Smart Beta: Managing Diversification of Minimum Variance Portfolios

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1 The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Lyxor Asset Management.
Main result

The difference in ex-post performance is mainly explained by the ex-ante level of volatility reduction targeted by smart beta portfolios. The choice of the diversification metric is marginal.

⇒ Two consequences:

1. Management report
2. Performance attribution
Comparing the trade-off relationships
Managing the Diversification
Understanding the behavior of smart beta portfolios
Dynamic smart beta strategies

Risk-based portfolios
Main objective

The EW portfolio
\[ x_i = x_j \]
Weights are equal.

The ERC portfolio
\[ RC_i = RC_j \]
Risk contributions are equal.

The GMV portfolio
\[ \min \frac{1}{2} x^\top \Sigma x \]
Minimize the volatility.

The MDP portfolio
\[ \max \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}} \]
Maximize the diversification ratio.
Comparing the trade-off relationships
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Risk-based portfolios
GMV optimization program

$$\arg\min \frac{1}{2} x^\top \Sigma x$$

u.c. $\begin{cases} 1^\top x = 1 \\ \sum_{i=1}^{n} x_i^2 \leq c_1 \\ x \geq 0 \end{cases}$

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.
Comparing the trade-off relationships
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Risk-based portfolios
ERC optimization program

Risk-based portfolios

ERC optimization program

\[
\arg\min_{x} \frac{1}{2} x^T \Sigma x \\
\text{u.c. } \begin{cases} 
1^T x = 1 \\
\sum_{i=1}^{n} x_i^2 \leq c_1 \\
\sum_{i=1}^{n} \ln x_i \geq c_2 \\
x \geq 0
\end{cases}
\]

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.

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Risk-based portfolios

MDP optimization program

\[
\begin{align*}
\text{arg min} \quad & \frac{1}{2} x^\top \Sigma x \\
\text{u.c.} \quad & 1^\top x = 1 \\
& \sum_{i=1}^n x_i^2 \leq c_1 \\
& \sum_{i=1}^n x_i \sigma_i \geq c_3 \\
& x \geq 0
\end{align*}
\]

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.
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Risk-based portfolios
Diversification profile of risk-based portfolios

Diversification profile of risk-based portfolios

**Figure:** The case of Euro Stoxx 50 Index in February 2013
Each risk-based portfolio is a minimum variance portfolio under a specific constraint:

\[
\begin{align*}
    1^\top x &= 1 \quad \text{(GMV)} \\
    \sum_{i=1}^{n} x_i^2 &\leq c_1 \quad \text{(EW)} \\
    \sum_{i=1}^{n} \ln x_i &\geq c_2 \quad \text{(ERC)} \\
    \sum_{i=1}^{n} x_i \sigma_i &\geq c_3 \quad \text{(MDP)}
\end{align*}
\]

We can combine these different constraints to obtain better diversified risk-based portfolios. The first and fourth constraints allow the GMV portfolio and the MDP respectively to be obtained. The second and third constraints manage the diversification in terms of weights and risk contributions.
Mixing the constraints
An example (EW – MDP)

\[ \text{arg min} \quad \frac{1}{2} x^\top \Sigma x \]

\[ \text{u.c.} \quad \begin{cases} 
\sum_{i=1}^{n} x_i^2 \leq c_1 \\
\sum_{i=1}^{n} x_i \sigma_i \geq c_3 \\
x \geq 0 
\end{cases} \]

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.*
We can write the constrained problem using Lagrange multipliers:

\[
x^* = \arg\min \frac{1}{2} x^\top \Sigma x - \\
\lambda_{\text{gmv}} \left( \sum_{i=1}^{n} x_i \right) + \lambda_{\text{h}} \left( \sum_{i=1}^{n} x_i^2 \right) - \\
\lambda_{\text{erc}} \left( \sum_{i=1}^{n} \ln x_i \right) - \lambda_{\text{mdp}} \left( \sum_{i=1}^{n} x_i \sigma_i \right) \\
\text{u.c. } x \geq 0
\]

Remark

The previous framework can be extended by replacing the variance minimization problem by the tracking error minimization problem. In this case, Problem (1) must include a new penalty function which is equal to:

\[-\lambda_{\text{te}} \left( \sum_{i=1}^{n} x_i (\Sigma x_{cw})_i \right) = -\lambda_{\text{te}} \beta (x | x_{cw}) \sigma^2 (x_{cw})\]
A unified optimization framework

The first-order condition is:
\[
\frac{\partial \mathcal{L}(x)}{\partial x_i} = (\sum x)_i - \lambda_{\text{gmv}} + 2\lambda_{\text{h}} x_i - \frac{\lambda_{\text{erc}}}{x_i} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\sum x_{\text{cw}})_i = 0
\]

The solution is the positive root of the second degree (convex) equation:
\[
x_i^2 \left( \sigma_i^2 + 2\lambda_{\text{h}} \right) + x_i \left( \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmv}} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\sum x_{\text{cw}})_i \right) - \lambda_{\text{erc}} = 0
\]

We finally obtain the following CCD numerical solution:
\[
x_i^* = \frac{\lambda_{\text{gmv}} + \lambda_{\text{mdp}} \sigma_i + \lambda_{\text{te}} (\sum x_{\text{cw}})_i - \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{2 \left( \sigma_i^2 + 2\lambda_{\text{h}} \right)}
\]  
\[
+ \sqrt{\left( \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmv}} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\sum x_{\text{cw}})_i \right)^2 + 4 \left( \sigma_i^2 + 2\lambda_{\text{h}} \right) \lambda_{\text{erc}} \left( \sigma_i^2 + 2\lambda_{\text{h}} \right)}
\]
A unified optimization framework

It is not possible to match all the diversification constraints

- Only a subset of Lagrange multipliers is interesting from a mathematical (and financial) point of view

This is equivalent to imposing the following constrained structure:

\[
x^* = \arg\min_{x} \frac{1}{2} x^\top \Sigma x \\
\text{u.c.} \begin{cases} 
D(x; \gamma) \geq c_1 \\
B(x; \delta) = c_2 \\
x \geq 0
\end{cases}
\]

where \(D(x; \gamma)\) and \(B(x; \delta)\) are the diversification and budget constraints:

\[
D(x; \gamma) = \gamma \sum_{i=1}^{n} \ln x_i - (1 - \gamma) \sum_{i=1}^{n} x_i^2 \quad \text{(ERC / EW)}
\]

\[
B(x; \delta) = \delta \sum_{i=1}^{n} x_i + (1 - \delta) \sum_{i=1}^{n} x_i \sigma_i \quad \text{(GMV / MDP)}
\]
The parameter $\gamma \in [0, 1]$ controls the trade-off between weight and risk diversification whereas the parameter $\delta \in [0, 1]$ controls the budget allocation.

We can then restrict $(c_1, c_2)$ by considering this optimization problem:

$$x^* (\lambda, \gamma, \delta) = \arg\min \frac{1}{2} x^\top \Sigma x - \lambda \mathcal{D}(x; \gamma) + (\lambda - 1) \mathcal{B}(x; \delta)$$

u.c. $x \geq 0$

where $\lambda \geq 0$ controls the impact on the diversification.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>GMV</th>
<th>EW</th>
<th>ERC</th>
<th>MDP</th>
<th>RP</th>
<th>BP</th>
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<td>0</td>
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<td>$+\infty$</td>
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<td>0</td>
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<tr>
<td>$\delta$</td>
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<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$\Rightarrow$ Extension to the tracking-error volatility ($\Rightarrow \mathcal{B}(x; \delta)$).
Examples

Figure: New families of smart beta portfolios

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Smart Beta: Managing Diversification of MV Portfolios

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Rule 1

There is no free lunch in smart beta. In particular, it is not possible to target a high volatility reduction, to be highly diversified and to take low beta risk.

Figure: Relationship between the volatility reduction and the beta
Rule 2
The smart beta portfolios have a time-varying objective of volatility reduction and tracking error.

Figure: Boxplot of the volatility reduction (in %)
Comparing the trade-off relationships
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Dynamic smart beta strategies

Ex-ante volatility reduction explains ex-post behavior

Rule 3
When we impose the same objective of volatility reduction $\eta^*$, smart beta portfolios become comparable.

<table>
<thead>
<tr>
<th>Index</th>
<th>$\eta^*$</th>
<th>VR</th>
<th>TE</th>
<th>$\beta$</th>
<th>$D_w$</th>
<th>$D_{rc}$</th>
<th>$D_{p}$</th>
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<td>92.8</td>
<td>83.0</td>
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<td>99.6</td>
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</tbody>
</table>

Table: Average correlation between risk-based portfolios (in %)
Rule 4

The performance of smart beta portfolios depends on the market risk premium.

Figure: Jul. 2007-Feb. 2009

Figure: Mar. 2009-Dec. 2013
The previous rules can be used to build dynamic smart beta strategies.

When risk is perceived as high/low, we expect a lower/higher risk premium:

1. High level of volatility reduction;
2. High level of risk diversification.

We link the parameter \( \lambda \) in Problem (2) to the **market sentiment**, which is approximated by the cross-section (CS) volatility:

\[
\lambda = 1 - \phi \frac{\sigma_{t}^{\text{cs}} - \sigma_{t}^{-}}{\sigma_{t}^{+} - \sigma_{t}^{-}}
\]

and we impose that \( \gamma = 1 \) (ERC) and \( \delta = 1 \) (GMV).
Empirical results

Risk-off: High $\sigma_t^{cs} \Rightarrow \lambda = 0 \Rightarrow \text{GMV}$ / Risk-on: Low $\sigma_t^{cs} \Rightarrow \lambda = 1 \Rightarrow \text{ERC}$.

$D_{#1}$ corresponds to the case $\phi = 1$ and $\lambda \in [0, 1]$.

$D_{#2}$ corresponds to the case $\phi = 0.85$ and $\lambda \in [0.15, 1]$.

Table: Comparing GMV, ERC and dynamic smart beta strategies (2001-2014)

<table>
<thead>
<tr>
<th></th>
<th>CW</th>
<th>GMV</th>
<th>ERC</th>
<th>D_{#1}</th>
<th>D_{#2}</th>
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<tr>
<td>$\mu$ (x) (in %)</td>
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<td>3.4</td>
<td>5.1</td>
<td>4.7</td>
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<td>6.3</td>
<td>3.3</td>
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<td>$\sigma$ (x) (in %)</td>
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<td>23.1</td>
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<td>19.8</td>
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<tr>
<td>SR (x)</td>
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<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
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