Liquidity Stress Testing in Asset Management – Part 2. Modeling the Asset Liquidity Risk

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Liquidity Stress Testing in Asset Management Part 2. Modeling the Asset Liquidity Risk*

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Abstract

This article is part of a comprehensive research project on liquidity risk in asset management, which can be divided into three dimensions. The first dimension covers liability liquidity risk (or funding liquidity) modeling, the second dimension focuses on asset liquidity risk (or market liquidity) modeling, and the third dimension considers the asset-liability management of the liquidity gap risk (or asset-liability matching). The purpose of this research is to propose a methodological and practical framework in order to perform liquidity stress testing programs, which comply with regulatory guidelines (ESMA, 2019, 2020) and are useful for fund managers. The review of the academic literature and professional research studies shows that there is a lack of standardized and analytical models. The aim of this research project is then to fill the gap with the goal of developing mathematical and statistical approaches, and providing appropriate answers.

In this second article focused on asset liquidity risk modeling, we propose a market impact model to estimate transaction costs. After presenting a toy model that helps to understand the main concepts of asset liquidity, we consider a two-regime model, which is based on the power-law property of price impact. Then, we define several asset liquidity measures such as liquidity cost, liquidation ratio and shortfall or time to liquidation in order to assess the different dimensions of asset liquidity. Finally, we apply this asset liquidity framework to stocks and bonds and discuss the issues of calibrating the transaction cost model.

Keywords: Asset liquidity, stress testing, bid-ask spread, market impact, transaction cost, participation rate, power law, liquidation cost, liquidation ratio, liquidation shortfall, time to liquidation.

JEL classification: C02, G32.

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1 Introduction

Since September 2020, the European Securities and Markets Authority (ESMA) has required asset managers to adopt a liquidity stress testing (LST) policy for their investment funds (ESMA, 2020). More precisely, each asset manager must assess the liquidity risk factors across their funds in order to ensure that stress testing is tailored to the liquidity risk profile of each fund. The issue of liquidity stress testing is that the analysis should include both sides of the equation: liability (or funding) liquidity and asset (or market) liquidity. This issue is not specific to the asset management industry, because it is a general problem faced by financial firms including the banking industry:

> "A liquidity stress test is the process of assessing the impact of an adverse scenario on institution's cash flows as well as on the availability of funding sources, and on market prices of liquid assets" (BCBS, 2017, page 60).

However, the main difference between the asset management and banking sectors is that banks have a longer experience than asset managers, both in the field of stress testing and liquidity management (BCBS, 2013b). Another difference is that the methodology for computing the liquidity coverage ratio and the monitoring tools are precise, comprehensive and very detailed by the regulator (BCBS, 2013a). This is not the case for the redemption coverage ratio, since the regulatory text only contains guidelines and no methodological aspects. Certainly, these differences can be explained by the lack of maturity of this topic in the asset management industry.

The aim of this research is to provide a methodological support for managing liquidity risk of investment funds. Since it is a huge project, we have divided it into three dimensions: (1) liability liquidity risk modeling, (2) asset liquidity risk measurement and (3) asset-liability liquidity risk management. This article only covers the second dimension and proposes a framework for assessing the liquidity of a portfolio given a redemption scenario[1].

Assessing the asset liquidity risk is equivalent to measuring the transaction cost of liquidating a portfolio. This means estimating the bid-ask spread component, the price impact of the transaction, the time to liquidation, the implementation shortfall, etc. This also implies defining a liquidation policy. Contrary to the liability liquidity risk where the academic literature is poor and not helpful, there are many quantitative works on the aspects of asset liquidity risk. This is particularly true for the modeling of transaction costs, much less for liquidation policies. The challenge is then to use the most interesting studies that are relevant from a professional point of view, and to cast them into a practical stress testing framework. This means simplifying and defining a few appropriate parameters that are useful to assess the asset liquidity risk.

This paper is organized as follows. Section Two deals with transaction cost modeling. A toy model will be useful to define the concepts of price impact and liquidation policies. Then, we consider a two-regime transaction cost model based on the power-law property of the price impact. In Section Three, we present the asset liquidity measures such as the liquidation ratio, the time to liquidation or the implementation shortfall. The implementation of a stress testing framework is developed in Section Four. In particular, we consider an approach that distinguishes invariant parameters and risk parameters that are impacted by a stress regime. We also discuss the portfolio distortion that may be induced by a liquidation policy, which does not correspond to the proportional rule. Finally, Section Five applies the analytical framework to stocks and bonds, and Section Six offers some concluding remarks.

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[1]The liability liquidity risk is studied in Roncalli *et al.* (2020), whereas the asset-liability management tools are presented in Roncalli *et al.* (2021)

2 Transaction cost modeling

In this section, we develop a transaction cost model that incorporates both the bid-ask spread and the market impact. For that, we first define these two concepts and explain the difference between real and nominal variables. Then, we present a toy model that allows to understand the main characteristics of a transaction cost function. Using the power-law property of price impact, we derive the square-root-linear model and show how this model can be calibrated.

2.1 Definition

2.1.1 Unit transaction cost

In what follows, we break down the unit transaction cost into two parts:

$$\boldsymbol{c}(x) = s + \boldsymbol{\pi}(x) \tag{1}$$

where s does not depend on the trade size and represents half of the bid-ask spread of the security, and $\boldsymbol{\pi}(x)$ depends on the trade size x and represents the price impact (or PI) of the trade. The trade size x is an invariant variable and is the ratio between the number of traded shares q (sold or purchased) and the daily trading volume v:

$$x = \frac{q}{v} \tag{2}$$

It is also called the participation rate.

Remark 1 *If we express the quantities in nominal terms, we have:*

$$x = \frac{Q}{V} = \frac{q \cdot P}{v \cdot P} = \frac{q}{v}$$

where P is the security price that is observed for the current date, and $Q = q \cdot P$ and $V = v \cdot P$ are the nominal values of q and v (expressed in USD or EUR). In the sequel, lowercase symbols generally represent quantities or numbers of shares whereas uppercase symbols are reserved for nominal values. For example, the unit transaction cost $\boldsymbol{C}(Q,V)$ is defined by:

$$\boldsymbol{C}(Q,V) = \boldsymbol{c}\left(\frac{Q}{V}\right) = \boldsymbol{c}\left(\frac{q}{v}\right) \tag{3}$$

2.1.2 Total transaction cost

The total transaction cost of the trade is the product of the unit transaction cost and the order size expressed in dollars:

$$\mathcal{TC}(q) = q \cdot P \cdot \boldsymbol{c}(x) = Q \cdot \boldsymbol{c}(x) \tag{4}$$

where P is the price of the security. Again, we can break down $\mathcal{TC}(q)$ into two components:

$$\mathcal{TC}(q) = \mathcal{BAS}(q) + \mathcal{PI}(q) \tag{5}$$

where $\mathcal{BAS}(q) = Q \cdot s$ is the trading cost due to the bid-ask spread and $\mathcal{PI}(q) = Q \cdot \boldsymbol{\pi}(x)$ is the trading cost due to the market impact.

Remark 2 *By construction, we have:*

$$\mathcal{TC}(Q,V) = Q \cdot \boldsymbol{C}(Q,V) \tag{6}$$

2.1.3 Trading limit

The previous framework only assumes that $x \geq 0$. However, this is not realistic since we cannot trade any values of x in practice. From a theoretical point of view, we have $q \leq v$, meaning that $x \leq 1$ and x is a participation rate. From a practical point of view, q is an ex-post quantity whereas v is an ex-ante quantity, implying that x is a relative trading size and can be larger than one. Nevertheless, it is highly unlikely that the fund manager will trade a quantity larger than the ex-ante daily trading volume. It is more likely that the asset manager's trading policy imposes a trading limit x^+ beyond which the fund manager cannot trade:

$$0 \leq x \leq x^+ < 1 \tag{7}$$

This is equivalent to say that the unit transaction cost becomes infinite when the trade size is larger than the trading limit. It follows that the unit transaction cost may be designed in the following way:

$$\boldsymbol{c}(x) = \begin{cases} s + \boldsymbol{\pi}(x) & \text{if } x \in [0, x^+] \\ +\infty & \text{if } x > x^+ \end{cases} \tag{8}$$

In this case, the concept of total transaction cost (or trading cost) only makes sense if the trade size x is lower than the trading limit x^+. Therefore, we will see later that the trading (or liquidation) cost must be completed by liquidation measures such as liquidation ratio or liquidation time.

Remark 3 *The trading limit x^+ is expressed in %. For instance, it is generally set at 10% for equity trading desks. This means that the trader can sell any volume up to 10% of the average daily volume without any permissions. Above the 10% trading limit, the trader must inform the risk manager and obtain authorization to execute its sell order. This trading limit x^+ can be expressed as a maximum number of shares q^+. The advantage of this trading policy is that it does not depend on the daily volume, which is time-varying. Another option is to express the trading limit in nominal terms. Let Q^+ be the nominal trading limit. We have the following relationship:*

$$x^+ = \frac{q^+}{v} = \frac{Q^+}{V} \tag{9}$$

2.2 A toy model of transaction cost

Let us consider a simple model where the unit transaction cost has the functional form given in Figure 1. In this toy model, we assume that the unit transaction cost corresponds to the bid-ask spread if the selling amount x is lower than a threshold $\tilde{x}$. Beyond this normal market size, the transaction cost includes a market impact. This market impact is linear and is an increasing function of x. Moreover, we generally assume that market impact becomes infinite if the selling amount is larger than x^+, which is known as the maximum trading size or the trading limit. It follows that the unit transaction cost may be parameterized by this function:

$$\boldsymbol{c}'(x) = \begin{cases} s & \text{if } x \leq \tilde{x} \\ s + \alpha(x - \tilde{x}) & \text{if } \tilde{x} \leq x \leq x^+ \\ +\infty & \text{if } x > x^+ \end{cases} \tag{10}$$

It depends on four parameters: the bid-ask spread s, the slope α of the market impact and two thresholds: the normal size $\tilde{x}$ and the maximum trading size x^+. For example, we obtain Figure 1 with the following set of parameters: $s = 2$ bps, $\alpha = 2\%$, $\tilde{x} = 2\%$ and $x^+ = 8\%$. The unit transaction cost is equal to 2 bps for small orders and reaches 14 bps when the trade size equals to the trading limit that is equal to 8%.

Figure 1: Simple modeling of unitary transaction costs
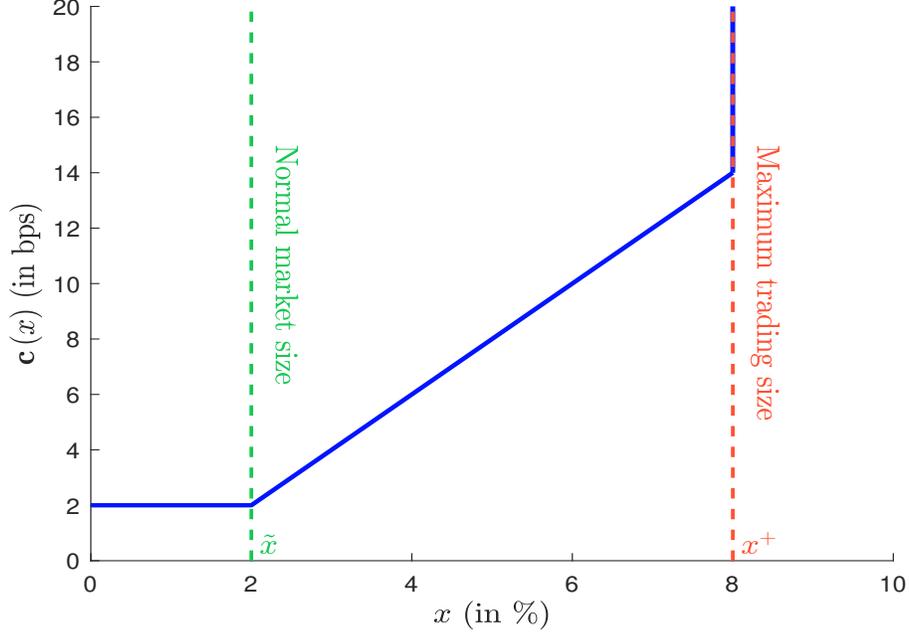
For each security i, the unit transaction cost is then defined by the 4-tuple $\left(s_i, \alpha_i, \tilde{x}_i, x_i^+\right)$ where s_i is a security-specific parameter and α_i is a model parameter. This means that α_i is the same for all securities that belong to the same liquidity bucket $\mathcal{LB}_j$. For instance, $\mathcal{LB}_j$ may group all large cap US stocks. $\tilde{x}_i$ and x_i^+ may be security-specific parameters, but they are generally considered as model parameters in order to simplify the calibration of the unit transaction cost.

The previous approach may be simplified by considering that the market impact begins at $x = \tilde{x} = 0$. In this case, the unit transaction cost becomes:

$$\boldsymbol{c}''(x) = \left\{ \begin{array}{ll} s + \alpha x & \text{if } x \leq x^+ \\ +\infty & \text{if } x > x^+ \end{array} \right. \tag{11}$$

The interest of this parametrization is to reduce the number of parameters since this unit transaction cost function is then defined by the triplet $\left(s_i, \alpha_i, x_i^+\right)$ for each security i. An example is provided in Figure 29 on page 79.

Remark 4 *The parameterization $\boldsymbol{c}''(x)$ allows us to use the traditional mean-variance framework based on QP optimization (Chen et al., 2019). This explains the practitioners' great interest in the function $\boldsymbol{c}''(x)$ because it is highly tractable and is compatible with the Markowitz approach with low computational complexity[2].*

Remark 5 *In Appendix B.1 on page 71, we show how to transform the function $\boldsymbol{c}'(x)$ into the function $\boldsymbol{c}''(x)$, and vice versa. However, the right issue is to estimate $\hat{\boldsymbol{c}}''(x)$ or more precisely the slope $\hat{\alpha}$ of the market impact. In this case, we use Equations (61) and (62) on page 72 to transform $\hat{\alpha}$ into α for the functions $\boldsymbol{c}'(x)$ and $\boldsymbol{c}''(x)$.*

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[2]Nevertheless, this parameterization is less frequent than the simple approach that only considers the bid-ask spread (Scherer, 2007): $\boldsymbol{c}'''(x) = s$.

2.3 The power-law model of price impact

2.3.1 General formula for the market impact

The previous trading cost model is useful for portfolio optimization, but price impact is certainly too simple from a trading or risk management perspective. Nevertheless, price impact has been extensively studied by academics[3], and it is now well-accepted that market impact is power-law:

$$\boldsymbol{\pi}(x) := \boldsymbol{\pi}(x; \gamma) = \varphi_\gamma \sigma x^\gamma \tag{12}$$

where $\gamma > 0$ is a scalar, σ is the daily volatility of the security[4] and φ_γ is a scaling factor[5]. In particular, Equation (12) is valid under a no-arbitrage condition (Jusselin and Rosenbaum, 2020). Empirical studies showed that $\gamma \in [0.3, 0.7]$. For example, the seminal paper of Loeb (1983) has been extensively used by Torre (1997) to develop the MSCI Barra market impact model, which considers that $\gamma = 0.5$. Almgren et al. (2005) concluded that $\gamma = 3/5$ is a better figure than $\gamma = 1/2$. On the contrary, Engle et al. (2012) found that $\gamma \approx 0.43$ for NYSE stocks and $\gamma \approx 0.37$ for NASDAQ stocks, while Frazzini et al. (2018) estimated that the average exponent is equal to 0.35 for developed equity markets. Bacry et al. (2015) confirmed a square root temporary impact in the daily participation and observed a power-law pattern with an exponent between 0.5 and 0.8. However, the results obtained by academics are generally valid for small values of x. For instance, the median value of x is equal to 0.6% in Almgren et al. (2005), Tóth et al. (2011) have used trades[6], which are smaller than 0.01%, Zarinelli et al. (2015) have considered a database of seven million metaorders, implying that data with small values of x dominate data with large values of x, etc.

Even though there is an academic consensus[7] that $\gamma \approx 0.5$, this assumption is not satisfactory from a practical point of view when we have to sell or buy a large order ($x \gg 0.5\%$). Some academics have also exhibited that γ is an increasing function of x. For instance, Moro et al. (2009) found that γ is equal to 0.64 for LSE stocks when there is a low fraction of market orders, but γ is equal to 0.72 when there is a high fraction of market orders. Similarly, Cont et al. (2014) estimated that γ is equal to 1 when we aggregate trades and consider order flow imbalance instead of single trade sizes. Breen et al. (2002) used a linear regression model for estimating the price impact. We also recall that the seminal paper of Kyle (1985) assumes that $\gamma = 1$. In fact, these two concepts of transaction cost are not necessarily exclusive:

> "Empirically, both a linear model and a square root model explain transaction costs well. A square-root model explains transaction costs for orders in the 90th to 99th percentiles better than a linear model; a linear model explains transaction costs for the largest 1% of orders slightly better than the square-root model" (Kyle and Obizhaeva, 2016, page 1347).

This finding is shared by Boussema et al. (2002) and D'Hondt and Giraud (2008), who observed that market impact increases significantly when trade size is greater than 1% or turnover is lower than 0.03%.

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[3]See for instance the survey articles of Bouchaud (2010) and Kyle and Obizhaeva (2018).

[4]The daily volatility is equal to the annualized volatility divided by the factor $\sqrt{260}$. In the sequel, we use the symbol σ to name both the daily and annualized volatilities. When the volatility is used in a transaction cost formula, it corresponds to a daily volatility. In the text, the volatility is always expressed on an annual basis.

[5]The value of φ_γ depends on the value taken by the exponent γ.

[6]See Figure 1 in Tóth et al. (2011).

[7]For instance, the square-root model is used by Gârleanu and Pedersen (2013), Frazzini et al. (2018) and Briere et al. (2020).

Remark 6 *According to [Bucci et al. (2019)](#), the relationship between trade size and market impact is close to a square-root function for intermediate trading volumes (i.e. when $0.1\% \leq x \leq 10\%$), but shows an approximate linear behavior for smaller trading volumes (i.e. when $0.001\% \leq x \leq 0.1\%$). These different results demonstrate that there is no consensus on a unique functional form for computing the price impact.*

Figure 2: Convexity measure of the power-law model
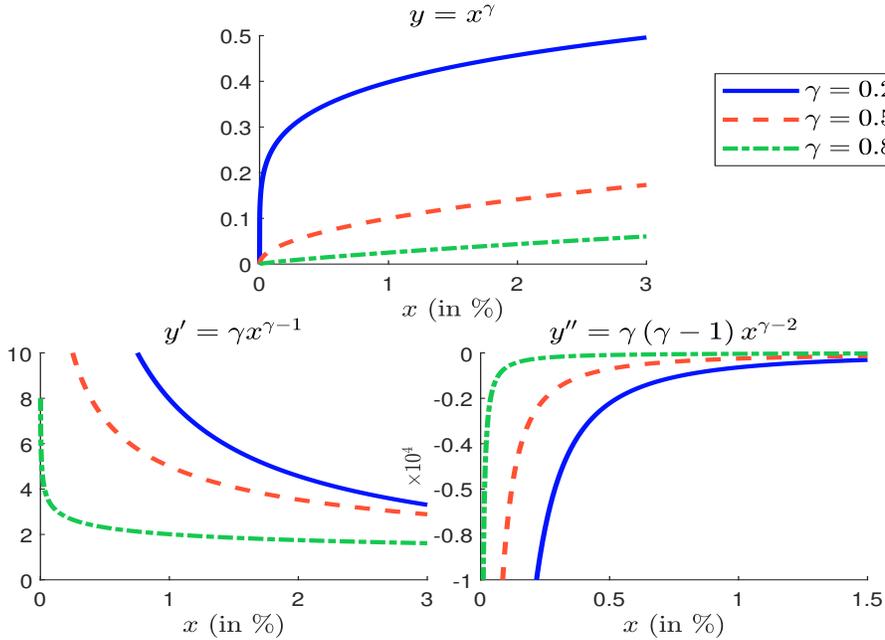
In Figure 2, we report the power function $y = x^{\gamma}$ and its first and second derivatives for three exponents γ. We deduce that the concavity is larger for low values of γ and x. When x is equal to 1, the power function converges to the same value $y = 1$ whatever the value of γ. It follows that the choice of γ primarily impacts small trading sizes.

2.3.2 Special cases

From Equation (12), we deduce the two previous competing approaches of [Loeb (1983)](#) and [Kyle (1985)](#), and also the constant (or bid-ask spread) model:

- The square-root model ($\gamma = 1/2$):

$$\boldsymbol{\pi}\left(x; 1/2\right) \approx \varphi_{1/2}\sigma\sqrt{x} \tag{13}$$

  Generally, we assume that the scaling factor $\varphi_{1/2}$ is close to one, implying that the multiplicative factor is equal to the daily volatility.

- The linear model ($\gamma = 1$):

$$\boldsymbol{\pi}\left(x; 1\right) \approx \varphi_1\sigma x \tag{14}$$

In this case, the scaling factor φ_1 may be calibrated with respect to $\varphi_{1/2}$ by considering that the two price impact functions coincide at a threshold $\tilde{x}$. We deduce that[8]:

$$\boldsymbol{\pi}(x;1) \approx \varphi_{1/2}\sigma \frac{x}{\sqrt{\tilde{x}}} \tag{15}$$

• The constant model ($\gamma = 0$):
$$\boldsymbol{\pi}(x;0) \approx \varphi_0 \sigma \tag{16}$$

By assuming that $\varphi_0 = 0$, we obtain the bid-ask spread model:

$$\boldsymbol{c}(x) = s$$

In Tables 1 and 2, we have reported the values taken by the price impact function $\boldsymbol{\pi}(x)$ for different values of the annualized volatility σ and trade size x. We assume that $\varphi_{1/2} = 1$ and $\tilde{x} = 1\%$. It follows that $\varphi_1 = 10$. Results must be read as follows: a trade size of 0.50% has a price impact of 4.4 bps when the asset volatility is 10% in the case of the square-root model, whereas the price impact becomes 3.1 bps if we consider the linear model.

Table 1: Price impact in bps when $\gamma = 1/2$ (square-root model)

	x	0.01%	0.05%	0.10%	0.50%	1%	2%	5%	10%	15%
	1%	0.1	0.1	0.2	0.4	0.6	0.9	1.4	2.0	2.4
	5%	0.3	0.7	1.0	2.2	3.1	4.4	6.9	9.8	12.0
	10%	0.6	1.4	2.0	4.4	6.2	8.8	13.9	19.6	24.0
	15%	0.9	2.1	2.9	6.6	9.3	13.2	20.8	29.4	36.0
σ	20%	1.2	2.8	3.9	8.8	12.4	17.5	27.7	39.2	48.0
	25%	1.6	3.5	4.9	11.0	15.5	21.9	34.7	49.0	60.0
	30%	1.9	4.2	5.9	13.2	18.6	26.3	41.6	58.8	72.1
	50%	3.1	6.9	9.8	21.9	31.0	43.9	69.3	98.1	120.1

Table 2: Price impact in bps when $\gamma = 1$ (linear model)

	x	0.01%	0.05%	0.10%	0.50%	1%	2%	5%	10%	15%
	1%	0.0	0.0	0.1	0.3	0.6	1.2	3.1	6.2	9.3
	5%	0.0	0.2	0.3	1.6	3.1	6.2	15.5	31.0	46.5
	10%	0.1	0.3	0.6	3.1	6.2	12.4	31.0	62.0	93.0
	15%	0.1	0.5	0.9	4.7	9.3	18.6	46.5	93.0	139.5
σ	20%	0.1	0.6	1.2	6.2	12.4	24.8	62.0	124.0	186.1
	25%	0.2	0.8	1.6	7.8	15.5	31.0	77.5	155.0	232.6
	30%	0.2	0.9	1.9	9.3	18.6	37.2	93.0	186.1	279.1
	50%	0.3	1.6	3.1	15.5	31.0	62.0	155.0	310.1	465.1

Figure 3 shows the differences between the two models when the annualized volatility is set to 10%. First, we notice that the concavity of the square-root model is mainly located

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[8]We have:

$$\boldsymbol{\pi}(\tilde{x};1) = \boldsymbol{\pi}(\tilde{x};1/2) \quad \Leftrightarrow \quad \varphi_1 \sigma \tilde{x} = \varphi_{1/2}\sigma\sqrt{\tilde{x}}$$
$$\Leftrightarrow \quad \varphi_1 = \frac{\varphi_{1/2}}{\sqrt{\tilde{x}}}$$

Figure 3: Square-root model versus linear model ($\sigma = 10\%$)
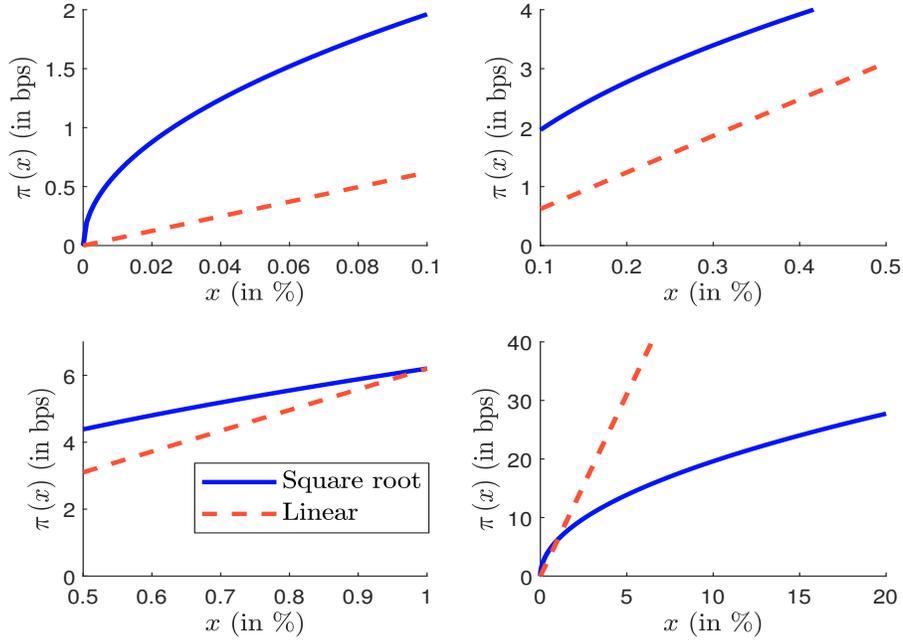
for small values of x, since the trading cost function $\boldsymbol{\pi}\left(x;\sfrac{1}{2}\right)$ may be approximated by a piecewise linear function with only three or four knots. Second, the square-root model implies higher trading costs than the linear model when trade sizes are *small*, and we verify that $\boldsymbol{\pi}\left(x;\sfrac{1}{2}\right) \geq \boldsymbol{\pi}\left(x;1\right)$ when $x \leq \tilde{x} = 1\%$. For large trade sizes, it is the linear model that produces higher trading costs compared to the square-root model[9]: $\boldsymbol{\pi}\left(x;1\right) \gg \boldsymbol{\pi}\left(x;\sfrac{1}{2}\right)$.

2.4 A two-regime transaction cost model

2.4.1 General formula

In the toy model, we distinguish two market impact regimes. The first one corresponds to small trading sizes — $x \in [0, \tilde{x}]$, which generate a low price impact. In the second regime, trading sizes are larger — $x \in [\tilde{x}, x^+]$, and the price impact has a significant contribution to the transaction cost. The research studies on the power-law model also show that there may be several regimes of market impact depending on the value of γ. Therefore, we can generalize the toy model where the two regimes correspond to two power functions:

$$\boldsymbol{\pi}\left(x\right) = \begin{cases} \varphi_1 \sigma x^{\gamma_1} & \text{if } x \leq \tilde{x} \\ \varphi_2 \sigma x^{\gamma_2} & \text{if } \tilde{x} \leq x \leq x^+ \\ +\infty & \text{if } x > x^+ \end{cases} \tag{17}$$

where γ_1 and γ_2 are the exponents of the two market impact regimes. Moreover, the scalars φ_1 and φ_2 are related since the cost function $\boldsymbol{\pi}\left(x\right)$ is continuous. This implies that $\varphi_2 = \varphi_1 \tilde{x}^{\gamma_1 - \gamma_2}$. In this case, the price impact model is defined by the 5-tuple $(\varphi_1, \gamma_1, \gamma_2, \tilde{x}, x^+)$ since φ_2 is computed from these parameters. An alternative approach is to define the model

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[9]This large difference between square-root and linear models has been already observed by Frazzini *et al.* (2018).

by the parameter set $(\gamma_1, \gamma_2, \tilde{x}, x^+, \boldsymbol{\pi}(\tilde{x}))$. Here, we fix the market impact at the inflection point, and we have $\varphi_1 = \sigma^{-1}\tilde{x}^{-\gamma_1} \cdot \boldsymbol{\pi}(\tilde{x})$ and $\varphi_2 = \varphi_1\tilde{x}^{\gamma_1-\gamma_2}$.

**Remark 7** *Another parameterization of the two-regime model may be:*

$$\boldsymbol{\pi}(x) = \begin{cases} \varphi_1\sigma x^{\gamma_1} & \text{if } x \leq \tilde{x} \\ \boldsymbol{\pi}(\tilde{x}) + \varphi_2\sigma(x - \tilde{x})^{\gamma_2} & \text{if } \tilde{x} < x \leq x^+ \\ +\infty & \text{if } x > x^+ \end{cases} \tag{18}$$

where $\boldsymbol{\pi}(\tilde{x}) = \varphi_1\sigma\tilde{x}^{\gamma_1}$. This model is defined by the parameter set $(\varphi_1, \gamma_1, \varphi_2, \gamma_2, \tilde{x}, x^+)$.

Remark 8 *The model of Bucci et al. (2019) is obtained with the two parameterizations by setting $\gamma_1 = 1$, $\tilde{x} = 0.1\%$ and $\gamma_2 = 1/2$.*

In Figure 4, we report three examples of the two-regime model. The first two examples correspond to the first parameterization, whereas the last example uses the second parameterization. In this last case, we observe a step effect due to the high concavity[10] applied to the small values of $x - \tilde{x}$. Therefore, it is better to use the first parameterization.

Figure 4: Two-regime model (annualized volatility $\sigma = 10\%$)
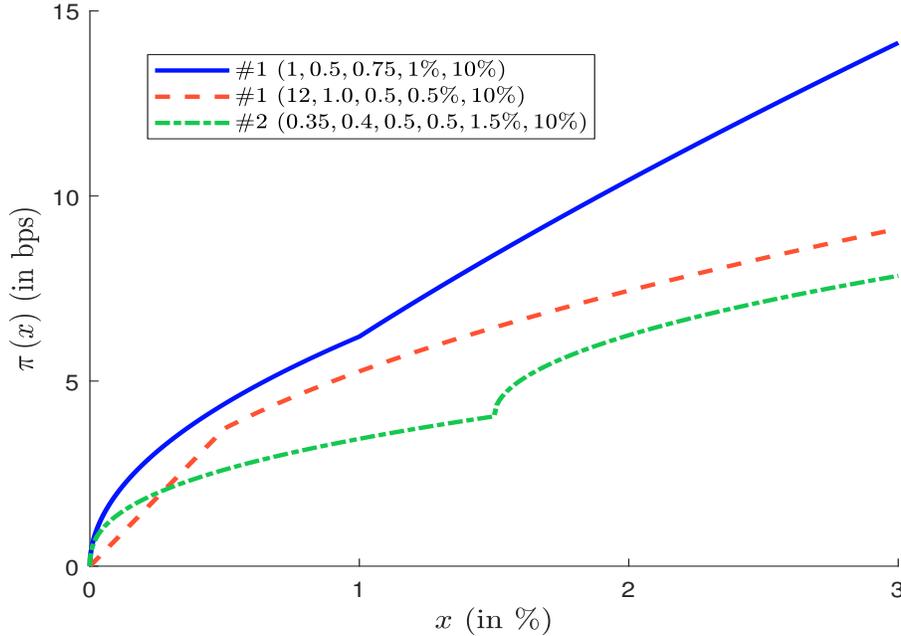
Remark 9 *One of the questions which emerges with the calibration of the two-regime model is the effective difference between the two regimes. In particular, we have the choice between $\gamma_1 > \gamma_2$ and $\gamma_2 > \gamma_1$. In other words, we have the choice to decrease or increase the convexity beyond the inflection point $\tilde{x}$. The "small size effect" described by Bucci et al. (2019) is not really an issue, because the impact is so small. Indeed, the order of magnitude of the price impact for $x \leq 0.1\%$ is one or two basis points in the power-law model[11]. The significant*

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[10]This step effect has been illustrated in Figure 2 on page 7.

[11]For instance, we have $\boldsymbol{\pi}(0.01\%) = 0.62$ bps and $\boldsymbol{\pi}(0.1\%) = 1.92$ bps when $\sigma = 10\%$ and $\gamma = 0.5$.

issue is more to have a coherent approach when the trading size is close to the trading limit x^+. An example is provided in Figure 5 when the annualized volatility σ is 10% and $\varphi_1 = 1$. We recall that $\boldsymbol{\pi}(x) = \infty$ when $x > x^+$ because of the order execution policy imposed by the asset manager. Therefore, it is obvious that the right choice is $\gamma_2 > \gamma_1$, implying that the convexity must increase. Otherwise, it is not consistent to impose a low convexity below x^+ and an infinite convexity beyond x^+.

Figure 5: Two-regime model ($\sigma = 10\%, \varphi_1 = 1$)
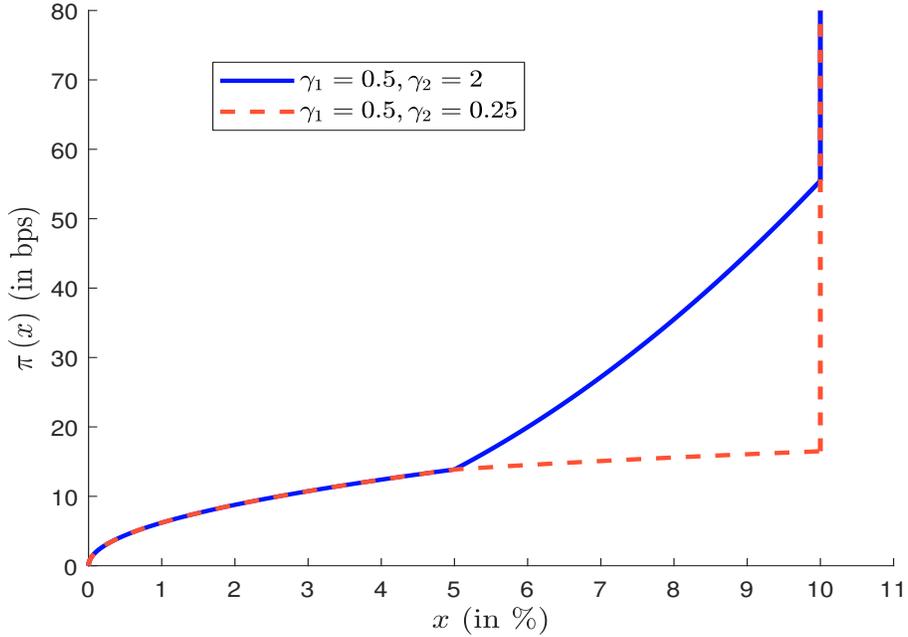
2.4.2 The square-root-linear model

From the two-regime model, we can define the square-root-linear (SQRL) model which has been suggested by Kyle and Obizhaeva (2016):

$$\boldsymbol{\pi}(x) = \begin{cases} \varphi_1 \sigma \sqrt{x} & \text{if } x \leq \tilde{x} \\ \varphi_1 \sigma \dfrac{x}{\sqrt{\tilde{x}}} & \text{if } \tilde{x} \leq x \leq x^+ \\ +\infty & \text{if } x > x^+ \end{cases} \qquad (19)$$

In this case, we assume that the square-root model is valid for *small* trade sizes ($x \leq \tilde{x}$), whereas the linear model is better for *large* trade sizes ($\tilde{x} \leq \tilde{x} \leq x^+$). However, beyond the threshold value x^+, we consider that trading costs are prohibitive and infinite. As for the toy model, the value x^+ may be interpreted as a trading limit. We have represented the SQRL model in Figure 6 for the previous parameters ($\sigma = 10\%$ and $\varphi_1 = 1$) when the inflection point $\tilde{x}$ is equal to 1%.

In Table 3, we report the price impact of this model for several values of the annualized volatility σ. We can compare these figures with those given in Tables 1 and 2 on page 8. Let us consider the case when the volatility is equal to 20%, which corresponds to the typical

Figure 6: Square-root-linear model ($\sigma = 10\%$)
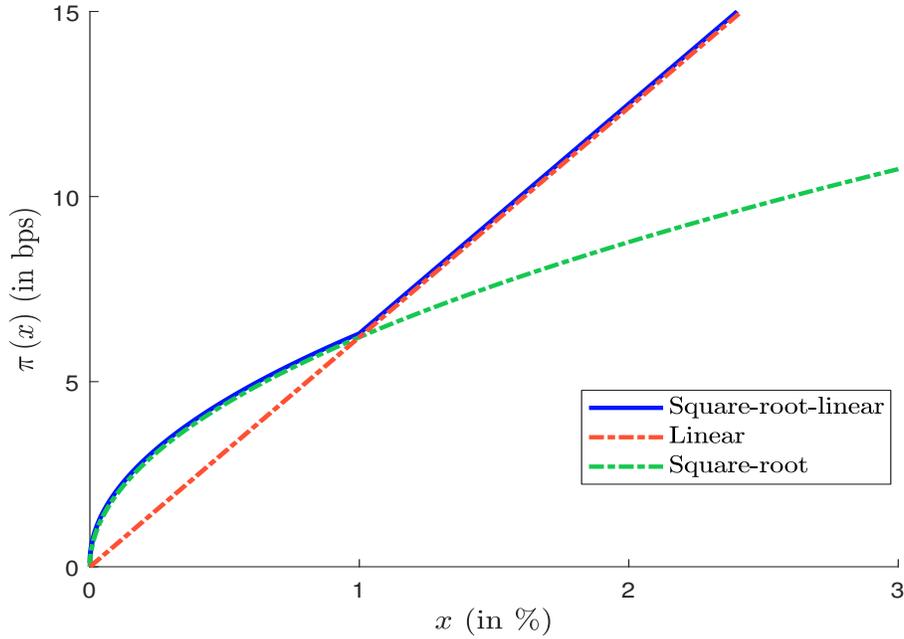
volatility observed for single stocks. We observe that there is an acceleration of the price impact beyond the inflection point. For instance, the price impact is equal to 24.8 bps for $x = 2\%$, 62.0 bps for $x = 5\%$, etc.

Table 3: Price impact in bps (square-root-linear model)

	x	0.01%	0.05%	0.10%	0.50%	1%	2%	5%	10%	15%
	1%	0.1	0.1	0.2	0.4	0.6	1.2	3.1	6.2	9.3
	5%	0.3	0.7	1.0	2.2	3.1	6.2	15.5	31.0	46.5
	10%	0.6	1.4	2.0	4.4	6.2	12.4	31.0	62.0	93.0
	15%	0.9	2.1	2.9	6.6	9.3	18.6	46.5	93.0	139.5
σ	20%	1.2	2.8	3.9	8.8	12.4	24.8	62.0	124.0	186.1
	25%	1.6	3.5	4.9	11.0	15.5	31.0	77.5	155.0	232.6
	30%	1.9	4.2	5.9	13.2	18.6	37.2	93.0	186.1	279.1
	50%	3.1	6.9	9.8	21.9	31.0	62.0	155.0	310.1	465.1

Remark 10 *The SQRL model and more generally the two-regime model can be used as an incentive trading model, since trades are penalized when they are larger than $\tilde{x}$. In this case, x^+ is a hard threshold limit while $\tilde{x}$ can be considered as a soft threshold limit. Indeed, the asset manager does not explicitly prohibit the fund manager from trading between $\tilde{x}$ and x^+, but he is clearly not encouraged to trade, because the transaction costs are high[12]. This is particularly true if the asset manager has a centralized trading desk and ex-ante trading costs are charged to the fund manager.*

---

[12]For instance, the price impact is equal to 35.1 bps for $x = 2\%$ and 138.7 bps for $x = 5\%$ when we use a two-regime model with the following parameters: $\sigma = 20\%$, $\varphi_1 = 1$, $\gamma_1 = 1/2$, $\gamma_2 = 3/2$ and $\tilde{x} = 1\%$.

3 Asset liquidity measures

Since liquidity is a multi-faceted concept, we must use several measures in order to encompass the different dimensions (Roncalli, 2020, page 347). If we focus on asset liquidity, we generally distinguish two types of measurement. The first category assesses the liquidity risk profile and includes the liquidation ratio, the time to liquidation and the liquidation shortfall. The second category concerns liquidity costs such as transaction costs and effective costs. The main difference between the two categories is that the first one focuses on the volume, while the second one mixes both volume and price dimensions.

3.1 Redemption scenario

In Roncalli *et al.* (2020), we have developed several methods and tools in order to define a redemption shock $\mathcal{R}$ for a given investment fund. This redemption shock is expressed as a percentage of the fund's total net asset TNA. Therefore, we can deduce the stress liability outflow:

$$\mathcal{F}^-(t) := \mathbb{R} = \mathcal{R} \cdot \mathrm{TNA}(t)$$

The asset structure of the fund is given by the vector $\omega = (\omega_1, \ldots, \omega_n)$ where ω_i is the number of shares of security i and n is the number of assets that make up the asset portfolio. By construction, we have:

$$\mathrm{TNA}(t) = \sum_{i=1}^{n} \omega_i \cdot P_i(t)$$

where $P_i(t)$ is the current price of security i. The redemption shock $\mathcal{R}$ must be translated into the redemption scenario $q = (q_1, \ldots, q_n)$, where q_i is the number of shares of security i that must be sold. After the sell order, we must have the following equality[13]:

$$\mathrm{TNA}(t^+) := \mathrm{TNA}(t) - \mathcal{F}^-(t) = \sum_{i=1}^{n} (\omega_i - q_i) \cdot P_i(t) \tag{20}$$

where t^+ means $t + \mathrm{d}t$ and $\mathrm{d}t$ is a small time step. Generally, we assume that the portfolio composition remains the same, meaning that:

$$\frac{q_i \cdot P_i(t)}{q_j \cdot P_j(t)} = \frac{\omega_i \cdot P_i(t)}{\omega_j \cdot P_j(t)}$$

It follows that the solution is simple and is equal to the proportional rule:

$$q_i = \mathcal{R} \cdot \omega_i \tag{21}$$

It is called the vertical slicing approach (or pro-rata liquidation). Nevertheless, since $\omega_i - q_i$ must be a natural number, q_i must also be a natural number. Therefore, due to round-off errors, the final redemption shock may not match the proportional rule.

Remark 11 *In Section 4.3 on page 32, we discuss the construction of the redemption scenario in more detail, in particular how to manage the distortion of the portfolio allocation weights.*

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[13]We notice that the dollar value of the redemption is equal to $\sum_{i=1}^{n} q_i \cdot P_i(t)$.

3.2 Liquidity risk profile

We first consider volume-related liquidity measures. One of the most popular measures is the liquidation ratio $\mathcal{LR}(q; h)$, which measures the proportion of a portfolio q that can be liquidated after h trading days. This statistic depends on the size of each exposure q_i and the liquidation policy, which is defined by the trading limit q_i^+. Another interesting statistic is the liquidation time (or time to liquidation) $\mathcal{LT}(q; p)$, which is the inverse function of the liquidity ratio. It indicates the number of required trading days in order to liquidate a proportion p of the portfolio.

3.2.1 Liquidation ratio

For each security that makes up the portfolio, we recall that q_i^+ denotes the maximum number of shares that can be sold during a trading day for the asset i. The number of shares $q_i(h)$ liquidated after h trading days is defined as follows:

$$q_i(h) = \min\left(\left(q_i - \sum_{k=0}^{h-1} q_i(k)\right)^+, q_i^+\right) \tag{22}$$

where $q_i(0) = 0$. The liquidation ratio $\mathcal{LR}(q; h)$ is then the proportion of the redemption scenario q that is liquidated after h trading days:

$$\mathcal{LR}(q; h) = \frac{\sum_{i=1}^{n} \sum_{k=1}^{h} q_i(k) \cdot P_i}{\sum_{i=1}^{n} q_i \cdot P_i} \tag{23}$$

By definition, $\mathcal{LR}(q; h)$ is between 0 and 1. For instance, $\mathcal{LR}(q; 1) = 50\%$ means that we can fulfill 50% of the redemption on the first trading day, $\mathcal{LR}(q; 5) = 80\%$ means that we can fulfill 80% of the redemption after five trading days, etc.

We consider a portfolio, which is made up of 5 assets. The redemption scenario is defined below by the number of shares q_i that have to be sold:

Asset	1	2	3	4	5
q_i	4 351	2 005	755	175	18
q_i^+	1 000	1 000	200	200	200
P_i	89	102	67	119	589

We also indicate the trading limit q_i^+ and the current price P_i of each asset. In Table 4, we report the number of liquidated shares $q_i(h)$ and the liquidation ratio $\mathcal{LR}(q; h)$. After the first trading day, we have liquidated 1 000 shares of Asset #1 because of the trading policy that imposes a trading limit of 1 000. We notice that we need 5 trading days in order to sell 4 351 shares of Asset #1. If we consider the liquidation ratio, we obtain $\mathcal{LR}(q; 1) = 35\%$, $\mathcal{LR}(q; 2) = 65.34\%$, etc.

Remark 12 *The liquidation period $h^+ = \inf\{h : \mathcal{LR}(q; h) = 1\}$ indicates how many trading days we need to liquidate the redemption scenario q. In the previous example, h^+ is equal to 5, meaning that the liquidation of this redemption scenario requires five trading days.*

We can break down the liquidation ratio as follows:

$$\mathcal{LR}(q; h) = \frac{1}{\sum_{i=1}^{n} q_i \cdot P_i} \sum_{k=1}^{h} \sum_{i=1}^{n} \mathcal{LA}_{i,k}(q)$$

Table 4: Number of liquidated shares $q_i(h)$

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	$\mathcal{LR}(q;h)$
1	1 000	1 000	200	175	18	35.00%
2	1 000	1 000	200	0	0	65.34%
3	1 000	5	200	0	0	80.61%
4	1 000	0	155	0	0	95.36%
5	351	0	0	0	0	100.00%
Total	4 351	2 005	755	175	18	

where $\mathcal{LA}_{i,k}(q) = q_i(k) \cdot P_i$ is the liquidation amount for security i and trading day k. It follows that:

$$\mathcal{LR}(q;h) = \sum_{k=1}^{h} \sum_{i=1}^{n} \mathcal{LC}_{i,k}(q) = \sum_{k=1}^{h} \mathcal{LC}_k(q)$$

where $\mathcal{LC}_{i,k}(q)$ is the liquidation contribution for security i and trading day k:

$$\mathcal{LC}_{i,k}(q) = \frac{\mathcal{LA}_{i,k}(q)}{\sum_{i=1}^{n} q_i \cdot P_i}$$

and $\mathcal{LC}_k(q) = \sum_{i=1}^{n} \mathcal{LC}_{i,k}(q)$ is the liquidation contribution for trading day k. Another useful decomposition is to consider the break-down by security:

$$
\begin{aligned}
\mathcal{LR}(q;h) &= \sum_{i=1}^{n} \frac{q_i \cdot P_i}{\sum_{i=1}^{n} q_i \cdot P_i} \frac{\sum_{k=1}^{h} \mathcal{LA}_{i,k}(q)}{q_i \cdot P_i} \\
&= \sum_{i=1}^{n} w_i \cdot \mathcal{LR}(q_i;h) \\
&= \sum_{i=1}^{n} \mathcal{LC}_i(q;h)
\end{aligned}
$$

where w_i is the relative weight of security i in portfolio q and $\mathcal{LR}(q_i;h)$ is the liquidation ratio applied to the selling order q_i:

$$\mathcal{LR}(q_i;h) = \frac{\sum_{k=1}^{h} \mathcal{LA}_{i,k}(q)}{q_i \cdot P_i}$$

$\mathcal{LC}_i(q;h) = w_i \cdot \mathcal{LR}(q_i;h)$ is the liquidation contribution of asset i.

We consider the previous example. Table 5 shows the values taken by the liquidation contribution $\mathcal{LC}_{i,h}(q)$. For instance, $\mathcal{LC}_{1,2}(q) = 15.14\%$ means that the liquidation of 1 000 shares of the second asset during the first trading day represents 15.14% of the redemption scenario. The sum of each row h corresponds to the liquidation contribution $\mathcal{LC}_h(q)$. For instance, we have $13.21\% + 15.14\% + 1.99\% + 3.09\% + 1.57\% = 35.00\%$. The sum of each column corresponds to the weights w_i because we have[14] $w_i = \sum_{k=1}^{h^+} \mathcal{LC}_{i,k}(q)$. The weights w_i and the liquidation ratios $\mathcal{LR}(q_i;h)$ are given in Table 6. We observe that the

---

[14]This result comes from the following identity:

$$\sum_{k=1}^{h^+} \mathcal{LC}_{i,k}(q) = \sum_{k=1}^{h^+} \frac{\mathcal{LA}_{i,k}(q)}{\sum_{j=1}^{n} q_j \cdot P_j} = \sum_{k=1}^{h^+} \frac{q_i(k) \cdot P_i}{\sum_{j=1}^{n} q_j \cdot P_j} = \frac{q_i \cdot P_i}{\sum_{j=1}^{n} q_j \cdot P_j} = w_i$$

Table 5: Liquidation contribution $\mathcal{LC}_{i,h}(q)$ by trading day

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	$\mathcal{LC}_h(q)$
1	13.21%	15.14%	1.99%	3.09%	1.57%	35.00%
2	13.21%	15.14%	1.99%	0.00%	0.00%	30.34%
3	13.21%	0.08%	1.99%	0.00%	0.00%	15.27%
4	13.21%	0.00%	1.54%	0.00%	0.00%	14.75%
5	4.64%	0.00%	0.00%	0.00%	0.00%	4.64%
Total	57.47%	30.35%	7.51%	3.09%	1.57%	100.00%

Table 6: Weight w_i and liquidation ratio $\mathcal{LR}(q_i; h)$ of the assets

	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5
$\mathcal{LR}(q_i; 1)$	22.98%	49.88%	26.49%	100.00%	100.00%
$\mathcal{LR}(q_i; 2)$	45.97%	99.75%	52.98%	100.00%	100.00%
$\mathcal{LR}(q_i; 3)$	68.95%	100.00%	79.47%	100.00%	100.00%
$\mathcal{LR}(q_i; 4)$	91.93%	100.00%	100.00%	100.00%	100.00%
$\mathcal{LR}(q_i; 5)$	100.00%	100.00%	100.00%	100.00%	100.00%
w_i	57.47%	30.35%	7.51%	3.09%	1.57%

Table 7: Liquidation contribution $\mathcal{LC}_i(q; h)$ by asset

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	$\mathcal{LR}(q; h)$
1	13.21%	15.14%	1.99%	3.09%	1.57%	35.00%
2	26.42%	30.28%	3.98%	3.09%	1.57%	65.34%
3	39.63%	30.35%	5.97%	3.09%	1.57%	80.61%
4	52.84%	30.35%	7.51%	3.09%	1.57%	95.36%
5	57.47%	30.35%	7.51%	3.09%	1.57%	100.00%

assets are respectively liquidated in five, three, three, four, one and one trading days. If we multiply the weights w_i by the liquidation ratios $\mathcal{LR}(q_i; h)$, we obtain the liquidation contribution $\mathcal{LC}_i(q; h)$ by asset. If we sum the elements of each row, we obtain the liquidity ratio $\mathcal{LR}(q; h)$.

As explained by Roncalli and Weisang (2015a), the liquidation ratio will depend on three factors: the liquidity of the portfolio to sell, the amount to sell and the liquidation policy. They illustrated the impact of these factors using several index portfolios. For instance, we report in Figure 7 the example of the EUROSTOXX 50 index portfolio. We notice that the liquidation ratio is different if we consider a selling order of $1, $10 or $50 bn. It is also different if the trading limit is equal to 10% or 30% of the average daily volume[15] (ADV). In Figure 8, we compare the liquidation ratio for different index portfolios when the trading limit is set to 10% of ADV. We notice that the liquidity profile is better for the S&P 500 Index and a size of $50 bn than for the EUROSTOXX 50 Index and a size of $10 bn. We also observe that liquidating $1 bn of the MSCI INDIA Index is approximately equivalent to liquidating $10 bn of the EUROSTOXX 50 Index. Of course, these results may differ from one period to another, because the liquidity is time-varying. Nevertheless, we observe that the liquidity of the portfolio is different depending on whether we consider small cap stocks or large cap stocks. The liquidity ratio also decreases with the amount to sell. Finally, the liquidity ratio also depends on the trading constraints or the liquidation policy.
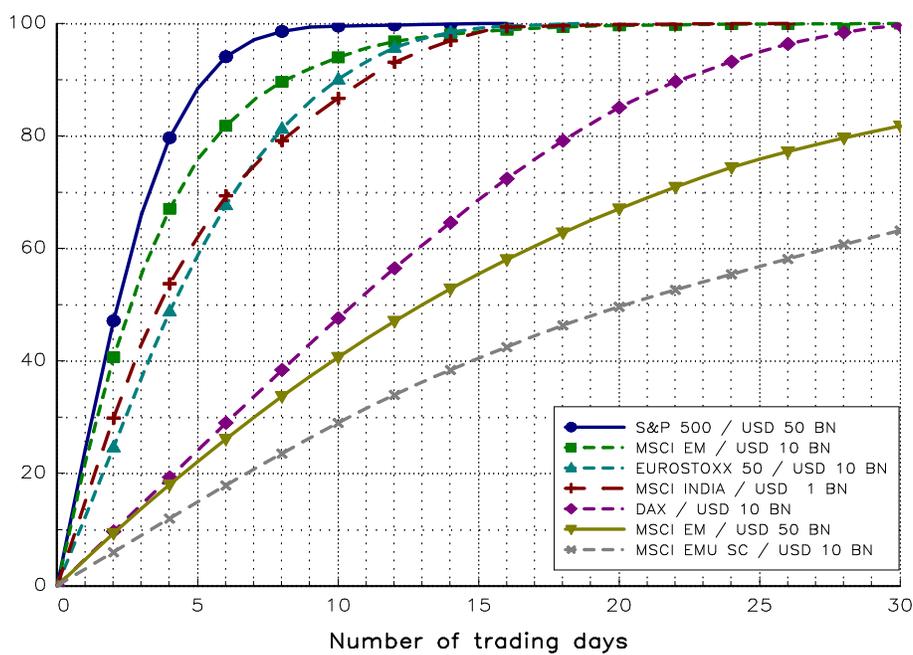
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[15]Roncalli and Weisang (2015a) used the three-month average daily volume computed by Bloomberg.

Figure 7: Liquidation ratio (in %) of the EUROSTOX 50 index portfolio
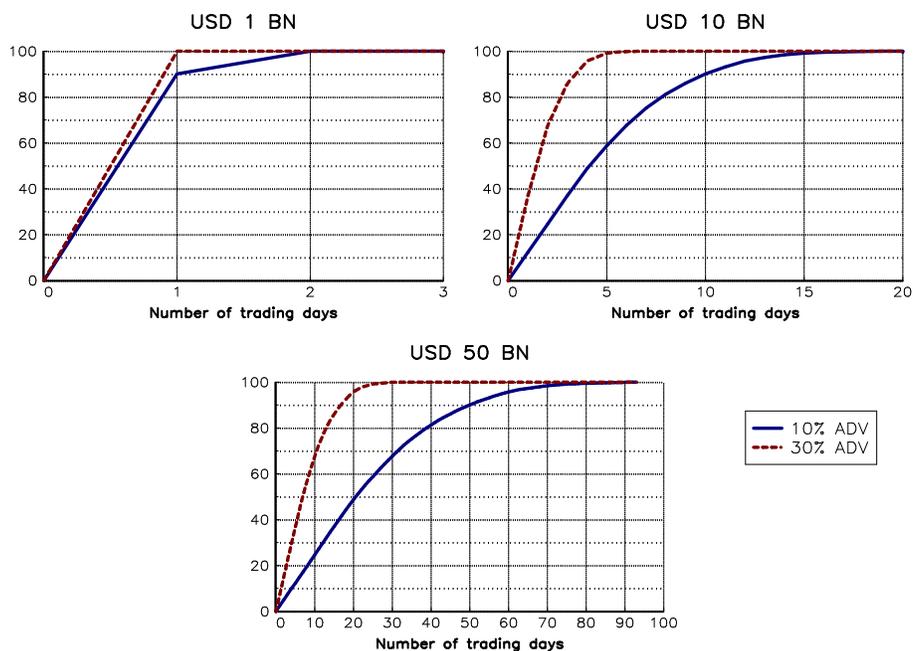
Source: Roncalli and Weisang (2015a, Figure 15, page 50), data as of April 30, 2015.

Figure 8: Comparing the liquidation ratio (in %) between equity index portfolios

Source: Roncalli and Weisang (2015a, Figure 5, page 28), data as of April 30, 2015.

3.2.2 Time to liquidation

The liquidation time is the inverse function of the liquidation ratio:

$$
\begin{aligned}
\mathcal{LT}(q;p) &= \mathcal{LR}^{-1}(q;p) \\
&= \inf\{h : \mathcal{LR}(q;h) \geq p\}
\end{aligned}
$$

For instance, $\mathcal{LT}(q;75\%) = 8$ means that we need 8 trading days to fulfill 75% of the redemption. The liquidation time is a step function because $\mathcal{LT}(q;p)$ is an integer. If we consider the previous example, we have $\mathcal{LR}(q;0) = 0$, $\mathcal{LR}(q;1) = 35\%$, $\mathcal{LR}(q;2) = 65.34\%$, etc. We deduce that $\mathcal{LT}(q;p) = 0$ if $p < 35\%$, $\mathcal{LT}(q;p) = 1$ if $35\% \leq p < 65.34\%$, etc.

In Table 8, we report some figures of liquidation time that were calculated by Roncalli and Weisang (2015a). The size of the equity index portfolio is set to \$10 bn, and two liquidation policies are tested (10% and 30% of the average daily volume). In the case of the S&P 500 Index, liquidating 90% of a \$10 bn equity index portfolio takes two trading days with a trading limit of 10% of the ADV and one trading day with a trading limit of 30% of the ADV. In the case of the MSCI EMU Small Cap Index, these liquidation times becomes 74 and 25 trading days.

Table 8: Time to liquidation (size = \$10 bn)

Index	S&P 500	ES 50	DAX	NASDAQ	MSCI EM	MSCI INDIA	MSCI EMU SC
p (in %)				10% of ADV			
50	1	5	11	2	3	37	21
75	1	7	17	3	5	71	43
90	2	10	23	3	9	110	74
99	2	15	29	5	17	156	455
p (in %)				30% of ADV			
50	1	2	4	1	1	13	7
75	1	3	6	1	2	24	15
90	1	4	8	1	3	37	25
99	1	5	10	2	6	52	152

Source: Roncalli and Weisang (2015a, Tables 6 and 7, page 26), data as of April 30, 2015.

Remark 13 *The liquidation risk profile of the redemption scenario q can be defined by the function $h \mapsto \mathcal{LR}(q;h)$ or the function $p \mapsto \mathcal{LT}(q;p)$. As shown by Roncalli and Weisang (2015a), it depends on the asset liquidity, the liquidation policy and the portfolio composition.*

3.2.3 Liquidation shortfall

The liquidation shortfall $\mathcal{LS}(q)$ is defined as the remaining redemption that cannot be fulfilled after one trading day:

$$
\mathcal{LS}(q) = 1 - \mathcal{LR}(q;1) \tag{24}
$$

For instance, it is equal to 65% for the previous example described on page 14. The liquidation shortfall is an increasing function of the order size. An illustration is given in Figure 9 by considering three different liquidation policies.

Figure 9: Liquidation shortfall with respect to the portfolio notional
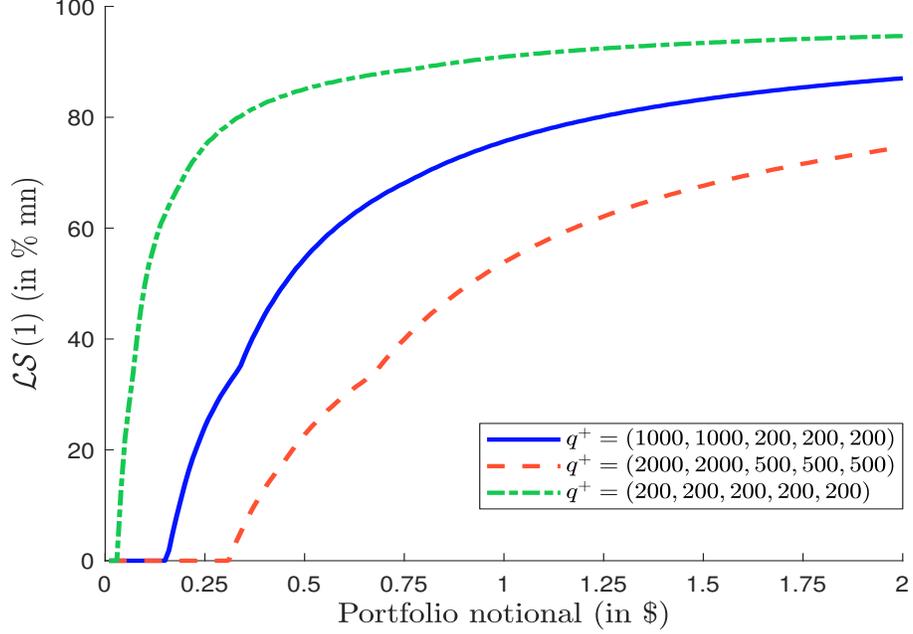
3.3 Liquidity cost

We now turn to liquidity measures that incorporate the price (or cost) dimension. Generally, we measure the liquidity cost by the transaction cost. However, in a liquidity stress testing program, this measure is merely theoretical since it is based on the transaction cost model. Therefore, it can be completed by the ex-post liquidity cost, which is also called the effective cost.

3.3.1 Transaction cost

We define the transaction cost of the redemption scenario $q = (q_1, \ldots, q_n)$ as the product of the unit costs and the dollar volumes:

$$\mathcal{TC}(q) = \sum_{i=1}^{n} q_i \cdot P_i \cdot \boldsymbol{c}_i(x_i) = \sum_{i=1}^{n} Q_i \cdot \boldsymbol{c}_i(x_i) \tag{25}$$

where $Q_i = q_i \cdot P_i$ is the nominal volume (expressed in \$), $x_i = v_i^{-1} q_i$ is the participation rate when selling security i and $\boldsymbol{c}_i(x)$ is the unit transaction cost associated with security i. We can then break down the liquidity cost into two parts[16]:

$$\mathcal{TC}(q) = \mathcal{BAS}(q) + \mathcal{PI}(q)$$

where the bid-ask spread component is equal to:

$$\mathcal{BAS}(q) = \sum_{i=1}^{n} Q_i \cdot s_i$$

---

[16]We have:

$$\boldsymbol{c}_i(x) = s_i + \boldsymbol{\pi}_i(x)$$

and the market impact cost is given by:

$$\mathcal{PI}(q) = \sum_{i=1}^{n} Q_i \cdot \boldsymbol{\pi}_i(x_i)$$

The previous analysis assumes that we can sell the portfolio q instantaneously or during the same day. However, Equation (25) is only valid if the volumes q_i are less than the trading limits $q_i^+ = x_i^+ \cdot v_i$. Otherwise, we have:

$$\mathcal{TC}(q) = \sum_{i=1}^{n} \sum_{h=1}^{h^+} \mathbb{1}\{q_i(h) > 0\} \cdot q_i(h) \cdot P_i \cdot \boldsymbol{c}_i\left(\frac{q_i(h)}{v_i}\right) \qquad (26)$$

In this case, the bid-ask spread component has the same expression, but the market impact component is different. Indeed, we have[17]:

$$
\begin{aligned}
\mathcal{BAS}(q) &= \sum_{i=1}^{n} \sum_{h=1}^{h^+} \mathbb{1}\{q_i(h) > 0\} \cdot q_i(h) \cdot P_i \cdot s_i \\
&= \sum_{i=1}^{n} Q_i \cdot s_i \qquad (27)
\end{aligned}
$$

but:

$$\mathcal{PI}(q) = \mathcal{TC}(q) - \mathcal{BAS}(q) \neq \sum_{i=1}^{n} Q_i \cdot \boldsymbol{\pi}_i(x_i) \qquad (28)$$

Remark 14 *We assume that $q_i \leq q_i^+$. We have $q_i(1) = q_i$ and $q_i(h) = 0$ for $h > 1$. We obtain:*

$$
\begin{aligned}
\mathcal{TC}(q) &= \sum_{i=1}^{n} q_i(1) \cdot P_i \cdot \boldsymbol{c}_i\left(\frac{q_i(1)}{v_i}\right) \\
&= \sum_{i=1}^{n} q_i \cdot P_i \cdot \boldsymbol{c}_i(x_i)
\end{aligned}
$$

We retrieve the expression given in Equation (25).

Remark 15 *Since the transaction cost is measured in dollars, it may be useful to express it as a percentage of the redemption value:*

$$\mathcal{TC}_r(q) = \frac{\mathcal{TC}(q)}{\sum_{i=1}^{n} q_i \cdot P_i}$$

An alternative measure is to compare the total transaction cost with the bid-ask spread component:

$$\mathcal{TC}_s(q) = \frac{\mathcal{TC}(q)}{\mathcal{BAS}(q)}$$

---

[17]Because of the following identity:

$$\sum_{h=1}^{h^+} \mathbb{1}\{q_i(h) > 0\} \cdot q_i(h) = q_i$$

We consider the previous example. We recall the characteristics of the redemption portfolio:

Asset	1	2	3	4	5
q_i	4 351	2 005	755	175	18
q_i^+	1 000	1 000	200	200	200
P_i (in \$)	89	102	67	119	589
$\boldsymbol{\pi}_i(x)$	SQRL model with $\varphi_1 = 1$, $\tilde{x} = 5\%$ and $x^+ = 10\%$				
σ_i (in %)	25	20	18	30	20
s_i (in bps)	4	4	5	5	5
v_i	10 000	10 000	2 000	2 000	2 000

We also indicate the transaction cost function. It is given by the SQRL model with $\varphi_1 = 1$, $\tilde{x} = 5\%$ and $x^+ = 10\%$. For each asset i, we also indicate the annualized volatility σ_i, the value of the bid-ask spread s_i and the daily volume v_i.

The value of the redemption portfolio is equal to \$673 761. The total transaction cost is equal to $\mathcal{TC}(q) = \$4 373.55$ with the following breakdown: $\mathcal{BAS}(q) = \$277.71$ and $\mathcal{PI}(q) = \$4 095.85$. These figures represent respectively 64.9, 4.1 and 60.8 bps of the portfolio value. We deduce that the price impact explains 93.7% of the transaction cost. The contribution of each asset is respectively equal to 34.6%, 30.5% and 16.6%, 16.0% and 2.4%. More results can be found in Tables 37–41 on page 80.

3.3.2 Implementation shortfall and effective cost

The previous analysis assumes that the transaction cost is calculated with a model. Therefore, Equation (26) defines an ex-ante transaction cost. In practice, this ex-ante transaction cost will differ from the effective transaction cost. In order to define the latter, we must reintroduce the time index t in the analysis. The current value of the redemption scenario is equal to:

$$\mathbb{V}^{\mathrm{mid}}(q) = \sum_{i=1}^{n} q_i(t) \cdot P_i^{\mathrm{mid}}(t)$$

where $q_i(t)$ and $P_i^{\mathrm{mid}}(t)$ are the number of shares to sell and the mid-price for the security i at the current time t. The value of the liquidated portfolio is equal to:

$$\mathbb{V}^{\mathrm{liquidated}}(q) = \sum_{i=1}^{n} \sum_{t_k \geq t} q_i(t_k) \cdot P_i^{\mathrm{bid}}(t_k)$$

where $q_i(t_k)$ and $P_i^{\mathrm{bid}}(t_k)$ are the number of shares that were sold and the bid price for the security i at the execution time t_k. The effective cost is then the difference between $V^{\mathrm{mid}}(t)$ and $V^{\mathrm{liquidated}}(t)$:

$$\mathcal{IS}(q) = \max\left(\mathbb{V}^{\mathrm{mid}}(q) - \mathbb{V}^{\mathrm{liquidated}}(q), 0\right) \tag{29}$$

The effective cost[18] $\mathcal{IS}(q)$ is called by Perold (1988) the implementation shortfall, which measures the difference in price between the time a portfolio manager makes an investment decision and the actual traded price. Therefore, $\mathbb{V}^{\mathrm{mid}}(q)$ is the benchmark price, $\mathbb{V}^{\mathrm{liquidated}}(q)$ is the traded price and $\mathcal{IS}(q)$ is the total amount of slippage.

---

[18]Since $\mathbb{V}^{\mathrm{liquidated}}(q)$ can be higher than $\mathbb{V}^{\mathrm{mid}}(q)$, $\mathcal{IS}(q)$ is floored at zero. This situation occurs when execution times t_k are very different than the current time t and market prices have gone up — $P_i^{\mathrm{bid}}(t_k) \geq P_i^{\mathrm{mid}}(t)$.

4 Implementing the stress testing framework

In this section, we detail the general approach for implementing the liquidity stress testing program on the asset side. We will see that it is based on three steps. First, we have to correctly define the asset liquidity buckets (or asset liquidity classes). Each asset liquidity bucket is associated with a unique unit transaction cost function and a given liquidation policy. Second, we have to calibrate the parameters of the transaction cost function that are related to a given liquidity bucket. Third, we must define the appropriate estimation method of the security-specific parameters. Nevertheless, before presenting the three-step approach, we must understand how stress testing impacts transaction costs. Does stress testing modify the conventional transaction cost function? Does stress testing change the liquidation policy? What parameters are impacted? This analysis will help to justify the three-step approach of asset liquidity stress testing. Finally, the last part of this section is dedicated to an issue that generally occurs when implementing the LST program. This concerns the distortion of the redemption scenario on the asset side. In this article, we only present general considerations, but this issue will be extensively studied in our third article dedicated to liquidity stress testing in asset management (Roncalli *et al.*, 2021).

4.1 How does stress testing impact transaction costs?

If we consider the two-regime model, we have:

$$
c\left(\frac{q}{v}\right) = \begin{cases}
s + \varphi_1 \sigma \left(\dfrac{q}{v}\right)^{\gamma_1} & \text{if } q \leq \tilde{x} \cdot v \\
s + \varphi_1 \tilde{x}^{\gamma_1 - \gamma_2} \sigma \left(\dfrac{q}{v}\right)^{\gamma_2} & \text{if } \tilde{x} \cdot v \leq q \leq x^+ \cdot v \\
+\infty & \text{if } q > x^+ \cdot v
\end{cases}
$$

The parameters of the transaction cost model are s, φ_1, σ, γ_1, γ_2, $\tilde{x}$ and x^+. The question is whether we need two sets of parameters:

1. $\left(s^{\text{normal}}, \varphi_1^{\text{normal}}, \sigma^{\text{normal}}, \gamma_1^{\text{normal}}, \gamma_2^{\text{normal}}, \tilde{x}^{\text{normal}}, x^{+\text{normal}}\right)$ for normal periods;

2. $\left(s^{\text{stress}}, \varphi_1^{\text{stress}}, \sigma^{\text{stress}}, \gamma_1^{\text{stress}}, \gamma_2^{\text{stress}}, \tilde{x}^{\text{stress}}, x^{+\text{stressl}}\right)$ for stress periods.

This is equivalent having two different transaction cost functions: $c^{\text{normal}}(x)$ and $c^{\text{stress}}(x)$. This is not satisfactory because this means that we need to calibrate many parameters in the stress period. Moreover, we do not distinguish between parameters that are related to the security and parameters that are related to the liquidity bucket. Clearly, we can assume that the parameters $(\varphi_1, \gamma_1, \gamma_2, \tilde{x}, x^+)$ are the same for all the assets belonging to the same liquidity bucket. They can change, but at a low frequency, for instance because of the annual calibration exercise or a change to the liquidation policy. The other parameters s and σ are defined at the security level and can change daily[19]. Therefore, the unit transaction cost function must be written as $c_x(x; s_{i,t}, \sigma_{i,t})$ because $s_{i,t}$ and $\sigma_{i,t}$ change with the security and the time. We notice that this transaction cost function uses the participation ratio x, which is the ratio between the order size q and the daily volume v. However, v is another related-security parameter since it changes every day. This is not equivalent to selling 1 000 shares in the market if the daily volume is 10 000 or 20 000. It follows that the unit transaction cost function must be written as $c_q(q; s_{i,t}, \sigma_{i,t}, v_{i,t})$ because $s_{i,t}$, $\sigma_{i,t}$ and $v_{i,t}$ change with the security and the time. The q-approach to the unit transaction cost $c(q; s_{i,t}, \sigma_{i,t}, v_{i,t})$ differs then from the x-approach to the unit transaction cost $c(x; s_{i,t}, \sigma_{i,t})$ because it has an additional parameter, which is the daily volume.

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[19]Indeed, the spread and the volatility of the security change every day because of market conditions.
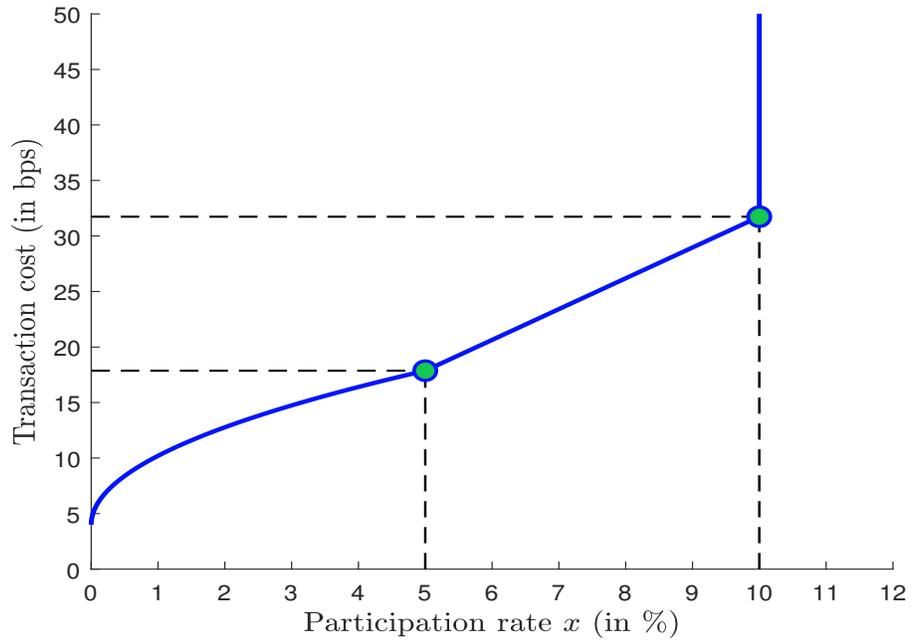
Figure 10: The x-approach of the unit transaction cost

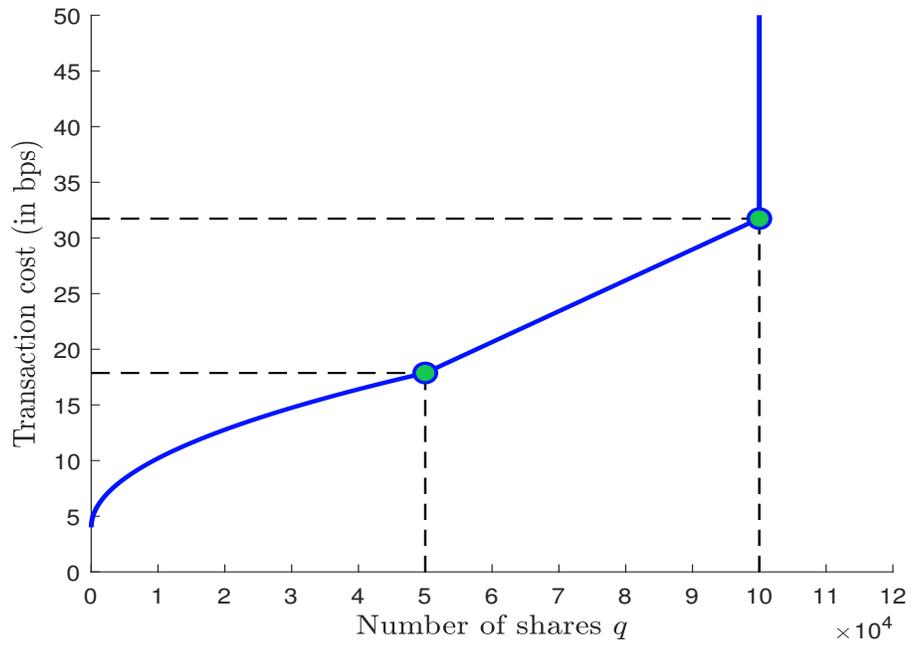Figure 11: The q-approach of the unit transaction cost

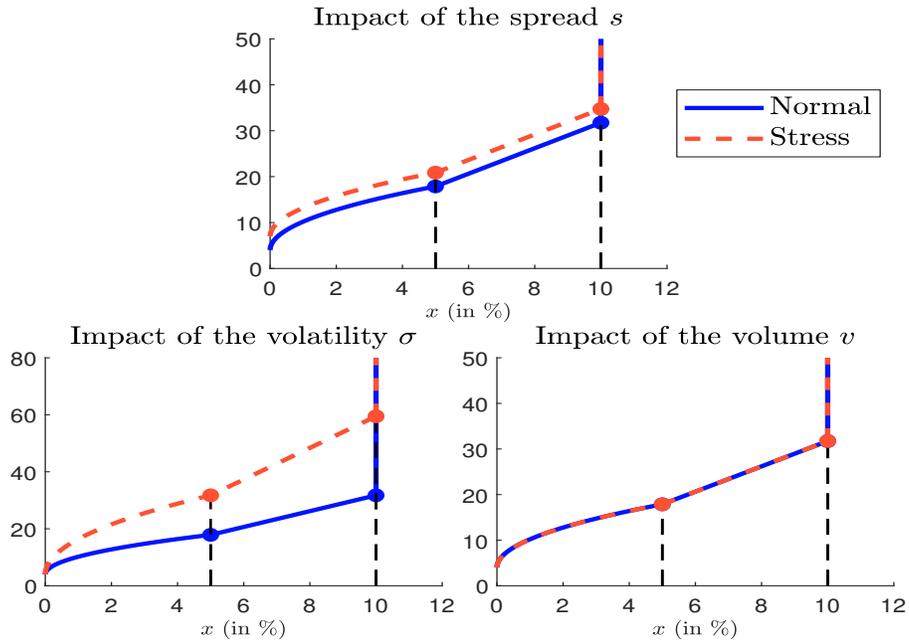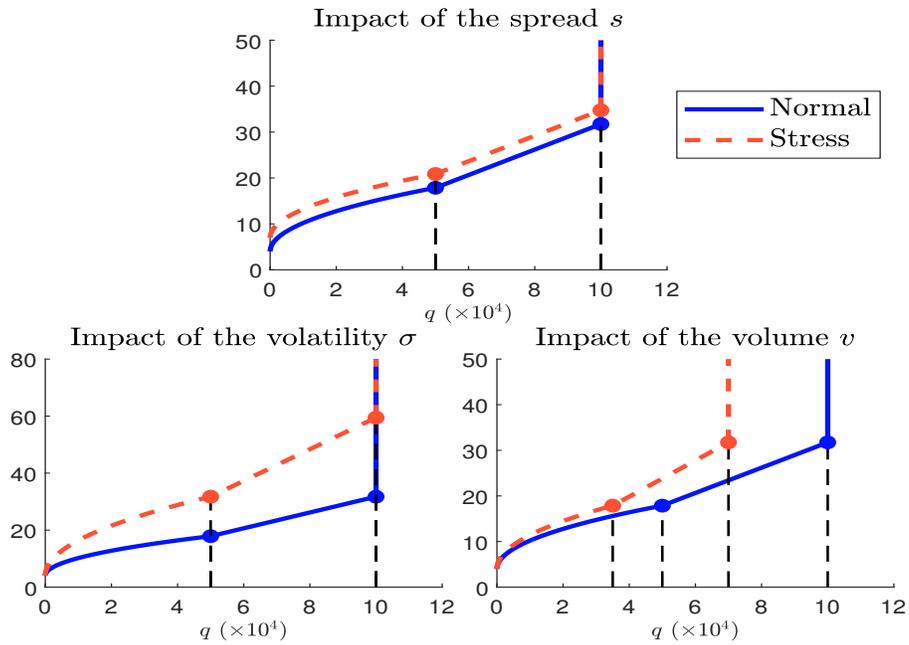Figure 12: Impact of security-specific parameters in the x-approach

Figure 13: Impact of security-specific parameters in the q-approach

At first sight, it seems that introducing the volume is a subtle distinction. For instance, we have reported in Figures 10 and 11 the functions $\boldsymbol{c}_x \left(x; s_{i,t}, \sigma_{i,t} \right)$ and $\boldsymbol{c}_q \left(q; s_{i,t}, \sigma_{i,t}, v_{i,t} \right)$ when the price impact is given by the SQRL model[20], the security-related parameters are equal to $s_{i,t} = 4$ bps, $\sigma_{i,t} = 10\%$ and $v_{i,t} = 100\,000$, and the liquidation policy is set to $x^+ = 10\%$. The two figures have exactly the same shape and we have the following correspondence:

$$\boldsymbol{c}_q \left(q; s_{i,t}, \sigma_{i,t}, v_{i,t} \right) := \boldsymbol{c}_x \left(x = \frac{q}{v_{i,t}}; s_{i,t}, \sigma_{i,t} \right)$$

Let us now see the impact of changing the parameters $s_{i,t}$, $\sigma_{i,t}$ and $v_{i,t}$. In a stress period, we generally observe an increase in the bid-ask spread and the asset volatility, and a reduction in the daily volume that is traded in the market. In the top panel in Figures 12 and 13, we show the difference between the two unit transaction costs when the bid-ask spread increases from 4 bps to 7 bps. We observe that the functions $\boldsymbol{c}_x$ and $\boldsymbol{c}_q$ are both shifted up, but they are the same. In the bottom/left panel, we report the impact when the volatility in the stress period is twice the volatility in the normal period[21]. We notice that the higher volatility has shifted the trading cost upward and it has also changed the shape of the unit transaction cost function. But again, the two functions $\boldsymbol{c}_x$ and $\boldsymbol{c}_q$ are the same using the equivalence relationship $q = x \cdot v_{i,t}$. We now consider the impact of the volume. Generally, the daily volume is reduced in stress periods. In the bottom/right panel in Figures 12 and 13, we assume that the daily volume is equal to $v_{i,t} = 100\,000$ in the normal period and $v_{i,t} = 70\,000$ in the stress period. Contrary to the parameters $s_{i,t}$ and $\sigma_{i,t}$, we observe that the two functions $\boldsymbol{c}_x$ and $\boldsymbol{c}_q$ are not equivalent in this case. Indeed, $v_{i,t}$ has no impact on $\boldsymbol{c}_x$ whereas it completely changes the shape of $\boldsymbol{c}_q$ because the inflection point $\tilde{q}$ and the trading limit q^+ are different. It follows that the invariance with respect to x does not imply the invariance with respect to q.

In the case of a liquidity stress program, we have to consider the combination of the three effects. Results are reported in Figure 14. We recall that the normal period is defined by $s_{i,t} = 4$ bps, $\sigma_{i,t} = 10\%$ and $v_{i,t} = 100\,000$, while the stress period is defined by $s_{i,t} = 7$ bps, $\sigma_{i,t} = 20\%$ and $v_{i,t} = 70\,000$. During the stress period, the transaction cost is higher because the spread is larger, the volatility has shifted the trading cost upward and the lower volume has moved the inflection point to the left. This is the primary effect. For instance, selling $40\,000$ shares of the security costs 16.40 bps during the normal period and 38.70 bps during the stress period (see Table 9). The secondary effect is on the liquidation profile, because the trading limit q^+ expressed as a number of shares is reduced in the stress period even if the liquidation policy does not change. This is because the liquidation policy is defined in terms of the maximum participation rate x^+. For instance, $100\,000$ shares of the security can be sold in one trading day in the normal period. This is no longer true in the stress period, and the position is liquidated in two trading days (see Table 9). It follows that the stress testing program has a negative, non-linear impact both on the transaction cost and the liquidation profile. In Table 9, we have $\boldsymbol{c}\,(40\,000) = 38.70$ bps, $\boldsymbol{c}\,(80\,000) = 57.39$ bps and $\boldsymbol{c}\,(100\,000) = 53.53$ bps. We observe that $\boldsymbol{c}\,(q)$ is not necessarily an increasing function of q because of the liquidation policy. Indeed, in the last case, $70\,000$ shares are sold at 62.47 bps during the first trading day and $30\,000$ shares are sold at 32.68 bps during the second trading day. The relative cost of selling $100\,000$ shares is lower than the relative cost of selling $70\,000$ shares, because the price impact is not at its maximum during the second day. In Figure 30 on page 79, we report the two functions, the relative (or unit) transaction cost $\boldsymbol{c}\,(q)$ and the total transaction cost $\mathcal{TC}\,(q)$, by assuming that the price is equal to \$1. We notice that the maximum relative cost is equal to 62.47 bps and is reached when the number

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[20]We assume that $\varphi_1 = 1$ and $\tilde{x} = 5\%$.
[21]The annualized volatility $\sigma_{i,t}$ increases from 10% to 20%.

Figure 14: Comparing the unit transaction cost in the normal and stress periods
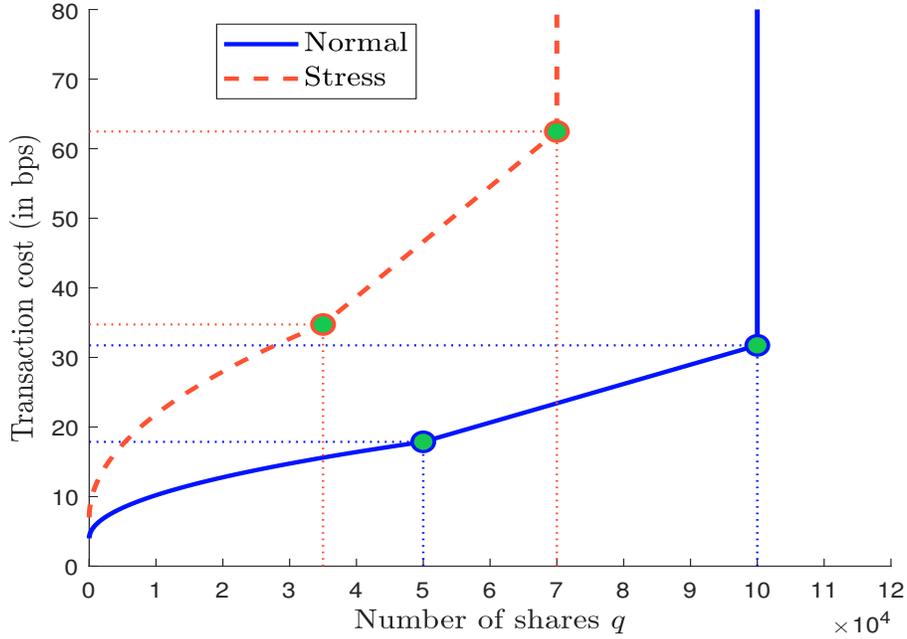
of shares is a multiple of $70\,000$, which is the trading limit. Therefore, $\boldsymbol{c}(q)$ is not increasing because of the averaging effect. Of course, this is not the case for the total transaction cost, which is an increasing function of q.

Table 9: Computation of the unit transaction cost

	Normal	Stress	Normal	Stress	Normal	Stress	Normal	Stress
q	$10\,000$		$40\,000$		$80\,000$		$100\,000$	
$q(h)$	$10\,000$	$10\,000$	$40\,000$	$40\,000$	$80\,000$	$70\,000$ $10\,000$	$100\,000$	$70\,000$ $30\,000$
s	4.00	7.00	4.00	7.00	4.00	7.00 7.00	4.00	7.00 7.00
$\boldsymbol{\pi}(q(h))$	6.20	14.82	12.40	31.70	22.19	55.47 14.82	27.74	55.47 25.68
$\boldsymbol{c}(q(h))$	10.20	21.82	16.40	38.70	26.19	62.47 21.82	31.74	62.47 32.68
$\boldsymbol{c}(q)$	10.20	21.82	16.40	38.70	26.19	57.39	31.74	53.53

Remark 16 *The previous analysis shows that we do not need a new transaction cost function for the stress period, because there is no reason for the functional form to change and the impact of the security-specific parameters are sufficient to implement the asset liquidity stress testing program.*

4.2 A three-step approach

As explained above, implementing an asset liquidity stress testing program involves three steps. In the first step, we define liquidity buckets. The second step corresponds to the estimation of the transaction cost function for a given liquidity bucket. Finally, the third step consists in calibrating the security-specific parameters.

4.2.1 Liquidity bucketing

Table 10: An example of classification matrix of liquidity buckets

Level 1	Level 2	Level 3	Level 4	HQLA Class
Equity	Large cap	DM	Region	1
		EM		1
	Small cap	DM	Region	2
		EM		2
	Derivatives	Futures	Turnover	1
		Options		2
	Private			5
Fixed-income	Sovereign	DM	Region	1/2
		EM		2/3
	Municipal			2
	Inflation-linked	DM	Region	1/2
		EM		2/3
	Corporate	IG	Currency	3
		HY		4
	Securitization	ABS	US/Non-US	2/3/4
		CLO		
		CMBS		
		RMBS		
	Derivatives	Caps/floors	Turnover	1/2/3
		Futures		
		Options		
		Swaps		
	CDS	Single-name	Turnover	3
		Multi-name		2
Currency	G10			1
	Others			1/2/3
Commodity	Agriculture	Grain & Oilseed		4
		Livestock		4
		Soft		4
	Energy	Electricity		2
		Gas		2
		Oil		2
	Metal	Gold		1
		Industrial		2/4
		Precious		2

Classification matrix A liquidity bucket is a set of homogenous securities such that they share the same functional form of the unit transaction cost[22]. For instance, we may consider that equities and bonds correspond to two liquidity buckets, meaning that we need two different functions. But we can also split equities between large cap and small cap equities.

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[22]Most of the time, they also share the same liquidation policy.

An example of matrix classification is provided in Table 10. There are several levels depending on the requirements of the asset manager and the confidence level on the calibration. Generally, Level 2 is sufficiently granular and enough to implement a liquidity stress testing program. For instance, it is extensively used by external providers of LST solutions (MSCI LiquidityMetrics, Bloomberg Liquidity Assessment (LQA), StateStreet Liquidity Risk Solution, etc.). Nevertheless, the asset manager may wish to go beyond Level 2 and adopt Level 3 for some buckets. For example, it could make sense to distinguish the functional form for DM and EM sovereign bonds. Level 4 is the ultimate level and differentiates securities by region, currency or turnover[23]. For example, if we consider the DM large cap stocks, we may split this category by region, e.g., North America, Eurozone, Japan and Europe-ex-EMU. In the case of corporate IG bonds, one generally splits these securities by currency, e.g., USD IG bonds, EUR IG bonds, GBP IG bonds, etc. For derivatives, one may build two categories depending on the turnover value, e.g., the most liquid contacts and the other derivative products.

HQLA classes   In this article, we focus on the transaction cost. The asset-liability management will be studied in the third part of our comprehensive research project on liquidity risk in asset management (Roncalli *et al.*, 2021). Nevertheless, we notice that the asset manager must develop two asset liquidity classification matrices: liquidity buckets and HQLA classes. The term HQLA refers to the liquidity coverage ratio (LCR) introduced in the Basel III framework (BCBS, 2010, 2013a). An asset is considered to be a high-quality liquid asset if it can be easily converted into cash. Therefore, the concept of HQLA is related to asset quality and asset liquidity. It is obvious that the LST regulation is inspired by the liquidity management regulation developed by the Basel Committee on Banking Supervision. For instance, the redemption coverage ratio (RCR) for asset managers is related to the liquidity coverage ratio for banks. According to ESMA (2020), the redemption coverage ratio is "*a measurement of the ability of a fund's assets to meet funding obligations arising from the liabilities side of the balance sheet, such as a redemption shock*". In Roncalli *et al.* (2021), we will see that it is helpful to define another asset liquidity classification matrix that is complementary to the previous liquidity buckets. This new classification matrix uses HQLA classes, whose goal is to group assets by their relative liquidity risk. For instance, such asset liquidity classification matrix is already used in the US with the Rule 22e-4(b) (Roncalli *et al.*, 2020, page 5), which considers four classes: (1) highly liquid investments, (2) moderately liquid investments, (3) less liquid investments and (4) illiquid investments. Here is an example based on five HQLA classes[24]:

- Tier 1: Sovereign bonds (EUR, USD, GBP, AUD, JPY, SEK, CAD and domestic currency of the asset manager), large cap equities, specified currency pairs[25], bond futures, equity index futures, etc.

- Tier 2: Other IG sovereign bonds, municipal bonds, small cap equities, other IG currency pairs, multi-name CDS, commodity futures (energy, precious metals, non-ferrous metals), equity options, etc.

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[23]The turnover is defined as "*the gross value of all new deals entered into during a given period and is measured in terms of the nominal or notional amount of the contracts. It provides a measure of market activity and can also be seen as a rough proxy for market liquidity*" (Bank for International Settlements, 2014).

[24]It is derived from the liquidity period buckets defined in the Basel III capital requirements for market risk (BCBS, 2019).

[25]They correspond to the 20 most liquid currencies: USD, EUR, JPY, GBP, AUD, CAD, CHF, MXN, CNY, NZD, RUB, HKD, SGD, TRY, KRW, SEK, ZAR, INR, NOK and BRL.

- Tier 3: IG corporate bonds, HY sovereign bonds, HY currency pairs, single-name CDS, etc.

- Tier 4: HY corporate bonds, other commodity futures, etc.

- Tier 5: Private equities, real estate, etc.

For derivatives on interest rates, we can map them with respect to sovereign bonds. For instance, interest rate swaps on EUR, USD, GBP, AUD, JPY, SEK and CAD are assigned to Tier 1, interest rate swaps on IG currencies are assigned to Tier 2, interest rate swaps on HY currencies are assigned to Tier 3, etc. For securitization products, the best approach is to classify them with respect to their external credit rating.

4.2.2 Defining the unit transaction cost function

We consider that the two-regime model is the appropriate function to estimate the transaction cost of a redemption scenario. Nevertheless, we introduce some slight modifications, because the power-law model has been mainly investigated in the stock market. These modifications are necessary when we consider fixed-income products and derivatives.

The econometric model We assume that Security i belongs the jth liquidity bucket $\mathcal{LB}_j$ and rewrite the two-regime model as follows[26]:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) = \beta_j^{(s)} s_{i,t} + \beta_j^{(\boldsymbol{\pi})} \sigma_{i,t} \boldsymbol{\pi}_j^{\star}\left(q_i; v_{i,t}\right) \tag{30}$$

where:

$$\boldsymbol{\pi}_j^{\star}\left(q_i; v_{i,t}\right) = \begin{cases} \left(\dfrac{q_i}{v_{i,t}}\right)^{\gamma_{1,j}} & \text{if } q_i \leq \tilde{q}_{i,t} \\ \left(\dfrac{\tilde{q}_{i,t}}{v_{i,t}}\right)^{\gamma_{1,j}}\left(\dfrac{q_i}{\tilde{q}_{i,t}}\right)^{\gamma_{2,j}} & \text{if } \tilde{q}_{i,t} \leq q_i \leq q_{i,t}^{+} \\ +\infty & \text{if } q_i > q_{i,t}^{+} \end{cases} \tag{31}$$

The total transaction cost of selling q_i shares is then equal to[27]:

$$\mathcal{TC}\left(q_i\right) = \alpha_i q_i + Q_i \boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right)$$

Compared to the conventional two-regime model, we notice the introduction of two new parameters: α_i and $\beta_j^{(s)}$. For some securities (e.g., derivatives), we have to pay a fixed cost for each share, which motivates the addition of the term $\alpha_i q_i$. The introduction of the scaling factor $\beta_j^{(s)}$ is motivated because quoted bid-ask spreads are not always available for some liquidity buckets $\mathcal{LB}_j$. In this case, we can use an empirical model for computing $s_{i,t}$. From a theoretical point of view, we should have $\beta_j^{(s)} = 1$. This is the case for equities for instance, but not necessarily the case for some fixed-income securities. The reason is that asset managers do not necessarily face the same bid-ask spread costs. Therefore, $\beta_j^{(s)}$ may be less or greater than one.

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[26] We have the following relationship: $\beta_j^{(\boldsymbol{\pi})} = \varphi_1$. We also recall that $\sigma_{i,t}$ corresponds to the daily volatility in the transaction cost formula.

[27] In the case of a redemption scenario $q = (q_1, \dots, q_n)$, we obtain:

$$\mathcal{TC}\left(q\right) = \sum_{i=1}^{n} \sum_{h=1}^{h^+} \mathbb{1}\left\{q_i\left(h\right) > 0\right\} \cdot \left(\alpha_i q_i\left(h\right) + q_i\left(h\right) P_i \boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right)\right)$$

The model parameters The calibration of the functional form consists in estimating at least four parameters: $\beta_j^{(s)}$, $\beta_j^{(\boldsymbol{\pi})}$, $\gamma_{1,j}$ and $\gamma_{2,j}$. We can use the method of non-linear least squares. But we generally prefer to consider a two-stage approach by first determining the exponents $\gamma_{1,j}$ and $\gamma_{2,j}$ and then running a linear regression in order to obtain the OLS estimates of $\beta_j^{(s)}$ and $\beta_j^{(\boldsymbol{\pi})}$.

Remark 17 *The parameters $\tilde{q}_{i,t}$ and $q_{i,t}^+$ are particular. From a theoretical point of view, they are equal to $\tilde{q}_{i,t} = \tilde{x}_j v_{i,t}$ and $q_{i,t}^+ = x_j^+ v_{i,t}$, meaning that we have two other parameters $\tilde{x}_j$ and x_j^+ that are related to $\mathcal{LB}_j$. Nevertheless, for some liquidity buckets, the asset manager may choose to define the trading limit $q_{i,t}^+$ at the security level, meaning that we have $q_{i,t}^+ = x_i^+ v_{i,t}$. For instance, if we consider the category of DM sovereign bonds, trading limits may be fixed by country and maturity. Therefore, the liquidation policy may be different if we consider 10Y US, German, French and UK government bonds. When the inflection point $\tilde{x}_j$ (or $\tilde{x}_i$) is difficult to estimate, it can be a fraction of the trading limit x_j^+ (or x_i^+). The most frequent cases are $\tilde{x}_j = x_j^+/2$ (or $\tilde{x}_i = x_i^+/2$), and $\tilde{x}_j = x_j^+$ (or $\tilde{x}_i = x_i^+$) if we prefer to consider only one regime.*

The security-specific parameters They correspond to the bid-ask spread $s_{i,t}$, the volatility $\sigma_{i,t}$ and the daily volume $v_{i,t}$. Contrary to the model parameters, these parameters[28] depend on the time t. They are the key elements of the stress testing program, since their values will differ in normal and stress regimes.

Concerning the parameter $s_{i,t}$, we can consider an average of the bid-ask spread observed during a normal period (e.g., the last month) or we can use the daily quoted bid-ask spread in the case of stocks. For some fixed-income securities (e.g., corporate bonds, securitization products, etc.), quoted bid-ask spreads are not always available. In this case, we can use a statistical model that depends on the characteristics of the security. A simple model may distinguish bid-ask spreads by credit ratings[29]. A more sophisticated model may use intrinsic bond features such as maturity, notional outstanding, coupon value, credit rating, industrial sector, etc. (Ben Slimane and de Jong, 2017; Jurksas, 2018; Feldhütter and Poulsen, 2018; Guo *et al.*, 2019).

The parameter $\sigma_{i,t}$ measures the volatility of the asset i at time t. In the normal regime, $\sigma_{i,t}$ is measured with the historical volatility. We can consider a long-term volatility using a study period of three months, or we can consider a short-term estimator such as the exponentially weighted moving average (EWMA) volatility, the two-week empirical volatility or the GARCH volatility. In this last case, the volatility rapidly changes on a daily basis, and we can observe jumps in the transaction cost for the same securities from one day to the next. Therefore, we think that it is better to use a long-term estimator, in particular because the stress regime will incorporate these abnormal high-volatility regimes. For some securities, the daily volatility is not the most appropriate measure for measuring their risk. Therefore, it may be convenient to define $\sigma_{i,t}$ as a function of the security characteristics. For instance, we show in Appendix B.3 on page 76 that the main component of a corporate bond's volatility is the duration-times-spread (or DTS) of the bond[30].

The third security-specific parameter is the daily volume $v_{i,t}$. As for the volatility, we can use a short-term or a long-term measure. For instance, we can use the daily volume

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[28]In some cases, they also include $\tilde{q}_{i,t}$ and $q_{i,t}^+$.

[29]In this case, we assume that the bid-ask spread decreases with the credit quality, implying that the bid-ask spread of AAA-rated bonds is less than the bid-ask spread of BBB-rated bonds. Generally, credit ratings are grouped in order to form three or four categories.

[30]See Section 5.2.3 on page 48.

of the previous day. However, there is a consensus to use a longer period and to consider the three-month average daily volume. Again, we can alternatively use a statistical model when the data of daily volumes are not available. For instance, it can be a function of the outstanding amount for bonds, the turnover for derivatives, etc.

The trading limit $q_{i,t}^{+}$ has a particular status because it may be either a security-specific parameter or a model parameter. When it is a security-specific parameter, the asset manager defines $q_{i,t}^{+}$ at a low frequency, for instance every year or when there is a market change for trading the security i. However, the most frequent case is to consider $q_{i,t}^{+}$ as a model parameter: $q_{i,t}^{+} = x_{j}^{+} v_{i,t}$. In this situation, the asset manager generally uses the traditional rule of thumb $x_{j}^{+} = \mathcal{LR}_{j}^{+}$ where $\mathcal{LR}_{j}^{+}$ is the liquidation policy ratio of the liquidity bucket $\mathcal{LB}_{j}$. A typical value is 10% in the case of the stock market.

4.2.3 Calibration of the risk parameters in the stress regime

According to Roncalli (2020), there are three main approaches to generate a stress scenario: historical, macro-economic and probabilistic. However, in the case of asset management, the first two categories are more relevant, because asset managers do not have the same experience as banks in this domain, and data on transaction costs under stress periods are scarce. In this case, it is better to implement the probabilistic approach using the method of multiplicative factors.

As explained previously, the values of the security-specific parameters allow to distinguish the normal period and the stress period. The model parameters do not change, meaning that we use the same unit transaction cost function whatever the study period. It follows that the risk parameters are the bid-ask spread, the volatility and the volume. Therefore, asset liquidity stress testing leads to stressing the values of these three parameters.

Historical stress scenarios   The underlying idea of historical stress testing is to define the triple $(s_{i}^{\text{stress}}, \sigma_{i}^{\text{stress}}, v_{i}^{\text{stress}})$ from the sample $\{(s_{i,t}, \sigma_{i,t}, v_{i,t}), t \in T^{\text{stress}}\}$ where T^{stress} is the stress period and then to compute the stress transaction cost function:

$$\boldsymbol{c}_{i}^{\text{stress}}(q_{i}) := \boldsymbol{c}_{i}\left(q_{i}; s_{i}^{\text{stress}}, \sigma_{i}^{\text{stress}}, v_{i}^{\text{stress}}\right) \tag{32}$$

For instance, we can consider the empirical mean or the empirical quantile[31] at the confidence level α (e.g., $\alpha = 99\%$). Since this method seems to be very simple, we face a drawback because the triple $(s_{i}^{\text{stress}}, \sigma_{i}^{\text{stress}}, v_{i}^{\text{stress}})$ does not necessarily occur at the same trading day. A more coherent approach consists in computing the trading cost for all days that make up the stress period and taking the supremum:

$$\boldsymbol{c}_{i}^{\text{stress}}(q_{i}) := \sup_{t \in T^{\text{stress}}} \boldsymbol{c}_{i}\left(q_{i}; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) \tag{33}$$

Remark 18 *An alternative approach is to implement the worst-case scenario. The underlying idea is to consider one stress period or several stress periods and to consider the worst-case value:* $s_{i}^{\text{wcs}} = \max_{t \in T^{\text{stress}}} s_{i,t}$, $\sigma_{i}^{\text{wcs}} = \max_{t \in T^{\text{stress}}} \sigma_{i,t}$ *and* $v_{i}^{\text{wcs}} = \min_{t \in T^{\text{stress}}} v_{i,t}$. *By construction, we verify the relationship* $\boldsymbol{c}_{i}^{\text{wcs}}(q_{i}) \geq \boldsymbol{c}_{i}^{\text{stress}}(q_{i})$.

Remark 19 *According to ESMA (2020, page 12, §31), "historical scenarios for LST could include the 2008-2010 global financial crisis or the 2010-2012 European debt crisis".*

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[31] For the volume, we consider the empirical quantile $1 - \alpha$.

Conditional stress scenarios In the case of macro-economic (or conditional) stress testing, the goal is to estimate the relationship between risk parameters and risk factors that define a stress scenario, and then deduce the stress value of these risk parameters (Roncalli, 2020, page 909). Let p_i be a parameter (s_i, σ_i or v_i). First, we consider the linear factor model:

$$p_{i,t} = \beta_0 + \sum_{k=1}^{m} \beta_k \mathcal{F}_{k,t} + \varepsilon_{i,t} \tag{34}$$

where $\varepsilon_{i,t} \sim \mathcal{N}\left(0, \sigma_{\varepsilon_i}^2\right)$ and $(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ is the set of risk factors at time t. Then, the estimates $\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_m\right)$ are deduced from the method of ordinary least squares or the quantile regression. Finally, we translate the stress scenario on the risk factors $(\mathcal{F}_1^{\text{stress}}, \ldots, \mathcal{F}_m^{\text{stress}})$ into a stress scenario on the risk parameter:

$$p_i^{\text{stress}} = \hat{\beta}_0 + \sum_{k=1}^{m} \hat{\beta}_k \mathcal{F}_k^{\text{stress}} \tag{35}$$

Remark 20 *From a practical point of view, pooling the data for the same liquidity class offers a more robust basis for estimating the coefficients $(\beta_0, \beta_1, \ldots, \beta_m)$. This is why the estimation may use the panel data analysis with fixed effects instead of the classic linear regression.*

Remark 21 *Concerning risk factors, we can use those provided by the "Dodd-Frank Act stress testing" (DFAST) that was developed by the Board of Governors of the Federal Reserve System (Board of Governors of the Federal Reserve System, 2017). They concern activity, interest rates, inflation and market prices of financial assets.*

The method of multiplicative factors Conditional stress testing is the appropriate approach for dealing with hypothetical stress scenarios. Nevertheless, it is not obvious to find an empirical relationship between the risk factors $(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ and the risk parameters $(s_{i,t}, \sigma_{i,t}, v_{i,t})$. This is why it is better to use the method of multiplicative factors to generate hypothetical scenarios. This approach assumes that there is a relationship between the stress parameter and its normal value:

$$p_i^{\text{stress}} = m_p p_i^{\text{normal}} \tag{36}$$

where m_p is the multiplicative factor. Therefore, defining the hypothetical stress scenario is equivalent to applying the multiplicative factors to the current values of the risk parameters:

$$\left(s_i^{\text{stress}}, \sigma_i^{\text{stress}}, v_i^{\text{stress}}\right) := (m_s s_{i,t}, m_\sigma \sigma_{i,t}, m_v v_{i,t}) \tag{37}$$

In this approach, the hypothetical stress scenario is determined by the triple (m_s, m_σ, m_v).

4.3   Measuring the portfolio distortion

If we consider the proportional rule $q \propto \omega$ (vertical slicing approach), the portfolio distortion is equal to zero, but we may face high liquidation costs because of some illiquid securities. On the contrary, we can concentrate the liquidation on the most liquid securities (waterfall approach), but there is a risk of a high portfolio distortion. Therefore, we have a trade-off between the liquidation cost and the portfolio distortion.

In Appendix B.2.3 on page 75, we show that the optimal portfolio liquidation can be obtained using the following optimization problem:

$$q^{\star}(\lambda) = \arg\min \frac{1}{2}\sigma^2(q \mid \omega) + \lambda c(q \mid \omega) \tag{38}$$

$$\text{s.t.} \begin{cases} \mathbf{1}_n^{\top} w(\omega - q) = 1 \\ w^-(\omega - q) \le w(\omega - q) \le w^+(\omega - q) \end{cases}$$

where $\sigma(q \mid \omega)$ is the tracking error due to the redemption and $c(q \mid \omega)$ is the liquidation cost. The portfolio distortion is then measured by the tracking error between the portfolio before the redemption and the portfolio after the redemption. Using the optimization problem, we can find liquidation portfolios that induce a lower transaction cost than the proportional rule for the same redemption amount $\mathbb{R}$. The downside is that they also generate a tracking error. Let us illustrate this trade-off with the following example[32]:

Asset	1	2	3	4	5
ω_i	20 000	20 000	18 000	9 000	8 000
P_i (in \$)	80	100	130	120	90
σ_i (in %)	30	30	30	15	15
s_i (in bps)	10	10	10	5	5
v_i	10 000	10 000	10 000	20 000	20 000

The transaction cost function is given by the SQRL model with $\varphi_1 = 1$, $\tilde{x} = 5\%$ and $x^+ = 10\%$. In Figure 15, we report the efficient frontier of liquidation. We notice that the proportional rule implies a transaction cost of 88 bps. In order to reduce this cost, we must accept a tracking error risk. For instance, if we reduce the transaction cost to 70 bps, the liquidation has generated 22 bps of tracking error risk.

Therefore, managing the asset liquidity risk is not only a question of transaction cost, but also a question of portfolio management. Indeed, the fund manager may choose to change the portfolio allocation in a stress period by selling the most liquid assets in order to fulfill the redemptions. The fund manager may also choose to maintain an exposure on some assets in the event of a liquidity crisis. In these situations, the proportional rule is not optimal and depends on the investment constraints. For instance, the definition of the optimal liquidation policy is not the same for active managers and passive managers. This is why liquidity stress testing on the asset side is not only a top-down approach, but must also be completed by a bottom-up approach.

Remark 22 *The liquidation tracking error is the right measure for assessing the portfolio distortion in the case of an equity portfolio:*

$$\mathcal{D}(q \mid \omega) = \sigma(q \mid \omega)$$
$$= \sqrt{(w(\omega) - w(\omega - q))^{\top} \Sigma (w(\omega) - w(\omega - q))}$$

where $w(\omega)$ is the vector of portfolio weights before the redemption, $w(\omega - q)$ is the vector of portfolio weights after the redemption and Σ is the covariance matrix of stock returns.

---

[32]The correlation matrix of asset returns is equal to:

$$\rho = \begin{pmatrix} 100\% \\ 10\% & 100\% \\ 40\% & 70\% & 100\% \\ 50\% & 40\% & 80\% & 100\% \\ 30\% & 30\% & 50\% & 50\% & 100\% \end{pmatrix}$$

Figure 15: Optimal portfolio liquidation
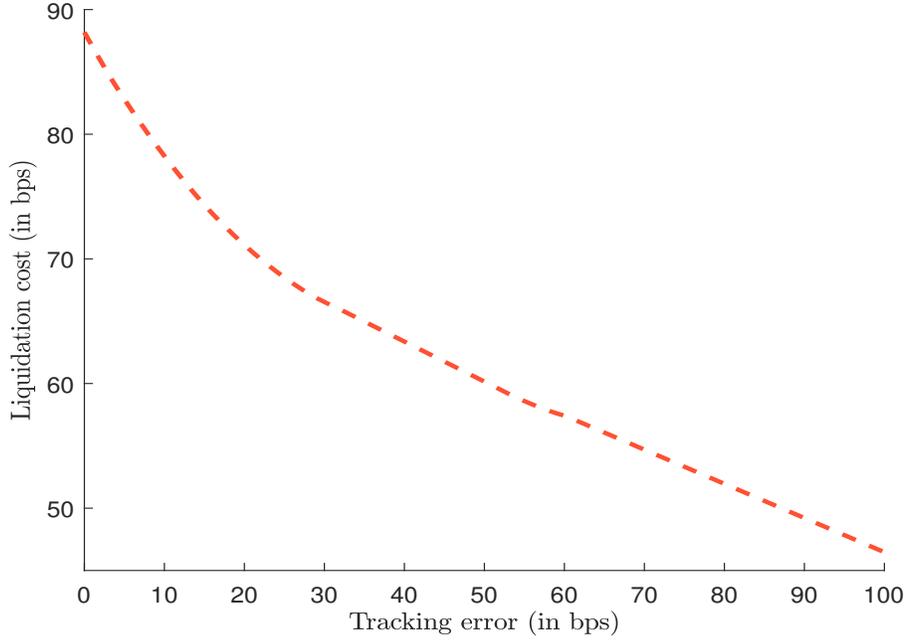
For a bond portfolio, it can be replaced by the liquidation active risk, which measures the active risk due to the redemption:

$$\mathcal{D}\left(q \mid \omega\right) = \mathcal{AR}\left(q \mid \omega\right)$$

The active risk can be measured with respect to the modified duration (MD) or the duration-times-spread (DTS). We can also use a hybrid approach by considering the average of the MD and DTS active risks:

$$
\begin{aligned}
\mathcal{AR}\left(q \mid \omega\right) \quad = \quad & \frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\sum_{i\in\mathcal{S}ector_j}\left(w_i\left(\omega-q\right)-w_i\left(\omega\right)\right)\mathrm{MD}_i\right)^2 + \\
& \frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\sum_{i\in\mathcal{S}ector_j}\left(w_i\left(\omega-q\right)-w_i\left(\omega\right)\right)\mathrm{DTS}_i\right)^2
\end{aligned}
$$

where $n_{\mathcal{S}ector}$ is the number of sectors, MD_i is the modified duration of Bond i and DTS_i is the duration-times-spread of Bond i.

5 Application to stock and bond markets

The accuracy of the model and the calibration is an issue. Indeed, we may wonder which accuracy we must target for a liquidity stress testing exercise, given that there are multiple unknowns in a liquidity crisis. In particular, the LST model may be different from the proprietary pre-trade model and less precise because of two main reasons. First, in a liquidity stress testing exercise, we are more interested in the global figures at the fund manager level,

the asset class level and the asset manager level, and less interested in the figures at the portfolio (or security) level. Second, the model must be simple in order to identify the stress parameters. This is why the LST market impact model used by the risk department may be less accurate than the pre-trade model used by the trading desk, because the challenges are very different. The framework presented above is not complex enough for order execution[33], but it is sufficiently flexible and accurate to give the right order of magnitude for liquidity stress testing purposes.

In our analytical framework, we recall that the backbone of the LST exercise on the asset side is given by Equations (30) and (31) on page 29, and Equation (32) on page 31:

1. for each liquidity bucket $\mathcal{LB}_j$, we have to estimate the parameters $\beta_j^{(s)}$, $\beta_j^{(\boldsymbol{\pi})}$, $\gamma_{1,j}$ and $\gamma_{2,j}$ of the unit transaction cost model;

2. for each security i, we have to define the bid-ask spread $s_{i,t}$, the volatility $\sigma_{i,t}$ and the daily volume $v_{i,t}$;

3. we also have to specify the inflection point $\tilde{q}_{i,t} = \tilde{x}_j v_{i,t}$:

   (a) we generally estimate $\tilde{x}_j$ at the level of the liquidity bucket;
   (b) if $\tilde{q}_{i,t} = q_{i,t}^+$, there is only one regime, implying that the parameters $\gamma_{2,j}$ and $\tilde{x}_j$ vanish;

4. we then have to specify the trading limit $q_{i,t}^+$ for each security; except for large cap equities and some sovereign bonds, we use the proportional rule $q_{i,t}^+ = x_j^+ v_{i,t}$, where x_j^+ is the maximum trading limit of the liquidity bucket $\mathcal{LB}_j$ defined by the asset manager's risk department;

5. finally, we have to specify how the three security parameters are stressed: s_i^{stress}, σ_i^{stress} and v_i^{stress}.

It is obvious that the key challenge of the LST calibration is data availability. Since the LST model may include a lot of parameters, we suggest proceeding step by step. For instance, as a first step, we may calibrate the model for all global equities. Then, we may distinguish between large cap and small cap equities. Next, we may consider an LST model region by region (e.g., US, Eurozone, UK, Japan, etc.), and so on. In the early stages, we may also use expert judgement in order to fix some parameters, for instance $\gamma_{2,j}$, $\tilde{x}_j$, etc. Some parameters are also difficult to observe. For instance, the bid-ask spread $s_{i,t}$ and the trading volume $v_{i,t}$ are not available for many bonds. This is why we use a model or an approximation formula. For example, we can replace the trading volume $v_{i,t}$ by the notional outstanding amount n_i. The volume-based participation rate $x_i = v_{i,t}^{-1} q_i$ is then replaced by the outstanding-based participation rate $y_i = n_i^{-1} q_i$, implying that we have to calibrate the scaling factor $\beta_j^{(\boldsymbol{\pi})}$ in order to take into account this new parameterization. We can also use the rule $V_{i,t} = \xi \mathcal{M}_{i,t}$ where ξ is the proportionality factor between volume and outstanding amount. Moreover, the volatility parameter is not always pertinent in the case of bonds, and it may be better to use the duration-times-spread (DTS).

Remark 23 *In this section, we remove the reference to the liquidity bucket $\mathcal{LB}_j$ in order to reduce the amount of notation when it is possible. This concerns the four parameters $\beta_j^{(s)}$,*

---

[33]Nevertheless, Curato *et al.* (2017) tested different pre-trade order models and concluded that "*a fully satisfactory and practical model of market impact [...] seems to be still lacking*". As such, pre-trade models are not yet completely accurate, except perhaps for large cap equities.

$\beta_j^{(\boldsymbol{\pi})}$, $\gamma_{1,j}$ and $\gamma_{2,j}$. *Moreover, we consider the calibration of the single-regime model as a first step:*

$$\begin{aligned} \boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) &= \beta^{(s)} s_{i,t} + \beta^{(\boldsymbol{\pi})} \sigma_{i,t}\left(\frac{q_i}{v_{i,t}}\right)^{\gamma_1} \\ &= \beta^{(s)} s_{i,t} + \beta^{(\boldsymbol{\pi})} \sigma_{i,t} x_{i,t}^{\gamma_1} \end{aligned} \tag{39}$$

The second regime is calibrated during the second step as shown in Section 5.3 on page 50. We also assume that the annualized volatility is scaled by the factor $1/\sqrt{260}$ in order to represent a daily volatility measure. This helps to understand the magnitude of the parameter $\beta^{(\boldsymbol{\pi})}$. By default, we can then consider that $\beta^{(\boldsymbol{\pi})} \approx 1$.

5.1 The case of stocks

5.1.1 Large cap equities

We consider the dataset described in Appendix C.1 on page 77. We filter the data in order to keep only the stocks that belong to the MSCI USA and MSCI Europe indices. For each observation i, we have the transaction cost $\boldsymbol{c}_i$, the (end-of-day) bid-ask spread s_i, the participation rate x_i and the daily volatility σ_i. We first test a highly constrained statistical model:

$$\boldsymbol{c}_i = s_i + \sigma_i\sqrt{x_i} + \varepsilon_i \tag{40}$$

where $\varepsilon_i \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$. We obtain $R^2 = 53.47\%$ and $R_c^2 = 15.87\%$. Since we observe a large discrepancy between R^2 and R_c^2, we must be careful about the interpretation of the statistical models. This means that the average cost $\bar{\boldsymbol{c}}$ explains a significant part of the trading cost, implying that the dispersion of trading costs is not very large.

In order to improve the explanatory power of the transaction cost function, we consider two alternative models:

$$\boldsymbol{c}_i = \beta^{(s)} s_i + \beta^{(\boldsymbol{\pi})} \sigma_i \sqrt{x_i} + \varepsilon_i \tag{41}$$

and:

$$\boldsymbol{c}_i = \beta^{(s)} s_i + \beta^{(\boldsymbol{\pi})} \sigma_i x_i^{\gamma_1} + \varepsilon_i \tag{42}$$

Model (41) can be seen as a special case of Model (42) when the exponent γ_1 is set to $1/2$. Using the method of non-linear least squares, we estimate the parameters, and the results are reported in Tables 11 and 12. We notice that the assumptions $(\mathcal{H}_1)$ $\beta^{(s)} = 1$ and $(\mathcal{H}_2)$ $\beta^{(\boldsymbol{\pi})} = 1$ are both rejected. When the estimation of γ_1 is not constrained, its optimal value is equal to 0.5873, which is a little bit higher than 0.5. Nevertheless, we observe that the explanatory powers are very close for the constrained and unconstrained models. The fact that $\beta^{(\boldsymbol{\pi})}$ is larger for the unconstrained model (0.2970 versus 0.1898) indicates a bias in our dataset. The model tends to overfit the lowest values of x_i and not the highest value of x_i, which are certainly not sufficiently represented in the dataset.

Table 11: Non-linear least squares estimation of Model (41)

Parameter	Estimate	Stderr	t-student	p-value
$\beta^{(s)}$	1.4465	0.0014	1049.9020	0.0000
$\beta^{(\boldsymbol{\pi})}$	0.1898	0.0030	62.7720	0.0000
γ_1	0.5000	0.0053	93.5817	0.0000
	$R^2 = 98.41\%$		$R_c^2 = 97.12\%$	

Table 12: Non-linear least squares estimation of Model (42)

Parameter	Estimate	Stderr	t-student	p-value
$\beta^{(s)}$	1.4468	0.0012	1213.2593	0.0000
$\beta^{(\boldsymbol{\pi})}$	0.2970	0.0039	76.0394	0.0000
γ_1	0.5873	0.0044	132.7093	0.0000

$$R^2 = 98.81\% \qquad R_c^2 = 97.84\%$$

Figure 16: Histogram of estimated parameters
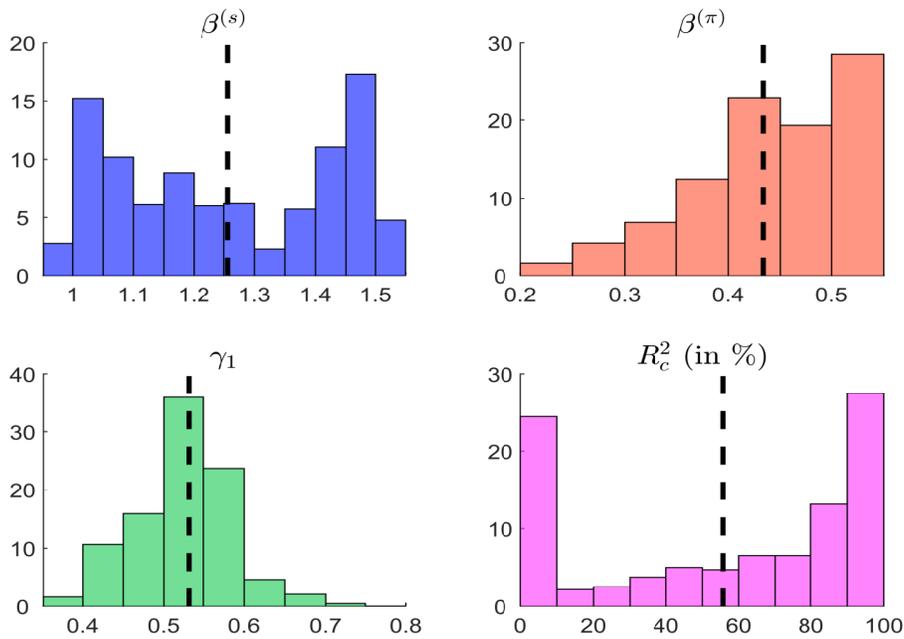
Table 13: Descriptive statistics of the estimates

Parameter	Mean	Median	Min.	$Q\,(10\%)$	$Q\,(25\%)$	$Q\,(75\%)$	$Q\,(90\%)$	Max.
$\beta^{(s)}$	1.256	1.234	0.992	1.001	1.082	1.443	1.487	1.558
$\beta^{(\boldsymbol{\pi})}$	0.434	0.448	-0.209	0.330	0.391	0.500	0.510	0.527
γ_1	0.531	0.525	0.368	0.446	0.488	0.563	0.597	1.676
R_c^2	0.557	0.681	0.000	0.000	0.094	0.916	0.961	0.992

The figures taken by R^2 and R_c^2 are extremely high and not realistic. This confirms that there is a bias in our dataset. To better understand this issue, we estimate Model (42) for each stock. Results are reported in Figure 16 and Table 13. On average, R_c^2 is equal to 55.7%, which is far from the previous result. We observe that the model presents a high explanatory power for some stocks and a low explanatory power for other stocks (bottom/right panel in Figure 16). These results highlight the heterogeneity of the database. Therefore, estimating a transaction cost model is not easy when mixing small and large values of transaction costs and participation rates. Finally, we propose the following benchmark formula for the transaction cost model:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) = 1.25 \cdot s_{i,t} + 0.40 \cdot \sigma_{i,t} \sqrt{x_{i,t}} \tag{43}$$

The price impact of this function is reported in Figure 17 in the case where the annualized volatility of the stock return is equal to 30%.

Figure 17: Estimated price impact (in bps)
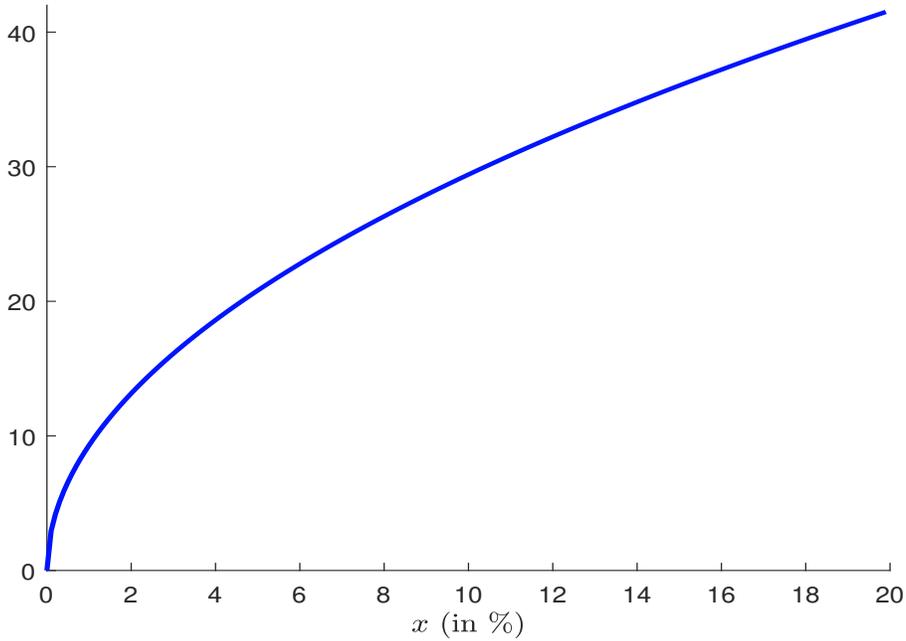
Remark 24 *We notice sensitivity of the results when we filter the data with respect to the participation rate. For instance, we obtain:*

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) = 1.51 \cdot s_{i,t} + 0.56 \cdot \sigma_{i,t} x_{i,t}^{0.78}$$

when we only consider the observations with a participation rate larger than 0.5%.

5.1.2 Small cap equities

In this analysis, we consider all the stocks that belong to the MSCI USA, MSCI Europe, MSCI USA Small Cap and MSCI Europe Small cap indices. This means that the dataset

Figure 18: Relationship between the market capitalization and the parameter $\beta^{(s)}$
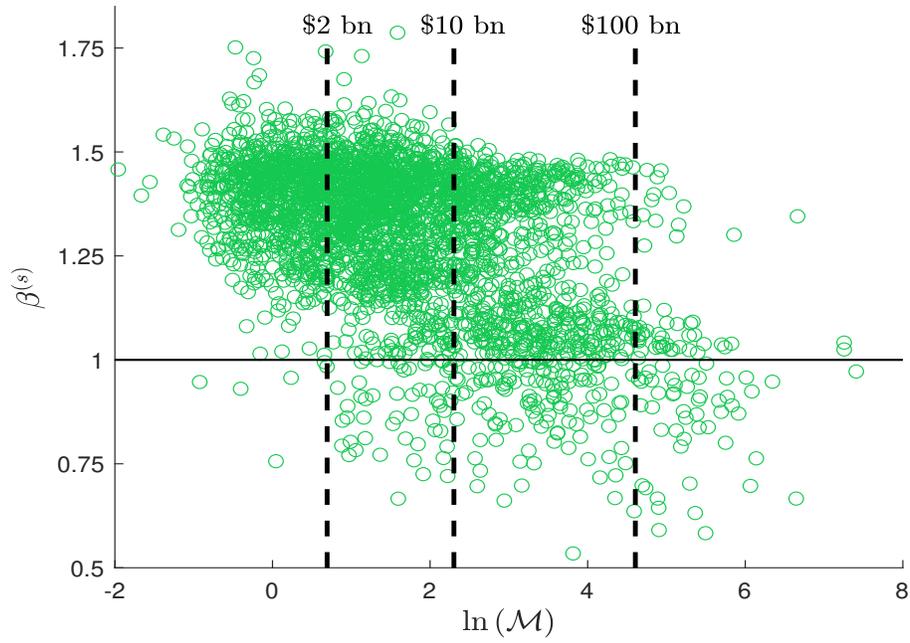
Figure 19: Relationship between the market capitalization and the parameter $\beta^{(\boldsymbol{\pi})}$
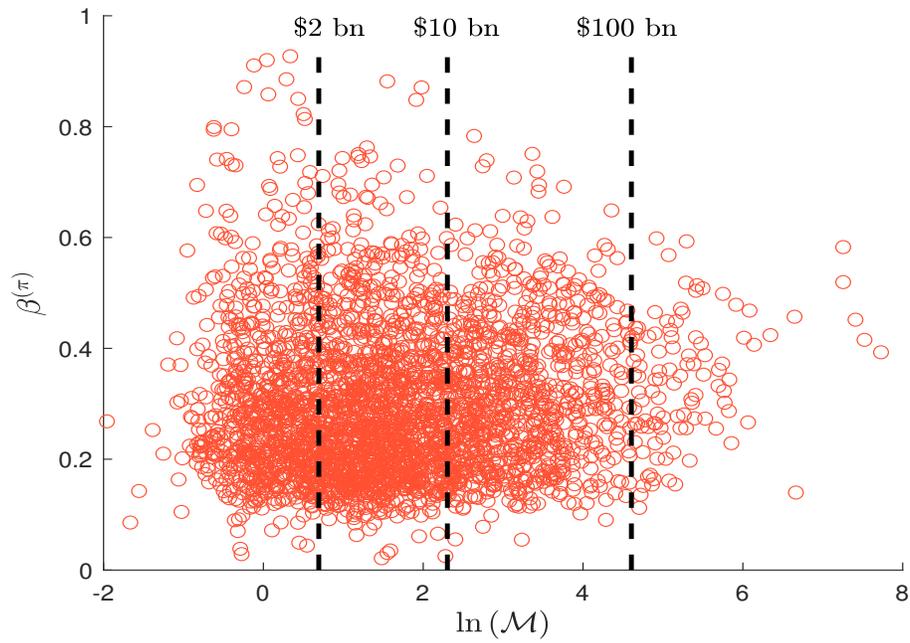
Figure 20: Ratio of the parameters $\beta^{(s)}$ and $\beta^{(\boldsymbol{\pi})}$ with respect to the values of the large cap class
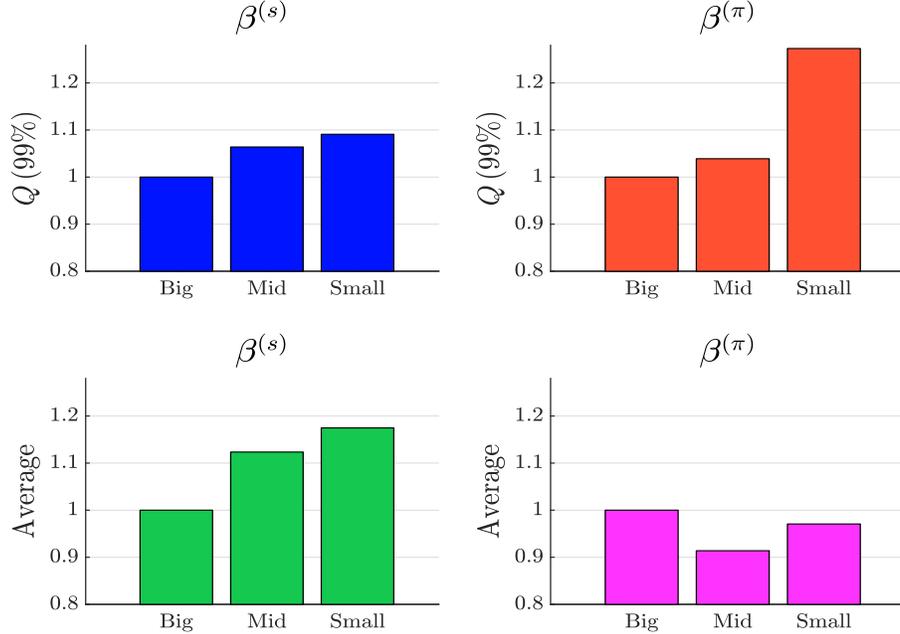
corresponds to large cap and small cap stocks. We run the linear regression (41) for the different stocks and estimate the parameters $\beta^{(s)}$ and $\beta^{(\boldsymbol{\pi})}$. In Figures 18 and 19, we report the scatterplot between the market capitalization[34] and these parameters. On average, the estimate of $\beta^{(s)}$ is higher when the market capitalization is low than when the market capitalization is high. In a similar way, we observe more dispersion of the estimate $\beta^{(\boldsymbol{\pi})}$ for small cap stocks. In order to verify that small cap stocks are riskier than large cap stocks, we split the stock universe into three buckets according to market capitalization[35]. In Figure 20, we plot the ratio of the estimates $\beta^{(s)}$ and $\beta^{(\boldsymbol{\pi})}$ with the values obtained for the large cap class. We notice that the two parameters are larger for small cap stocks, especially if we consider the 99% quantile. To take into account this additional risk, we propose the following benchmark formula for small cap stocks:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) = 1.40 \cdot s_{i,t} + 0.50 \cdot \sigma_{i,t}\sqrt{x_{i,t}} \tag{44}$$

If we compare this function with Equation (43), we notice that the parameter $\beta^{(s)}$ is equal to 1.40 instead of 1.25, implying an additional fixed transaction cost of $+12\%$ for small cap stocks. For the parameter $\beta^{(\boldsymbol{\pi})}$, the value is equal to 0.50 instead of 0.40, implying that the price impact is 25% higher for small cap stocks.

Remark 25 *A conservative approach consists in using the highest values of $\beta^{(\boldsymbol{\pi})}$. For instance, we can define $\beta^{(\boldsymbol{\pi})} = 0.50$ for large cap stocks and $\beta^{(\boldsymbol{\pi})} = 0.75$ for small cap stocks. In this case, the price impact is 50% higher for small cap stocks.*

---

[34]In order to obtain an easy-to-read graph, the x-axis corresponds to the logarithm of the market capitalization, which is expressed in billions of US dollars.

[35]We use the following classification: $+\$10$ bn for large caps, $\$2 - \$10$ bn for mid caps and $-\$2$ bn for small caps.

5.2 The case of bonds

5.2.1 Defining the participation rate

The key variable of the transaction cost formula is the participation rate:

$$x = \frac{q}{v} = \frac{Q}{V}$$

where q is the number of shares to trade and v is the daily trading volume (expressed in number of shares). We can also formulate the participation rate with the nominal values Q and V expressed in USD or EUR. In the case of bonds, the daily trading volume is not observed. Moreover, this statistic is not always relevant because some bonds are traded infrequently. To illustrate this phenomenon, we can use the zero-trading days statistic, which is defined as the ratio between the number of days with zero trades and the total number of trading days within the period. For instance, Hotchkiss and Jostova (2017) report that 79.4% of US IG bonds and 84.1% of US HY bonds are not traded monthly between January 1995 to December 1999. Dick-Nielsen *et al.* (2012) find that the median number of zero-trading days was equal to 60.7% on a quarterly basis from Q4 2004 to Q2 2009 in the US corporate bond market.

The turnover is a measure related to the trading volume. It is the ratio between the nominal trading volume V and the market capitalization $\mathcal{M}$ of the security, or between the trading volume v and the number of issued shares[36] n:

$$\boldsymbol{\tau} = \frac{V}{\mathcal{M}} = \frac{v}{n}$$

In the case of bonds, $\mathcal{M}$ and n correspond to the outstanding amount and the number of issued bonds. It follows that $V = \boldsymbol{\tau}\mathcal{M}$ and:

$$x = \frac{Q}{\boldsymbol{\tau}\mathcal{M}} = \frac{q}{\boldsymbol{\tau}n}$$

We deduce that the volume-based participation rate x is related to the outstanding-based participation rate y:

$$y = \frac{q}{n}$$

The scaling factor between y and x is then exactly equal to the daily turnover ratio $\boldsymbol{\tau}$.

According to SIFMA (2021a), the daily turnover ratio is equal to 0.36% for US corporate bonds in 2019. This figure is relatively stable since it is in the range $0.30\% - 0.36\%$ between 2005 and 2019, except in 2008 where we observe a turnover of 0.26%. However, it was highest before 2005. For instance, it was equal to 0.44% in 2002. If we make the distinction between IG and HY bonds, it seems that the turnover ratio is greater for the latter. For instance, we obtain a turnover ratio of 0.27% for US IG bonds and 0.65% for HY bonds. In the case of US treasury securities, the five-year average daily turnover figure is 4.6% for bills, 1.2% for TIPS and 3.5% for notes and bonds (SIFMA, 2021b).

In the case of European bonds, statistics are only available for government bonds. We can classify the countries into three categories (AFME, 2020):

- The daily turnover ratio is above 1% and close to 1.5% for Germany, Spain and UK.

- The daily turnover ratio is between 0.5% and 1.0% for Belgium, France, Ireland, Italy, Netherlands, and Portugal.

---

[36]The market capitalization is equal to the number of shares times the price: $\mathcal{M} = nP$.

• The daily turnover ratio is lower than 0.5% for Denmark and Greece.

These different figures show that the turnover ratio cannot be considered as constant. Therefore, the single-regime transaction cost function becomes:

$$
\begin{aligned}
\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, v_{i,t}\right) &= \beta^{(s)} s_{i,t} + \beta^{(\boldsymbol{\pi})} \sigma_{i,t} \left(\frac{q_i}{\boldsymbol{\tau}_{i,t} n_i}\right)^{\gamma_1} \\
&= \beta^{(s)} s_{i,t} + \beta_{i,t}^{(\boldsymbol{\pi})} \sigma_{i,t} y_i^{\gamma_1}
\end{aligned}
\tag{45}
$$

where $y_i = n_i^{-1} q_i$ is the outstanding-based participation rate and $\beta_{i,t}^{(\boldsymbol{\pi})}$ is the scaling factor of the price impact:

$$
\beta_{i,t}^{(\boldsymbol{\pi})} = \frac{\beta^{(\boldsymbol{\pi})}}{\boldsymbol{\tau}_{i,t}^{\gamma_1}}
\tag{46}
$$

Since the turnover ratio is time-varying and depends on the security, it follows that $\beta_{i,t}^{(\boldsymbol{\pi})}$ depends on the time t and the security i. Equation (45) for bonds is then less attractive than Equation (39) for equities. However, we can make two assumptions:

1. the turnover ratio $\boldsymbol{\tau}_{i,t}$ is stable on long-run periods;

2. the turnover ratio $\boldsymbol{\tau}_{i,t}$ computed at the security level is not representative of its trading activity.

We notice that turnover ratios are generally computed for a group of bonds, for instance all German government bonds or all US corporate IG bonds. The reason lies again in the fact that the daily turnover of a given bond may be equal to zero very often because of the zero-trading days effect. Nevertheless, if one bond is not traded at all for a given period (e.g., a day or a week), it does not mean that it is perfectly illiquid during this period. This may be due to a very low supply or demand during this period. In a bullish market, if no investors want to sell some bonds because there is strong demand and low supply, these investors are rational to keep their bonds. Since buy-and-hold strategies dominate in bond markets, trading a bond is a signal that the bond is not priced fairly. In this framework, the fundamental price of a bond must change in order to observe a trading activity on this bond. The situation in the stock market is different because the computation of the fair price uses a more short-term window and buy-and-hold strategies do not dominate.

Therefore, we can assume that the turnover ratio is equal for the same family of bonds, implying that:

$$
\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = \beta^{(s)} s_{i,t} + \tilde{\beta}^{(\boldsymbol{\pi})} \sigma_{i,t} y_i^{\gamma_1}
\tag{47}
$$

This equation is similar to Equation (39) for equities. Nevertheless, there is a difference between the two scaling coefficients $\beta^{(\boldsymbol{\pi})}$ and $\tilde{\beta}^{(\boldsymbol{\pi})}$. The last one is more sensitive because we have:

$$
\tilde{\beta}^{(\boldsymbol{\pi})} = \frac{\beta^{(\boldsymbol{\pi})}}{\boldsymbol{\tau}^{\gamma_1}}
$$

The underlying idea is then to consider more granular liquidity buckets $\mathcal{LB}_j$ for the bond asset class than the equity asset class in order to be sure that the securities belonging to the same liquidity bucket have a similar turnover ratio $\boldsymbol{\tau}$. In Figure 21, we report the relationship between $\boldsymbol{\tau}$ and $\tilde{\beta}^{(\boldsymbol{\pi})}$ for several values of the exponent γ_1. When γ_1 is low, the impact of $\boldsymbol{\tau}$ on $\tilde{\beta}^{(\boldsymbol{\pi})}$ is very low, meaning that we can consider $\tilde{\beta}^{(\boldsymbol{\pi})}$ as a constant. However, when γ_1 is high (greater than 0.25), the turnover may have a high impact and $\tilde{\beta}^{(\boldsymbol{\pi})}$ cannot be assumed to be a constant. In the first case, the estimation of $\beta^{(s)}$ and $\tilde{\beta}^{(\boldsymbol{\pi})}$ is robust. In the second case, the estimation of $\tilde{\beta}^{(\boldsymbol{\pi})}$ only makes sense if the turnover is comparable between the securities of the liquidity bucket $\mathcal{LB}_j$.

Figure 21: Relationship between the turnover τ and the scaling factor $\tilde{\beta}^{(\pi)}$
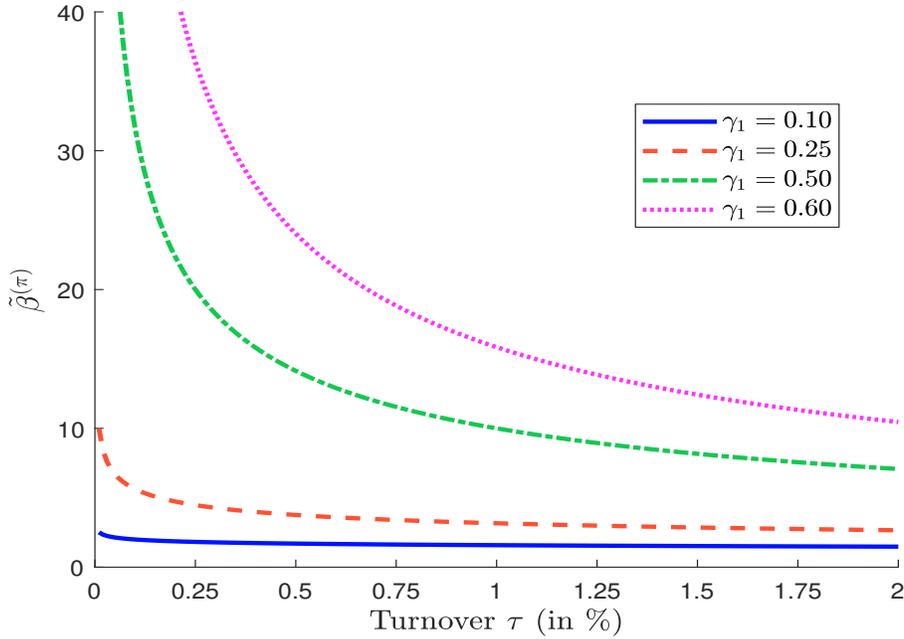
Remark 26 *In Table 14, we report the values of the outstanding-based participation rate with respect to the volume-based participation rate x and the daily turnover τ. For example, if $x = 30\%$ and $\tau = 4\%$, we obtain a participation rate of 1.2%. While volume-based participation rates are expressed in %, we conclude that outstanding-based participation rates are better expressed in bps.*

Table 14: Outstanding-based participation rate (in bps) with respect to x and τ

τ (in %)	x (in %)								
	0.01	0.05	0.10	0.50	1	5	10	20	30
0.5	0.005	0.025	0.05	0.25	0.5	2.5	5	10	15
1.0	0.010	0.050	0.10	0.50	1.0	5.0	10	20	30
2.0	0.020	0.100	0.20	1.00	2.0	10.0	20	40	60
4.0	0.040	0.200	0.40	2.00	4.0	20.0	40	80	120

5.2.2 Sovereign bonds

We consider a dataset of sovereign bond trades, whose description is given in Appendix C.2 on page 77. For each observation i, we have the transaction cost c_i, the spread s_i, the outstanding-based participation rate y_i and the daily volatility σ_i. We run a two-stage regression model:

$$\begin{cases} \ln(c_i - s_i) - \ln \sigma_i = c_\gamma + \gamma_1 \ln y_i + u_i & \text{if } c_i > s_i \\ c_i = c_\beta + \beta^{(s)} s_i + \mathcal{D}_i^{(\pi)} \tilde{\beta}^{(\pi)} \sigma_i y_i^{\gamma_1} + v_i \end{cases} \quad (48)$$

Table 15: Two-stage estimation of the sovereign bond transaction cost model

Parameter	Estimate	Stderr	t-student	p-value
c_γ	0.3004	0.0500	6.0096	0.0000
γ_1	0.2037	0.0046	44.6050	0.0000
c_β	0.0002	0.0000	15.7270	0.0000
$\beta^{(s)}$	0.9099	0.0109	83.3412	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	2.1521	0.0153	140.6059	0.0000
	$R^2 = 39.87\%$		$R_c^2 = 28.94\%$	

where c_γ and c_β are two intercepts, and u_i and v_i are two residuals. Since the transaction cost can be lower than the bid-ask spread[37], we introduce the dummy variable $\mathcal{D}_i^{(\boldsymbol{\pi})} = \mathbb{1}\{\boldsymbol{c}_i > s_i\}$. We estimate the exponent γ_1 using the first linear regression model. Then, we estimate the parameters $\beta^{(s)}$ and $\tilde{\beta}^{(\boldsymbol{\pi})}$ using the second linear regression by considering the OLS estimate of γ_1. Results are given in Table 15. We obtain $\gamma_1 = 0.2037 \ll 0.5$, which is lower than the standard value for equities. We also obtain $\beta^{(s)} = 0.9099$ and $\tilde{\beta}^{(\boldsymbol{\pi})} = 2.1521$. Curiously, the value of $\beta^{(s)}$ is less than one. One possible explanation is that we use trades from a big asset manager that may have a power to negotiate and the capacity to trade inside the bid-ask spreads when the participation rate is low. Nevertheless, the explanatory power of the model is relatively good. Indeed, we obtain $R^2 = 39.87\%$ and $R_c^2 = 28.94\%$.

Another approach for calibrating the model is to consider a grid-search process. In this case, we estimate the linear regression:

$$\boldsymbol{c}_i = c_\beta + \beta^{(s)} s_i + \mathcal{D}_i^{(\boldsymbol{\pi})} \tilde{\beta}^{(\boldsymbol{\pi})} \sigma_i y_i^{\gamma_1} + v_i$$

by considering several values of γ_1. The optimal model corresponds then to the linear regression that maximizes the coefficient of determination R_c^2. Figure 22 illustrates the grid search process. The optimal solution is reached for $\gamma_1 = 0.0925$, and we obtain the results given in Table 16. The explanatory power is close to the one calibrated with the two-stage approach (30.56% versus 28.94%). However, the two calibrated models differ if we compare the parameters γ_1 and $\tilde{\beta}^{(\boldsymbol{\pi})}$. In order to understand the differences, we draw the estimated price impact function in Figure 23 when the annualized volatility of the sovereign bond is equal to 4.36%, which is the median volatility of our dataset. We conclude that the two estimated functions are in fact very close[38].

Table 16: Grid-search estimation of the sovereign bond transaction cost model

Parameter	Estimate	Stderr	t-student	p-value
γ_1	0.0925			
c_β	0.0000	0.0000	0.9309	0.3519
$\beta^{(s)}$	0.9556	0.0107	89.4426	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	0.8482	0.0057	149.2147	0.0000
	$R^2 = 41.24\%$		$R_c^2 = 30.56\%$	

---

[37]We recall that the bond market is not an electronic market. Bid-ask spreads are generally declarative and not computed with quoted bid and ask prices.

[38]See Figure 31 on page 81 for a logarithmic scale. We note that the grid-search estimate is more conservative for very low participation rates.

Figure 22: Parameter estimation using the grid-search approach
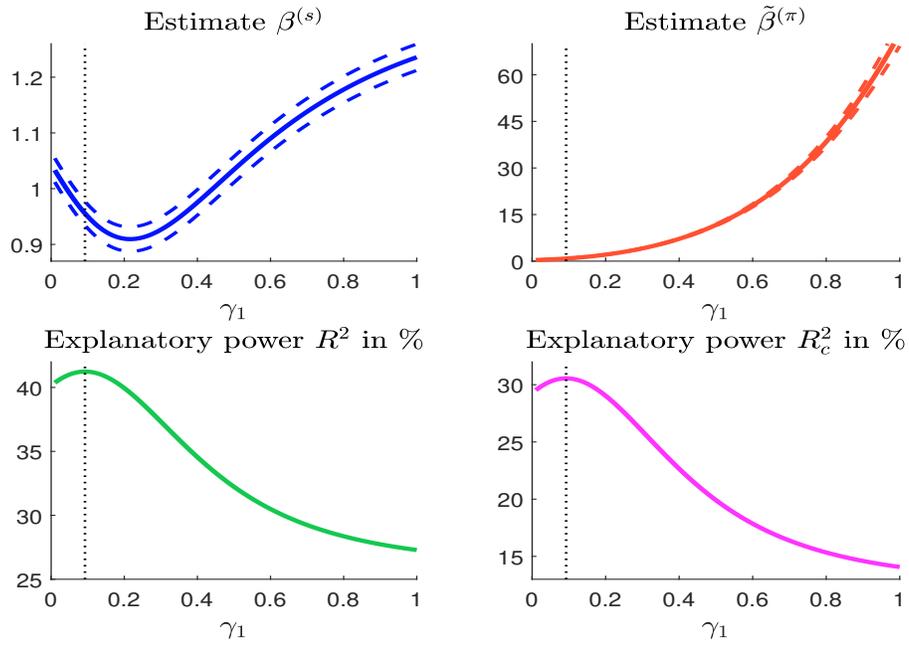
Figure 23: Estimated price impact (in bps)
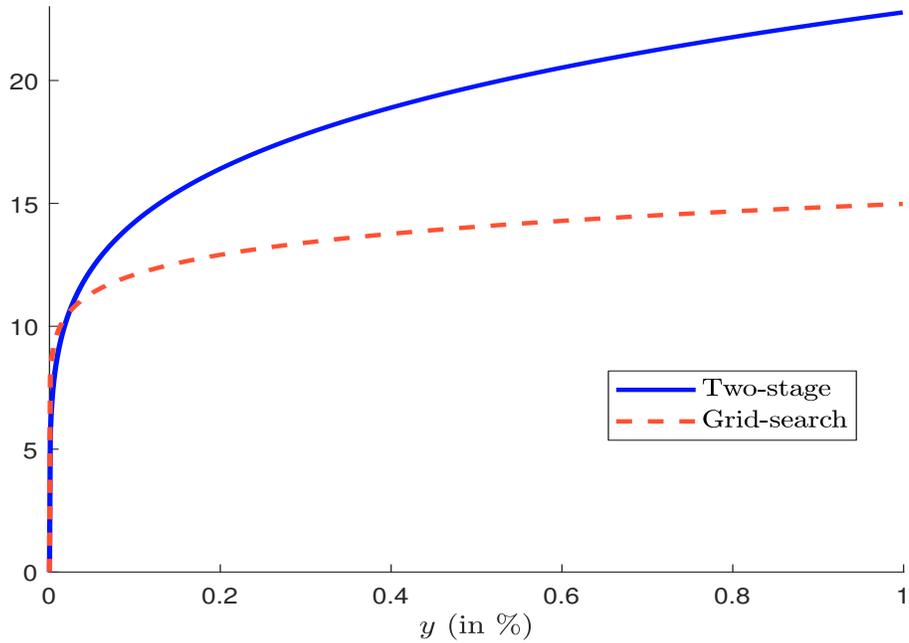
Figure 24: Estimated price impact (in bps) with respect to the volume-based participation rate
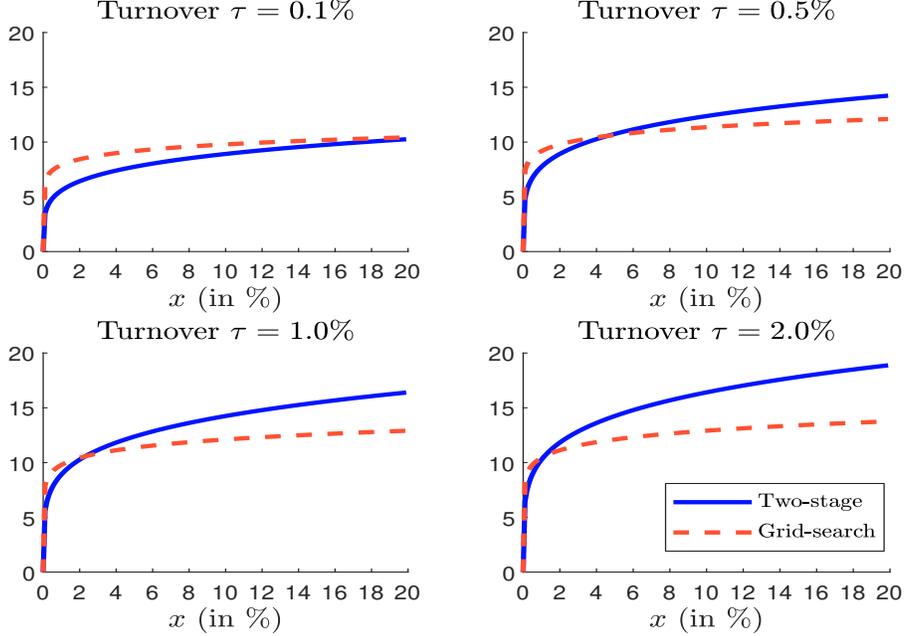
In order to better understand the transaction cost function, we consider the parameterization with respect to the volume-based participation rate by using the following relationship:

$$x = \frac{y}{\tau}$$

Results are given in Figure 24 for different assumptions of the daily turnover τ. Again, it is very difficult to prefer one of the two estimated models. Therefore, we perform an implicit analysis. Using the estimates of the parameters, we can compute the implied scaling factor:

$$\hat{\beta}^{(\pi)} = \tau^{\gamma_1} \tilde{\beta}^{(\pi)}$$

for a given value of the daily turnover. We can also compute the implied turnover:

$$\hat{\tau} = \left(\frac{\beta^{(\pi)}}{\tilde{\beta}^{(\pi)}} \right)^{\frac{1}{\gamma_1}}$$

for a given scaling factor $\beta^{(\pi)}$. If we analyze the results reported in Table 17, it is obvious that the two-stage estimated model is more realistic than the grid-search estimated model. Indeed, when $\beta^{(\pi)}$ is set to 0.80, the implicit turnover $\hat{\tau}$ is respectively equal to 0.78% and 53.13%. This second figure is not realistic if we compare it to the empirical statistics of daily turnover.

The previous model can be easily improved by considering more liquidity buckets. For instance, if we calibrate[39] the model by issuer or currency, we obtain the results reported in Tables 18 and 19. We observe that $\gamma_1 \in [0.05, 0.29]$. We also notice that $\beta^{(s)} < 1$ in most cases, except for Italy, Spain and the US. Moreover, we observe a large dispersion of the parameter $\tilde{\beta}^{(\pi)}$. In a similar way, we can propose a parameterization of $\tilde{\beta}^{(\pi)}$:

$$\tilde{\beta}^{(\pi)} = f(\mathcal{F}_1, \ldots, \mathcal{F}_m)$$

---

[39]We use the two-stage estimation approach.

Table 17: Implicit analysis

$\beta^{(\pi)}$		0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10
$\hat{\tau}$ (in %)	Two-stage	0.03	0.08	0.19	0.40	0.78	1.38	2.32	3.71
	Grid-search	0.03	0.33	2.37	12.54	53.13	189.81	592.91	1661.42
τ (in %)		0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.50
$\hat{\beta}^{(\pi)}$	Two-stage	0.70	0.73	0.76	0.78	0.80	0.82	0.84	0.91
	Grid-search	0.51	0.52	0.53	0.54	0.54	0.55	0.55	0.58

where $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ are a set of bond characteristics (Ben Slimane and de Jong, 2017). For instance, if we assume that the parameters γ_1 and $\beta^{(s)}$ are the same for all the bonds, we observe that $\tilde{\beta}^{(\pi)}$ is an increasing function of the credit spread, the duration and the issue date (or the age of the bond).

Table 18: Two-stage estimation of the sovereign bond transaction cost model by issuer

Issuer	γ_1	c_β	$\beta^{(s)}$	$\tilde{\beta}^{(\pi)}$	R^2 (in %)	R_c^2 (in %)
Austria	0.2255	-0.0002	0.8599	3.1385	54.1	48.4
Belgium	0.2482	-0.0000	0.8097	3.3974	44.0	32.5
EM	0.0519	0.0010	0.6828	0.4473	74.9	47.4
Finland	0.2894	0.0000	0.7002	4.0287	46.3	31.8
France	0.2138	0.0000	0.8794	3.0087	40.1	29.7
Germany	0.2415	0.0001	0.9811	2.7007	51.6	38.7
Ireland	0.2098	0.0001	0.5403	2.4097	43.9	26.7
Italy	0.1744	-0.0004	2.7385	1.9030	31.3	22.3
Japan	0.0657	0.0001	0.4700	0.6407	79.5	56.4
Netherlands	0.2320	-0.0000	0.7640	3.7709	46.9	34.2
Portugal	0.2318	0.0001	0.9250	3.0248	49.6	33.0
Spain	0.2185	0.0000	1.2547	2.0758	40.9	26.7
United Kingdom	0.2194	0.0003	0.6837	2.3367	51.2	30.3
USA	0.1252	0.0001	1.0626	1.2866	53.8	40.9

Table 19: Two-stage estimation of the sovereign bond transaction cost model by currency

Currency	γ_1	c_β	$\beta^{(s)}$	$\tilde{\beta}^{(\pi)}$	R^2 (in %)	R_c^2 (in %)
EUR	0.2262	0.0000	1.0233	2.9122	35.2	25.7
GBP	0.2117	0.0002	1.3602	2.0878	48.8	30.2
JPY	0.0834	0.0001	0.4811	0.8553	75.6	50.9
USD	0.1408	0.0004	0.8430	1.0121	61.5	46.9

Remark 27 *If we perform the linear regression without the intercept c_β, we obtain the results reported in Tables 42 and 43 on page 81. We notice that the impact on the coefficients $\beta^{(s)}$ and $\tilde{\beta}^{(\pi)}$ is weak.*

The choice of the value of γ_1 is not obvious. Finally, we decide to fix its value at 0.25. Based on the results given in Table 20, $\beta^{(s)} = 1.00$ seems to be a good choice. If we consider the results given in Tables 42 and 43, $\beta^{(s)} = 1.25$ is more appropriate. We have used end-of-day bid-ask spreads, which are generally lower than intra-day bisk-ask spreads. Therefore,

Table 20: Estimation of the sovereign bond transaction cost model when γ_1 is set to 0.25

Parameter	Estimate	Stderr	t-student	p-value
γ_1	0.2500			
$\beta^{(s)}$	1.0068	0.0103	97.9041	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	3.1365	0.0214	146.6939	0.0000
	$R^2 = 38.35\%$		$R_c^2 = 27.15\%$	

to reflect this risk, it may be more prudent to assume that $\beta^{(s)} = 1.25$. Finally, we propose the following benchmark formula for computing the transaction cost for sovereign bonds:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = 1.25 \cdot s_{i,t} + 3.00 \cdot \sigma_{i,t} y_i^{0.25} \tag{49}$$

If we compare this expression with Equation (43), we notice that the coefficient of the bid-ask spread is the same and the price impact exponent is lower (0.25 versus 0.50 for stocks), implying a lower liquidity risk.

5.2.3 Corporate bonds

We estimate Model (48) by using a dataset of corporate bond trades, whose description is given in Appendix C.3 on page 77. Results are given in Table 21. We notice that all the estimates are significant at the 99% confidence level and the explanatory power is relatively high since we have $R^2 = 64.77\%$ and $R_c^2 = 41.66\%$.

Table 21: Two-stage estimation of the corporate bond transaction cost model with the volatility risk measure

Parameter	Estimate	Stderr	t-student	p-value
c_γ	0.3652	0.0338	10.8119	0.0000
γ_1	0.1168	0.0045	26.1322	0.0000
c_β	0.0008	0.0000	77.4368	0.0000
$\beta^{(s)}$	0.7623	0.0042	183.1617	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	0.9770	0.0044	224.1741	0.0000
	$R^2 = 64.77\%$		$R_c^2 = 41.66\%$	

The previous model's good results should be considered cautiously because of two reasons. The first one is that the explanatory power depends on the maturity of the bonds. For instance, if we focus on short-term corporate bonds when the time-to-maturity is less than two years, we obtain $R_c^2 = 18.86\%$, which is low compared to the previous figure of 41.66%. The second reason is that the volatility data is not always available. This is particularly true when the age of corporate bonds is very low. On average, we do not have the value of the historical volatility for 20.95% of observations. Moreover, we recall that the asset risk is measured by the daily volatility σ_i in the model. However, we know that the price volatility is not a good measure for measuring the risk of a bond when the bond is traded at a very low frequency. This is why we observe a poor explanatory power when we consider bonds that present a high ratio of zero-trading days or a low turnover. This is the case of some EM corporate bonds or some mid-cap issuers. Therefore, we propose replacing the transaction cost function (47) with the following function:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = \beta^{(s)} s_{i,t} + \tilde{\beta}^{(\boldsymbol{\pi})} \mathcal{R}_{i,t} y_i^{\gamma_1} \tag{50}$$

where $\mathcal{R}_{i,t}$ is a better risk measure than the bond return volatility.

Table 22: Two-stage estimation of the corporate bond transaction cost model with the DTS risk measure

Parameter	Estimate	Stderr	t-student	p-value
c_γ	-3.4023	0.0309	-109.9488	0.0000
γ_1	0.0796	0.0041	19.5020	0.0000
c_β	0.0005	0.0000	55.7256	0.0000
$\beta^{(s)}$	0.7153	0.0034	207.4743	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	0.0356	0.0001	300.5100	0.0000

$$R^2 = 68.64\% \qquad R_c^2 = 46.45\%$$

In Appendix B.3 on page 76, we show that the corporate bond risk is a function of the duration-times-spread or DTS. Therefore, we consider the following transaction cost function:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = \beta^{(s)} s_{i,t} + \tilde{\beta}^{(\boldsymbol{\pi})} \, \mathrm{DTS}_{i,t} \, y_i^{\gamma_1} \tag{51}$$

Using our dataset of bond rates, we estimate the parameters by using the two-stage method:

$$\begin{cases} \ln\left(\boldsymbol{c}_i - s_i\right) - \ln \mathrm{DTS}_i = c_\gamma + \gamma_1 \ln y_i + u_i & \text{if } \boldsymbol{c}_i > s_i \\ \boldsymbol{c}_i = c_\beta + \beta^{(s)} s_i + \mathcal{D}_i^{(\boldsymbol{\pi})} \tilde{\beta}^{(\boldsymbol{\pi})} \, \mathrm{DTS}_i \, y_i^{\gamma_1} + v_i \end{cases} \tag{52}$$

Results are given in Table 22. We notice that the results are a little bit better since the explanatory power R_c^2 is equal to 46.45% instead of 41.66%, and all estimated coefficients are significant at the 99% confidence level. Moreover, if we focus on corporate bonds where the time-to-maturity is less than two years, we obtain $R_c^2 = 38.21\%$ or an absolute improvement of 20%! Nevertheless, the value of γ_1 is equal to 0.0796, which is a low value. This result is disappointing because the model does not depend on the participation rate when $\gamma_1 \approx 0$:

$$\lim_{\gamma_1 \to 0} \boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = \beta^{(s)} s_{i,t} + \tilde{\beta}^{(\boldsymbol{\pi})} \, \mathrm{DTS}_{i,t}$$

This type of model is not useful and realistic when performing liquidity stress testing since the liquidity cost does not depend on the trade size!

The asset manager that provided the data uses a trading/dealing desk with specialized bond traders in order to minimize trading impacts and transaction costs. In particular, we observe that bond traders may be very active. For example, they may decide to not sell or buy the bond if the transaction cost is high. In this case, with the agreement of the fund manager, they can exchange the bond of an issuer with another bond of the same issuer[40], a bond of another issuer or a basket of bonds in order to reduce the transaction cost. More generally, they execute a sell or buy order of a bond with a high participation rate only if the trading impact is limited, implying that these big trades are opportunistic and not systematic contrary to small and medium trades. In a similar way, bond traders may know the inventory or the axis of the brokers and market markers. They can offer to fund managers to initiate a trade because the trade impact will be limited or even because the transaction cost is negative! We conclude that the behavior of bond traders is different depending on whether the trade is small/medium or large.

Since the goal of bond traders is to limit sensitivity to high participation rates, it is normal that we obtain a low value for the coefficient γ_1. We decide to force the coefficient

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[40]With other characteristics such as the maturity.

Table 23: Estimation of the corporate bond transaction cost model when γ_1 is set to 0.25

Parameter	Estimate	Stderr	t-student	p-value
γ_1	0.2500			
$\beta^{(s)}$	0.8979	0.0028	323.2676	0.0000
$\tilde{\beta}^{(\boldsymbol{\pi})}$	0.1131	0.0004	293.5226	0.0000

$$R^2 = 66.24\% \qquad R_c^2 = 42.35\%$$

γ_1 and to use the standard value of 0.25 that has been chosen for the sovereign bond model. Based on the results reported in Table 23, we finally propose the following benchmark formula to compute the transaction cost for corporate bonds:

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = 1.50 \cdot s_{i,t} + 0.125 \cdot \mathrm{DTS}_{i,t}\, y_i^{0.25} \tag{53}$$

If we compare this expression with Equation (49), we notice that the coefficient of the bid-ask spread is larger (1.50 versus 1.25 for sovereign bonds), because of the larger uncertainty on the quoted spreads in the corporate bond universe. Concerning the price impact exponent, we use the same value.

Remark 28 *In order to compare sovereign and corporate bonds, we can transform Equation (49) by considering the relationship between the DTS and the daily volatility. In our sample[41], the average ratio is equal to* 30.3. *We deduce that the equivalent transaction cost formula based on the DTS measure for sovereign bonds is equal to:*

$$\boldsymbol{c}_i\left(q_i; s_{i,t}, \sigma_{i,t}, n_i\right) = 1.25 \cdot s_{i,t} + 0.10 \cdot \mathrm{DTS}_{i,t}\, y_i^{0.25} \tag{54}$$

We notice that the price impact is +25% *higher for corporate bonds compared to sovereign bonds.*

5.3 Extension to the two-regime model

As explained in Section 2.1.3 on page 4, the asset manager generally imposes a trading limit, because it is not possible to have a 100% participation rate. In Figure 25, we have reported the estimated price impact for corporate bonds[42]. Panel (a) corresponds to the estimated raw function. From a mathematical point of view, the price impact is defined even if the participation rate is larger than 100%. In the case of stocks, a 150% volume-based participation rate is plausible, but it corresponds to a very big trade. In the case of bonds, a 150% outstanding-based participation rate is impossible, because this trade size is larger than the issued size! As such, imposing a trading limit is a first modification to obtain a realistic transaction cost function. However, as explained in Section 2.4 on page 9, this is not sufficient. For instance, we use a trading limit of 300 bps in Panel (b). Beyond this trading limit, the price impact is infinite. But if we trade exactly 300 bps, the price impact is equal to 34 bps, and we obtain a concave price impact before this limit. It is better to introduce a second regime (see Equation 17 on page 9), implying the following function for

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[41]Figure 32 on page 82 reports the relationship between the volatility and the duration-times-spread of sovereign bonds.

[42]we recall that $\tilde{\beta}^{(\boldsymbol{\pi})} = 0.125$ and $\gamma_1 = 0.25$ for corporate bonds.

the price impact:

$$
\boldsymbol{\pi}\left(y\right) = \begin{cases} \tilde{\beta}^{(\boldsymbol{\pi})} \operatorname{DTS} y^{\gamma_1} & \text{if } y \leq \tilde{y} \\ \left(\tilde{\beta}^{(\boldsymbol{\pi})} \dfrac{\tilde{y}^{\gamma_1}}{\tilde{y}^{\gamma_2}}\right) \operatorname{DTS} y^{\gamma_2} & \text{if } \tilde{y} \leq y \leq y^{+} \\ +\infty & \text{if } y > y^{+} \end{cases}
$$

In Panel (c), the inflection point $\tilde{y}$ and the power γ_2 are set to 200 bps and 1. We have two areas. The grey area indicates that the trading is prohibitive beyond 300 bps. The red area indicates that the trading is penalized between 200 bps and 300 bps, because trading costs are no longer concave, but convex. Of course, we can use a larger value of γ_2 to penalize this area of participation rates (for example $\gamma_2 = 2$). Finally, we obtain the final transaction cost function in Panel (d).

Figure 25: From the single-regime model to the two-regime model (corporate bonds)
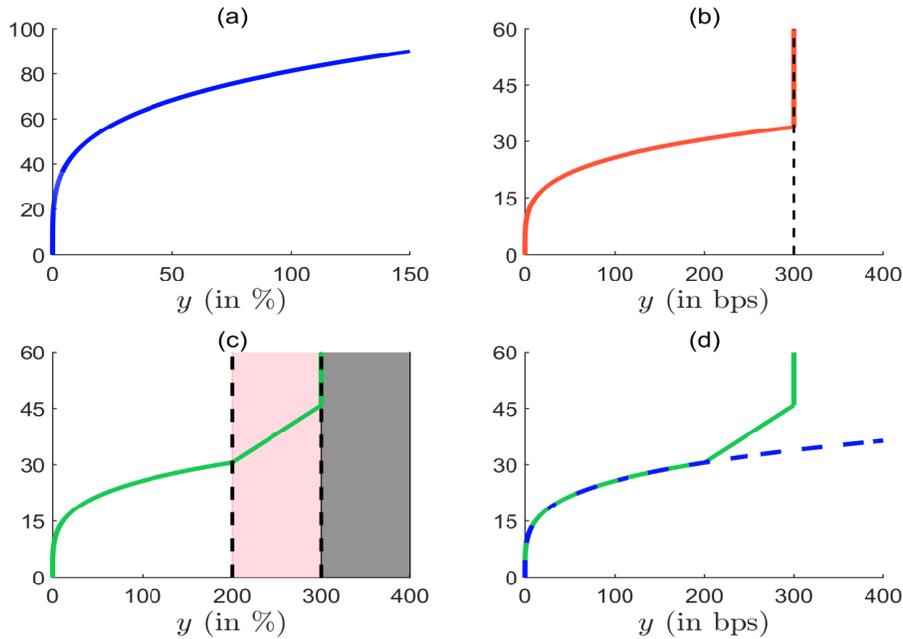
The issue of using a two-regime model is the calibration of the second regime. However, as said previously, it is unrealistic to believe that we can estimate the inflection point and the parameter γ_2 from data. Indeed, asset managers do not experience sufficient big trades and do not have enough data to calibrate the second regime. We are in an uncertain area, and it is better that these values are given by experts. For instance, we can use $\gamma_2 = 1$ or $\gamma_2 = 2$ to force the convexity of the second regime. The inflection point can be equal to $3/4$ or $2/3$ of the trading limit.

5.4 Stress testing of security-specific parameters

In this section, we conduct a stress testing program in order to define the transaction cost function in a stress regime. We first define the methodological framework based on the extreme value theory (EVT). Then, we apply the EVT approach to the security-specific parameters. Finally, we give the transaction cost function in the case of a LST program for equity funds.

5.4.1 Methodological aspects

Following Roncalli (2020, Chapters 12 and 14), we consider the extreme value theory for performing stress testing. We summarize this framework below and provide the main results[43].

The block maxima (BM) approach We note $X \sim \mathbf{F}$ a continuous random variable and $X_{i:n}$ the ith order statistic in the sample[44] $\{X_1, \ldots, X_n\}$. The maximum order statistic is defined by $X_{n:n} = \max(X_1, \ldots, X_n)$. We can show that $\mathbf{F}_{n:n}(x) = \mathbf{F}(x)^n$. If there exist two constants a_n and b_n and a non-degenerate distribution function $\mathbf{G}$ such that $\lim_{n \to \infty} \mathbf{F}_{n:n}(a_n x + b_n) = \mathbf{G}(x)$, the Fisher-Tippett theorem tells us that $\mathbf{G}$ can only be a Gumbel, Fréchet or Weibull probability distribution. In practice, these three distributions are replaced by the GEV distribution $\mathcal{GEV}(\mu, \sigma, \xi)$:

$$\mathbf{G}(x; \mu, \sigma, \xi) = \exp\left(-\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}\right)$$

defined on the support $\Delta = \left\{x : 1 + \xi\sigma^{-1}(x - \mu) > 0\right\}$. The parameters $\theta = (\mu, \sigma, \xi)$ can be calibrated by maximizing the log-likelihood function[45]:

$$\hat{\theta} = \arg\max \sum_t -\frac{1}{2}\ln\sigma^2 - \left(\frac{1 + \xi}{\xi}\right)\ln\left(1 + \xi\left(\frac{x_t - \mu}{\sigma}\right)\right) - \left(1 + \xi\left(\frac{x_t - \mu}{\sigma}\right)\right)^{-1/\xi}$$

where x_t is the observed maximum for the tth block maxima period[46]. By assuming that the length of the block maxima period is equal to n_{BM} trading days, the stress scenario associated with the random variable X for a given return time $\mathcal{T}$ is equal to:

$$\mathbb{S}(\mathcal{T}) = \mathbf{G}^{-1}\left(\alpha; \hat{\mu}, \hat{\sigma}, \hat{\xi}\right)$$

where:

$$\alpha = 1 - \frac{n_{\text{BM}}}{\mathcal{T}}$$

and $\mathbf{G}^{-1}$ is the quantile function:

$$\mathbf{G}^{-1}(\alpha; \mu, \sigma, \xi) = \mu - \frac{\sigma}{\xi}\left(1 - (-\ln\alpha)^{-\xi}\right)$$

Finally, we obtain:

$$\mathbb{S}(\mathcal{T}) = \hat{\mu} - \frac{\hat{\sigma}}{\hat{\xi}}\left(1 - \left(-\ln\left(1 - \frac{n_{\text{BM}}}{\mathcal{T}}\right)\right)^{-\hat{\xi}}\right) \tag{55}$$

---

[43]See Roncalli (2020, pages 753-777 and 904-909) for a detailed presentation of extreme value theory and its application to stress testing and scenario analysis.

[44]We assume that the random variables are *iid*.

[45]We recall that the probability density function of the GEV distribution is equal to:

$$g(x; \mu, \sigma, \xi) = \frac{1}{\sigma}\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-(1+\xi)/\xi} \exp\left(-\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}\right)$$

[46]The block maxima approach consists of dividing the observation period into non-overlapping periods of fixed size and computing the maximum of each period.

The peak over threshold (POT) approach In this approach, we are interested in estimating the distribution of exceedance over a certain threshold u:

$$\mathbf{F}_u(x) = \Pr\{X - u \leq x \mid X > u\}$$

where $0 \leq x < x_0 - u$ and $x_0 = \sup\{x \in \mathbb{R} : \mathbf{F}(x) < 1\}$. We notice that:

$$\mathbf{F}_u(x) = \frac{\mathbf{F}(u+x) - \mathbf{F}(u)}{1 - \mathbf{F}(u)}$$

For very large u, $\mathbf{F}_u(x)$ follows a generalized Pareto distribution $\mathcal{GPD}(\sigma, \xi)$:

$$\begin{aligned}
\mathbf{F}_u(x) &\approx \mathbf{H}(x; \sigma, \xi) \\
&= 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{-1/\xi}
\end{aligned}$$

defined on the support $\Delta = \{x : 1 + \xi\sigma^{-1}x > 0\}$.

Remark 29 *In fact, there is a strong link between the block maxima approach and the peak over threshold method. Suppose that $X_{n:n} \sim \mathcal{GEV}(\mu, \sigma, \xi)$. Using the fact that $\mathbf{F}_{n:n}(x) = \mathbf{F}(x)^n$, we can show that (Roncalli, 2020, page 774):*

$$\begin{aligned}
\mathbf{F}_u(x) &\approx 1 - \left(1 + \frac{\xi x}{\sigma + \xi(u - \mu)}\right)^{-1/\xi} \\
&= \mathbf{H}(x; \sigma + \xi(u - \mu), \xi)
\end{aligned}$$

Therefore, we obtain a duality between GEV and GPD distribution functions.

The parameters $\theta = (\sigma, \xi)$ are estimated by the method of maximum likelihood[47] once the threshold u_0 is found. To determine u_0, we use the mean residual life plot, which consists in plotting u against the empirical mean $\hat{e}(u)$ of the excess:

$$\hat{e}(u) = \frac{\sum_{i=1}^n (x_i - u)^+}{\sum_{i=1}^n \mathbb{1}\{x_i > u\}}$$

For any value $u \geq u_0$, we must verify that the mean residual life is a linear function of u since we have:

$$\mathbb{E}[X - u \mid X > u] = \frac{\sigma + \xi u}{1 - \xi}$$

The threshold u_0 is then found graphically.

To compute the stress scenario $\mathbb{S}(T)$, we recall that:

$$\mathbf{F}_u(x) = \frac{\mathbf{F}(u+x) - \mathbf{F}(u)}{1 - \mathbf{F}(u)} \approx \mathbf{H}(x)$$

where $\mathbf{H} \sim \mathcal{GPD}(\sigma, \xi)$. We deduce that:

$$\begin{aligned}
\mathbf{F}(x) &= \mathbf{F}(u) + (1 - \mathbf{F}(u)) \cdot \mathbf{F}_u(x - u) \\
&\approx \mathbf{F}(u) + (1 - \mathbf{F}(u)) \cdot \mathbf{H}(x - u)
\end{aligned}$$

---

[47]The probability density function of the GPD distribution is equal to:

$$h(x; \sigma, \xi) = \frac{1}{\sigma}\left(1 + \frac{\xi x}{\sigma}\right)^{-(1+\xi)/\xi}$$

We consider a sample of size n. We note n' as the number of observations whose value x_i is larger than the threshold u_0. The non-parametric estimate of $\mathbf{F}(u_0)$ is then equal to:

$$\hat{\mathbf{F}}(u_0) = 1 - \frac{n'}{n}$$

Therefore, we obtain the following semi-parametric estimate of $\mathbf{F}(x)$ for x larger than u_0:

$$
\begin{aligned}
\hat{\mathbf{F}}(x) &= \hat{\mathbf{F}}(u_0) + \left(1 - \hat{\mathbf{F}}(u_0)\right) \cdot \hat{\mathbf{H}}(x - u_0) \\
&= \left(1 - \frac{n'}{n}\right) + \frac{n'}{n}\left(1 - \left(1 + \frac{\hat{\xi}(x - u_0)}{\hat{\sigma}}\right)^{-1/\hat{\xi}}\right) \\
&= 1 - \frac{n'}{n}\left(1 + \frac{\hat{\xi}(x - u_0)}{\hat{\sigma}}\right)^{-1/\hat{\xi}}
\end{aligned}
$$

We can interpret $\hat{\mathbf{F}}(x)$ as the historical estimate of the probability distribution tail that is improved by the extreme value theory. We have[48]:

$$\hat{\mathbf{F}}^{-1}(\alpha) = u_0 + \frac{\hat{\sigma}}{\hat{\xi}}\left(\left(\frac{n}{n'}(1 - \alpha)\right)^{-\hat{\xi}} - 1\right)$$

We recall that the stress scenario of the random variable X associated with the return time $\mathcal{T}$ is equal to $\mathbb{S}(\mathcal{T}) = \hat{\mathbf{F}}^{-1}(\alpha)$ where $\alpha = 1 - \mathcal{T}^{-1}$. Finally, we deduce that:

$$\mathbb{S}(\mathcal{T}) = u_0 + \frac{\hat{\sigma}}{\hat{\xi}}\left(\left(\frac{n}{n'\mathcal{T}}\right)^{-\hat{\xi}} - 1\right) \tag{56}$$

5.4.2 Application to asset liquidity

We assume that the current date t is not a stress period. Let $p_{i,t}$ be a security-specific parameter observed at time t. We would like to compute its stress value $p_{i,t+h}^{\text{stress}}$ for a given time horizon h. As explained in Section 4.2.3 on page 31, we can use a multiplicative shock:

$$p_{i,t+h}^{\text{stress}} = m_p \cdot p_{i,t}$$

where m_p is the multiplier factor. Depending on the nature of the parameter, we can also use an additive shock:

$$p_{i,t+h}^{\text{stress}} = p_{i,t} + \Delta_p$$

where Δ_p is the additive factor. For instance, we can assume that a multiplicative shock is relevant for the trading volume, but an additive shock is more appropriate for the credit spread. Using a sample $\{p_{i,1}, \dots, p_{i,T}\}$ of the parameter p, we compute $m_t = \frac{p_{i,t+h}}{p_{i,t}}$ or $m_t = p_{i,t+h} - p_{i,t}$. Then, we apply the previous EVT framework to the time series $\{m_1, \dots, m_T\}$ and estimate the stress scenario m_p or Δ_p for a given return time $\mathcal{T}$ and a holding period h. We notice that two periods are used to define the stress scenario. The time horizon h indicates the frequency of the stress scenario. It is different to compute a daily, weekly or monthly stress. The return time $\mathcal{T}$ indicates the severity of the stress scenario. If $\mathcal{T}$ is set to one year, we observe this stress scenario every year on average. Again, it is different to compute a stress with a return time of one year, two years or five years. In some sense, h corresponds to the holding period whereas $\mathcal{T}$ measures the occurrence probability.

---

[48]The quantile function of the GPD distribution is equal to:
$$\mathbf{H}^{-1}(\alpha; \sigma, \xi) = \frac{\sigma}{\xi}\left((1 - \alpha)^{-\xi} - 1\right)$$

Market risk   We consider the VIX index from January 1990 to February 2021. We have a sample of 7 850 observations. In Figure 26, we report the histogram of the VIX index and the multiplicative factor m_σ for three time horizons (one day, one week and one month). The estimates of the GEV and GPD distributions are reported in Table 24. Using Equations 55 and 55, we deduce the stress scenarios associated with m_σ and Δ_σ for three time horizons (1D, 1W and 1M) and five return times (6M, 1Y, 2Y, 5Y, 10Y and 50Y) in Tables 25 and 26.

Figure 26: Empirical distribution of the multiplicative factor m_σ
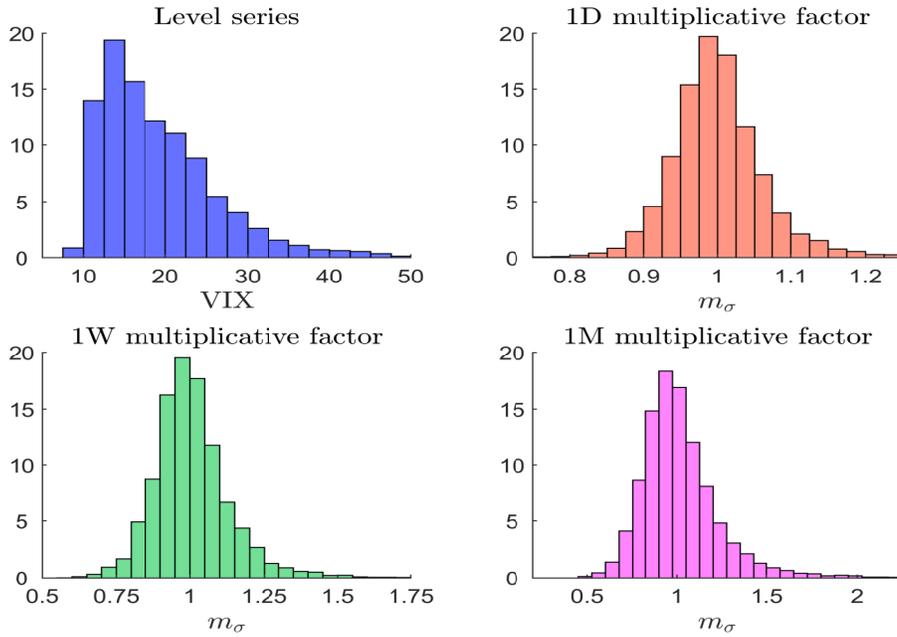
Table 24: EVT estimates of the VIX index

		GEV			GPD		
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\xi}$	u_0	$\hat{\sigma}$	$\hat{\xi}$
m_σ	1D	1.103	0.049	0.299	1.229	0.096	0.138
	1W	1.157	0.101	0.229	1.460	0.203	0.243
	1M	1.138	0.185	0.238	1.960	0.425	0.410
Δ_σ	1D	1.739	1.036	0.424	4.943	2.560	0.238
	1W	2.568	1.821	0.322	2.950	2.022	0.291
	1M	2.277	3.179	0.201	16.830	11.522	0.008

How should we interpret these results? For example, the multiplicative weekly stress scenario is equal to 1.50 if we consider a return time of one year and the BM/GEV approach. For the additive scenario, we obtain a figure of 9.66%. This means that the volatility can be multiplied by 1.50 or increased by 9.66% in one week, and we observe this event (or an equivalent more severe event) every year. If we average the historical, BM/GEV and POT/GPD approaches, the 2Y weekly stress scenario is respectively ×1.80 (multiplicative

stress) and $+17\%$ (additive stress). If we focus on the monthly stress scenario, these figures become $\times 2.66$ and $+29\%$.

Table 25: Multiplicative stress scenarios of the volatility

$\mathcal{T}$ (in years)		0.385	$^1/_2$	1	2	5	10	50
α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
	Historical	1.23	1.25	1.32	1.43	1.50	1.57	
1D	BM/GEV	1.20	1.22	1.29	1.37	1.51	1.65	2.09
	POT/GPD	1.23	1.25	1.33	1.41	1.52	1.62	1.90
	Historical	1.46	1.51	1.70	1.89	2.26	2.56	
1W	BM/GEV	1.34	1.38	1.50	1.64	1.86	2.06	2.66
	POT/GPD	1.46	1.51	1.68	1.87	2.18	2.47	3.35
	Historical	1.96	2.05	2.44	2.99	4.23	5.08	
1M	BM/GEV	1.47	1.55	1.78	2.04	2.46	2.83	3.99
	POT/GPD	1.96	2.08	2.45	2.96	3.88	4.86	8.53

Table 26: Additive stress scenarios of the volatility

$\mathcal{T}$ (in years)		0.385	$^1/_2$	1	2	5	10	50
α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
	Historical	4.94	5.50	7.72	10.77	14.15	18.22	
1D	BM/GEV	3.91	4.51	6.42	8.94	13.59	18.50	37.34
	POT/GPD	4.93	5.58	7.12	8.42	9.85	10.74	12.31
	Historical	9.49	10.88	14.50	20.43	24.56	27.97	
1W	BM/GEV	6.08	6.97	9.66	12.95	18.53	23.96	42.34
	POT/GPD	9.57	10.65	13.92	17.92	24.61	30.99	51.86
	Historical	16.83	19.04	27.22	35.62	46.59	61.40	
1M	BM/GEV	7.84	9.13	12.74	16.80	23.03	28.54	44.68
	POT/GPD	16.64	19.67	27.70	35.77	46.51	54.70	73.88

Trading volume Dealing with volatility is relatively simple thanks to the availability of the VIX. In the case of the trading volume, we face more difficulties because there is not a standard index that measures the market depth. This means that we must use the trading volume of the stocks. From a robustness point of view, it is obvious that computing a stress for each stock is not relevant. Therefore, given the times series of $v_{i,t}$ for several stocks, we would like to compute a synthetic stress scenario that is valid for all stocks. The first idea is to compute the multipliers for each stock and to pool all the data. The second idea is to compute the multipliers for each date and to average the data by date. For the BM/GEV approach, we compute the maximum for each block and each stock, and then we average the maxima by block.

We consider the 30-day average daily volume of the stocks that make up[49] the EuroStoxx 50 Index from January 2010 to December 2020. At each date, we compute the multiplicative factor of the trading volume[50]. Then, we apply the previous pooling and averaging

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[49]Since the composition changes from one month to another, we have 73 stocks during the period. Nevertheless, at each date, we only consider the 50 stocks that are valid at the first trading day of the month.

[50]In fact, it is a reductive factor since the risk is not that daily volumes increase, but that they decrease.

approaches to these data[51]. Results are given in Table 27. If we average the historical, BM/GEV and POT/GPD approaches, the 2Y weekly and monthly stress scenarios are respectively ×0.75 and ×0.48. This means that the daily volume is approximately reduced by 25% if we consider a one-week holding period and 50% if we consider a one-month holding period.

Table 27: Multiplicative stress scenarios of the trading volume

	$\mathcal{T}$ (in years)		0.385	$^1/_2$	1	2	5	10	50
	α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
	Historical		0.93	0.93	0.91	0.88	0.84	0.80	0.71
	BM/GEV	Pooling	0.94	0.94	0.92	0.90	0.87	0.85	0.80
1W	POT/GPD	Pooling	0.95	0.94	0.91	0.88	0.84	0.80	0.70
	BM/GEV	Averaging	0.94	0.94	0.93	0.92	0.91	0.90	0.89
	POT/GPD	Averaging	0.93	0.92	0.92	0.91	0.91	0.90	0.89
	Historical		0.79	0.77	0.72	0.67	0.61	0.55	0.48
	BM/GEV	Pooling	0.86	0.85	0.81	0.78	0.74	0.71	0.65
1W	POT/GPD	Pooling	0.87	0.83	0.75	0.68	0.61	0.56	0.47
	BM/GEV	Averaging	0.87	0.86	0.84	0.82	0.79	0.77	0.73
	POT/GPD	Averaging	0.82	0.81	0.79	0.78	0.76	0.75	0.72
	Historical		0.50	0.48	0.41	0.36	0.31	0.29	0.26
	BM/GEV	Pooling	0.72	0.69	0.62	0.56	0.50	0.46	0.39
1M	POT/GPD	Pooling	0.40	0.38	0.36	0.33	0.31	0.29	0.26
	BM/GEV	Averaging	0.75	0.73	0.68	0.63	0.58	0.55	0.49
	POT/GPD	Averaging	0.62	0.60	0.57	0.54	0.50	0.48	0.42

Bid-ask spread   We have seen that stress scenarios of the daily volume are more difficult to compute than stress scenarios of the volatility. This issue is even more important with bid-ask spreads because of the data quality. Ideally, we would like to obtain the weighted average bid-ask spread adjusted by the volume for each stock and each trading day. However, this information is not easily available or is expensive. This is why databases of asset managers and trading platforms generally report the end-of-day bid-ask spread. However, unlike the closing price, which corresponds to the security's end-of-day transaction price observed during a regular market trading period, there is no standard definition of the bid and ask end-of-day prices. In particular, it is not obvious that the end-of-day bid-ask spread corresponds to the last bid-ask spread observed during the regular market trading period. Rather, our experience shows that the end-of-day bid-ask spread may be impacted by after-hours trading orders. It seems that this synchronization bias between regular trading and after-hours trading only impacts bid-ask spreads and not closing prices.

To illustrate this issue, we report the end-of-day bid-ask spread of the BNP Paribas stock between January 2010 and December 2020 in Figure 27. During this period, the stock's median bid-ask spread is equal to 1.22 bps. This value is relatively low, however, we observe many trading days where the bid-ask spread is larger than 20 bps[52]. Therefore, the bid-ask spread may jump from 2 bps to 80 bps in one day. It is obvious that these extreme variations are not realistic and no institutional investor has paid a bid-ask spread of 80 bps for the BNP Paribas stock during the period. These extreme points are not unusual

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[51]We can also transform these stress scenarios on the trading volume into stress scenarios on the participation rate using the following formula: $m_x = \frac{1}{m_v}$. Results are reported in Table 44 on page 84.

[52]These observations correspond to the red bars in 27

Figure 27: Historical bid-ask spread of BNP Paribas (in bps)
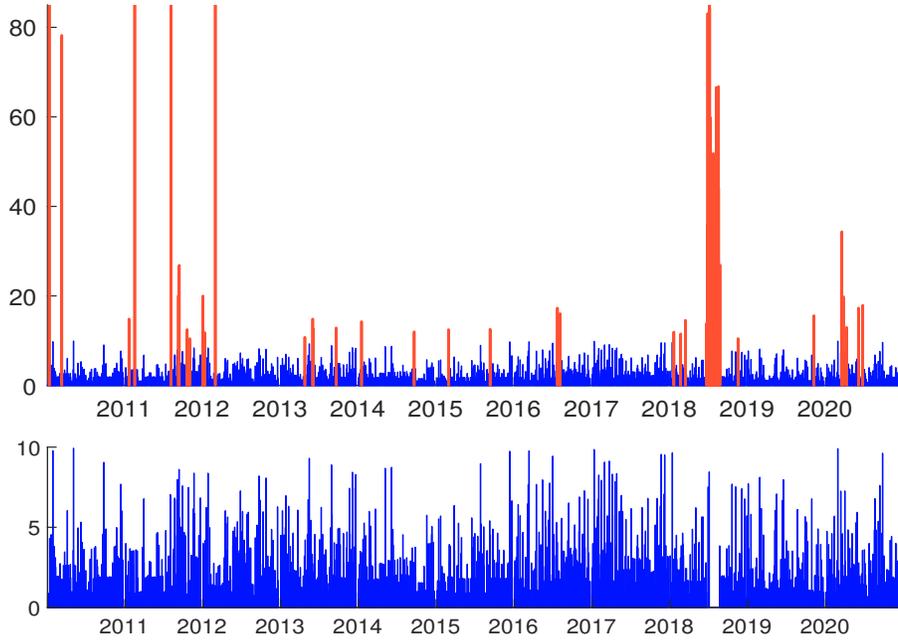
as illustrated by the figures reported in Table 28. For the 50 stocks of the Eurostoxx 50 Index, we have computed the frequency at which the bid-ask spread is negative, the daily multiplicative factor is greater than 5 or 10, and the absolute variation is greater than 25 and 100 bps. We consider two well-known data providers, FactSet and Bloomberg, that are extensively used by equity portfolio managers. These results illustrate that reported bid and ask end-of-day prices may deviate substantially from the closing price because of the synchronization bias between regular and after-hours trading.

Table 28: Statistics of daily multiplicative and additive factors for the Eurostoxx 50 stocks $(2010 - 2020)$

Frequency	Factset	Bloomberg
$\Pr\{s < 0\}$	0.01%	0.24%
$\Pr\{m_s > 10\}$	0.77%	0.62%
$\Pr\{m_s > 5\}$	3.49%	3.12%
$\Pr\{|\Delta_s| > 100 \text{ bps}\}$	0.63%	0.44%
$\Pr\{|\Delta_s| > 25 \text{ bps}\}$	4.52%	3.05%

There are different ways to fix the previous problem. For example, we can consider a ten-day moving average of daily bid-ask spreads for each stock. Or we can calculate the weighted average of the bid-ask spreads for a given universe of stocks for each trading day. The first case corresponds to a time-series average, whereas the second case corresponds to a cross-section average. In both cases, the underlying idea is to apply a denoising filter in order to estimate the average trend. A variant of the second method is to consider the median bid-ask spread, and we apply this approach to the stocks of the Eurostoxx 50 Index from January 2010 to December 2020. As in the case of the daily volume, we only consider

the 50 stocks that are in the index at each trading day. The empirical distributions of m_s and Δ_s are given in Figures 35 and 36 on page 83. Using these data, we calibrate the GEV and GPD models, and we obtain the stress scenarios that are reported in Tables 29 and 30. If we average the historical, BM/GEV and POT/GPD approaches, the 2Y weekly stress scenario is respectively $\times 3$ (multiplicative stress) and $+6.5$ bps (additive stress).

Table 29: Multiplicative stress scenarios of the bid-ask spread

$\mathcal{T}$ (in years)		0.385	$^1/_2$	1	2	5	10	50
α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
1D	Historical	1.66	1.73	1.93	2.40	2.75	7.11	
	BM/GEV	1.63	1.70	1.92	2.19	2.64	3.08	4.56
	POT/GPD	1.65	1.71	1.94	2.32	3.24	4.49	11.70
1W	Historical	1.74	1.88	2.58	3.49	6.78	9.76	
	BM/GEV	1.67	1.76	2.05	2.41	3.07	3.75	6.27
	POT/GPD	1.81	1.93	2.41	3.22	5.20	7.92	23.78
1M	Historical	2.54	2.92	5.12	6.65	9.62	9.98	
	BM/GEV	1.75	1.86	2.18	2.58	3.25	3.90	6.12
	POT/GPD	2.40	2.64	3.52	4.85	7.72	11.21	27.90

Table 30: Additive stress scenarios of the bid-ask spread

$\mathcal{T}$ (in years)		0.385	$^1/_2$	1	2	5	10	50
α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
1D	Historical	1.67	1.82	2.93	6.46	10.94	18.14	
	BM/GEV	1.42	1.63	2.28	3.14	4.71	6.37	12.70
	POT/GPD	1.77	2.04	3.07	4.76	8.78	14.17	44.13
1W	Historical	1.98	2.37	5.19	10.10	12.36	19.11	
	BM/GEV	1.48	1.70	2.40	3.33	5.08	6.94	14.17
	POT/GPD	2.19	2.57	3.91	6.00	10.63	16.43	45.46
1M	Historical	3.36	3.98	7.90	10.60	16.04	21.36	
	BM/GEV	1.51	1.77	2.62	3.82	6.20	8.91	20.46
	POT/GPD	2.99	3.57	5.73	9.23	17.33	27.95	84.86

5.4.3 Definition of the stress transaction cost function

If we assume that $x^+ = 10\%$, $\tilde{x} = \frac{2}{3}x^+$ and $\gamma_2 = 1$, the transaction cost function for large cap stocks is equal to:

$$\boldsymbol{c}\left(q; s, \sigma, v\right) = \begin{cases} 1.25 \cdot s + 0.40 \cdot \sigma\sqrt{x} & \text{if } x \leq 6.66\% \\ 1.25 \cdot s + 1.55 \cdot \sigma x & \text{if } 6.66\% \leq x \leq 10\% \\ +\infty & \text{if } x > 10\% \end{cases} \tag{57}$$

We consider the following stress scenario[53]:

- $\Delta_s = 8$ bps

- $\Delta_\sigma = 20\%$

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[53]This stress scenario is approximatively the 2Y weekly stress scenario.

- $m_v = 0.75$

We deduce that the transaction cost function in the stress regime becomes:

$$
c\left(q;s,\sigma,v\right)=\begin{cases} 1.25 \cdot (s+8 \text{ bps}) + 0.40 \cdot \left(\sigma + \dfrac{20\%}{\sqrt{260}}\right)\sqrt{\dfrac{4}{3}x} & \text{if } x \leq 5\% \\[2ex] 1.25 \cdot (s+8 \text{ bps}) + 1.55 \cdot \left(\sigma + \dfrac{20\%}{\sqrt{260}}\right)\dfrac{4}{3}x & \text{if } 5\% \leq x \leq 7.5\% \\[2ex] +\infty & \text{if } x > 7.5\% \end{cases}
$$

Table 31: Stress testing computation

x	Case	Annualized volatility							Liquidation		
		10%	15%	20%	25%	30%	35%	40%	$\mathcal{LT}$	$\mathcal{LS}$	$\mathcal{LS}$
				$c\left(q;s,\sigma,v\right)$ (in bps)						one-day	two-day
0.00%	Normal	5.0	5.0	5.0	5.0	5.0	5.0	5.0	1	0%	0%
	Stress	15.0	15.0	15.0	15.0	15.0	15.0	15.0	1	0%	0%
0.01%	Normal	5.2	5.4	5.5	5.6	5.7	5.9	6.0	1	0%	0%
	Stress	15.9	16.0	16.1	16.3	16.4	16.6	16.7	1	0%	0%
0.05%	Normal	5.6	5.8	6.1	6.4	6.7	6.9	7.2	1	0%	0%
	Stress	16.9	17.2	17.6	17.9	18.2	18.5	18.8	1	0%	0%
0.10%	Normal	5.8	6.2	6.6	7.0	7.4	7.7	8.1	1	0%	0%
	Stress	17.7	18.2	18.6	19.1	19.5	20.0	20.4	1	0%	0%
0.50%	Normal	6.8	7.6	8.5	9.4	10.3	11.1	12.0	1	0%	0%
	Stress	21.1	22.1	23.1	24.1	25.1	26.1	27.2	1	0%	0%
1.00%	Normal	7.5	8.7	10.0	11.2	12.4	13.7	14.9	1	0%	0%
	Stress	23.6	25.0	26.5	27.9	29.3	30.8	32.2	1	0%	0%
5.00%	Normal	10.5	13.3	16.1	18.9	21.6	24.4	27.2	1	0%	0%
	Stress	34.2	37.4	40.6	43.8	47.0	50.2	53.4	1	0%	0%
7.50%	Normal	12.2	15.8	19.4	23.0	26.6	30.2	33.8	1	0%	0%
	Stress	43.8	48.6	53.4	58.2	63.0	67.8	72.6	1	0%	0%
10.00%	Normal	14.6	19.4	24.2	29.0	33.8	38.6	43.4	1	0%	0%
	Stress	40.0	44.2	48.4	52.5	56.7	60.9	65.0	2	2.5%	0%
20.00%	Normal	14.6	19.4	24.2	29.0	33.8	38.6	43.4	2	10%	0%
	Stress	41.4	45.8	50.2	54.6	59.0	63.4	67.8	3	12.5%	5.5%

In Table 31, we have reported an example of stress testing applied to single stocks. For each value of σ and x, we report the unit cost $c\left(q;s,\sigma,v\right)$ in bps for the normal and stress regimes. For instance, if the annualized volatility is equal to 30% and the liquidation of the exposure on the single stock represents 0.05% of the normal daily volume, the transaction cost is equal to 6.7 bps in the normal period. In the stress period, it increases to 18.2 bps, which is an increase of 171%. We have also reported the liquidation time, the one-day liquidation shortfall and the two-day liquidation shortfall. Let us consider a 10% liquidation. Because of the liquidity policy, we can liquidate 7.5% the first day and 2.5% the second day during the stress period, whereas we can liquidate the full exposure during the normal period. Therefore, the liquidation time, which is normally equal to one day, takes two days in the stress period. If we consider a 20% liquidation, the (one-day) liquidation shortfall is equal to 12.5% and the time-to-liquidation is equal to three days.

6 Conclusion and discussion

Liquidity stress testing is a recent topic in asset management, which has given rise to numerous publications from regulators (AMF, 2017; BaFin, 2017; ESMA, 2019, 2020; FSB, 2017; IOSCO, 2015, 2018). In particular, LST has been mandatory in Europe since September 2020. However, contrary to banks, asset managers have less experience conducting a liquidity stress testing program at the global portfolio level. Moreover, this topic has not been extensively studied by the academic research. Therefore, we are in a trial-and-error period where standard models are not really established, and asset managers use very different approaches to assess liquidity stress tests. The aim of this research project is to propose a simple LST approach that may become a benchmark for asset managers. In a previous paper, we have already developed a framework for modeling the liability liquidity risk (Roncalli *et al.*, 2020). In a forthcoming paper, we will propose several tools for managing the asset-liability liquidity gap. In this paper, we focus on measuring the asset liquidity risk.

Contrary to the first and third parts of this project, there is a large body of academic literature that has studied the estimation of transaction costs. In particular, we assume that price impact verifies the power-law property. This means that there is a concave relationship between the participation rate and the transaction cost. This model is appealing because (1) it has been proposed by the academic research in the case of stocks, (2) it is simple and (3) it is suitable for stress testing purposes. The first reason is important, because the model must be approved by the regulators. The fact that this model has academic roots is therefore a key element in terms of robustness and independent validation. The second reason is critical, because a complex transaction cost model with many parameters and variables may be not an industrial solution. This is particularly true if the calibration requires a large amount of data. In the case of our model, we have three parameters (spread sensitivity, price impact sensitivity and price impact exponent) and three explanatory variables (bid-ask spread, volatility risk and participation rate). If the asset manager does not have enough data, it can always use some internal experts to set the value of these parameters. Moreover, we have seen that this model can also be applied to bonds with some minor corrections. For instance, in the case of corporate bonds, it is better to use the DTS instead of the volatility in order to measure the market risk. Finally, the third reason is convenient when we perform stress testing programs. When applied to liquidity in asset management, they can concern the liability side and/or the asset side (Brunnermeier and Pedersen, 2009). For instance, the asset manager can assume that the liquidity crisis is due to funding issues. In this case, the stress scenario could be a severe redemption scenario. But it can also assume that the liquidity crisis is due to market issues. In this case, the stress scenario could be a market liquidity crisis with a substantial reduction in trading volumes and an increase in volatility risk. Therefore, it is important that a stress scenario of market liquidity risk could be implemented, and not only a stress scenario of funding liquidity risk. Our transaction cost model has three variables that can be stressed: the spread, the market risk and the trading volume (or the market depth). We think that these three transmission channels are enough to represent a market liquidity crisis. Nevertheless, the high concavity of the price impact function when the exponent is smaller than $1/2$ is not always relevant when we also impose trading policy limits. Therefore, we propose an extension of the previous model by considering two regimes with two power-law models where the second exponent takes a larger value than the first exponent. In this case, the transaction cost function has two additional parameters: the exponent of the second regime and the inflection point that separates the first and second regimes. Therefore, we can obtain a price impact which is more convex in the second regime when the participation rate is high. In terms of calibration, we propose using expert estimates, implying no more data analysis.

Table 32: Impact of size on the market impact

Size	Stocks			Bonds		
	Unit cost	Total cost	Average cost	Unit cost	Total cost	Average cost
$\times 1$	$\times 1.0$	$\times 1.0$	$+0\%$	$\times 1.0$	$\times 1.0$	$+0\%$
$\times 2$	$\times 1.4$	$\times 2.8$	$+41\%$	$\times 1.2$	$\times 2.4$	$+19\%$
$\times 3$	$\times 1.7$	$\times 5.2$	$+73\%$	$\times 1.3$	$\times 3.9$	$+32\%$
$\times 4$	$\times 2.0$	$\times 8.0$	$+100\%$	$\times 1.4$	$\times 5.7$	$+41\%$
$\times 5$	$\times 2.2$	$\times 11$	$+124\%$	$\times 1.5$	$\times 7.5$	$+50\%$
$\times 10$	$\times 3.2$	$\times 32$	$+216\%$	$\times 1.8$	$\times 18$	$+78\%$

We have proposed some formulas for large cap stocks, small cap stocks, sovereign bonds and corporate bonds[54]. This is an especially challenging exercise. Indeed, the calibrated formulas highly depend on the data[55]. Because we use a small sample on a particular period and this sample is specific to an asset manager, the data are not representative of the industry as a whole. Moreover, in the case of bonds, we have decided to exclude opportunistic trades with a negative transaction cost. This is why these calibrated formulas must be adjusted and validated by the asset manager before using them. On page 78, we have reported the values of the unit transaction cost. These tables can be used as a preliminary pricing grid that can be modified. For instance, the asset manager generally knows its average price impact, and can then change the values of β^s, β^{π} and γ_1 in order to retrieve its average cost. This pricing grid can also be modified by the trading desk cell by cell in order to avoid some unrealistic values[56]. One of the difficulties is to maintain some coherency properties between the different cells of the pricing grid. In the case of the power-law model, if we multiply the size by α, the unit cost is multiplied by α^{γ_1} while the total cost is multiplied by $\alpha^{1+\gamma_1}$. In Table 32, we have reported the impact of the size on the price impact when we consider our benchmark formulas[57]. For example, we notice that if we multiply the size of the trade by 5, the average cost due to the price impact increases by 124% for stocks and 50% for bonds. Quantifying these size effects is essential in a liquidity stress testing program because the risk in a stress period is mainly related to the size issue. And it is not always obvious to obtain a pricing grid that satisfies some basic coherency properties.

As explained in the introduction, our motivation is to propose a framework that can help asset managers to implement liquidity stress testing, which is a relative new topic for this industry. We are aware that it is challenging, and the final model can appear too simple to describe the transaction cost function of any stocks and bonds. This is true. For instance, it is not precise enough to calibrate swing prices. However, we reiterate that the goal is not to build a pre-trade system, but to implement a liquidity stress testing program from an industrial viewpoint. In a liquidity crisis, there are so many unknowns and uncertainties that a sophisticated model does not necessarily enable redemption issues to be managed better. An LST model must be sufficiently realistic and pragmatic in order to give the magnitude order of the stress severity and compare the different outcomes. We think that the model proposed here has some appealing properties to become a benchmark for asset managers. However, the road to obtain the same standardization that we encounter in the banking regulation of market, credit or counterparty risk is long. More research in this area from academics and professionals is needed.

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[54]These formulas correspond to Equations (43), (44), (49) and (53).

[55]For example, using Reuters bid-ask spreads instead of Bloomberg bid-ask spreads dramatically changes the parameter β^s for sovereign and corporate bonds.

[56]For instance, a price impact of 198 bps may be considered too high when the outstanding-based participation rate is set to 100 bps and the DTS of the corporate bond is equal to 5 000 bps.

[57]We recall that γ_1 is equal to 0.5 for stocks and 0.25 for bonds.

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Appendix

A   Glossary

Bid-ask spread

The bid-ask spread corresponds to the difference between the ask price and the bid price of a security divided by its mid-point price. It is a component of the liquidity cost, since the unit transaction cost depends on the half bid-ask spread. In this article, we use the term bid-ask spread in place of half bid-ask spread, and we denote it by s.

Break-even redemption scenario

The break-even redemption scenario is the maximum amount expressed in dollars that can be liquidated in one day:

$$
\begin{aligned}
\mathbb{R}^{\text{break}-\text{even}} &= \sup\left\{\mathbb{R} : \mathcal{LS}\left(\mathbb{R}\right) = 0\right\} \\
&= \inf\left\{\mathbb{R} : \mathcal{LR}\left(\mathbb{R}; 1\right) = 1\right\}
\end{aligned}
$$

HQLA class

The term HQLA refers to high-quality liquid asset. An HQLA class groups all the securities that present the same ability to be converted into cash. An HQLA class is different than a liquidity bucket, because this latter classification is used to define the unit transaction cost function. For instance, it does not make sense that a bond and a stock share the same transaction cost function. However, they can belong to the same HQLA class if they have the same conversion property into cash.

Implementation shortfall

The implementation shortfall measures the total amount of slippage, that is the difference in price between the time a portfolio manager makes an investment decision and the actual traded price. Its mathematical expression is:

$$
\mathcal{IS}\left(q\right) = \max\left(\mathbb{V}^{\text{mid}}\left(q\right) - \mathbb{V}^{\text{liquidated}}\left(q\right), 0\right)
$$

where $\mathbb{V}^{\text{mid}}\left(q\right)$ is the current value of the redemption scenario and $\mathbb{V}^{\text{liquidated}}\left(q\right)$ is the value of the liquidated portfolio.

Liquidation policy

See trading limit.

Liquidation ratio

The liquidation ratio $\mathcal{LR}\left(q; h\right)$ is the proportion of the redemption trade that is liquidated after h trading days. We generally focus on daily and weekly liquidation ratios $\mathcal{LR}\left(q; 1\right)$ and $\mathcal{LR}\left(q; 5\right)$. The liquidation ratio is also used to define the liquidation time (or time to liquidation), which is an important measure for managing the liquidity risk. We also use the notation $\mathcal{LR}\left(\mathbb{R}; h\right)$ where $\mathbb{R}$ is the dollar amount of the redemption scenario.

Liquidation shortfall

The liquidation shortfall is defined as the residual redemption that cannot be fulfilled after one trading day. It is expressed as a percentage of the redemption value. If it is equal to 0%, this means that we can liquidate the redemption in one trading day. More generally, its mathematical expression is:

$$\mathcal{LS}(q) = 1 - \mathcal{LR}(q; 1)$$

where $\mathcal{LR}(q; h)$ is the liquidation ratio. If the redemption scenario is expressed in dollars, we have:

$$\mathcal{LS}(\mathbb{R}) = 1 - \mathcal{LR}(\mathbb{R}; 1)$$

Liquidation time

See time to liquidation.

Liquidity bucket

A liquidity bucket defines a set of securities that share the same liquidity properties. Therefore, the securities have the same functional form of the unit transaction cost. Examples of liquidity buckets are large cap DM stocks, small cap stocks, sovereign bonds, corporate IG bonds, HY USD bonds, HY EUR bonds, EM bonds, energy commodities, soft commodities, metal commodities, agricultural commodities, G10 currencies, EM currencies, REITS, etc. The jth liquidity bucket is denoted by $\mathcal{LB}_j$.

Market impact

See price impact.

Outstanding-based participation rate

The outstanding-based participation rate is a normalization of the trade size:

$$y = \frac{q}{n}$$

where q is the number of shares that have been sold and n is the number of issued shares. The outstanding-based participation rate is a modification of the (volume-based) participation rate, because the trading volume cannot always be computed for some securities, for example bonds.

Participation rate

The participation rate is a normalization of the trade size:

$$x = \frac{q}{v}$$

where q is the number of shares that have been sold and v is the trading volume. The participation rate is used to define the unit transaction cost function $\boldsymbol{c}(x)$.

Price impact (unit)

The (unit) price impact $\pi(q)$ is the part of the unit transaction cost function which is not explained by the bid-ask spread.

Price impact (total)

The price impact (or market impact) $\mathcal{PI}(q)$ is the part of the transaction cost due to the trade size:

$$\mathcal{PI}(q) = \mathcal{TC}(q) - \mathcal{BAS}(q)$$

We generally expect that it is an increasing function of the redemption size.

Pro-rata liquidation

The pro-rata liquidation uses the proportional rule, implying that each asset is liquidated such that the structure of the portfolio is the same before and after the liquidation.

Redemption scenario

A redemption scenario q is defined by the vector $(q_1, \ldots, q_n)$ where q_i is the number of shares of security i to sell. This scenario can be expressed in dollars:

$$Q := (Q_1, \ldots, Q_n) = (q_1 P_1, \ldots, q_n P_n)$$

where P_i is the price of security i. The redemption scenario may also be defined by its dollar value $\mathbb{R}$:

$$\mathbb{R} = \mathbb{V}(q) = \sum_{i=1}^{n} q_i P_i$$

If we consider a portfolio defined by its weights $w = (w_1, \ldots, w_n)$, we have:

$$w_i = \frac{q_i P_i}{\mathbb{R}}$$

Time to liquidation

The time to liquidation is the inverse function of the liquidation ratio. It indicates the minimum number of days that it is necessary to liquidate the proportion p of the redemption. It is denoted by the function $\mathcal{LT}(q; p)$ or $\mathcal{LT}(\mathbb{R}; p)$.

Trading limit

The trading limit q^+ is the maximum number of shares that can be sold in one trading day. It can be expressed using the maximum participation rate:

$$x^+ = \frac{q^+}{v}$$

where v is the daily volume.

Transaction cost

The transaction cost of a redemption is made up of two components: the bid-ask spread cost and the price impact cost. It is denoted by $\mathcal{TC}(q)$.

Unit transaction cost

The unit transaction cost function $\boldsymbol{c}(x)$ is the percentage cost associated with the participation rate x for selling one share. It has two components:

$$\boldsymbol{c}(x) = s + \boldsymbol{\pi}(x)$$

where s is the half bid-ask spread and $\boldsymbol{\pi}(x)$ is the price impact. The total transaction cost of selling q shares is then:

$$\mathcal{TC}(q) = q \cdot P \cdot \boldsymbol{c}(x) = Q \cdot \boldsymbol{c}(x)$$

where P is the security price and $Q = q \cdot P$ is the nominal selling volume expressed in \$.

Valuation function

The valuation function $\mathbb{V}(\omega)$ gives the dollar value of the portfolio $\omega = (\omega_1, \ldots, \omega_n)$, which is expressed in number of shares:

$$\mathbb{V}(\omega) = \sum_{i=1}^{n} \omega_i P_i$$

The dollar value of the redemption is equal to $\mathbb{R} = \mathbb{V}(q) = \sum_{i=1}^{n} q_i P_i$, whereas the dollar value of the portfolio becomes $\mathbb{V}(\omega - q) = \sum_{i=1}^{n} (\omega_i - q_i) P_i$ after the liquidation of the redemption scenario.

Vertical slicing

See pro-rata liquidation.

Volume-based participation rate

See participation rate.

Waterfall liquidation

In this approach, the portfolio is liquidate by selling the most liquid assets first.

B Mathematical results

B.1 Relationship between the two unit cost functions in the toy model

We note:

$$\boldsymbol{c}'(x) = \begin{cases} s' & \text{if } x \le \tilde{x} \\ s' + \alpha'(x - \tilde{x}) & \text{if } \tilde{x} \le x < x^+ \\ +\infty & \text{if } x \ge x^+ \end{cases} \tag{58}$$

and:

$$\boldsymbol{c}''(x) = \begin{cases} s'' + \alpha''x & \text{if } x < x^+ \\ +\infty & \text{if } x \ge x^+ \end{cases} \tag{59}$$

If we assume that $\boldsymbol{c}'(0) = \boldsymbol{c}''(0)$ and $\boldsymbol{c}'(x^+) = \boldsymbol{c}''(x^+)$, we have the following relationships:

$$\alpha' = \alpha''\left(\frac{x^+}{x^+ - \tilde{x}}\right)$$

and:

$$\alpha'' = \alpha'\left(\frac{x^+ - \tilde{x}}{x^+}\right)$$

However, most of the time, we do not know the two analytical functions. Let us assume that the true model is given by $\boldsymbol{c}'(x)$, whereas we estimate the approximated model $\hat{\boldsymbol{c}}''(x)$, which is defined by:

$$\hat{\boldsymbol{c}}''(x) = \begin{cases} \hat{s}'' + \hat{\alpha}''x & \text{if } x < x^+ \\ +\infty & \text{if } x \ge x^+ \end{cases} \tag{60}$$

The least square estimates $\hat{s}''$ and $\hat{\alpha}''$ are equal to:

$$\hat{s}'' = \bar{\boldsymbol{c}}'(x) - \hat{\alpha}''\bar{x}$$

and:

$$\hat{\alpha}'' = \frac{\int_0^{x^+}(x - \bar{x})(\boldsymbol{c}'(x) - \bar{\boldsymbol{c}}'(x))\,\mathrm{d}x}{\int_0^{x^+}(x - \bar{x})^2\,\mathrm{d}x}$$

where $\bar{x}$ and $\bar{\boldsymbol{c}}'(x)$ are given by the mean value theorem:

$$\bar{x} = \frac{\int_0^{x^+} x\,\mathrm{d}x}{x^+} = \frac{x^+}{2}$$

and:

$$\bar{\boldsymbol{c}}'(x) = \frac{\int_0^{x^+}\boldsymbol{c}'(x)\,\mathrm{d}x}{x^+} = s' + \alpha'\frac{(x^+ - \tilde{x})^2}{2x^+}$$

We deduce that the least square estimates are:

$$\hat{\alpha}'' = \alpha'\left(1 + 2\left(\frac{\tilde{x}}{x^+}\right)^3 - 3\left(\frac{\tilde{x}}{x^+}\right)^2\right)$$

and:

$$\hat{s}'' = s' - \alpha'\tilde{x}\left(1 - \frac{\tilde{x}}{x^+}\right)^2$$

because we have:

$$
\begin{aligned}
\int_0^{x^+} (x - \bar{x})^2 \, \mathrm{d}x &= \int_0^{x^+} \left(x - \frac{x^+}{2} \right)^2 \mathrm{d}x \\
&= \frac{1}{3} \left[\left(x - \frac{x^+}{2} \right)^3 \right]_0^{x^+} \\
&= \frac{2}{3} \left(\frac{x^+}{2} \right)^3
\end{aligned}
$$

and[58]:

$$
\begin{aligned}
(*) &= \int_0^{x^+} (x - \bar{x}) \left(\boldsymbol{c}'(x) - \bar{c}'(x) \right) \mathrm{d}x \\
&= \int_0^{\tilde{x}} \left(x - \frac{x^+}{2} \right) \left(s' - s' - \alpha' \frac{(x^+ - \tilde{x})^2}{2x^+} \right) \mathrm{d}x + \\
&\quad \int_{\tilde{x}}^{x^+} \left(x - \frac{x^+}{2} \right) \left(s' + \alpha'(x - \tilde{x}) - s' - \alpha' \frac{(x^+ - \tilde{x})^2}{2x^+} \right) \mathrm{d}x \\
&= \alpha' \int_{\tilde{x}}^{x^+} \left(x - \frac{x^+}{2} \right) (x - \tilde{x}) \, \mathrm{d}x - \alpha' \frac{(x^+ - \tilde{x})^2}{2x^+} \int_0^{x^+} \left(x - \frac{x^+}{2} \right) \mathrm{d}x \\
&= \alpha' \int_{\tilde{x}}^{x^+} \left(x^2 - \left(\frac{2\tilde{x} + x^+}{2} \right) x + \frac{\tilde{x}x^+}{2} \right) \mathrm{d}x \\
&= \alpha' \left[\frac{x^3}{3} - \left(\frac{2\tilde{x} + x^+}{4} \right) x^2 + \frac{\tilde{x}x^+}{2} x \right]_{\tilde{x}}^{x^+} \\
&= \alpha' \left(\frac{1}{12} (x^+)^3 + \frac{1}{6} \tilde{x}^3 - \frac{1}{4} \tilde{x}^2 x^+ \right)
\end{aligned}
$$

In Figure 28, we illustrate how to transform one form of cost function into another form. In practice, we do not know the models $\boldsymbol{c}'(x)$ and $\boldsymbol{c}''(x)$. In fact, we estimate $\hat{\boldsymbol{c}}''(x)$. The right issue is then to transform $\hat{\boldsymbol{c}}''(x)$ into $\boldsymbol{c}'(x)$ or even $\boldsymbol{c}''(x)$. If we consider that the true model is $\boldsymbol{c}'(x)$, we have the following relationships:

$$
\alpha' = \hat{\alpha}'' \frac{(x^+)^3}{\left((x^+)^3 + 2\tilde{x}^3 - 3\tilde{x}^2 x^+ \right)} \tag{61}
$$

and:

$$
\alpha'' = \hat{\alpha}'' \frac{(x^+)^2 (x^+ - \tilde{x})}{\left((x^+)^3 + 2\tilde{x}^3 - 3\tilde{x}^2 x^+ \right)} \tag{62}
$$

If the true model is $\boldsymbol{c}''(x)$, we have $\alpha'' = \hat{\alpha}''$.

Remark 30 *In Figure 28, the parameters are equal to $s' = 2$ bps, $\alpha' = 2\%$, $\tilde{x} = 2\%$ and $x^+ = 8\%$. We find that $\alpha'' = 1.5\%$, while the OLS estimation gives $\hat{s}'' = -0.25$ bps and $\hat{\alpha}'' = 1.6875\%$.*

---

[58]We have:

$$
\int_0^{x^+} \left(x - \frac{x^+}{2} \right) \mathrm{d}x = 0
$$

Figure 28: Equivalence of cost models
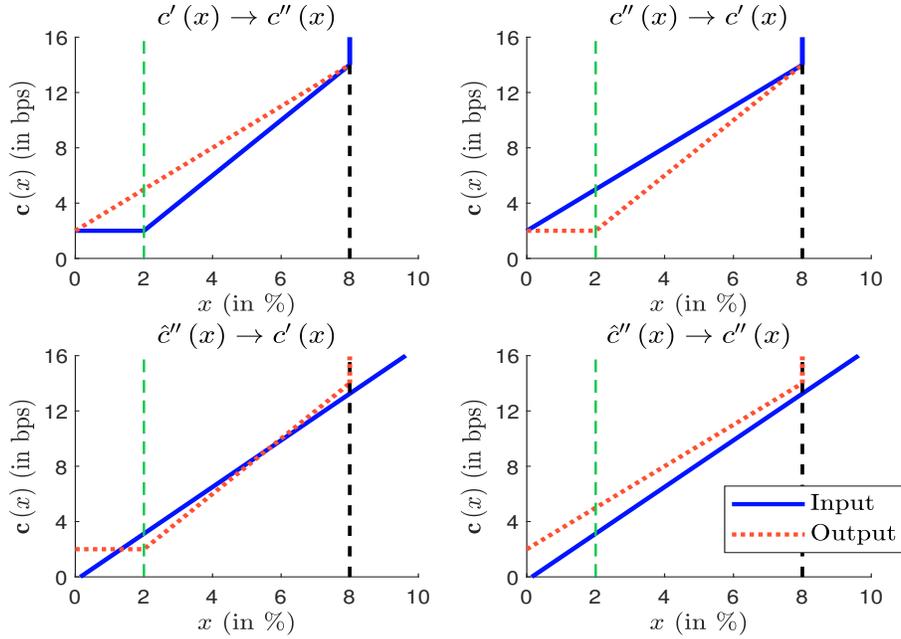
B.2 Analytics of portfolio distortion

B.2.1 Portfolio weights

We recall that the asset structure of the fund is given by the portfolio $\omega = (\omega_1, \ldots, \omega_n)$, where ω_i is the number of shares of security i. The portfolio weights are then equal to $w(\omega) = (w_1(\omega), \ldots, w_n(\omega))$ where:

$$w_i(\omega) = \frac{\omega_i P_i}{\sum_{j=1}^{n} \omega_j P_j} \tag{63}$$

and P_i is the current price of security i. Let $q = (q_1, \ldots, q_n)$ be the redemption scenario. It follows that the redemption weights are given by:

$$w_i(q) = \frac{q_i P_i}{\sum_{j=1}^{n} q_j P_j} \tag{64}$$

After the liquidation of q, the new asset structure is equal to $\omega - q$, and the new weights of the portfolio become:

$$w_i(\omega - q) = \frac{(\omega_i - q_i) P_i}{\sum_{j=1}^{n} (\omega_j - q_j) P_j} \tag{65}$$

We note $\mathbb{V}(\omega) = \sum_{j=1}^{n} \omega_j P_j$ and $\mathbb{V}(\omega - q) = \sum_{j=1}^{n} (\omega_j - q_j) P_j$ the dollar value of the portfolios before and after the liquidation. We notice that $\mathbb{V}(\omega) - \mathbb{V}(\omega - q)$ is exactly equal to the dollar value $\mathbb{R}$ of the redemption:

$$\mathbb{R} = \mathbb{V}(\omega) - \mathbb{V}(\omega - q) = \sum_{j=1}^{n} q_j P_j = \mathbb{V}(q)$$

We have:

$$
\begin{aligned}
w_i \left(\omega - q \right) & = \frac{\omega_i P_i}{\mathbb{V} \left(\omega \right) - \mathbb{R}} - \frac{q_i P_i}{\mathbb{V} \left(\omega \right) - \mathbb{R}} \\
& = \frac{\mathbb{V} \left(\omega \right)}{\mathbb{V} \left(\omega \right) - \mathbb{R}} w_i \left(\omega \right) - \frac{\mathbb{R}}{\mathbb{V} \left(\omega \right) - \mathbb{R}} w_i \left(q \right)
\end{aligned}
$$

The new weights $w_i \left(\omega - q \right)$ are a non-linear function of the portfolio weights $w_i \left(\omega \right)$, the redemption weights $w_i \left(q \right)$ and the redemption value $\mathbb{R}$. Except in the case[59] where $q_i \propto \omega_i$, computing the new weights is not straightforward because they depend on $\mathbb{R}$. From a theoretical point of view, we have $0 \leq q_i \leq \omega_i$ because the maximum we can sell is the number of shares in the portfolio. One problem is that the weights $w_i \left(\omega - q \right)$ are continuous whereas the number of shares q_i is an integer. This is why we prefer to consider the fuzzy constraint $-\epsilon \leq q_i \leq \omega_i + \epsilon$, where ϵ is typically equal to $1/2$. Since $\sum_{i=1}^{n} w_i \left(\omega - q \right) = 1$, we deduce that:

$$
-\epsilon \leq q_i \leq \omega_i + \epsilon \Leftrightarrow -\epsilon_i \leq w_i \left(\omega - q \right) \leq \min \left(\frac{\mathbb{V} \left(\omega \right)}{\mathbb{V} \left(\omega \right) - \mathbb{R}} w_i \left(\omega \right) + \epsilon_i, 1 \right)
$$

where:

$$
\epsilon_i = \frac{\epsilon P_i}{\mathbb{V} \left(\omega \right) - \mathbb{R}}
$$

We note the two bounds $w_i^{-} \left(\omega - q \right)$ and $w_i^{+} \left(\omega - q \right)$.

Remark 31 *From Equation (65), we deduce that:*

$$
q_i = \frac{\mathbb{V} \left(\omega \right) \left(w_i \left(\omega \right) - w_i \left(\omega - q \right) \right) + \mathbb{R} w_i \left(\omega - q \right)}{P_i}
$$

We can then compute q_i thanks to the previous equation when we know the portfolios weights $w_i \left(\omega \right)$ and $w_i \left(\omega - q \right)$.

B.2.2 Liquidation tracking error

We assume that the asset returns are normally distributed: $R = \left(R_1, \ldots, R_n \right) \sim \mathcal{N} \left(0, \Sigma \right)$. The random return of the portfolio ω is then equal to:

$$
\begin{aligned}
R \left(\omega \right) & = \frac{\sum_{i=1}^{n} \omega_i P_i R_i}{\sum_{j=1}^{n} \omega_j P_j} \\
& = \sum_{i=1}^{n} w_i \left(\omega \right) R_i \\
& = w_i \left(\omega \right)^{\top} R
\end{aligned}
$$

We conclude that:

$$
R \left(\omega \right) \sim \mathcal{N} \left(0, w \left(\omega \right)^{\top} \Sigma w \left(\omega \right) \right)
$$

If we consider the portfolio $\omega - q$, we have $R \left(\omega - q \right) = w \left(\omega - q \right)^{\top} R$ and:

$$
\begin{pmatrix} R \left(\omega \right) \\ R \left(\omega - q \right) \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} w \left(\omega \right)^{\top} \Sigma w \left(\omega \right) & w \left(\omega \right)^{\top} \Sigma w \left(\omega - q \right) \\ w \left(\omega - q \right)^{\top} \Sigma w \left(\omega \right) & w \left(\omega - q \right)^{\top} \Sigma w \left(\omega - q \right) \end{pmatrix} \right)
$$

---

[59] We have $w_i \left(\omega - q \right) = w_i \left(\omega \right)$.

Let e be the tracking error between the portfolios before and after the redemption. We have:

$$
\begin{aligned}
e &= R\left(\omega - q\right) - R\left(\omega\right) \\
&= \left(w\left(\omega\right) - w\left(\omega - q\right)\right)^{\top} R
\end{aligned}
$$

The standard deviation of e is called the "*liquidation tracking error*" and is denoted by $\sigma\left(q \mid \omega\right)$:

$$
\sigma\left(q \mid \omega\right) = \sqrt{\left(w\left(\omega\right) - w\left(\omega - q\right)\right)^{\top} \Sigma \left(w\left(\omega\right) - w\left(\omega - q\right)\right)}
$$

This is our measure of the portfolio distortion $\mathcal{D}\left(q \mid \omega\right)$.

Remark 32 *In the case where the redemption scenario does not modify the asset structure, we have $q_i = \mathcal{R}\omega_i$ and:*

$$
\begin{aligned}
w\left(\omega - q\right) &= \frac{\left(\omega_i - q_i\right) P_i}{\sum_{j=1}^{n}\left(\omega_j - q_j\right) P_j} \\
&= \frac{\left(\omega_i - \mathcal{R}\omega_i\right) P_i}{\sum_{j=1}^{n}\left(\omega_j - \mathcal{R}\omega_j\right) P_j} \\
&= \frac{\left(1 - \mathcal{R}\right) \omega_i P_i}{\sum_{j=1}^{n}\left(1 - \mathcal{R}\right) \omega_j P_j} \\
&= w_i
\end{aligned}
$$

We conclude that the portfolio distortion is equal to zero.

B.2.3 Optimal portfolio liquidation

Let $c\left(q \mid \omega\right)$ be the cost of liquidating the redemption scenario q. The problem of optimal portfolio liquidation is:

$$
\begin{aligned}
q^{\star} &= \arg\min_{q} c\left(q \mid \omega\right) & (66) \\
\text{s.t.} & \begin{cases} \sigma\left(q \mid \omega\right) \leq \mathcal{D}^{+} \\ \mathbf{1}_n^{\top} w\left(\omega - q\right) = 1 \\ w^{-}\left(\omega - q\right) \leq w\left(\omega - q\right) \leq w^{+}\left(\omega - q\right) \end{cases}
\end{aligned}
$$

where $\mathcal{D}^{+} \geq 0$ is the maximum portfolio distortion. If $\mathcal{D}^{+} = 0$, the optimal solution is $q^{\star} \propto \omega$. If $\mathcal{D}^{+} = \infty$, the distortion constraint vanishes, and the solution corresponds to the redemption scenario that presents the lower liquidating cost.

We can rewrite the previous problem as follows:

$$
\begin{aligned}
q^{\star}\left(\lambda\right) &= \arg\min \frac{1}{2}\sigma^2\left(q \mid \omega\right) + \lambda c\left(q \mid \omega\right) & (67) \\
\text{s.t.} & \begin{cases} \mathbf{1}_n^{\top} w\left(\omega - q\right) = 1 \\ w^{-}\left(\omega - q\right) \leq w\left(\omega - q\right) \leq w^{+}\left(\omega - q\right) \end{cases}
\end{aligned}
$$

This optimization problem is close to the γ-problem of mean-variance optimization (Roncalli, 2013). Nevertheless, this is not a QP problem, meaning that it is more complex to solve numerically. The underlying idea is then to write q as a function of $w\left(q\right)$ with $q_i = w_i\left(q\right) \mathbb{R}/P_i$ and minimizing the objective function (67) with respect to $w\left(q\right)$. Given a dollar value $\mathbb{R}$ of redemption, the set of optimal portfolio liquidations is given by $\{q^{\star}\left(\lambda\right), \lambda \in [0, \infty)\}$ and the efficient frontier corresponds to the parametric curve $\left(\sigma\left(q^{\star}\left(\lambda\right) \mid \omega\right), c\left(q^{\star}\left(\lambda\right) \mid \omega\right)\right)$.

B.3   Modeling the market risk of corporate bonds

Let $\mathfrak{s}_i(t)$ be the credit spread of the ith bond issuer. Following Roncalli (2013, pages 223-227), we assume that the credit spread follows a general diffusion process:

$$\mathrm{d}\mathfrak{s}_i(t) = \sigma_i^{\mathfrak{s}} \mathfrak{s}_i(t) \ \mathrm{d}W_i(t) \tag{68}$$

where $W_i(t)$ is a standard Brownian motion and $\sigma_i^{\mathfrak{s}}$ is a volatility parameter. We note $B_i(t, D_i)$ the zero-coupon bond price with maturity (or duration) D_i of the ith issuer. If we assume that the recovery date is equal to zero, we have:

$$\mathrm{d}\ln B_i(t, D_i) = -D_i \, \mathrm{d}r(t) - D_i \, \mathrm{d}\mathfrak{s}_i(t)$$

where $r(t)$ is the risk-free interest rate. If we assume that the credit spread is not correlated with the risk-free interest rate, we deduce that:

$$\begin{aligned} \sigma^2\left(\mathrm{d}\ln B_i(t, D_i)\right) &= D_i^2 \sigma^2\left(\mathrm{d}r(t)\right) + D_i^2 \sigma^2\left(\mathrm{d}\mathfrak{s}_i(t)\right) \\ &= D_i^2 \sigma^2\left(\mathrm{d}r(t)\right) + D_i^2 \left(\sigma_i^{\mathfrak{s}}\right)^2 \mathfrak{s}_i^2(t) \, \mathrm{d}t \end{aligned} \tag{69}$$

We deduce that the volatility of a bond has two parts: an interest rate component and a credit spread component.

If the credit risk component is sufficiently large with respect to the interest rate component, we obtain:

$$\begin{aligned} \sigma\left(\mathrm{d}\ln B_i(t, D_i)\right) &\approx \sigma_i^{\mathfrak{s}} \cdot D_i \cdot \mathfrak{s}_i(t) \\ &= \sigma_i^{\mathfrak{s}} \cdot \mathrm{DTS}_i(t) \end{aligned} \tag{70}$$

where $\mathrm{DTS}_i(t)$ is the duration-times-spread (or DTS) measure (Ben Dor *et al.*, 2007).

C Data

We consider the asset liquidity data provided by Amundi Asset Management. The database is called "*Amundi Liquidity Lab*" and contains the trades made by Amundi, but also other information such as order books for equities and the price quotations for bonds[60]. We filter the data in order to obtain a dataset with all the available characteristics, which are representative of normal trading. For instance, we exclude bond trades that are initiated by the counterparty. We also exclude equity trades that are made by an index fund manager when the transaction concerns a basket of stocks that replicate the index. Indeed, in this case, the transaction cost is generally related to the index, and does not necessarily reflect the transaction cost of each component. Finally, we use a subset of the data.

C.1 Equities

We use a sample of trades for the stocks that belong to the MSCI USA, MSCI Europe, MSCI USA Small Cap and MSCI Europe Small Cap indices. We also complete this database with pre-trade transaction costs computed by the BECS system (Citigroup, 2020) when we observe few observations for a given stock. Finally, we have a sample of 149 896 trades.

C.2 Sovereign bonds

We use a sample of 196 286 trades from January 2018 to December 2020 with the following split by currency:

Currency	EUR	USD	GBP	JPY	AUD	CAD	DKK
# of trades	129 904	34 965	7 354	6 831	4 277	3 586	1 409
Currency	SEK	MXN	PLN	MYR	SGD	ZAR	Other
# of trades	915	882	794	592	581	458	3 738

and the following split by the issuer's country:

Country	IT	FR	US	DE	ES	BE	GB
# of trades	31 870	23 033	20 798	19 587	16 668	8 961	7 646
Country	JP	NL	AT	AU	CA	PT	Other
# of trades	6 874	6 663	6 619	4 383	3 950	3 900	35 334

C.3 Corporate bonds

We use a sample of 258 153 trades from January 2018 to December 2020 with the following split by currency:

Currency	EUR	USD	GBP	SGD	AUD	CAD	CNH	Other
# of trades	204 724	46 620	5 791	307	194	138	128	251

and the following split by the issuer's country:

Country	US	FR	NL	GB	DE	IT	LU
# of trades	49 410	48 257	34 782	21 710	16 358	16 037	12 150
Country	ES	SE	IE	MX	AT	BE	Other
# of trades	11 797	5 857	4 775	3 799	3 289	3 173	27 709

---

[60]For each trade, we have at least three price quotations by three different banks and brokers.

D Price impact of the benchmark formulas

Table 33: Price impact (in bps) for large cap stocks

σ (in %)	x (in %)								
	0.01	0.05	0.10	0.50	1	5	10	20	30
10	0.2	0.6	0.8	1.8	2	6	8	11	14
20	0.5	1.1	1.6	3.5	5	11	16	22	27
30	0.7	1.7	2.4	5.3	7	17	24	33	41
40	1.0	2.2	3.1	7.0	10	22	31	44	54
50	1.2	2.8	3.9	8.8	12	28	39	55	68
60	1.5	3.3	4.7	10.5	15	33	47	67	82

Table 34: Price impact (in bps) for small cap stocks

σ (in %)	x (in %)								
	0.01	0.05	0.10	0.50	1	5	10	20	30
10	0.3	0.7	1.0	2.2	3	7	10	14	17
20	0.6	1.4	2.0	4.4	6	14	20	28	34
30	0.9	2.1	2.9	6.6	9	21	29	42	51
40	1.2	2.8	3.9	8.8	12	28	39	55	68
50	1.6	3.5	4.9	11.0	16	35	49	69	85
60	1.9	4.2	5.9	13.2	19	42	59	83	102

Table 35: Price impact (in bps) for sovereign bonds

σ (in %)	y (in bps)								
	0.01	0.10	1	2.5	5	10	20	50	100
1	0.6	1.0	1.9	2.3	2.8	3	4	5	6
2	1.2	2.1	3.7	4.7	5.6	7	8	10	12
3	1.8	3.1	5.6	7.0	8.3	10	12	15	18
5	2.9	5.2	9.3	11.7	13.9	17	20	25	29
10	5.9	10.5	18.6	23.4	27.8	33	39	49	59
15	8.8	15.7	27.9	35.1	41.7	50	59	74	88
20	11.8	20.9	37.2	46.8	55.6	66	79	99	118

Table 36: Price impact (in bps) for corporate bonds

DTS (in bps)	y (in bps)								
	0.01	0.10	1	2.5	5	10	20	50	100
50	0.2	0.4	0.6	0.8	0.9	1	1	2	2
100	0.4	0.7	1.3	1.6	1.9	2	3	3	4
250	1.0	1.8	3.1	3.9	4.7	6	7	8	10
500	2.0	3.5	6.3	7.9	9.3	11	13	17	20
1 000	4.0	7.0	12.5	15.7	18.7	22	26	33	40
2 500	9.9	17.6	31.3	39.3	46.7	56	66	83	99
5 000	19.8	35.1	62.5	78.6	93.5	111	132	166	198

E Additional results

Figure 29: Linear modeling of unit transaction costs
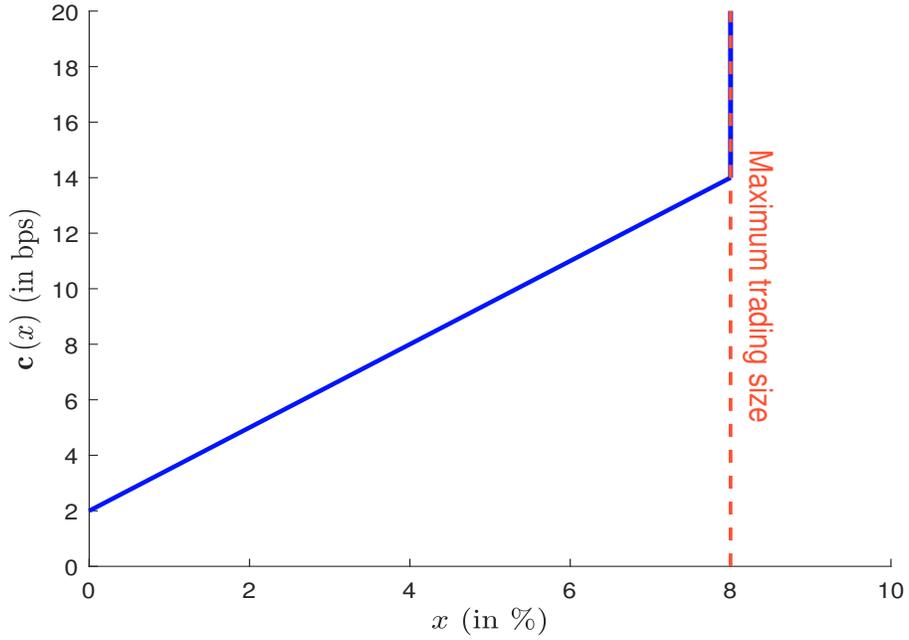
Figure 30: Comparing unit and total transaction costs in normal and stress periods
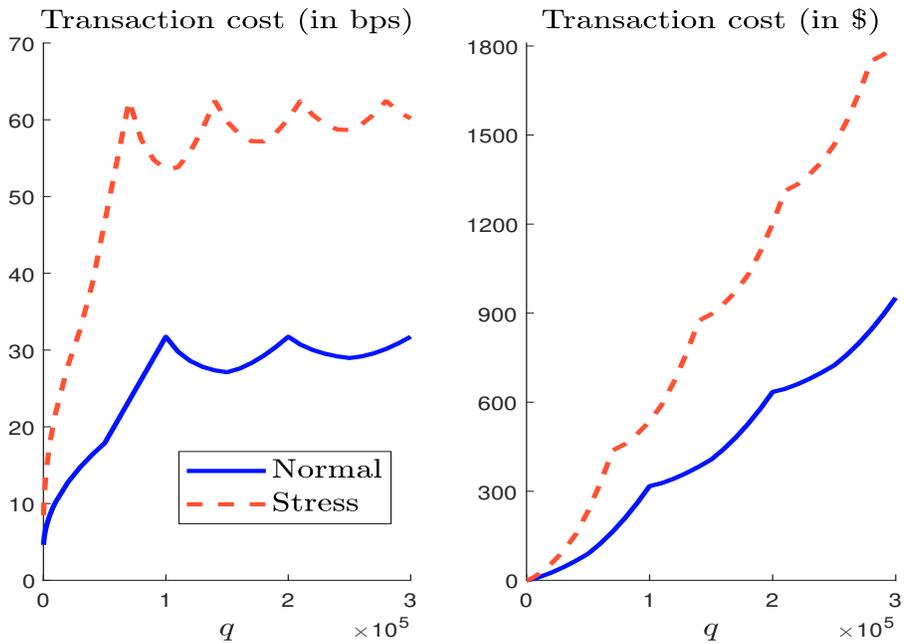
Table 37: Participation rate $x_i(h)$ (in %)

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5
1	10.00%	10.00%	10.00%	8.75%	0.90%
2	10.00%	10.00%	10.00%		
3	10.00%	0.05%	10.00%		
4	10.00%		7.75%		
5	3.51%				

Table 38: Notional $Q_i(h)$ (in $)

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5
1	89 000	102 000	13 400	20 825	10 602
2	89 000	102 000	13 400		
3	89 000	510	13 400		
4	89 000		10 385		
5	31 239				

Table 39: Bid-ask spread cost (in $)

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	Total
1	35.60	40.80	6.70	10.41	5.30	98.81
2	35.60	40.80	6.70			83.10
3	35.60		6.70			42.50
4	35.60		5.19			40.79
5	12.50					12.50
Total	154.90	81.80	25.29	10.41	5.30	277.71

Table 40: Price impact cost (in $)

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	Total
1	617.10	565.79	66.90	151.62	12.48	1 413.89
2	617.10	565.79	66.90			1 249.80
3	617.10	0.14	66.90			684.14
4	617.10		40.18			657.28
5	90.74					90.74
Total	2 559.16	1 131.73	240.87	151.62	12.48	4 095.85

Table 41: Transaction cost (in $)

h	Asset #1	Asset #2	Asset #3	Asset #4	Asset #5	Total
1	652.70	606.59	73.60	162.03	17.78	1 512.70
2	652.70	606.59	73.60			1 332.90
3	652.70	0.35	73.60			726.65
4	652.70		45.37			698.08
5	103.24					103.24
Total	2 714.05	1 213.53	266.16	162.03	17.78	4 373.55

Figure 31: Estimated price impact (in bps) — logarithmic scale
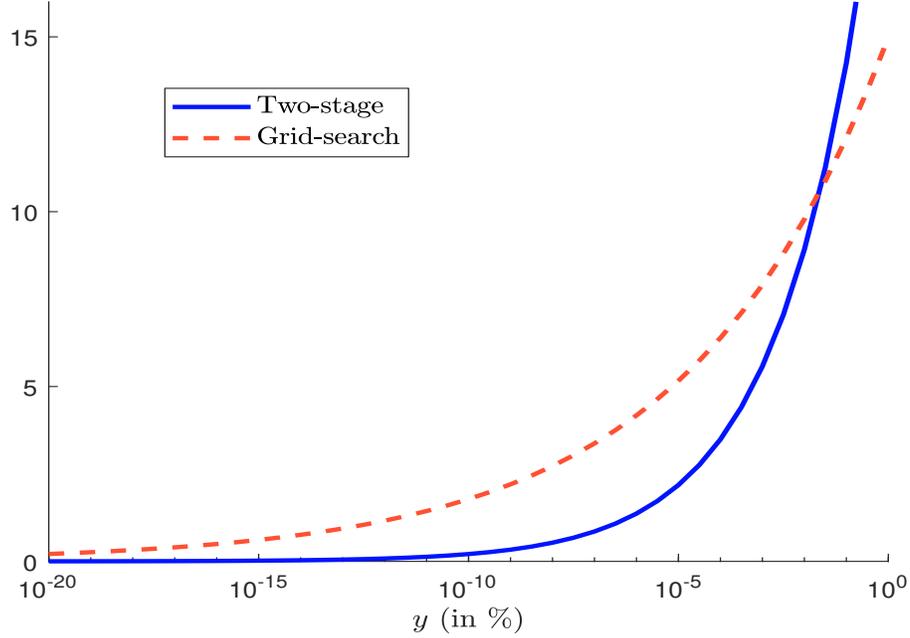
Table 42: Two-stage estimation of the sovereign bond transaction cost model without the intercept by issuer

Issuer	γ_1	$\beta^{(s)}$	$\tilde{\beta}^{(\pi)}$	R^2 (in %)	R_c^2 (in %)
Austria	0.2255	0.8023	3.0845	53.9	48.2
Belgium	0.2482	0.7789	3.3738	44.0	32.5
EM	0.0519	0.9158	0.4746	73.6	44.7
Finland	0.2894	0.7114	4.0416	46.3	31.8
France	0.2138	0.8942	3.0148	40.1	29.7
Germany	0.2415	1.0413	2.7838	51.5	38.5
Ireland	0.2098	0.6600	2.4977	43.8	26.4
Italy	0.1744	2.4706	1.7640	31.0	22.0
Japan	0.0657	0.5635	0.7315	78.0	53.4
Netherlands	0.2320	0.7219	3.7355	46.9	34.2
Portugal	0.2318	0.9693	3.0639	49.6	33.0
Spain	0.2185	1.3000	2.0990	40.8	26.7
United Kingdom	0.2194	0.9739	2.6262	49.9	28.5
USA	0.1252	1.1055	1.3395	53.6	40.7

Table 43: Two-stage estimation of the sovereign bond transaction cost model without the intercept by currency

Currency	γ_1	$\beta^{(s)}$	$\tilde{\beta}^{(\pi)}$	R^2 (in %)	R_c^2 (in %)
EUR	0.2262	1.0428	2.9347	35.2	25.7
GBP	0.2117	1.5328	2.2890	48.3	29.5
JPY	0.0834	0.5744	0.9771	74.2	48.2
USD	0.1408	0.9502	1.0906	60.4	45.4

Figure 32: Relationship between volatility and duration-times-spread (sovereign bonds)
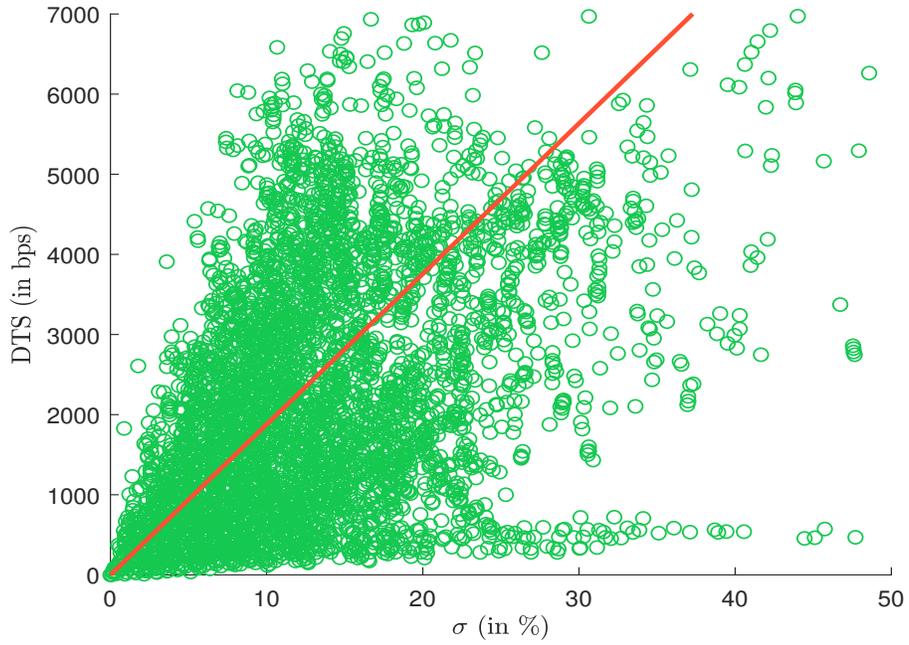
Figure 33: Empirical distribution of the additive factor Δ_σ
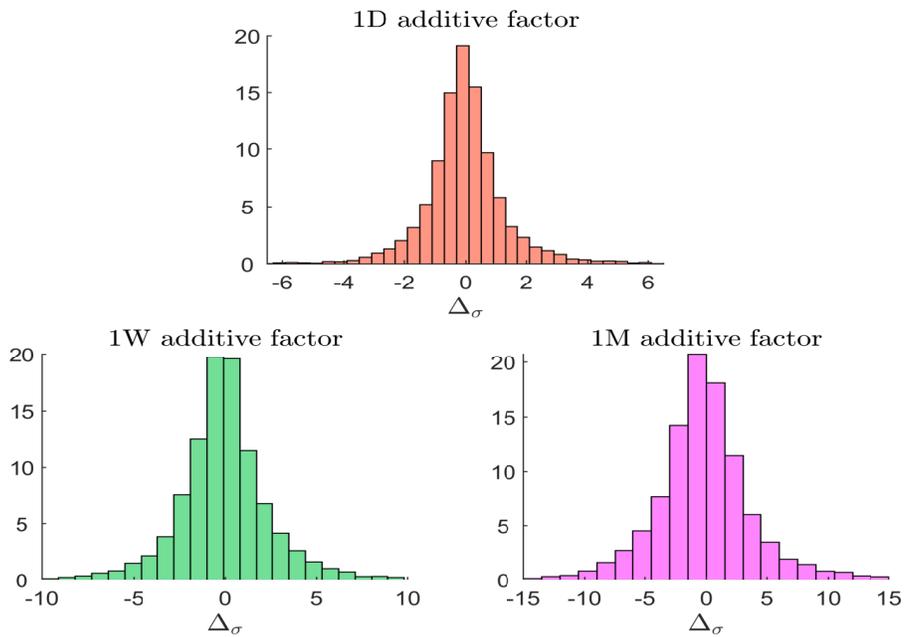
Figure 34: Empirical distribution of the multiplicative factor m_v
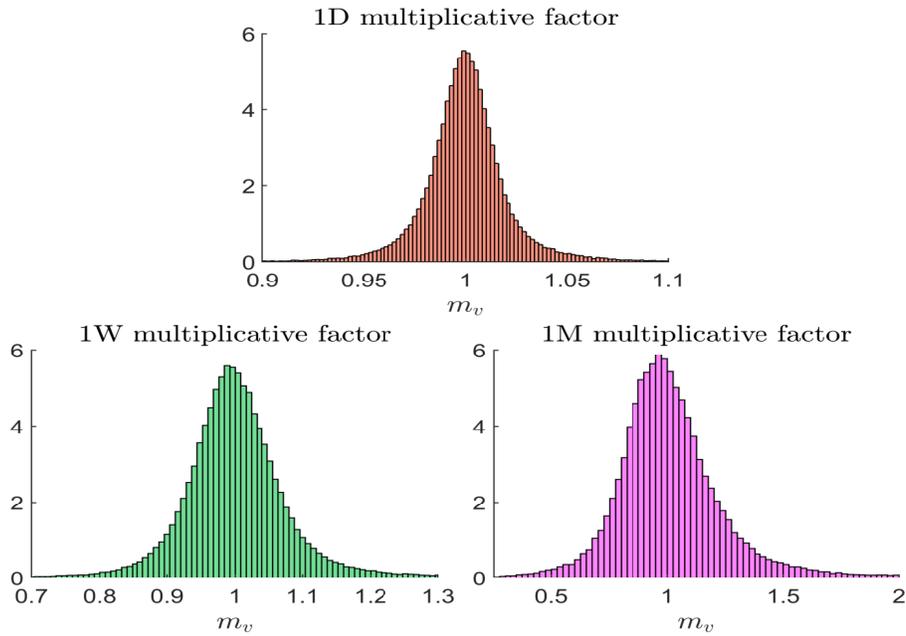
Figure 35: Empirical distribution of the multiplicative factor m_s
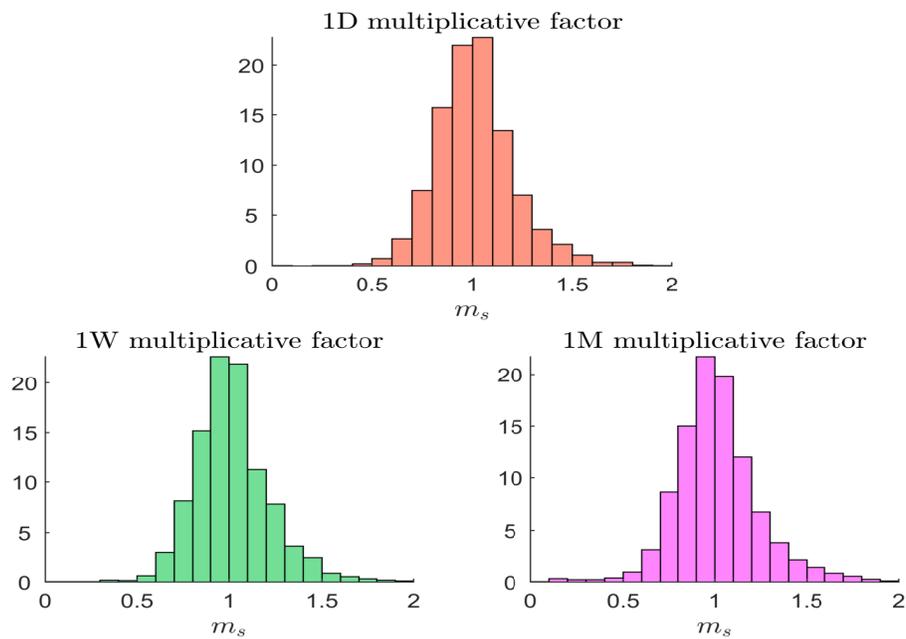
Figure 36: Empirical distribution of the additive factor Δ_s
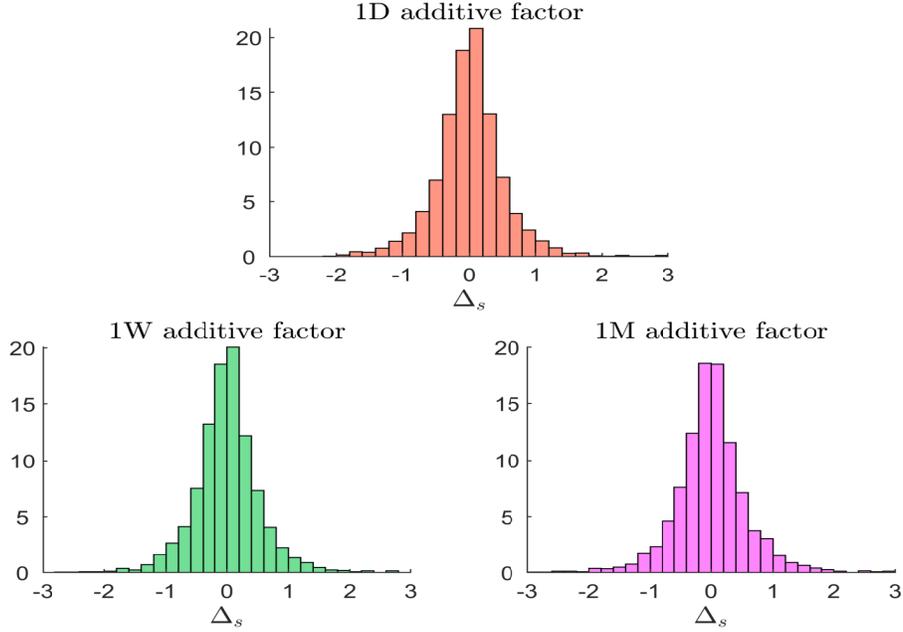
Table 44: Stress scenarios of the participation rate

			0.385	1/2	1	2	5	10	50
	$\mathcal{T}$ (in years)		0.385	1/2	1	2	5	10	50
	α (in %)		99.00	99.23	99.62	99.81	99.92	99.96	99.99
	Empirical		1.26	1.30	1.39	1.49	1.64	1.81	2.07
	BM/GEV	Pooling	1.16	1.18	1.23	1.28	1.35	1.40	1.53
1W	POT/GPD	Pooling	1.15	1.20	1.33	1.46	1.64	1.78	2.12
	BM/GEV	Averaging	1.14	1.16	1.19	1.22	1.27	1.30	1.37
	POT/GPD	Averaging	1.23	1.24	1.26	1.28	1.31	1.34	1.39
	Empirical		1.99	2.10	2.45	2.81	3.27	3.49	3.79
	BM/GEV	Pooling	1.39	1.45	1.61	1.78	1.99	2.15	2.55
1M	POT/GPD	Pooling	2.53	2.60	2.80	2.99	3.25	3.45	3.90
	BM/GEV	Averaging	1.34	1.38	1.48	1.58	1.71	1.81	2.04
	POT/GPD	Averaging	1.62	1.65	1.75	1.85	1.98	2.10	2.38

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