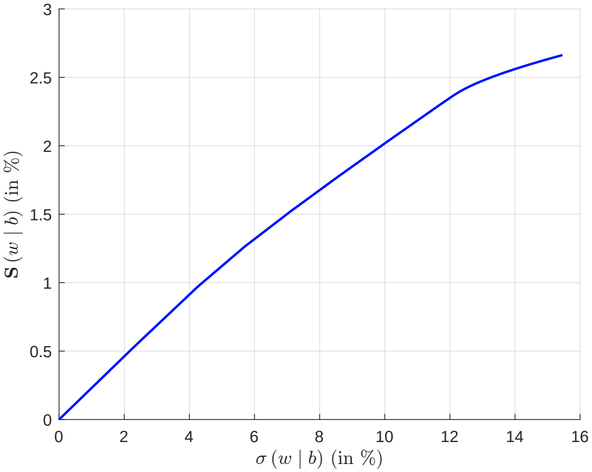
- We deduce that $Q = \Sigma$, $R = \Sigma b + \gamma\boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^- = \mathbf{0}_6$ and $w^+ = \mathbf{1}_6$

Question (f)

Draw the esg-variance efficient frontier $(\sigma(w^\star \mid b), \mathcal{S}(w^\star \mid b))$ where $w^\star$ is an optimal portfolio.

- We solve the QP problem for several values of $\gamma \in [0, 5\%]$ and obtain Figure 3

Figure 3: Efficient frontier of tracking a benchmark with an ESG score objective



Question (g)

Find the optimal portfolio $w^\star$ when we target a given tracking error volatility $\sigma^\star$. The values of $\sigma^\star$ are 0%, 1%, 2%, 3% and 4%.

- Using the QP numerical algorithm, we compte the optimal value $\sigma(w \mid b)$ for $\gamma = 0$ and $\gamma = 5\%$
- Then, we apply the bisection algorithm to find the optimal value $\gamma^\star$ such that:

$$\sigma(w \mid b) = \sigma^\star$$

Table 3: Solution of the σ-problem

Target $\sigma^{\star}$	0	1%	2%	3%	4%
$\gamma^{\star}$ (in bps)	0.000	4.338	8.677	13.015	18.524
$w^{\star}$ (in %)	15.000	15.175	15.350	15.525	14.921
	20.000	21.446	22.892	24.338	25.385
	19.000	23.084	27.167	31.251	35.589
	14.000	9.588	5.176	0.763	0.000
	15.000	12.656	10.311	7.967	3.555
	17.000	18.052	19.104	20.156	20.550
$\mathcal{S}(w^{\star} \mid b)$	0.000	0.230	0.461	0.691	0.915

Question (h)

Find the optimal portfolio $w^{\star}$ when we target a given excess score $\mathcal{S}^{\star}$. The values of $\mathcal{S}^{\star}$ are 0, 0.1, 0.2, 0.3 and 0.4.

- Same method as previously with the following equation:

$$\mathcal{S}\left(w \mid b\right) = \mathcal{S}^{\star}$$

- An alternative approach consists in solving the following optimization problem:

$$\begin{aligned} w^{\star} & = & \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) \\ & & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_{6}^{\top} w = 1 \\ \mathcal{S}\left(w \mid b\right) = \mathcal{S}^{\star} \\ \mathbf{0}_{6} \leq w \leq \mathbf{1}_{6} \end{array} \right. \end{aligned}$$

- We have: $Q = \Sigma$, $R = \Sigma b$, $A = \left(\begin{array}{c} \mathbf{1}_{6}^{\top} \\ \mathcal{S}^{\top} \end{array}\right)$, $B = \left(\begin{array}{c} 1 \\ \mathcal{S}^{\star} + \mathcal{S}^{\top} b \end{array}\right)$, $w^{-} = \mathbf{0}_{6}$ and $w^{+} = \mathbf{1}_{6}$

Table 4: Solution of the $\mathcal{S}$-problem

Target $\mathcal{S}^\star$	0	0.1	0.2	0.3	0.4
$\gamma^\star$ (in bps)	0.000	1.882	3.764	5.646	7.528
$w^\star$ (in %)	15.000	15.076	15.152	15.228	15.304
	20.000	20.627	21.255	21.882	22.509
	19.000	20.772	22.544	24.315	26.087
	14.000	12.086	10.171	8.257	6.343
	15.000	13.983	12.966	11.949	10.932
	17.000	17.456	17.913	18.369	18.825
$\sigma(w^\star \mid b)$ (in %)	0.000	0.434	0.868	1.301	1.735

Question (i)

We would like to compare the efficient frontier obtained in Question 3(f) with the efficient frontier when we implement a best-in-class selection or a worst-in-class exclusion. The selection strategy consists in investing only in the best three ESG assets, while the exclusion strategy implies no exposure on the worst ESG asset. Draw the three efficient frontiers. Comment on these results.

- For the best-in-class strategy, the optimization problem becomes:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) - \gamma \boldsymbol{\mathcal{S}}\left(w \mid b\right) \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_6^\top w = 1 \\ w_4 = w_5 = w_6 = 0 \\ \mathbf{0}_6 \leq w \leq \mathbf{1}_6 \end{array} \right. \end{aligned}$$

- The QP form is defined by $Q = \Sigma$, $R = \Sigma b + \gamma \boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^- = \mathbf{0}_6$ and $w^+ = \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0}_3 \end{pmatrix}$

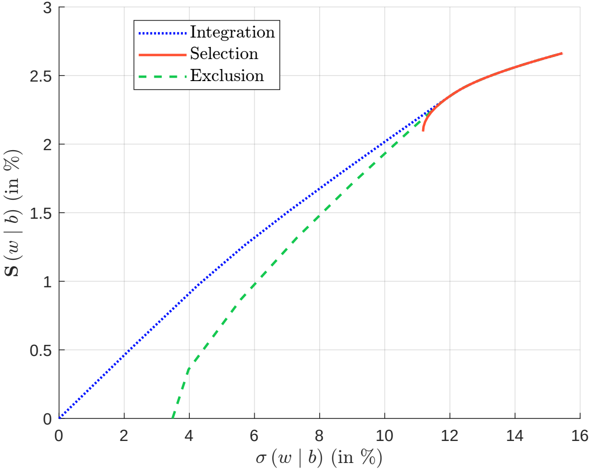
- For the worst-in-class strategy, the optimization problem becomes:

$$
\begin{aligned}
w^{\star} & = \arg\min \frac{1}{2}\sigma^2 \left(w \mid b\right) - \gamma \boldsymbol{\mathcal{S}} \left(w \mid b\right) \\
& \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_6^\top w = 1 \\ w_6 = 0 \\ \mathbf{0}_6 \leq w \leq \mathbf{1}_6 \end{array} \right.
\end{aligned}
$$

- The QP form is defined by $Q = \Sigma$, $R = \Sigma b + \gamma \boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^- = \mathbf{0}_6$ and $w^+ = \left(\begin{array}{c} \mathbf{1}_5 \\ 0 \end{array} \right)$

- The efficient frontiers are reported in Figure 4
- The exclusion strategy has less impact than the selection strategy
- The selection strategy implies a high tracking error risk

Figure 4: Comparison of the efficient frontiers (ESG integration, best-in-class selection and worst-in-class exclusion)



Question (j)

Which minimum tracking error volatility must the investor accept to implement the best-in-class selection strategy? Give the corresponding optimal portfolio.

- We solve the first problem of Question 3(i) with $\gamma = 0$
- We obtain:

$$\sigma\left(w \mid b\right) \geq 11.17\%$$

- The lower bound $\sigma\left(w^{\star} \mid b\right) = 11.17\%$ corresponds to the following optimal portfolio:

$$w^{\star} = \begin{pmatrix} 16.31\% \\ 34.17\% \\ 49.52\% \\ 0\% \\ 0\% \\ 0\% \end{pmatrix}$$

Remark

The impact of ESG scores on optimized portfolios depends on their relationship with expected returns, volatilities, correlations, beta coefficients, etc. In the previous exercise, the results are explained because the best-in-class assets are those with the lowest expected returns and beta coefficients while the worst-in-class assets are those with the highest expected returns and beta coefficients. For instance, we obtain a high tracking error risk for the best-in-class selection strategy, because the best-in-class assets have low volatilities and correlations with respect to worst-in-class assets, implying that it is difficult to replicate these last assets with the other assets.