Risk Parity: A (New) Tool for Asset Management[1]

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[1]The materials used in these slides are taken from RONCALLI T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.

[2]The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.

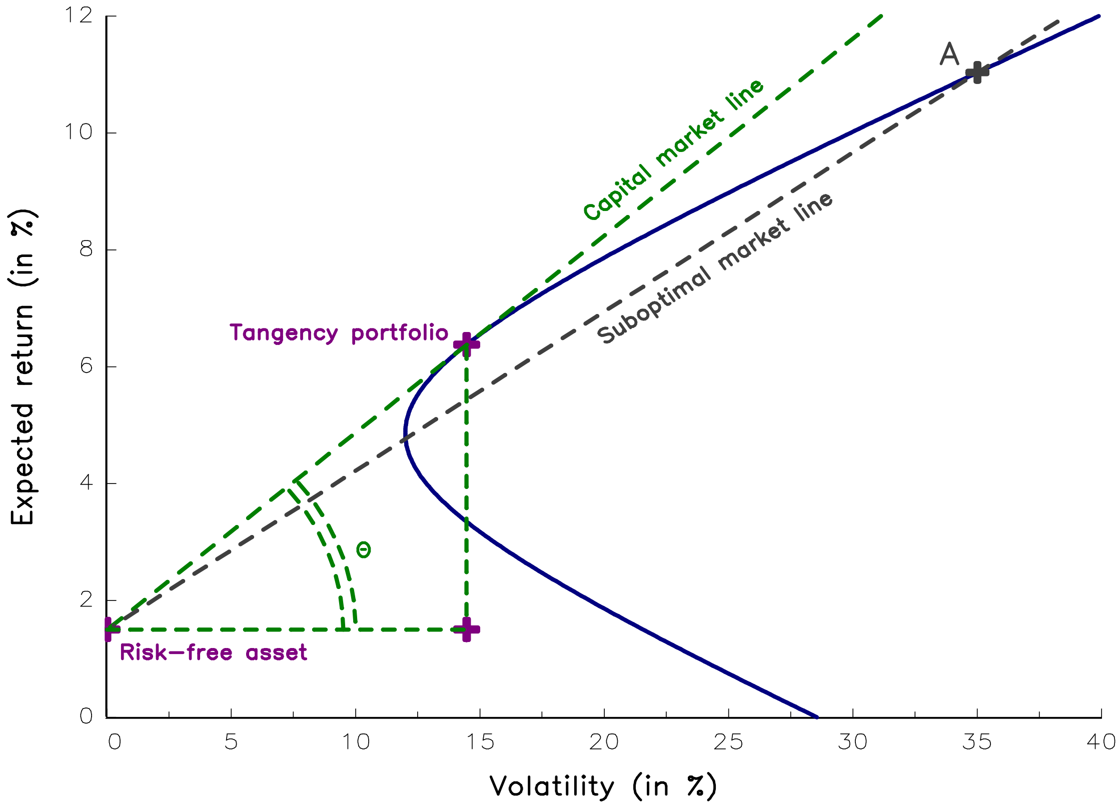
Outline

Some issues on portfolio optimization

- Portfolio optimization and asset management
- Stability issues
- Why regularization techniques are not sufficient
- The impact of the weight constraints

Portfolio optimization and passive management

We consider a universe of n assets. Let μ and Σ be the vector of expected returns and the covariance matrix of returns.



- The Markowitz portfolios are defined by:

$$\max \mu(x) = \mu^\top x$$
$$\text{u.c.} \quad \sigma(x) = \sqrt{x^\top \Sigma x} = \sigma^\star$$

- Tobin (1958) shows that the tangency portfolio dominates all the other optimized portfolios.
- If the market is efficient, the tangency portfolio is the market-cap portfolio (Sharpe, 1964).

We don't need portfolio optimization!!!

Portfolio optimization and active management

For active management, portfolio optimization continues to make sense.

However...

> *"The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with MV optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained MV optimization can yield results that are inferior to those of simple equal-weighting schemes" (Michaud, 1989).*

Are optimized portfolios optimal?

Stability issues
An illustration

- We consider a universe of 3 assets.
- The parameters are: $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.
- The objective is to maximize the expected return for a 15% volatility target.
- The optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$.

What is the sensitivity to the input parameters?

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	38.3	44.6	13.7	−8.0	60.6
x_2	20.2	25.9	8.9	56.1	74.1	−5.4
x_3	41.5	35.8	46.5	30.2	34.0	44.8

Stability issues

Solutions

In order to stabilize the optimal portfolio, we have to introduce some regularization techniques:

- regularization of the objective function by using resampling techniques
- regularization of the covariance matrix:
  - Factor analysis
  - Shrinkage methods
  - Random matrix theory
  - etc.
- regularization of the program specification by introducing some weight constraints

Why regularization techniques are not sufficient

On the importance of the information matrix

Optimized portfolios are solutions of the following quadratic program:

$$
\begin{aligned}
x^{\star} &= \arg\max x^{\top}\mu - \frac{\phi}{2} x^{\top}\Sigma x \\
\text{u.c.} &\quad \left\{ \begin{array}{l} \mathbf{1}^{\top}x = 1 \\ x \in \mathbb{R}^n \end{array} \right.
\end{aligned}
$$

We have:

$$
x^{\star}(\phi) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}} + \frac{1}{\phi} \cdot \frac{\left(\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}\right)\Sigma^{-1}\mu - \left(\mathbf{1}^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}}
$$

Optimal solutions are of the following form: $x^{\star} \propto f\left(\Sigma^{-1}\right)$.

The important quantity is then the information matrix $\mathcal{I} = \Sigma^{-1}$ and the eigendecomposition of $\mathcal{I}$ is:

$$
V_i(\mathcal{I}) = V_{n-i}(\Sigma) \quad \text{and} \quad \lambda_i(\mathcal{I}) = \frac{1}{\lambda_{n-i}(\Sigma)}
$$

Why regularization techniques are not sufficient

An illustration

We consider the example of Slide 6:
$\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.

The eigendecomposition of the covariance and information matrices is:

Asset / Factor	Covariance matrix Σ 1	2	3	Information matrix $\mathcal{I}$ 1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

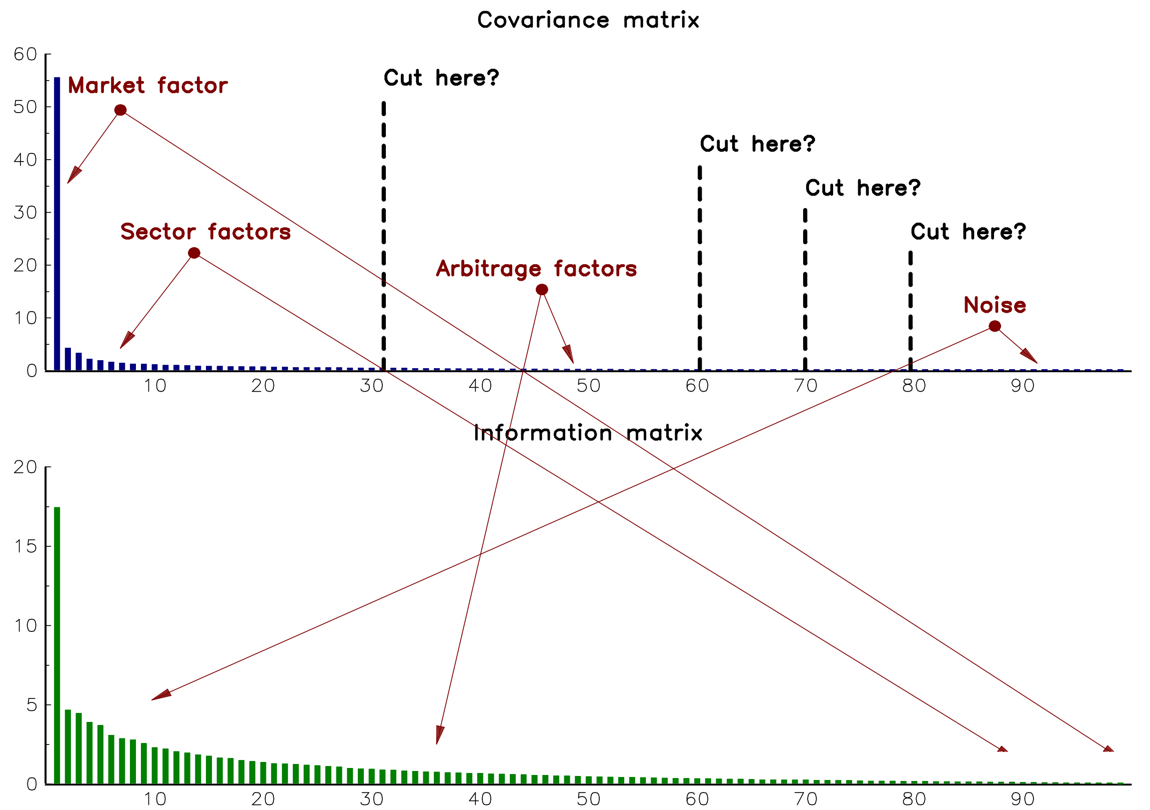
⇒ It means that the first factor of the information matrix corresponds to the last factor of the covariance matrix and that the last factor of the information matrix corresponds to the first factor.

⇒ Optimization on arbitrage risk factors, idiosyncratic risk factors and (certainly) noise factors!

Why regularization techniques are not sufficient

Working with a large universe of assets

Figure: Eigendecomposition of the FTSE 100 covariance matrix



⇒ Shrinkage is then necessary to eliminate the noise factors, but is not sufficient because it is extremely difficult to filter the arbitrage factors!

Shrinkage interpretation of weight constraints

The framework

We specify the optimization problem as follows:

$$\min \frac{1}{2} x^\top \Sigma x$$

$$\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^\top x = 1 \\ \mu^\top x \geq \mu^\star \\ x \in \mathcal{C} \end{array} \right.$$

where $\mathcal{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $x^\star$ or $x^\star(\mu, \Sigma)$:

$$\mathcal{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{x}$:

$$\mathcal{C}\left(x^-, x^+\right) = \left\{x \in \mathbb{R}^n : x_i^- \leq x_i \leq x_i^+\right\}$$

Shrinkage interpretation of weights constraints

Main result

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{x} = x^{\star}\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}^{\top} + \mathbf{1}\left(\lambda^{+} - \lambda^{-}\right)^{\top} \end{array} \right.$$

where λ^{-} and λ^{+} are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce a shrinkage method or to introduce some relative views (similar to the **Black-Litterman** approach).

Some examples

The minimum variance portfolio

Table: Specification of the covariance matrix Σ (in %)

σ_i	$\rho_{i,j}$			
15.00	100.00			
20.00	10.00	100.00		
25.00	40.00	70.00	100.00	
30.00	50.00	40.00	80.00	100.00

Given these parameters, the global minimum variance portfolio is equal to:

$$x^\star = \begin{pmatrix} 72.74\% \\ 49.46\% \\ -20.45\% \\ -1.75\% \end{pmatrix}$$

Some examples

The minimum variance portfolio

Table: Minimum variance portfolio when $x_i \geq 10\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.00	100.00			
23.805	0.000	0.000	20.00	10.00	100.00		
10.000	1.190	0.000	19.67	10.50	58.71	100.00	
10.000	1.625	0.000	23.98	17.38	16.16	67.52	100.00

Table: Minimum variance portfolio when $10\% \leq x_i \leq 40\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.915	20.20	100.00			
40.000	0.000	0.000	20.00	30.08	100.00		
10.000	0.915	0.000	21.02	35.32	61.48	100.00	
10.000	1.050	0.000	26.27	39.86	25.70	73.06	100.00

Myopic behavior of portfolio managers?

Weight constraints → Shrinkage methods

By using weight constraints, the portfolio manager may change (implicitly):

1. the value and/or the ordering of the volatilities;
2. the value, the sign and/or the ordering of the correlations;
3. the underlying assumption of the theory itself.

The question is then the following:

Is the portfolio manager aware of and in agreement with these changes?

The risk parity (or risk budgeting) approach

- Why the emergence of risk parity
- Definition
- Main properties
- Using risk factors instead of assets

Why the emergence of risk parity

- Use of Markowitz optimization:
  - Enhanced equity passive management $(+++)$
  - Equity active management $(+)$
  - Bond management $(---)$
  - Diversified multi-asset funds $(-)$
  - Absolute return funds $(+/++)$
  - Hedge funds $(+)$
  - Strategic asset allocation $(+++)$
- Alternative models: Black-Litterman, Robust optimization, etc.
- 2008 financial crisis & SAA
- The diversification puzzle

The rise of risk parity portfolios

- The place of risk management in asset management
- Be sensitive to Σ and not to Σ^{-1}
- The rise of heuristic approaches (EW, MV, ERC, MDP, etc.)
- A Marketing/Money Machine Battle ☺

Three methods to build a portfolio

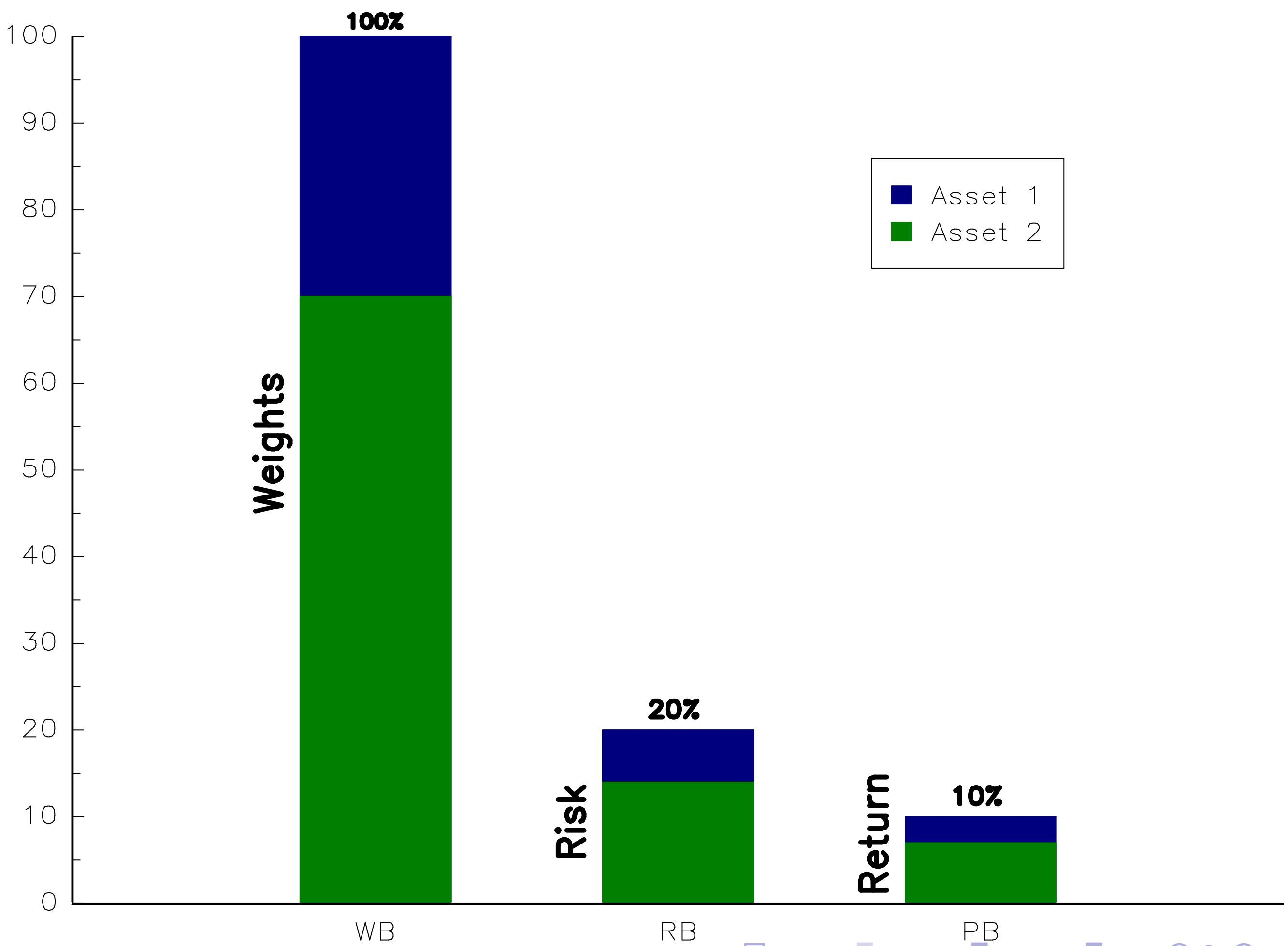
1. Weight budgeting (WB)
2. Risk budgeting (RB)
3. Performance budgeting (PB)

Ex-ante analysis
$\neq$
Ex-post analysis

Important result

RB = PB

Figure: The 30/70 rule



Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathcal{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned}\mathcal{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial \mathcal{R}(x_1, \ldots, x_n)}{\partial x_i} \\ &= \sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n)\end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

2. Risk budgeting (RB)

$$\mathrm{RC}_i = b_i \cdot \mathcal{R}(x_1, \ldots, x_n)$$

Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We assume that the risk measure $\mathcal{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$. We have:

$$\begin{aligned}
\frac{\partial\, \mathcal{R}(x)}{\partial\, x} &= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\
\mathrm{RC}_i(x_1, \ldots, x_n) &= x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
\sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sigma(x)
\end{aligned}$$

The risk budgeting portfolio is defined by this system of equations:

$$\begin{cases}
x_i \cdot (\Sigma x)_i = b_i \cdot \left(x^\top \Sigma x\right) \\
x_i \geq 0 \\
\sum_{i=1}^{n} x_i = 1
\end{cases}$$

An example

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1$^{\text{st}}$ asset and the 2$^{\text{nd}}$ asset, 50% between the 1$^{\text{st}}$ asset and the 3$^{\text{rd}}$ asset and 30% between the 2$^{\text{nd}}$ asset and the 3$^{\text{rd}}$ asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

Existence and uniqueness

We consider the following risk budgeting problem:

$$\begin{cases} \mathrm{RC}_i(x) = b_i \mathcal{R}(x) \\ x_i \geq 0 \\ \sum_{i=1}^n b_i = 1 \\ \sum_{i=1}^n x_i = 1 \end{cases}$$

Theorem

- The RB portfolio exists and is unique if the risk budgets are strictly positive (and if $\mathcal{R}(x)$ is bounded below)
- The RB portfolio exists and may be not unique if some risk budgets are set to zero
- The RB portfolio may not exist if some risk budgets are negative

These results hold for convex risk measures: volatility, Gaussian VaR & ES, elliptical VaR, non-normal ES, Kernel historical VaR, Cornish-Fisher VaR, etc.

The RB portfolio is a long-only minimum risk (MR) portfolio subject to a constraint of weight diversification

Let us consider the following minimum risk optimization problem:

$$\begin{array}{rcl} x^{\star}(c) & = & \arg\min \mathcal{R}(x) \\ & \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} b_i \ln x_i \geq c \\ \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{array} \right. \end{array}$$

- if $c = c^{-} = -\infty$, $x^{\star}(c^{-}) = x_{\mathrm{mr}}$ (no weight diversification)
- if $c = c^{+} = \sum_{i=1}^{n} b_i \ln b_i$, $x^{\star}(c^{+}) = x_{\mathrm{wb}}$ (no risk minimization)
- $\exists c^0 : x^{\star}(c^0) = x_{\mathrm{rb}}$ (risk minimization and weight diversification)

$\Longrightarrow$ if $b_i = 1/n$, $x_{\mathrm{rb}} = x_{\mathrm{erc}}$ (variance minimization, weight diversification and perfect risk diversification[3])

[3] The Gini coefficient of the risk measure is then equal to 0.

The RB portfolio is located between the MR portfolio and the WB portfolio

- The RB portfolio is a combination of the MR and WB portfolios:

$$\begin{array}{ll} x_i / b_i = x_j / b_j & \text{(wb)} \\ \partial_{x_i} \mathcal{R}(x) = \partial_{x_j} \mathcal{R}(x) & \text{(mr)} \end{array}$$

$$\mathrm{RC}_i / b_i = \mathrm{RC}_j / b_j \qquad \text{(rb)}$$

- The risk of the RB portfolio is between the risk of the MR portfolio and the risk of the WB portfolio:

$$\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{rb}}) \leq \mathcal{R}(x_{\mathrm{wb}})$$

  With risk budgeting, we always diminish the risk compared to the weight budgeting.

$\Rightarrow$ For the ERC portfolio, we retrieve the famous relationship:

$$\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{erc}}) \leq \mathcal{R}(x_{\mathrm{ew}})$$

Optimality of risk budgeting portfolios

If the RB portfolio is optimal (in the Markowitz sense), the ex-ante performance contributions are equal to the risk budgets:

Black-Litterman Approach

Budgeting the risk = budgeting the performance
(in an ex-ante point of view)

Let $\tilde{\mu}_i$ be the market price of the expected return. We have[4]:

$$\mathrm{PC}_i = x_i \tilde{\mu}_i \propto b_i$$

In the ERC portfolio, the (ex-ante) performance contributions are equal. The ERC portfolio is then the less concentrated portfolio in terms of risk contributions, but also in terms of performance contributions.

[4] If the risk measure is the volatility, we retrieve the famous result: $\mathrm{PC}_i = \mathrm{SR}\left(x \mid r\right) \mathrm{RC}_i$.

The case of the volatility risk measure

Some analytical solutions

- The case of uniform correlation[5] $\rho_{i,j} = \rho$:

$$x_i\left(-\frac{1}{n-1}\right) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}},\ x_i(0) = \frac{\sqrt{b_i}\sigma_i^{-1}}{\sum_{j=1}^n \sqrt{b_j}\sigma_j^{-1}},\ x_i(1) = \frac{b_i\sigma_i^{-1}}{\sum_{j=1}^n b_j\sigma_j^{-1}}$$

- The general case:

$$x_i = \frac{b_i\beta_i^{-1}}{\sum_{j=1}^n b_j\beta_j^{-1}}$$

  where β_i is the beta of the asset i with respect to the RB portfolio.

- The amount of beta is the same for all the assets that compose the ERC portfolio:

$$x_i\beta_i = x_j\beta_j$$

[5] The solution is noted $x_i(\rho)$.

The case of the volatility risk measure

RB portfolios vs MVO portfolios

With the example of Slide 6, the optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$ for a volatility of 15%. The corresponding risk contributions are 49.0%, 25.8% and 25.2%.

1. MVO: the objective is to target a volatility of 15%.
2. RB: the objective is to target the budgets $(49.0\%, 25.8\%, 25.2\%)$.

What is the sensitivity to the input parameters?

ρ			70%	90%		90%	
σ_2					18%	18%	
μ_1							9%
MVO	x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%
	x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%
	x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%
RB	x_1	38.3%	37.7%	38.9%	37.1%	37.7%	38.3%
	x_2	20.2%	20.4%	20.0%	22.8%	22.6%	20.2%
	x_3	41.5%	41.9%	41.1%	40.1%	39.7%	41.5%

$\Rightarrow$ RB portfolios are less sensitive to specification errors than optimized portfolios (Σ vs Σ^{-1}, RB shrinkage covariance matrix).

The case of the volatility risk measure

Solving the optimization problem

Cyclical coordinate descent method of Tseng (2001):

$$\arg\min f(x_1,\ldots,x_n) = f_0(x_1,\ldots,x_n) + \sum_{k=1}^{m} f_k(x_1,\ldots,x_n)$$

where f_0 is strictly convex and the functions f_k are non-differentiable. If we apply the CCD algorithm to the RB problem:

$$\mathcal{L}(x;\lambda) = \arg\min \sqrt{x^\top \Sigma x} - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

we obtain:

$$x_i^\star = \frac{-\sigma_i \sum_{j\neq i} x_j \rho_{i,j}\sigma_j + \sqrt{\sigma_i^2 \left(\sum_{j\neq i} x_j \rho_{i,j}\sigma_j\right)^2 + 4 b_i \sigma_i^2 \sigma(x)}}{2\sigma_i^2}$$

⇒ It always converges[6] (Theorem 5.1, Tseng, 2011).

[6] With an Intel T8400 3 GHz Core 2 Duo processor, computational times are 0.13, 0.45 and 1.10 seconds for a universe of 500, 1000 and 1500 assets.

Introducing expected returns in RB portfolios
The framework

We consider the standard deviation-based risk measure:

$$\mathcal{R}(x) = -\mu(x) + c \cdot \sigma(x)$$

It encompasses three well-known risk measures:

- Markowitz utility function with $c = \frac{\phi}{2}\sigma(x(\phi))$
- Gaussian value-at-risk with $c = \Phi^{-1}(\alpha)$
- Gaussian expected shortfall with $c = \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$

Theorem

The RB portfolio exists and is unique if[a]:

$$c > \mathrm{SR}(x^\star \mid r)$$

where $x^\star$ is the tangency portfolio.

[a] Because of the homogeneity property $\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x)$.

Introducing expected returns in RB portfolios

Relationship with the Markowitz theory

Volatility risk measure

$$\begin{aligned} x^\star(\kappa) &= \arg\min \frac{1}{2} x^\top \Sigma x \\ \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{array} \right. \end{aligned}$$

The RB portfolio is a minimum variance portfolio subject to a constraint of weight diversification.

Generalized risk measure

$$\begin{aligned} x^\star(\kappa) &= \arg\min -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} \\ \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{array} \right. \end{aligned}$$

The RB portfolio is a mean-variance portfolio subject to a constraint of weight diversification.

The factor model

- n assets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and m risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$.
- R_t is the $(n \times 1)$ vector of asset returns at time t and Σ its associated covariance matrix.
- $\mathcal{F}_t$ is the $(m \times 1)$ vector of factor returns at t and Ω its associated covariance matrix.
- We assume the following linear factor model:

$$R_t = A\mathcal{F}_t + \varepsilon_t$$

with $\mathcal{F}_t$ and ε_t two uncorrelated random vectors. The covariance matrix of ε_t is noted D. We have:

$$\Sigma = A\Omega A^\top + D$$

- The P&L of the portfolio x is:

$$\Pi_t = x^\top R_t = x^\top A\mathcal{F}_t + x^\top \varepsilon_t = y^\top \mathcal{F}_t + \eta_t$$

with $y = A^\top x$ and $\eta_t = x^\top \varepsilon_t$.

First route to decompose the risk

Let $B = A^\top$ and B^+ the Moore-Penrose inverse of B. We have therefore:

$$x = B^+ y + e$$

where $e = (I_n - B^+ B) x$ is a $(n \times 1)$ vector in the kernel of B.

We consider a convex risk measure $\mathcal{R}(x)$. We have:

$$\frac{\partial \mathcal{R}(x)}{\partial x_i} = \left(\frac{\partial \mathcal{R}(y,e)}{\partial y} B \right)_i + \left(\frac{\partial \mathcal{R}(y,e)}{\partial e} (I_n - B^+ B) \right)_i$$

Decomposition of the risk by m common factors and n idiosyncratic factors $\Rightarrow$ Identification problem!

Second route to decompose the risk

Meucci (2007) considers the following decomposition:

$$x = \begin{pmatrix} B^{+} & \tilde{B}^{+} \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \bar{B}^{\top} \bar{y}$$

where $\tilde{B}^{+}$ is any $n \times (n-m)$ matrix that spans the left nullspace of B^{+}.

Decomposition of the risk by m common factors and $n-m$ residual factors $\Rightarrow$ Better identified problem.

Euler decomposition of the risk measure

Theorem

The risk contributions of common and residual risk factors are:

$$\begin{aligned}
\mathrm{RC}\left(\mathcal{F}_j\right) &= \left(A^\top x\right)_j \cdot \left(A^+ \frac{\partial \mathcal{R}(x)}{\partial x}\right)_j \\
\mathrm{RC}\left(\tilde{\mathcal{F}}_j\right) &= \left(\tilde{B}x\right)_j \cdot \left(\tilde{B} \frac{\partial \mathcal{R}(x)}{\partial x}\right)_j
\end{aligned}$$

They satisfy the Euler allocation principle:

$$\sum_{j=1}^{m} \mathrm{RC}\left(\mathcal{F}_j\right) + \sum_{j=1}^{n-m} \mathrm{RC}\left(\tilde{\mathcal{F}}_j\right) = \mathcal{R}(x)$$

⇒ Risk contribution with respect to risk factors (resp. to assets) are related to marginal risk of assets (resp. of risk factors).

⇒ The main important quantity is marginal risk, not risk contribution!

An example

We consider 4 assets and 3 factors. The loadings matrix is:

$$A = \begin{pmatrix} 0.9 & 0 & 0.5 \\ 1.1 & 0.5 & 0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are equal to 20%, 10% and 10%. We consider a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Along assets $\mathcal{A}_1, \ldots, \mathcal{A}_n$

	x_i	$\mathrm{MR}(\mathcal{A}_i)$	$\mathrm{RC}(\mathcal{A}_i)$	$\mathrm{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	25.00%	18.81%	4.70%	21.97%
$\mathcal{A}_2$	25.00%	23.72%	5.93%	27.71%
$\mathcal{A}_3$	25.00%	24.24%	6.06%	28.32%
$\mathcal{A}_4$	25.00%	18.83%	4.71%	22.00%
$\sigma(x)$			21.40%	

Along factors $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and $\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_{n-m}$

	y_i	$\mathrm{MR}(\mathcal{F}_i)$	$\mathrm{RC}(\mathcal{F}_i)$	$\mathrm{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	100.00%	17.22%	17.22%	80.49%
$\mathcal{F}_2$	22.50%	9.07%	2.04%	9.53%
$\mathcal{F}_3$	35.00%	6.06%	2.12%	9.91%
$\tilde{\mathcal{F}}_1$	2.75%	0.52%	0.01%	0.07%
$\sigma(y)$			21.40%	

Beta contribution versus risk contribution

The linear model is:

$$\begin{pmatrix} R_{1,t} \\ R_{2,t} \\ R_{3,t} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \\ 0.3 & 0.5 \\ 0.8 & -0.2 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{1,t} \\ \mathcal{F}_{2,t} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

The factor volatilities are equal to 10% and 30%, while the idiosyncratic volatilities are equal to 3%, 5% and 2%.

If we consider the volatility risk measure, we obtain:

Portfolio	$(1/3, 1/3, 1/3)$		$(7/10, 7/10, -4/10)$	
Factor	β	$\mathrm{RC}^\star$	β	$\mathrm{RC}^\star$
$\mathcal{F}_1$	0.67	31%	0.52	3%
$\mathcal{F}_2$	0.33	69%	0.92	97%

The first portfolio has a bigger beta in factor 1 than in factor 2, but about 70% of its risk is explained by the second factor. For the second portfolio, the risk w.r.t the first factor is very small even if its beta is significant.

Matching the risk budgets

We consider the risk budgeting problem: $\mathrm{RC}(\mathcal{F}_j) = b_j \mathcal{R}(x)$. This problem is tricky, because the first-order conditions are PDE!

Some special cases

- Positive factor weights ($y \geq 0$) with $m = n \Rightarrow$ a unique solution.
- Positive factor weights ($y \geq 0$) with $m < n \Rightarrow$ at least one solution.
- Positive asset weights ($x \geq 0$ or long-only portfolio) $\Rightarrow$ zero, one or more solutions.

Applications

- Smart beta
- Bond portfolios management
- Multi-asset allocation
- Absolute return funds

Implementation of the risk parity approach

- Equities: SmartIX ERC indexes (Lyxor/FTSE), Eurostoxx 50 ERC ETF (Lyxor), Global Equity Risk Parity (LODH), Emerging Equity Risk Parity (LODH), etc.
- Bonds: RB EGBI index (Lyxor/Citigroup), RB Euro IG Corporate index (Lyxor/Citigroup), RB World Bond IG index (Lyxor/Citigroup), AC Risk Parity Bond Fund (Aquila Capital), etc.
- Commodities: Commodity Active Fund (Lyxor), Commodities Risk Parity (LODH), Risk Weighted Enhanced Commodity Ex Grains ETF (Ossiam), etc.
- Absolute return funds: All Weather fund (Bridgewater), IBRA (Invesco), ARMA (Lyxor), Global Risk Parity Fund (AQR), Risk Parity 7 Fund (Aquila Capital), Global Diversification (1741), Global Allocation Strategies Plus (Raiffeisen), etc.
- Strategic Asset Allocation: large and sophisticated pension funds.

Cap-weighted indexation and modern portfolio theory

Rationale of market-cap indexation

- **Markowitz Theory**: we maximize the expected return under constraint of a given level of volatility.
- **Separation Theorem**: there is one unique risky portfolio owned by investors called the tangency portfolio.
- **CAPM (1964)**: the tangency portfolio is the Market (Capitalization) portfolio, best represented by the capitalization- weighted index.
- **Jensen (1968)**: no alpha in equity mutual funds.
- Wells Fargo Bank (1971): **First (private) index fund**.
- Wells Fargo/American National Bank in Chicago (1973): **First S&P 500 index fund**.

Pros and cons of market-cap indexation

Pros of market-cap indexation

- A convenient and **recognized approach** to participate to broad equity markets.
- **Management simplicity**: low turnover & transaction costs.

Cons of market-cap indexation

- Trend-following strategy: momentum bias leads to bubble risk exposure as weight of best performers ever increases.
  $\Rightarrow$ Mid 2007, financial stocks represent 40% of the Eurostoxx 50 index.
- Growth bias as high valuation multiples stocks weight more than low-multiple stocks with equivalent realized earnings.
  $\Rightarrow$ Mid 2000, the 8 stocks of the technology/telecom sectors represent 35% of the Eurostoxx 50 index.
  $\Rightarrow$ $2^1/_2$ years later after the dot.com bubble, these two sectors represent 12%.
- Concentrated portfolios.
  $\Rightarrow$ The top 100 market caps of the S&P 500 account for around 70%.
- Lack of risk diversification and high drawdown risk: no portfolio construction rules leads to concentration issues (e.g. sectors, stocks).

Pros and cons of market-cap indexation

Some illustrations

- Mid 2000: 8 Technology/Telecom stocks represent 35% of the Eurostoxx 50 index
- In 2002: 7.5% of the Eurostoxx 50 index is invested into Nokia with a volatility of 70%
- Dec. 2006: 26.5% of the MSCI World index is invested in financial stocks
- June 2007: 40% of the Eurostoxx 50 is invested into Financials
- January 2013: 20% of the S&P 500 stocks represent 68% of the S&P 500 risk
- Over 10 years: two stocks contribute on average to more than 20% of the monthly performance of the Eurostoxx 50 index

Weight and risk concentration of CW indexes (June 2012)

$\mathcal{G}(x)$ = Gini coefficient, $\mathbb{L}(x)$ = Lorenz curve.

Ticker	Weights				Risk contributions			
	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%
SX5P	30.8	24.1	48.1	71.3	26.3	19.0	40.4	68.6
SX5E	31.2	23.0	46.5	72.1	31.2	20.5	44.7	73.3
INDU	33.2	23.0	45.0	73.5	35.8	25.0	49.6	75.9
BEL20	39.1	25.8	49.4	79.1	45.1	25.6	56.8	82.5
DAX	44.0	27.5	56.0	81.8	47.3	27.2	59.8	84.8
CAC	47.4	34.3	58.3	82.4	44.1	31.9	57.3	79.7
AEX	52.2	37.2	61.3	86.0	51.4	35.3	62.0	84.7
HSCEI	54.8	39.7	69.3	85.9	53.8	36.5	67.2	85.9
NKY	60.2	47.9	70.4	87.7	61.4	49.6	70.9	88.1
UKX	60.8	47.5	73.1	88.6	60.4	46.1	72.8	88.7
SXXE	61.7	49.2	73.5	88.7	63.9	51.6	75.3	90.1
SPX	61.8	52.1	72.0	87.8	59.3	48.7	69.9	86.7
MEXBOL	64.6	48.2	75.1	91.8	65.9	45.7	78.6	92.9
IBEX	64.9	51.7	77.3	90.2	68.3	58.2	80.3	91.4
SXXP	65.6	55.0	76.4	90.1	64.2	52.0	75.5	90.0
NDX	66.3	58.6	77.0	89.2	64.6	56.9	74.9	88.6
TWSE	79.7	73.4	86.8	95.2	79.7	72.6	87.3	95.7
TPX	80.8	72.8	88.8	96.3	83.9	77.1	91.0	97.3
KOSPI	86.5	80.6	93.9	98.0	89.3	85.1	95.8	98.8

Alternative-weighted indexation

Alternative-weighted indexation aims at building passive indexes where the weights are not based on market capitalization.

Two (or three) sets of responses:

1. Fundamental indexation (capturing *alpha*?)
    1. Dividend yield indexation
    2. RAFI indexation
2. Risk-based indexation (capturing *diversification*)
    1. Equally weighted (1/n)
    2. Minimum-variance portfolio
    3. ERC portfolio
    4. MDP portfolio
3. Risk factor indexation (capturing *normal* returns or beta $\neq$ abnormal returns or alpha)
    1. The market portfolio is not the only risk factor.
    2. Other factors: low beta (Black), value (Fama-French), small cap (Fama-French), momentum (Carhart), etc.

Capturing the equity risk premium

The aim of risk-based indexation is to capture the equity risk premium.

- How to build a passive indexation to capture this premium?
  - Differences between active and passive management
  - How the risk is rewarded?
  - What is the link between risk and performance?
- How the performance of risk-based indexation is explained?
  - Understanding the performance of the beta
  - Specific and arbitrage factors vs beta

Black-Litterman approach

If the portfolio is efficient, performance and risk are strongly related:

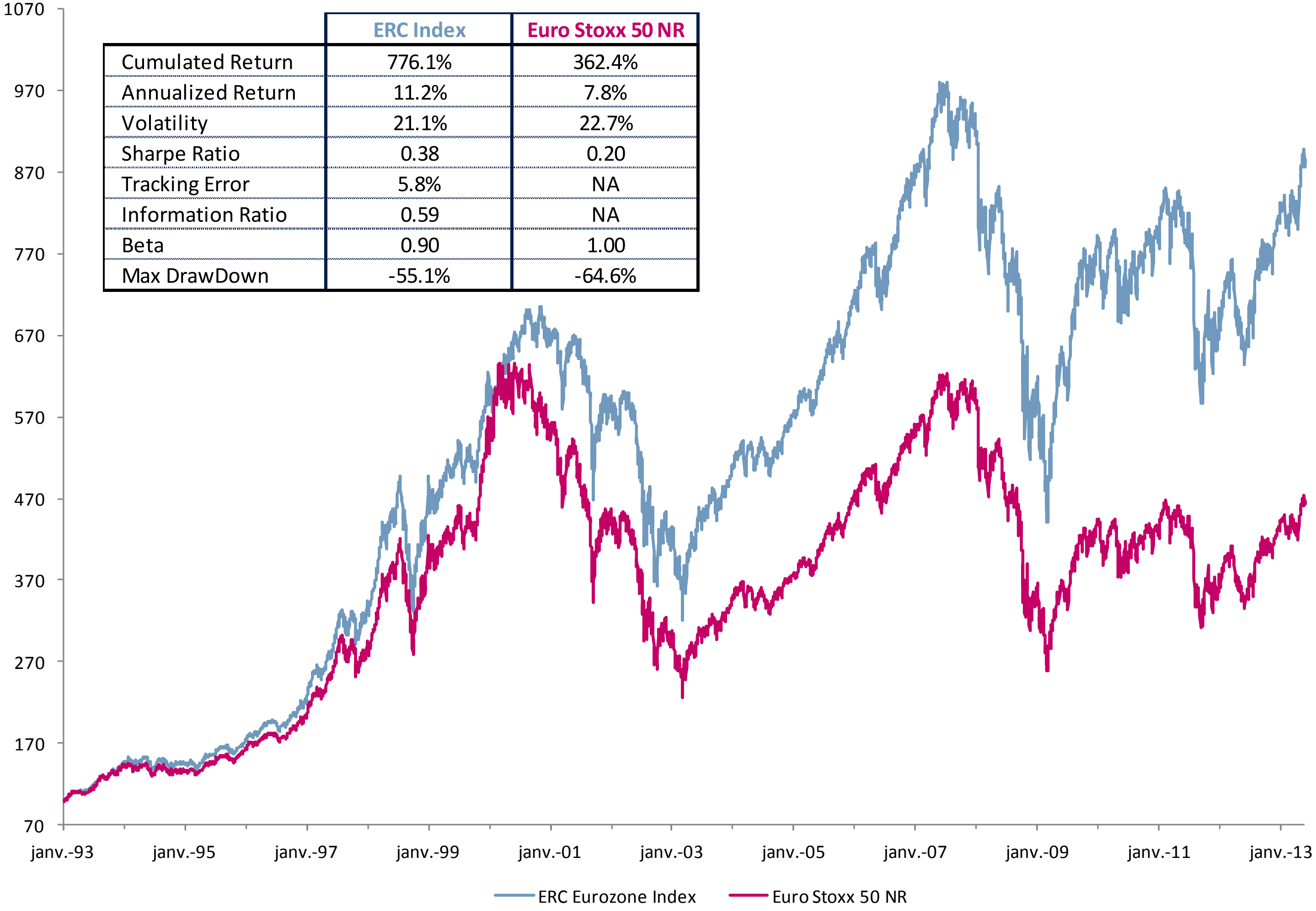
$$\text{Performance Contribution} = \text{Risk Contribution}$$

Capturing the equity risk premium

	APPLE	EXXON	MSFT	J&J	IBM	PFIZER	CITI	McDO
	Cap-weighted allocation (in %)							
Dec. 1999	1.05	12.40	38.10	7.94	12.20	12.97	11.89	3.46
Dec. 2004	1.74	22.16	19.47	12.61	11.00	13.57	16.76	2.70
Dec. 2008	6.54	35.03	14.92	14.32	9.75	10.30	3.15	5.98
Dec. 2010	18.33	22.84	14.79	10.52	11.29	8.69	8.51	5.02
Dec. 2012	26.07	20.55	11.71	10.12	11.27	9.62	6.04	4.61
Jun. 2013	20.78	19.80	14.35	11.64	11.36	9.51	7.79	4.77
	Implied risk premium (in %)							
Dec. 1999	5.96	2.14	8.51	3.61	5.81	5.91	6.19	2.66
Dec. 2004	3.88	2.66	2.79	2.03	2.32	3.90	3.02	1.86
Dec. 2008	9.83	11.97	10.48	6.24	7.28	8.06	17.15	6.28
Dec. 2010	5.38	3.85	4.42	2.29	3.66	3.76	6.52	2.54
Dec. 2012	5.87	2.85	3.58	1.44	2.80	1.77	5.91	1.88
Jun. 2013	5.59	2.79	3.60	1.55	2.92	1.91	5.24	1.82
	Performance contribution (in %)							
Dec. 1999	1.01	4.31	52.63	4.66	11.52	12.43	11.94	1.49
Dec. 2004	2.41	21.04	19.44	9.15	9.12	18.93	18.11	1.79
Dec. 2008	6.60	43.00	16.04	9.17	7.28	8.52	5.55	3.85
Dec. 2010	23.58	21.01	15.62	5.77	9.89	7.81	13.27	3.05
Dec. 2012	42.41	16.23	11.61	4.04	8.73	4.71	9.88	2.40
Jun. 2013	33.96	16.18	15.10	5.28	9.69	5.32	11.93	2.53

An example with the Eurostoxx 50 ERC index

Figure: Performance of the ERC Eurozone Index (Ticker: SGIXERCE Index)



	ERC Index	Euro Stoxx 50 NR
Cumulated Return	776.1%	362.4%
Annualized Return	11.2%	7.8%
Volatility	21.1%	22.7%
Sharpe Ratio	0.38	0.20
Tracking Error	5.8%	NA
Information Ratio	0.59	NA
Beta	0.90	1.00
Max DrawDown	-55.1%	-64.6%

Choosing the right smart beta

No explicit answer.

Depends (mainly) on the risk reduction targeted by the investor.

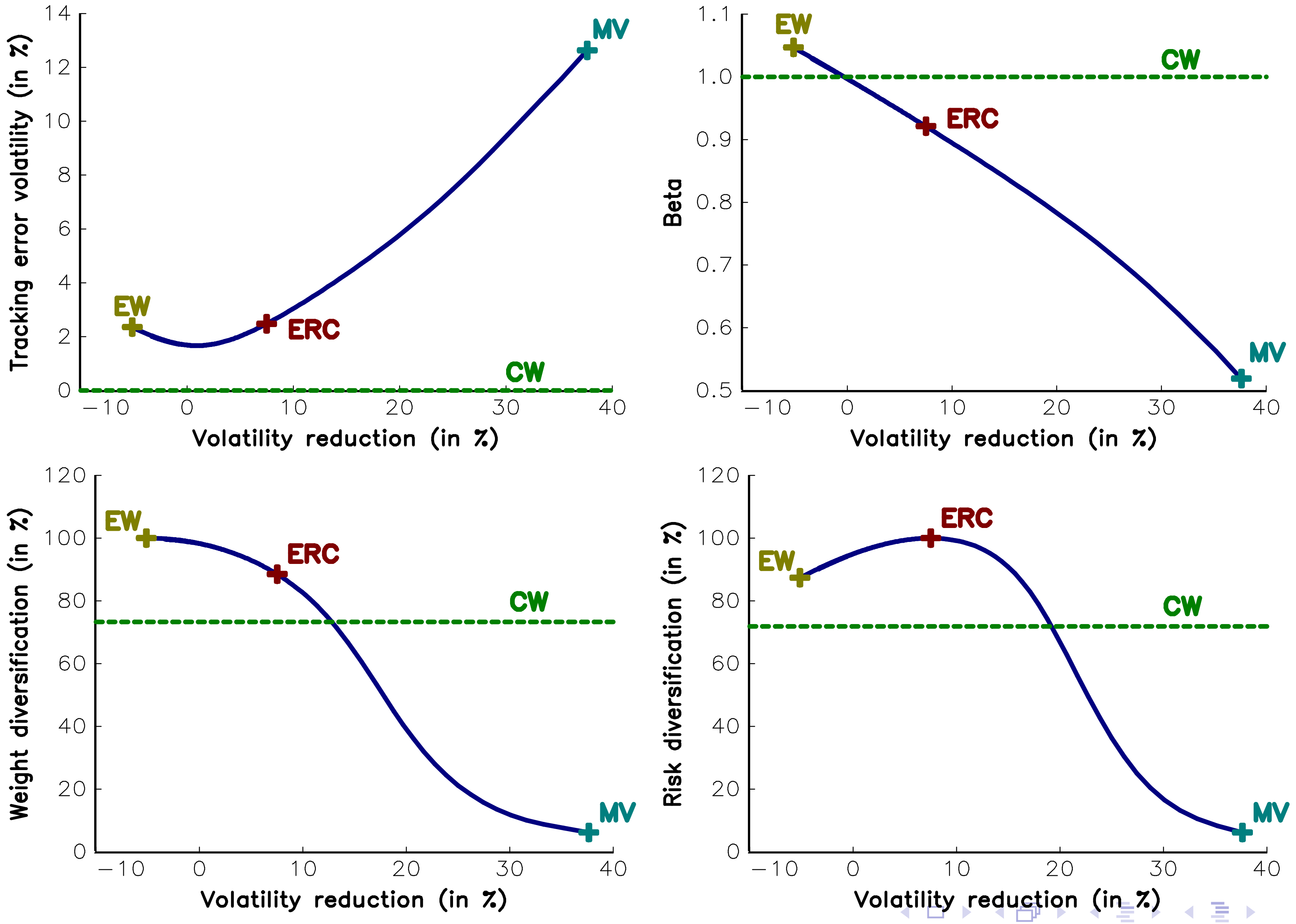
A trade-off problem between volatility reduction, diversification, benchmark risk (tracking error volatility, liquidity risk and investment capacity), factor return risk, etc.

The battle around the models becomes

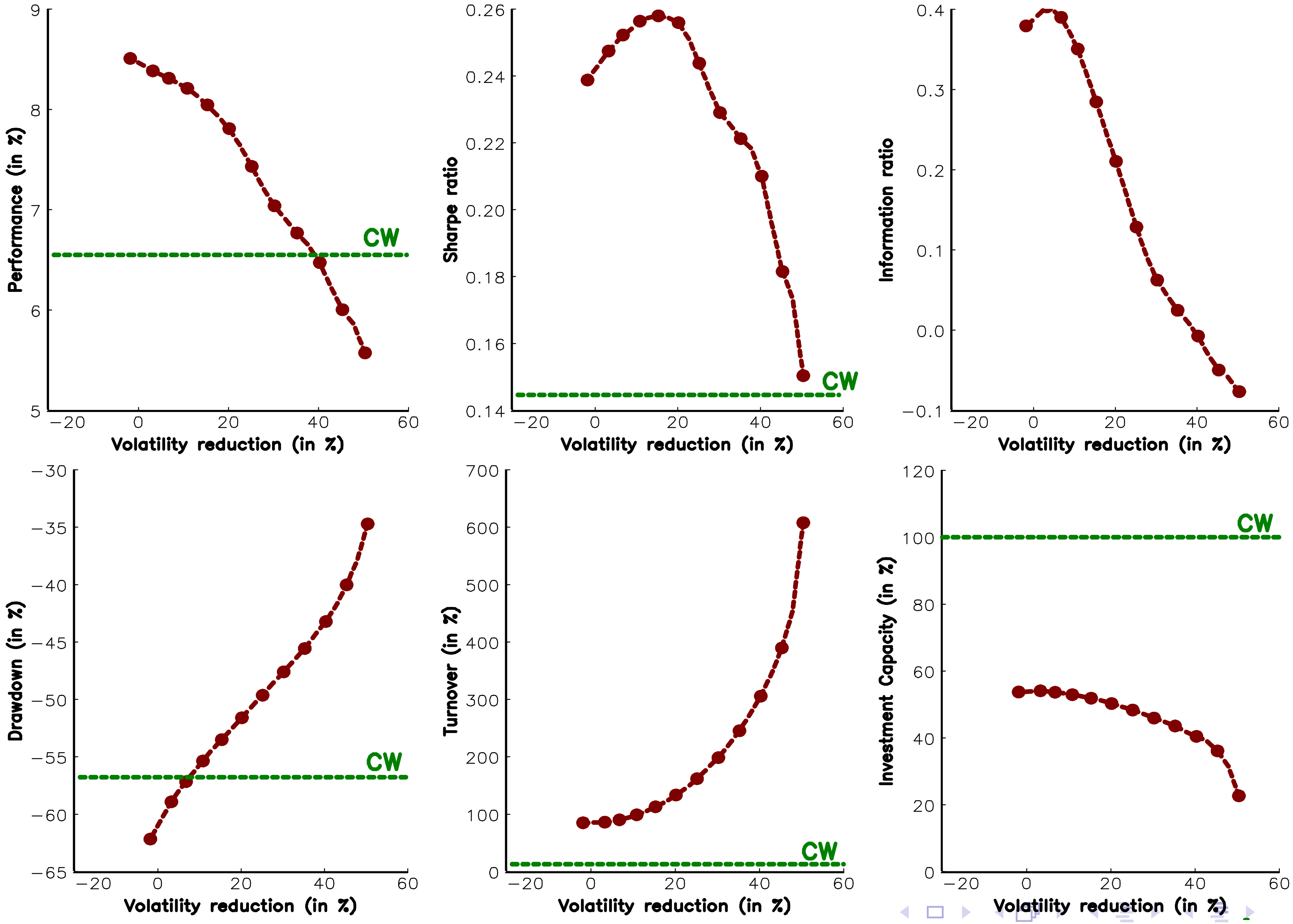
a battle around the volatility reduction.

$\Rightarrow$ Which model is consistent with a targeted volatility reduction?

The trade-off problem (ex-ante analysis)

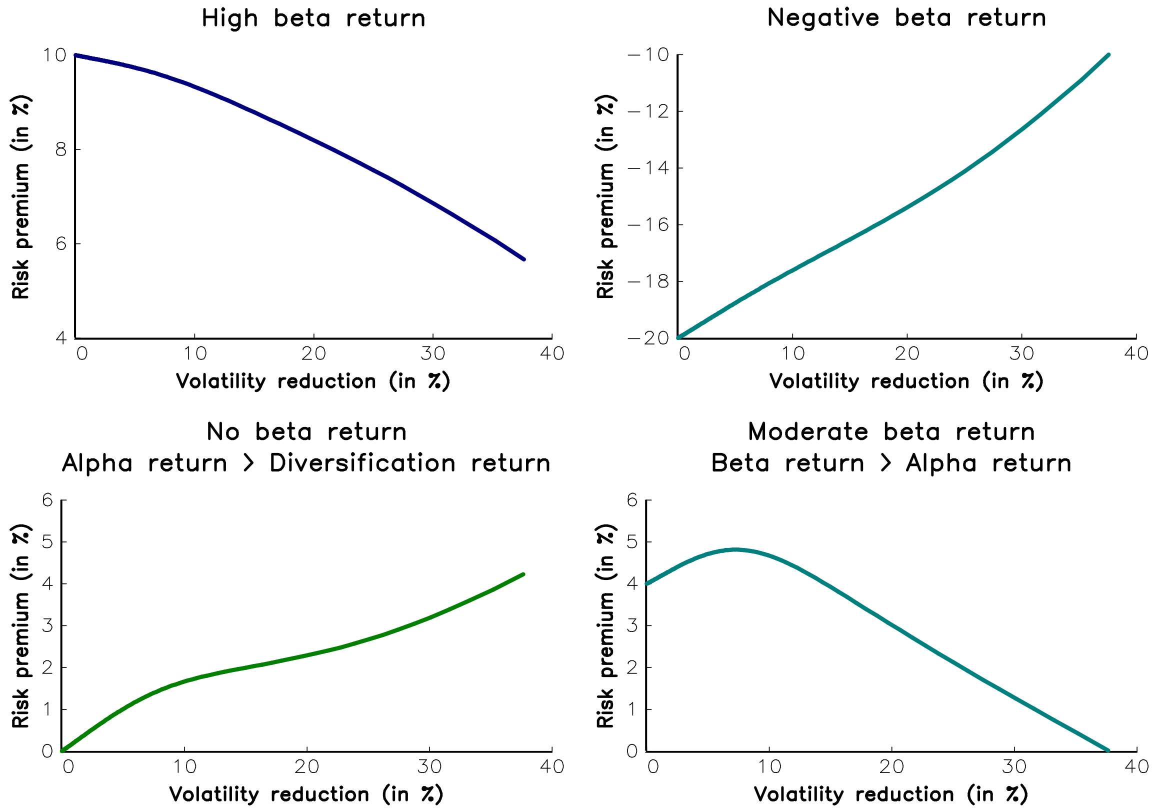


The trade-off problem (ex-post analysis)



Performance attribution of risk-based indexes

- Beta return (Sharpe, 1964)
- Diversification return (Booth and Fama, 1992)
- Alpha or low beta anomaly (Black, 1972)
  - Value factor
  - Small cap factor
  - Momentum factor (?)



Diversification return

Let V_t be the value of a rebalanced portfolio. We obtain[7]:

$$\begin{aligned} V_t &= V_0 e^{\frac{1}{2}\left(\sum_{i=1}^n x_i \sigma_i^2 - \sigma^2(x)\right)t} \prod_{i=1}^n \left(\frac{S_{i,t}}{S_{i,0}}\right)^{x_i} \\ &= V_0 e^{\mu_x t} U_x \end{aligned}$$

The difference between the return of the rebalanced portfolio $R(x)$ and the return of the buy-and-hold portfolio $\bar{R}$ is approximately equal to one-half of the diversification return minus the cross-section variance of asset returns:

$$R(x) - \bar{R} \simeq \frac{1}{2}\left(\mathfrak{d}(x) - \mathfrak{c}(x)\right)$$

with $\mathfrak{d}(x) = \sum_{i=1}^n x_i \sigma_i^2 - \sigma^2(x)$ and $\mathfrak{c}(x) = \sum_{i=1}^n x_i R_i^2 - \bar{R}^2$.

[7] For the ERC portfolio, we have $\mu_x \simeq \frac{1}{2}\bar{\sigma}_H(\bar{\sigma}_A - \rho\bar{\sigma}_H) \geq 0$.

Rationale of low beta (low volatility?) anomaly

- Low volatility anomaly = low beta anomaly
- Stylized facts (Black, Jensen & Scholes, 1972)
- Impact of constraints on CAPM (Black, 1972)
- BaB[8] factors (Frazzini & Pedersen, 2010)

CAPM model with borrowing constraints

- ϕ_j: risk aversion of the investor j
- m_j: borrowing constraint j (cannot use leverage if $m_j \leq 1$)
- λ_j: Lagrange coefficients associated to the borrowing constraint

We have:

$$\mathbb{E}_t\left[R_{i,t+1}\right] - r = \alpha_i + \beta_i\left(\mathbb{E}_t\left[R_{t+1}\left(x^\star\right)\right] - r\right)$$

where $\alpha_i = \psi\left(1-\beta_i\right)$, $\psi = \sum_{j=1}^m \phi\phi_j^{-1}\lambda_j m_j$ and $\phi = \left(\sum_{j=1}^m \phi_j^{-1}\right)^{-1}$.
We deduce that $\alpha_i > 0$ if $\beta_i < 1$ and $\alpha_i \leq 0$ if $\beta_i \geq 1$.

[8] Betting Against Beta

Illustration of the low beta anomaly

Example

We consider four assets with $\mu_1 = 5\%$, $\mu_2 = 6\%$, $\mu_3 = 8\%$, $\mu_4 = 6\%$, $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 25\%$, $\sigma_4 = 20\%$ and

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.20 & 0.60 & 1.00 & \\ 0.40 & 0.50 & 0.50 & 1.00 \end{pmatrix}$$

The risk-free rate is set to 2%. The market corresponds to two investors with $m_1 = 100\%$ and $m_2 = 50\%$.

No borrowing constraints

$\mu(x^\star) = 6.07\%$ and $\sigma(x^\star) = 13.77\%$.

Asset	$x_i^\star$	$\beta_i(x^\star)$	$\pi_i(x^\star)$
1	47.50%	0.74	3.00%
2	19.83%	0.98	4.00%
3	27.37%	1.47	6.00%
4	5.30%	0.98	4.00%

With borrowing constraints

$\mu(x^\star) = 6.30\%$ and $\sigma(x^\star) = 14.66\%$.

Asset	$x_i^\star$	α_i	$\beta_i(x^\star)$	$\pi_i(x^\star)$
1	42.21%	0.32%	0.62	2.68%
2	15.70%	0.07%	0.91	3.93%
3	36.31%	−0.41%	1.49	6.41%
4	5.78%	0.07%	0.91	3.93%

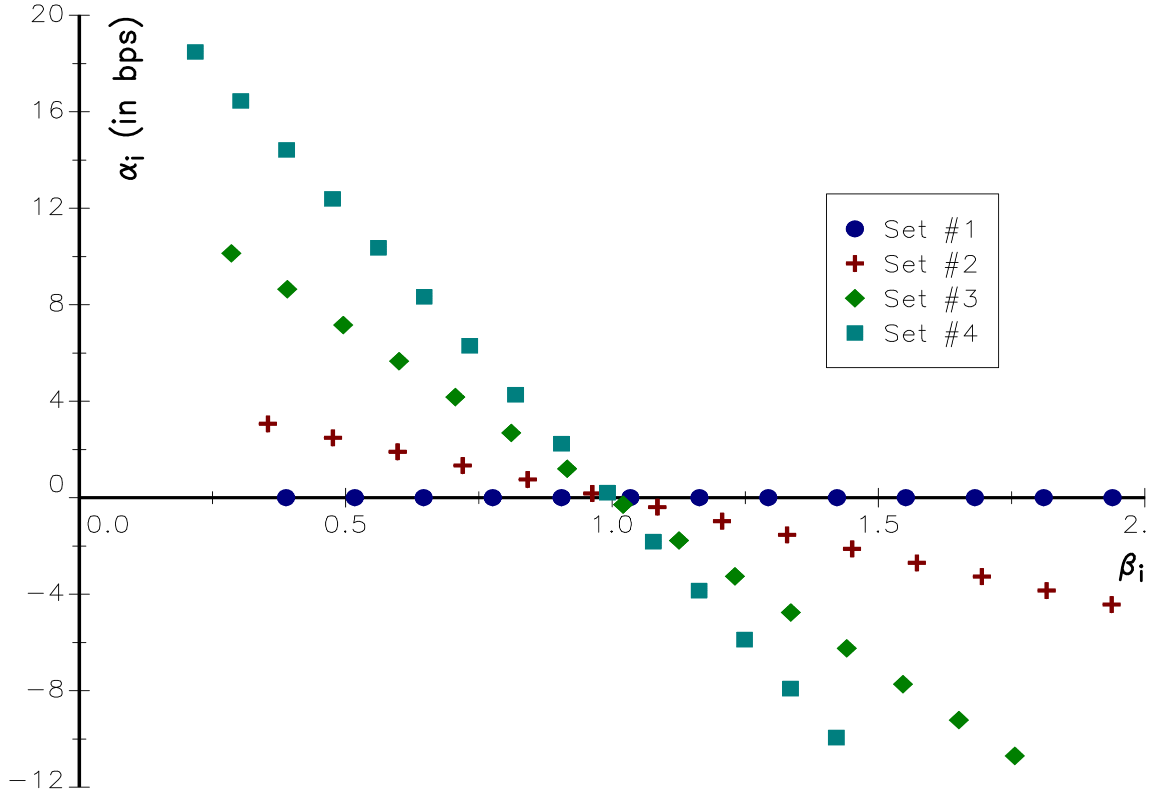
Illustration of the low beta anomaly

- $\sigma_i = 3\% : 17\%$
- $\rho = 50\%$
- $\mathrm{SR}_i = 0.5$
- $r = 2\%$

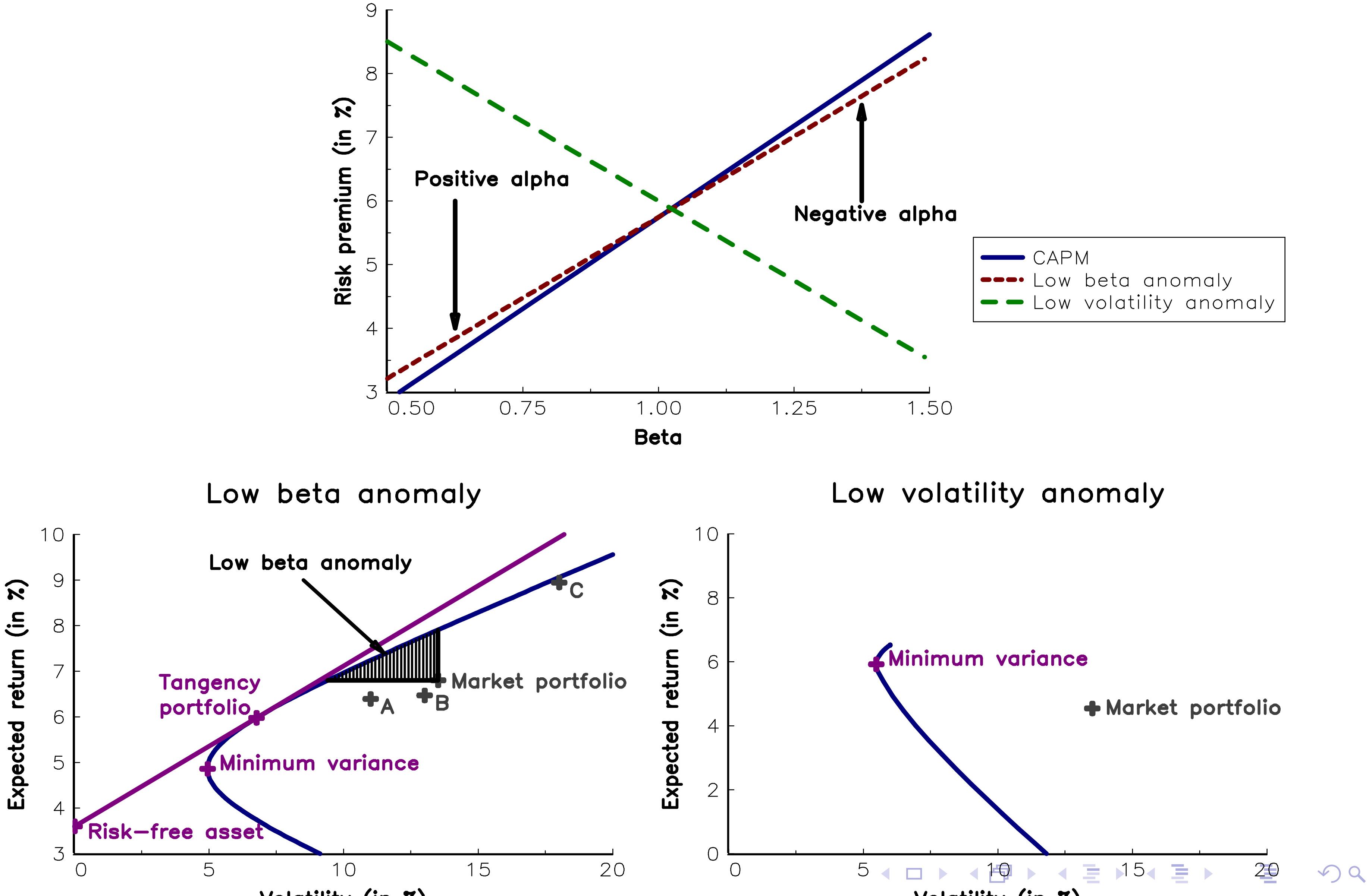
Table: Borrowing constraints m_j

Set	Investor j 1	2	3	4	5
#1	1.0	1.0	1.0	1.0	1.0
#2	0.8	0.8	1.0	1.0	1.0
#3	0.7	0.7	0.7	0.7	1.0
#4	0.5	0.5	0.5	0.5	1.0

Figure: Relationship between β_i and α_i



Comparing the low beta and low volatility anomalies



Choice of the AW index

Long-term investment

- 10% of volatility reduction costs 3.5% in average of tracking error volatility
- The optimal volatility reduction is between 5% and 20% in terms of risk/return profile
- ERC portfolio = a good candidate for smart beta indexing

Short-term investment

- In the short run, the choice of a smart beta depends on the views of the investor
- Most of the performance is explained by the beta return
- Two choices: CW (bull market) / MV (bear market).

Time to rethink the bond portfolios management

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position ≠ arbitrage position)

⇒ Time to rethink bond indexes? (Toloui, 2010)

We need to develop a framework to measure the credit risk of bond portfolios with two goals:

1. managing the credit risk of bond portfolios;
2. building alternative-weighted indexes.

For the application, we consider the euro government bond portfolios. The benchmark is the Citigroup EGBI index.

Defining the credit risk measure of a bond portfolio

- Volatility of price returns $\neq$ a good measure of credit risk
- Correlation of price returns $\neq$ a good measure of contagion
- A better measure is the asset swap spread, but it is an OTC data. That's why we use the CDS spread.

Our credit risk measure $\mathcal{R}(x)$ is the (integrated) volatility of the CDS basket which would perfectly hedge the credit risk of the bond portfolio.

Remark

$\mathcal{R}(x)$ depends on 3 "CDS" parameters $S_i(t)$ (the level of the CDS), σ_i^S (the volatility of the CDS) and $\Gamma_{i,j}$ (the cross-correlation between CDS) and two "portfolio" parameters x_i (the weight) and D_i (the duration).

Defining the credit risk measure of a bond portfolio

Let $B(t, D_i)$ be a zero-coupon risky bond of maturity (or duration) D_i. We have:

$$B(t, D_i) = e^{-(R(t)+S_i(t))\cdot D_i}$$

with $R(t)$ the risk-free rate and $S_i(t)$ the credit spread. It comes that:

$$\mathrm{d}\ln B(t, D_i) = -D_i \cdot \mathrm{d}R(t) - D_i \cdot \mathrm{d}S_i(t)$$

Let $x = (x_1, \ldots, x_n)$ be the weights of bonds in the portfolio. The risk measure is the volatility of the hedging (CDS) portfolio:

$$\mathcal{R}(x) = \sigma\left(\sum_{i=1}^{n} -x_i \cdot D_i \cdot \mathrm{d}S_i(t)\right)$$

We assume that:

$$\mathrm{d}S_i(t) = \sigma_i^S \cdot S_i(t)^{\beta_i} \cdot \mathrm{d}W_i(t)$$

with correlated Brownian motions $W_i(t)$ and $W_j(t)$.

Bond indexation

Debt weighting

It is defined by:

$$x_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

Two forms of debt-weighting are considered : DEBT (with the 11 countries) and DEBT* (without Greece after July 2010). This last one corresponds to the weighting scheme of the EGBI index.

Alternative weighting

1. Fundamental indexation
   The GDP-weighting is defined by:

$$x_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

2. Risk-based indexation
   The DEBT-RB and GDP-RB weightings are defined by:

$$b_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

$$b_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

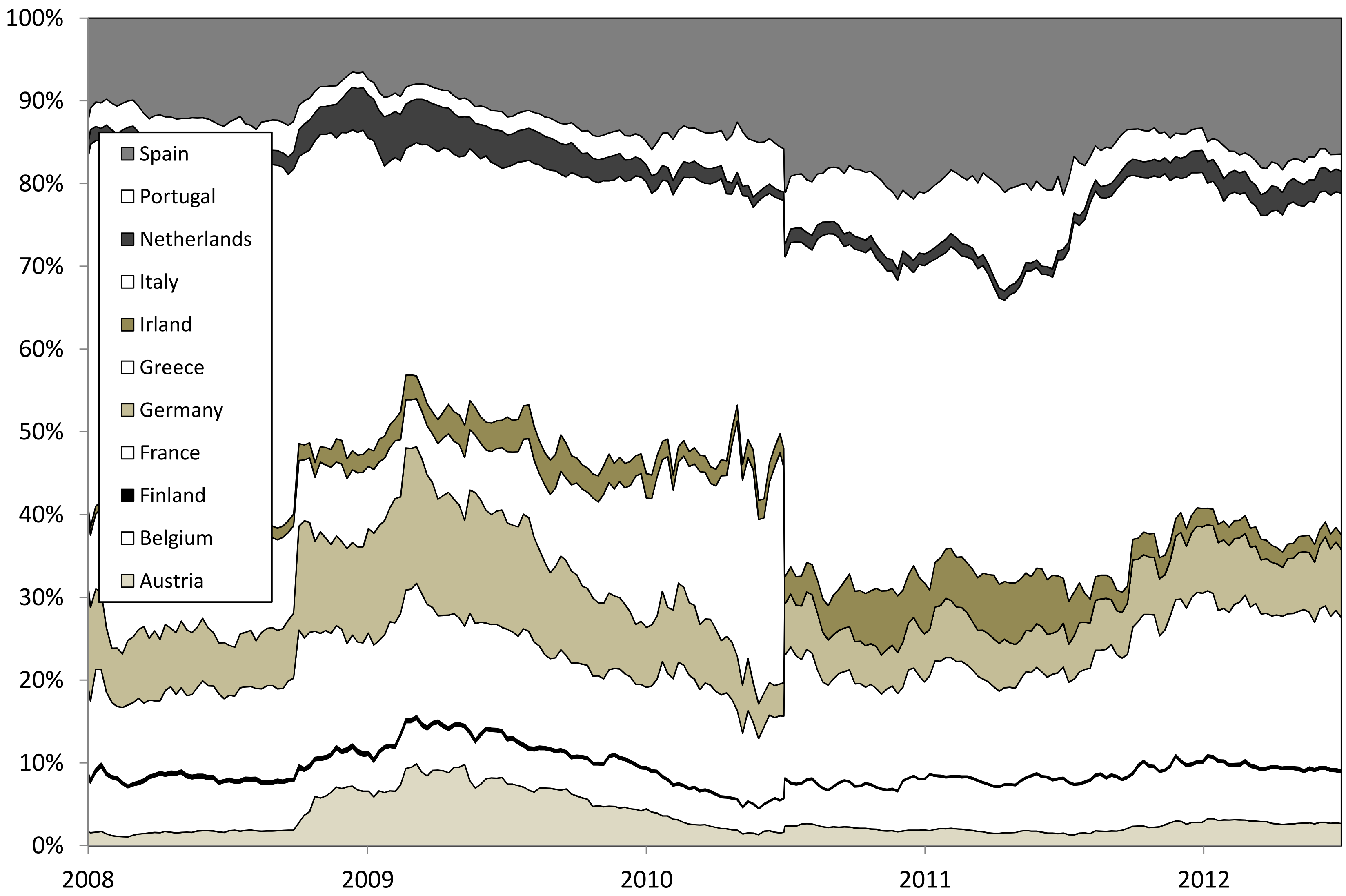
Weights and risk contribution of the EGBI portfolio (in %)

Country	Jan. 08		Jan. 10		Jan. 11		Jan. 12	
	x_i	$\mathcal{RC}_i^\star$	x_i	$\mathcal{RC}_i^\star$	x_i	$\mathcal{RC}_i^\star$	x_i	$\mathcal{RC}_i^\star$
Austria	3.9	1.7	3.8	4.5	4.0	1.9	4.2	2.8
Belgium	6.3	6.7	6.1	4.8	6.3	6.1	6.2	7.5
Finland	1.3	0.4	1.2	0.3	1.3	0.1	1.5	0.3
France	19.9	10.4	20.2	9.6	22.1	11.7	23.5	19.6
Germany	24.3	12.3	21.6	7.2	22.9	5.8	23.4	8.0
Greece	5.2	8.5	5.0	15.6	0.0	0.0	0.0	0.0
Ireland	1.0	1.0	1.9	3.0	2.1	6.2	1.7	2.2
Italy	22.6	42.1	23.1	35.2	23.4	38.3	20.8	40.3
Netherlands	5.5	1.8	5.3	2.1	6.1	1.4	6.5	2.6
Portugal	2.2	2.7	2.4	2.8	2.1	7.4	1.5	2.6
Spain	7.8	12.4	9.5	14.9	9.6	21.1	10.7	14.0
$\mathcal{R}(x)$	0.3		2.8		8.3		10.7	

$\Rightarrow$ Small changes in weights but large changes in risk contributions.

$\Rightarrow$ The credit risk measure has highly increased (the largest value 12.5% is obtained in November 25$^{\text{th}}$ 2011).

Dynamics of the risk contributions (EGBI portfolio)



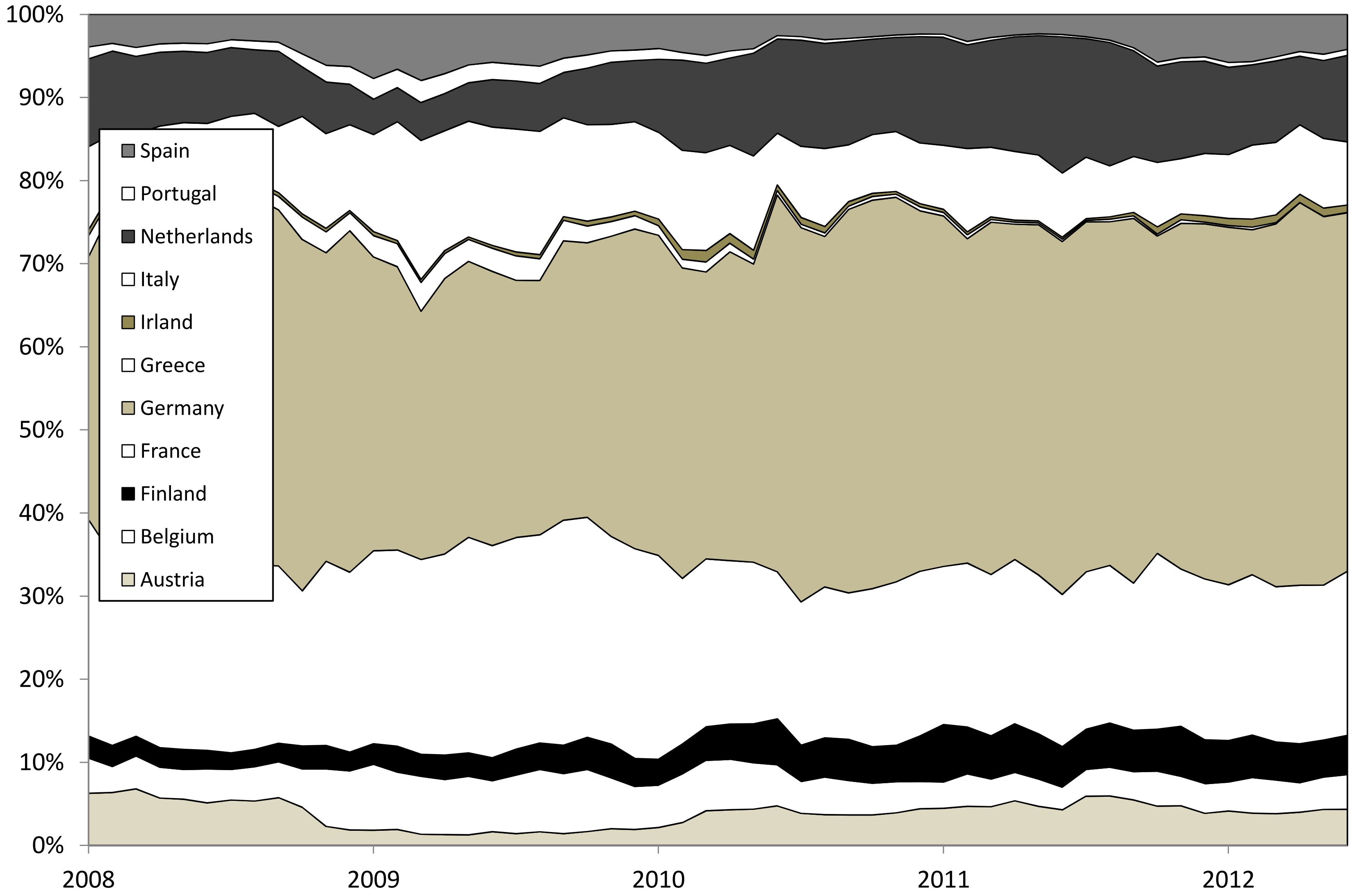
Weights and risk contribution of the DEBT-RB portfolio

Country	Jan. 08		Jan. 10		Jan. 11		Jan. 12	
	b_i	x_i	b_i	x_i	b_i	x_i	b_i	x_i
Austria	3.9	6.3	3.8	2.2	3.9	4.4	4.2	3.9
Belgium	6.3	4.2	6.1	5.1	6.0	3.3	6.1	3.6
Finland	1.3	2.6	1.2	3.1	1.2	5.5	1.5	5.3
France	19.9	26.1	20.2	24.5	21.2	19.8	23.3	19.3
Germany	24.3	31.6	21.6	38.5	21.9	43.4	23.2	42.7
Greece	5.2	2.5	5.0	1.1	4.3	0.5	1.0	0.2
Ireland	1.0	0.7	1.9	0.8	2.0	0.4	1.7	0.8
Italy	22.6	10.0	23.1	10.4	22.4	7.3	20.6	7.5
Netherlands	5.5	10.6	5.3	8.8	5.9	12.8	6.5	11.1
Portugal	2.2	1.4	2.4	1.3	2.0	0.3	1.5	0.5
Spain	7.8	3.9	9.5	4.1	9.2	2.4	10.6	5.1
$\mathcal{R}(x)$	0.2		1.8		4.4		7.3	

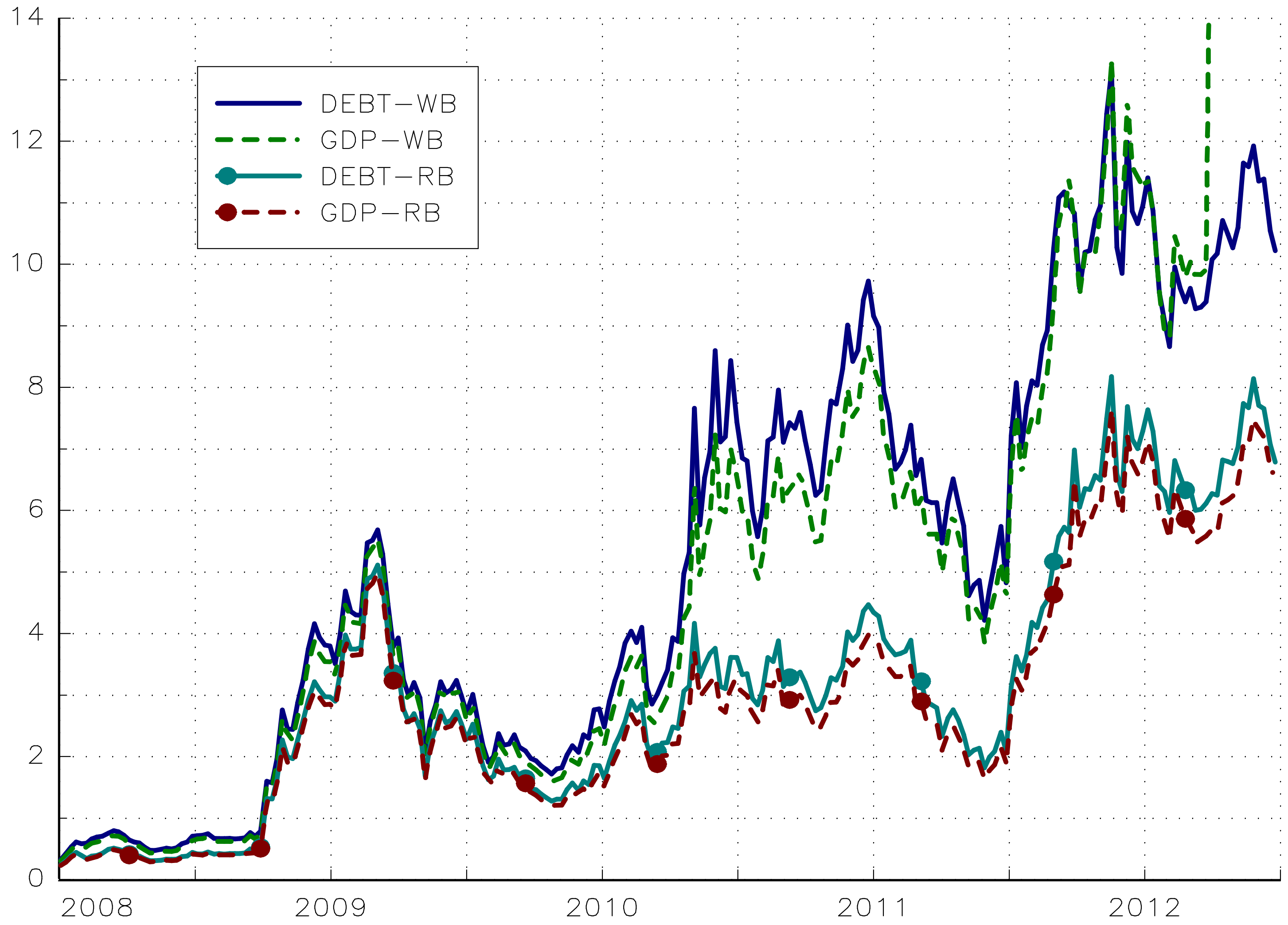
$\Rightarrow$ RB indexation is very different from WB indexation, in terms of weights, RC and credit risk measure.

$\Rightarrow$ The dynamics of the DEBT-RB is relatively smooth (yearly turnover = 89%).

Evolution of the weights (DEBT-RB indexation)

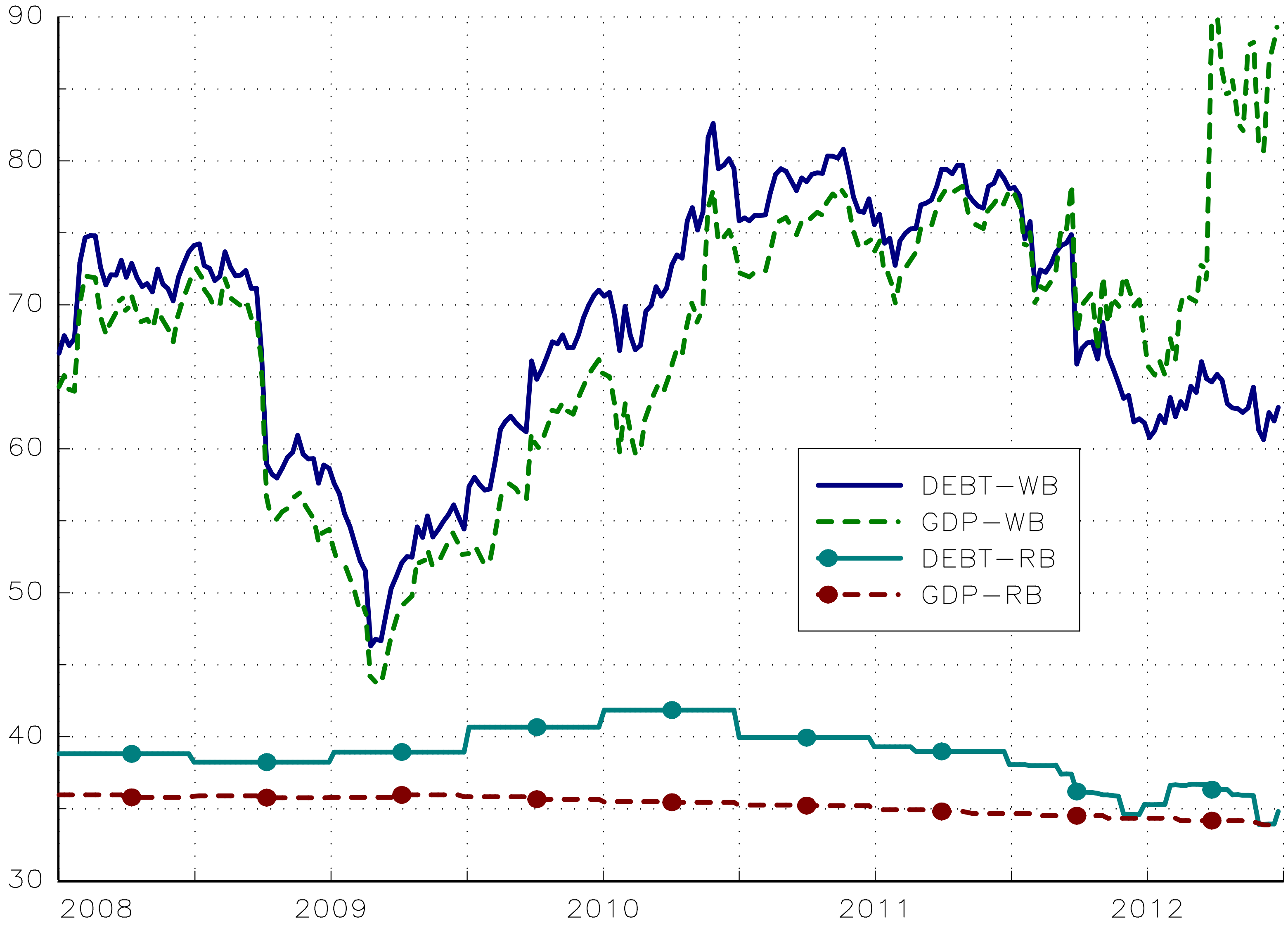


Dynamics of the credit risk measure (in %)

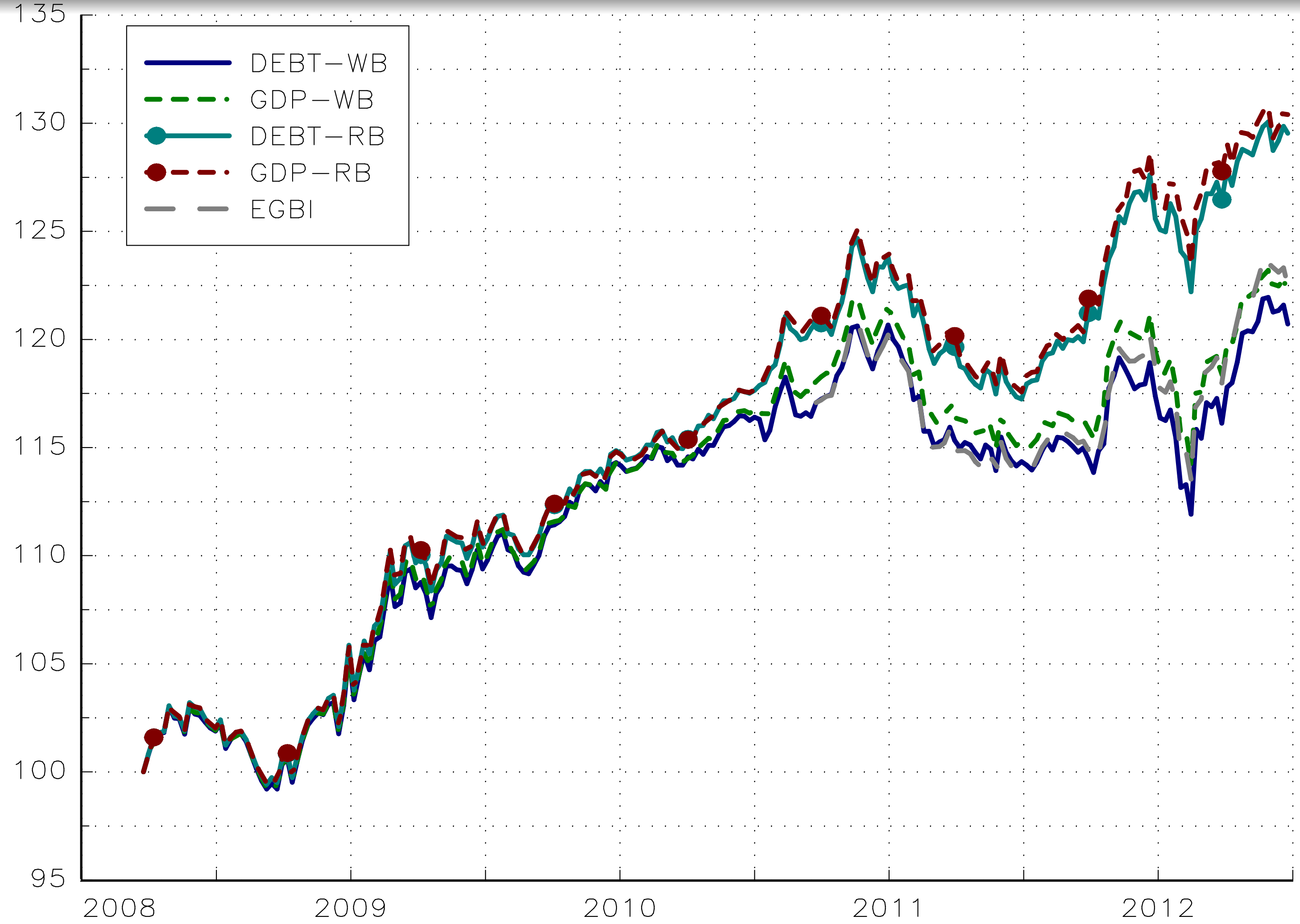


$\Rightarrow$ We verify the property: $\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{rb}}) \leq \mathcal{R}(x_{\mathrm{wb}})$.

Evolution of the GIIPS risk contribution (in %)

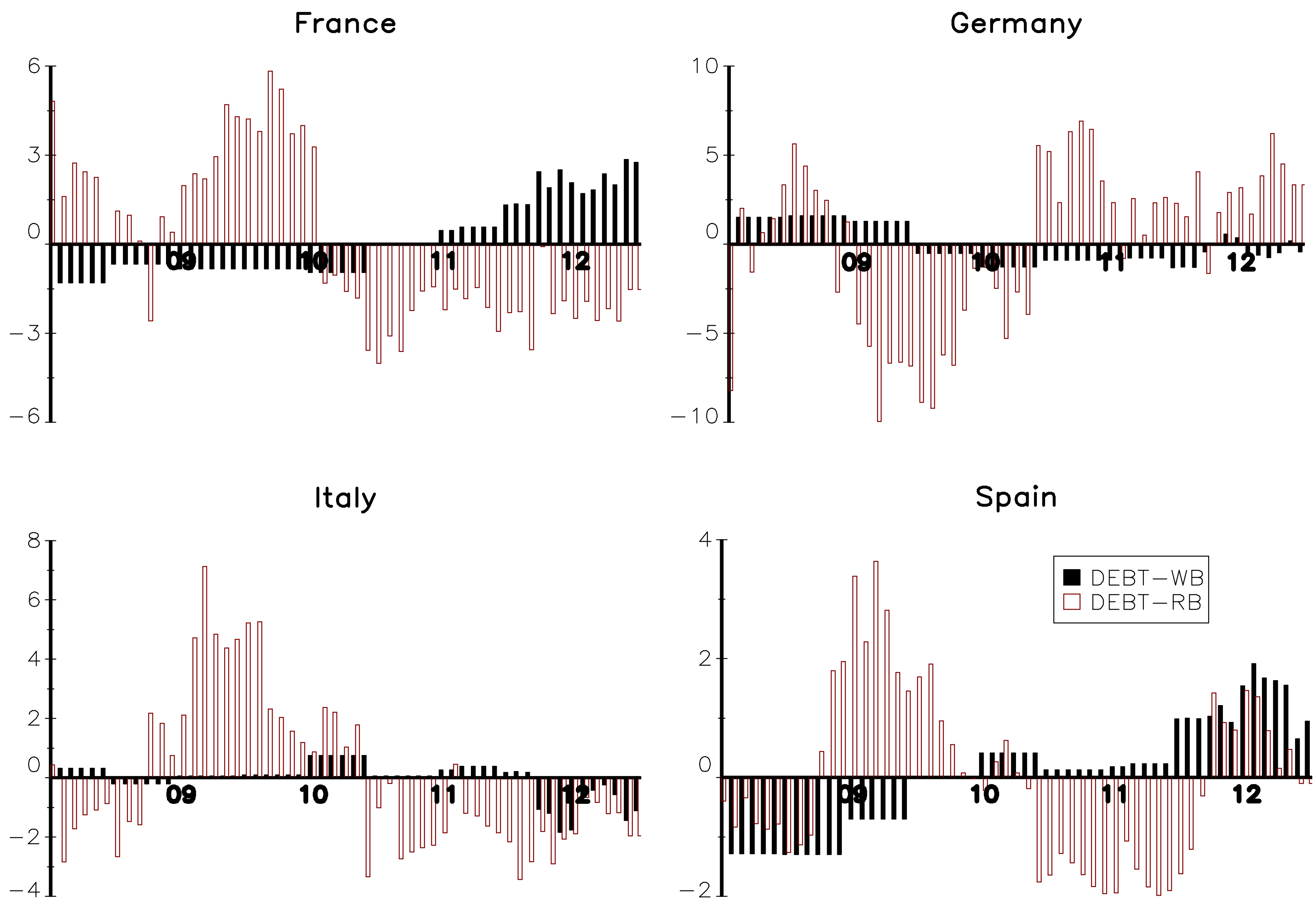


Simulated performance of the bond indexations



⇒ RB indexation / WB indexation = better performance, same volatility (credit risk ≠ interest risk) and smaller drawdowns.

Comparing the dynamic allocation for four countries



Justification of diversified funds

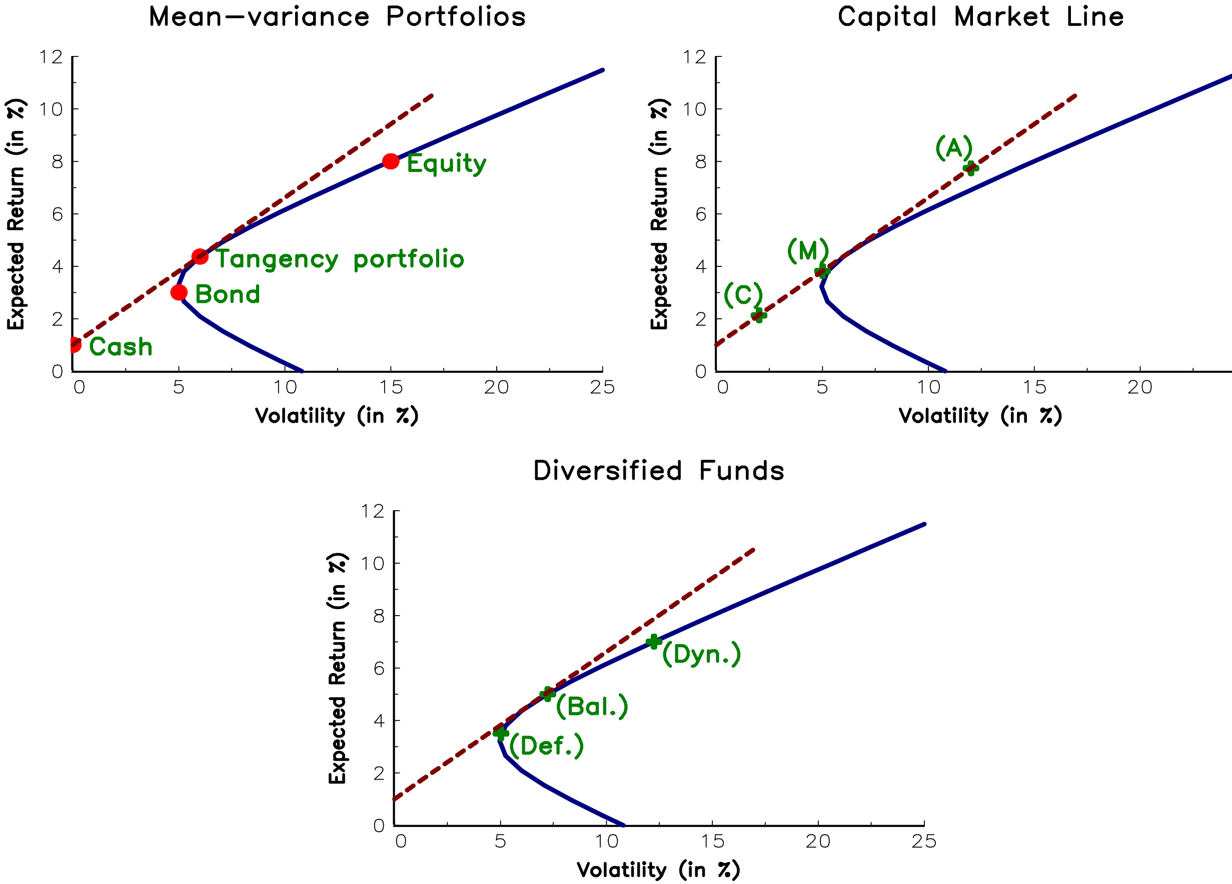
Investor Profiles

1. Conservative (low risk)
2. Moderate (medium risk)
3. Aggressive (high risk)

Fund Profiles

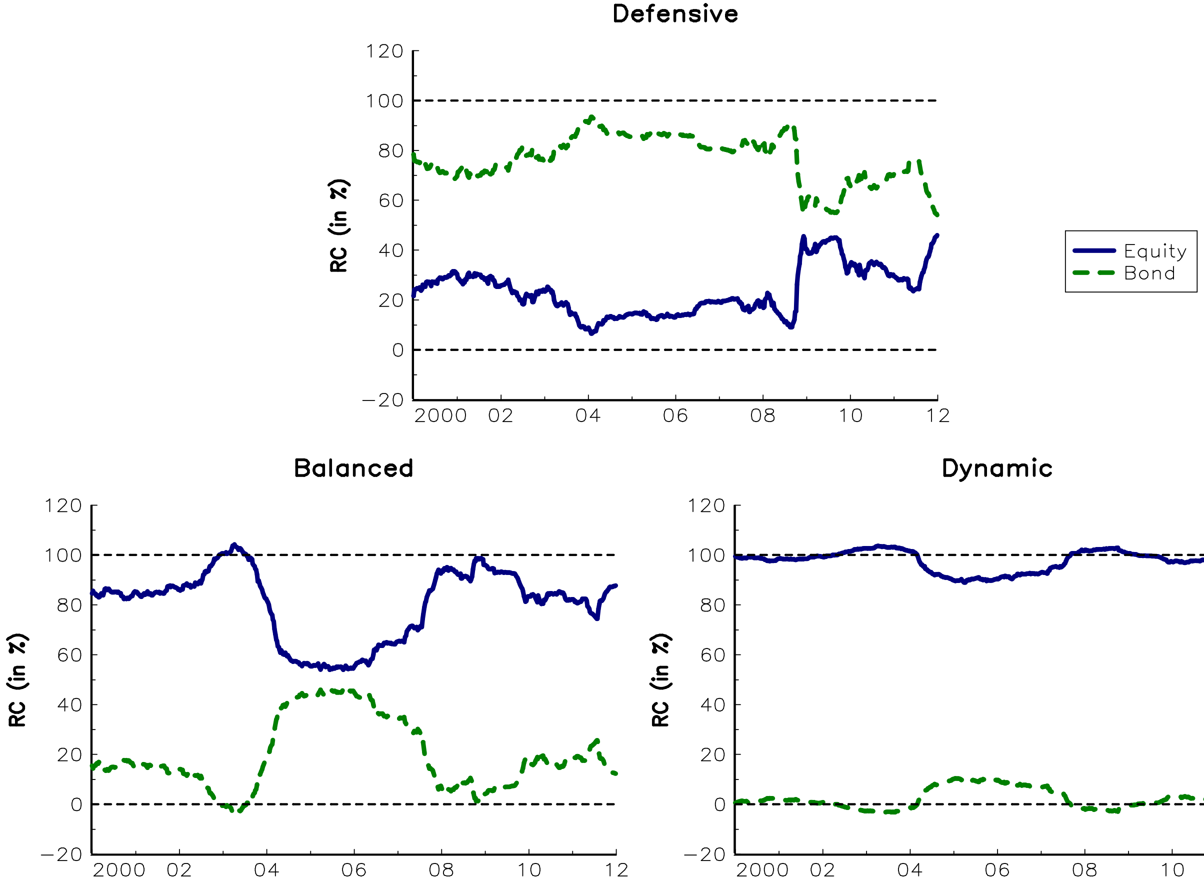
1. Defensive (20% equities and 80% bonds)
2. Balanced (50% equities and 50% bonds)
3. Dynamic (80% equities and 20% bonds)

Figure: The asset allocation puzzle



What type of diversification is offered by diversified funds?

Figure: Equity (MSCI World) and bond (WGBI) risk contributions



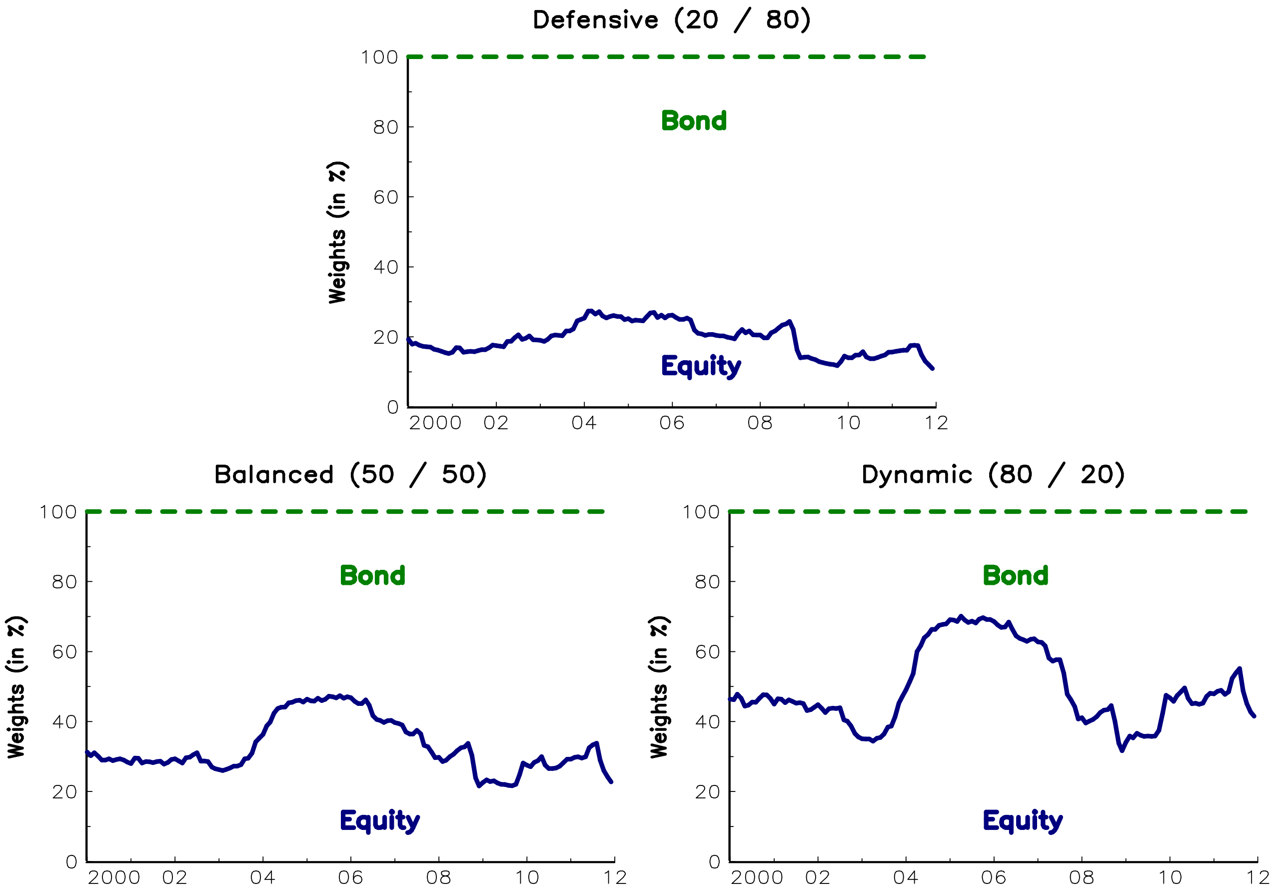
Diversified funds
=
Marketing idea?

- Contrarian constant-mix strategy
- Deleverage of an equity exposure
- Low risk diversification
- No mapping between fund profiles and investor profiles
- Static weights
- Dynamic risk contributions

Risk-balanced allocation

- Multi-dimensional target volatility strategy
- Trend-following portfolio (if negative correlation between return and risk)
- Dynamic weights
- Static risk contributions (risk budgeting)
- High diversification

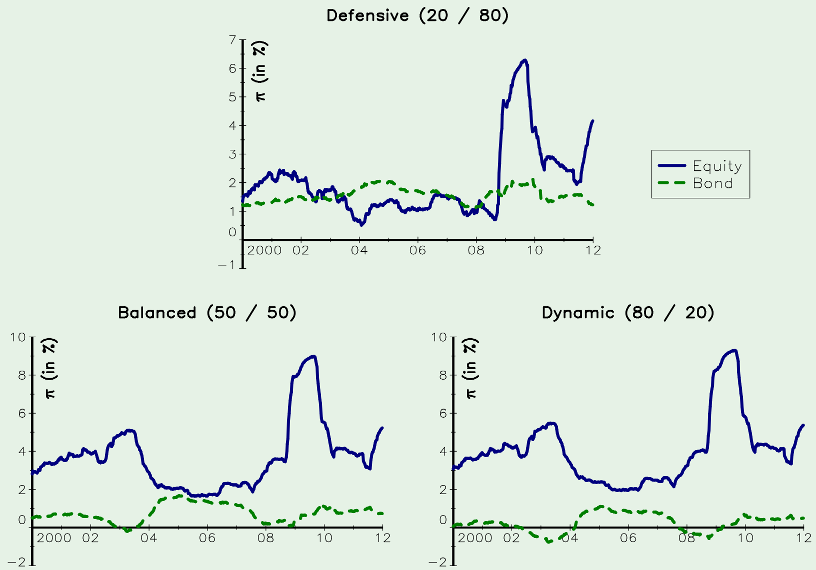
Figure: Equity and bond allocation



Are bonds growth or hedging assets?

Weight budgeting (diversified funds)

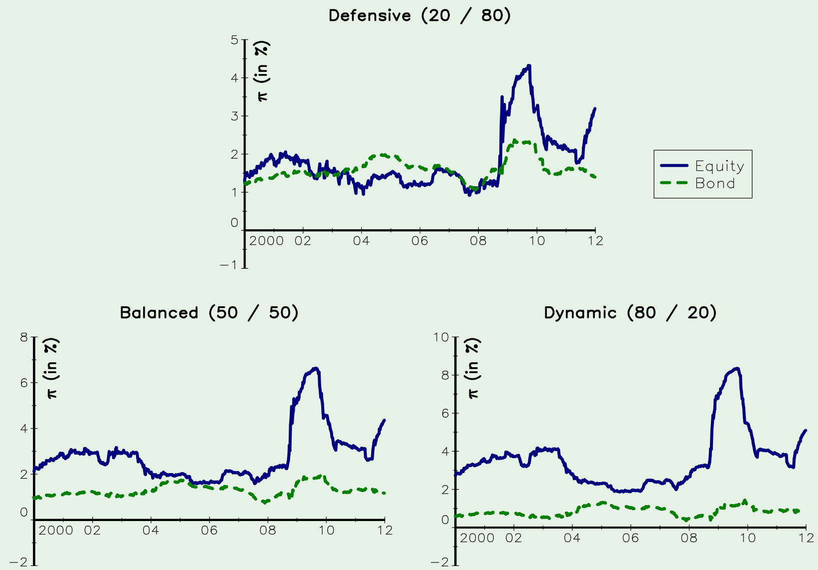
Equity and bond ex-ante risk premia



Asset	$\hat{\mu}(\tilde{\pi})$ Def.	Bal.	Dyn.
Equity	2.05	3.71	4.02
Bond	1.57	0.77	0.26

Risk budgeting (risk-balanced funds)

Equity and bond ex-ante risk premia



Asset	$\hat{\mu}(\tilde{\pi})$ Def.	Bal.	Dyn.
Equity	1.82	2.83	3.57
Bond	1.61	1.29	0.85

How to compare the performance of diversified and risk parity funds?

Table: Statistics of diversified and risk parity portfolios (2000-2012)

Portfolio	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2
Defensive	5.41	6.89	0.42	−17.23	0.19	2.67
Balanced	3.68	9.64	0.12	−33.18	−0.13	3.87
Dynamic	1.70	14.48	−0.06	−48.90	−0.18	5.96
Risk parity	5.12	7.29	0.36	−21.22	0.08	2.65
Static	4.71	7.64	0.29	−23.96	0.03	2.59
Leveraged RP	6.67	9.26	0.45	−23.74	0.01	0.78

- The 60/40 constant mix strategy is not the right benchmark.
- Results depend on the investment universe (number/granularity of asset classes).
- What is the impact of rising interest rates?

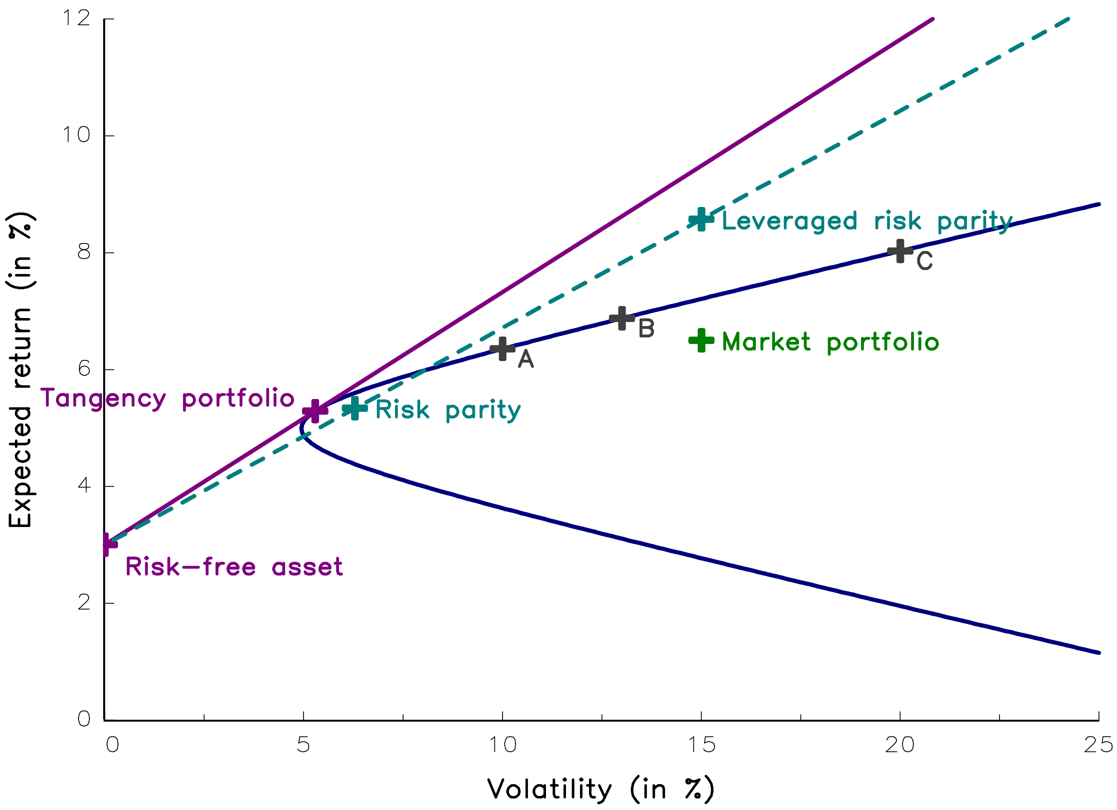
Leverage aversion theory and risk parity funds

- Borrowing constraints
- Cash constraints
- Impact on risk premia:

$$\mathbb{E}_t[R_{i,t+1}] - r = \alpha_i + \beta_i (\mathbb{E}_t[R_{t+1}(\bar{x})] - r)$$

- "*the alpha decreases in the beta* β_i"

Figure: Impact of leverage aversion on the efficient frontier



Strategic Asset Allocation

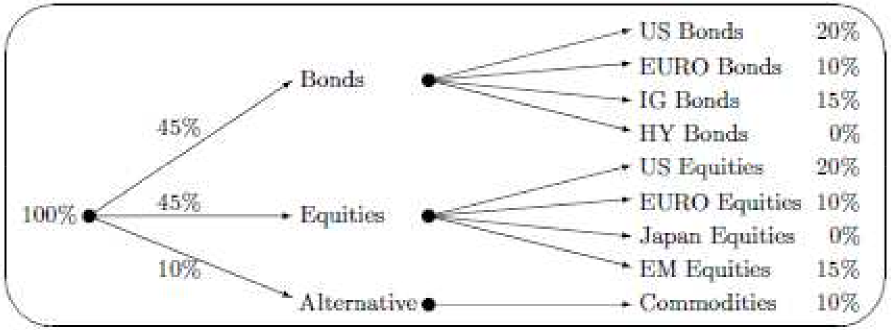
- Long-term investment policy (10-30 years)
- Capturing the risk premia of asset classes (equities, bonds, real estate, natural resources, etc.)
- Top-down macro-economic approach (based on short-run disequilibrium and long-run steady-state)

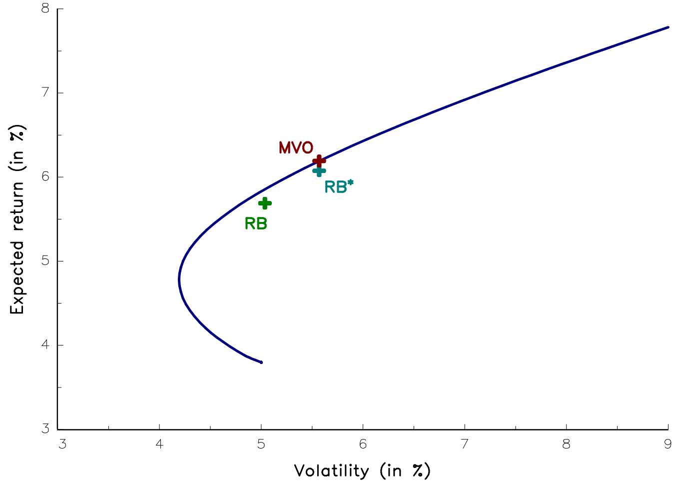
ATP Danish Pension Fund

> *"Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk" (Henrik Gade Jepsen, CIO of ATP, IPE, June 2012).*

These risk budgets are then transformed into asset classes' weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities (Source: FTfm, June 10, 2012).

Risk budgeting policy of a pension fund





Asset class	RB		RB*		MVO	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
US Bonds	36.8%	20.0%	45.9%	18.1%	66.7%	25.5%
EURO Bonds	21.8%	10.0%	8.3%	2.4%	0.0%	0.0%
IG Bonds	14.7%	15.0%	13.5%	11.8%	0.0%	0.0%
US Equities	10.2%	20.0%	10.8%	21.4%	7.8%	15.1%
Euro Equities	5.5%	10.0%	6.2%	11.1%	4.4%	7.6%
EM Equities	7.0%	15.0%	11.0%	24.9%	19.7%	49.2%
Commodities	3.9%	10.0%	4.3%	10.3%	1.5%	2.7%

RB* = A BL portfolio with a tracking error of 1% wrt RB / MVO = Markowitz portfolio with the RB* volatility

The framework of risk factor budgeting

- Combining the risk budgeting approach to define the asset allocation and the economic approach to define the factors.
- Following Eychenne *et al.* (2011), we consider 7 economic factors grouped into four categories:
  1. activity: gdp & industrial production;
  2. inflation: consumer prices & commodity prices;
  3. interest rate: real interest rate & slope of the yield curve;
  4. currency: real effective exchange rate.
- Quarterly data from Datastream.
- ML estimation using YoY relative variations for the study period Q1 1999 – Q2 2012.
- Risk measure: volatility.

Measuring risk factor contributions of SAA portfolios

- 13 AC: equity (US, EU, UK, JP), sovereign bonds (US, EU, UK, JP), corporate bonds (US, EU), High yield (US, EU) and US TIPS.
- Three given portfolios:
  - Portfolio #1 is a balanced stock/bond asset mix.
  - Portfolio #2 is a defensive allocation with 20% invested in equities.
  - Portfolio #3 is an aggressive allocation with 80% invested in equities.
- Portfolio #4 is optimized in order to take more inflation risk.

	Equity				Sovereign Bonds				Corp. Bonds		High Yield		TIPS
	US	EU	UK	JP	US	EU	UK	JP	US	EU	US	EU	US
#1	20%	20%	5%	5%	10%	5%	5%	5%	5%	5%	5%	5%	5%
#2	10%	10%			20%	15%	5%	5%	5%	5%	5%	5%	15%
#3	30%	30%	10%	10%	10%	10%							
#4	19.0%	21.7%	6.2%	2.3%		5.9%			24.1%	10.7%	2.6%	7.5%	

Factor	#1	#2	#3	#4
Activity	36.91%	19.18%	51.20%	34.00%
Inflation	12.26%	4.98%	9.31%	20.00%
Interest rate	42.80%	58.66%	32.92%	40.00%
Currency	7.26%	13.04%	5.10%	5.00%
Residual factors	0.77%	4.14%	1.47%	1.00%

Using the standard deviation-based risk measure

We remind that $\mathcal{R}(x) = -\mu(x) + c \cdot \sigma(x)$.

Table: Long-term strategic portfolios

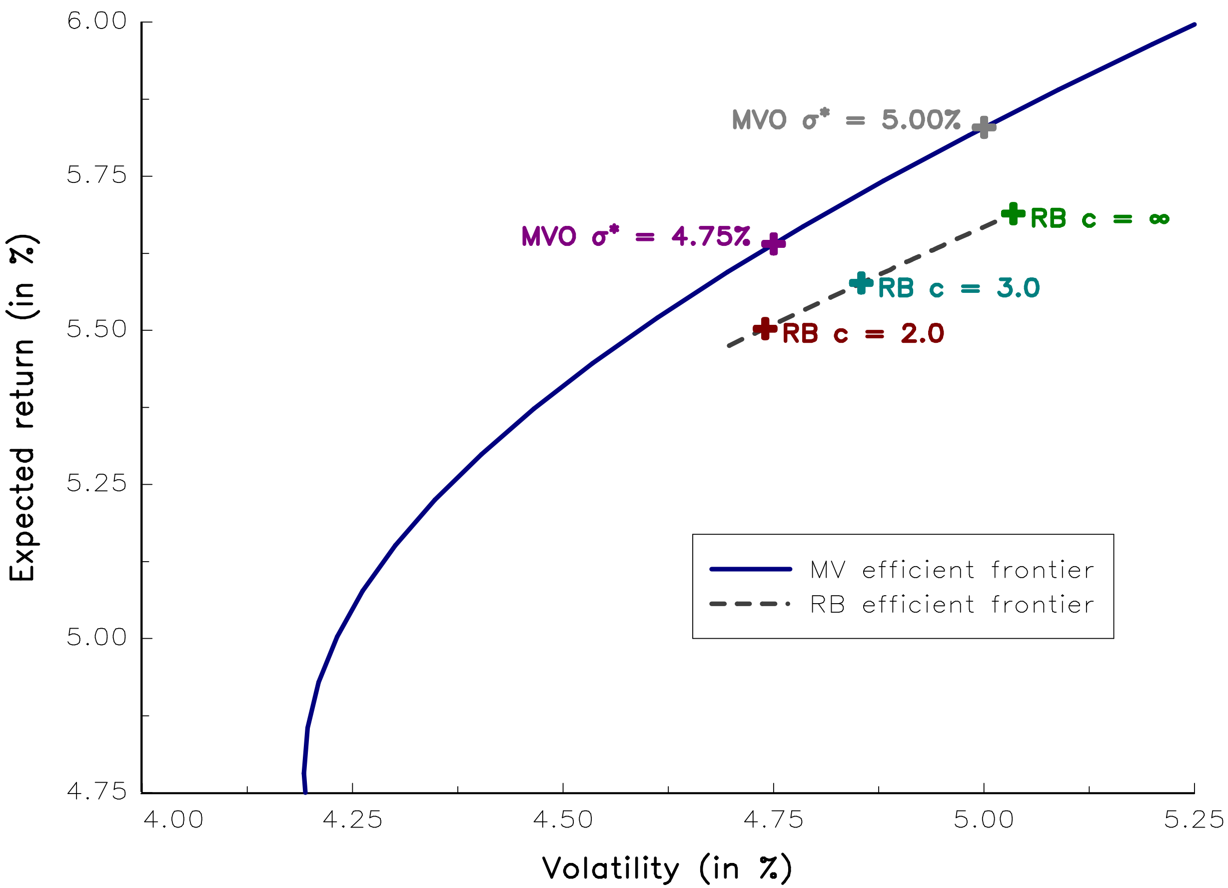
	RB						MVO			
	$c = \infty$		$c = 3$		$c = 2$		$\sigma^\star = 4.75\%$		$\sigma^\star = 5\%$	
	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$
(1)	36.8	20.0	38.5	23.4	39.8	26.0	60.5	38.1	64.3	34.6
(2)	21.8	10.0	23.4	12.3	24.7	14.1	14.0	7.4	7.6	3.2
(3)	14.7	15.0	13.1	14.0	11.7	12.8	0.0	0.0	0.0	0.0
(4)	10.2	20.0	9.5	18.3	8.9	17.1	5.2	10.0	5.5	10.8
(5)	5.5	10.0	5.2	9.2	4.9	8.6	5.2	9.2	5.5	9.8
(6)	7.0	15.0	6.9	14.5	7.0	14.4	14.2	33.7	16.0	39.5
(7)	3.9	10.0	3.4	8.2	3.0	6.9	1.0	1.7	1.1	2.1
$\mu(x)$	5.69		5.58		5.50		5.64		5.83	
$\sigma(x)$	5.03		4.85		4.74		4.75		5.00	
$\mathrm{SR}(x \mid r)$	1.13		1.15		1.16		1.19		1.17	

- RB portfolios have lower Sharpe ratios than MVO portfolios (by construction!), but the difference is small.
- RB portfolios are highly diversified, not MVO portfolios.
- Expected returns have a significant impact on the volatility contributions $\mathrm{VC}_i^\star$.

Using the standard deviation-based risk measure

- RB frontier is lower than MV frontier (because of the logarithmic barrier).
- $c = \infty$ corresponds to the RB portfolio with the highest volatility (and the highest expected return).
- $c \to \mathrm{SR}\left(x^\star \mid r\right)$ corresponds to the RB portfolio with the highest Sharpe ratio.

Figure: Efficient frontier of SAA portfolios



Risk parity and absolute return funds

The risk/return profile of risk parity funds is similar to that of diversified funds:

1. The drawdown is close to 20%
2. The Sharpe ratio is lower than 0.5

$\Rightarrow$ The (traditional) risk parity approach is not sufficient to build an absolute return fund.

How to transform it to an absolute return strategy?

1. By incorporating some views on economics and asset classes (global macro fund, e.g. the All Weather fund of Bridgewater)
2. By introducing trends and momentum patterns (long term CTA)
3. By defining a more dynamic allocation

From risk budgeting to tactical asset allocation

There are two traditional ways to incorporate the expected returns in risk parity portfolios:

1. The first method consists of defining the risk budgets according to the expected returns:
$$b_i = f(\mu_i)$$
where f is an increasing function. It implies that we allocate more risk to assets that have better expected returns.
2. The second method consists of modifying the weights of the RB portfolio. To do this, we generally use the Black-Litterman model or the tracking error (TE) model.

The risk budgeting solution

A more consistent solution is to choose a risk measure that depends on expected returns:
$$\mathcal{R}(x) = -\mu(x) + c \cdot \sigma(x)$$

Impact of expected returns on the allocation

- 3 assets.
- $\sigma_1 = 15\%$, $\sigma_2 = 20\%$ and $\sigma_3 = 25\%$.
- $\rho_{1,2} = 30\%$, $\rho_{1,3} = 50\%$ and $\rho_{2,3} = 70\%$.
- 5 parameter sets of expected returns:

Set	#1	#2	#3	#4	#5
μ_1	0%	0%	20%	0%	0%
μ_2	0%	10%	10%	−20%	30%
μ_3	0%	20%	0%	−20%	−30%

- ERC portfolios with $c = 2$

Set	#1	#2	#3	#4	#5
x_1	45.25	37.03	64.58	53.30	29.65
x_2	31.65	33.11	24.43	26.01	63.11
x_3	23.10	29.86	10.98	20.69	7.24
$\mathcal{VC}_1^\star$	33.33	23.80	60.96	43.79	15.88
$\mathcal{VC}_2^\star$	33.33	34.00	23.85	26.32	75.03
$\mathcal{VC}_3^\star$	33.33	42.20	15.19	29.89	9.09
$\sigma(x)$	15.35	16.22	14.11	14.89	16.00

Calibrating the scaling factor

In a TAA model, the risk measure is no longer static:

$$\mathcal{R}_t(x_t) = -x_t^\top \mu_t + c_t \cdot \sqrt{x_t^\top \Sigma_t x_t}$$

c_t can not be constant because:

1. the solution may not exist[9].
2. this rule is time-inconsistent (1Y $\neq$ 1M):

$$\begin{aligned}\mathcal{R}_t(x_t; c, h) &= -h \cdot x_t^\top \mu_t + c\sqrt{h} \cdot \sqrt{x_t^\top \Sigma_t x_t} \\ &= h \cdot \mathcal{R}_t(x_t; c', 1)\end{aligned}$$

with $c' = h^{-0.5} c$.

[9] There is no solution if $c = \Phi^{-1}(99\%)$ and the maximum Sharpe ratio is 3.

An illustration

$$c_t = \max\left(c_{\mathrm{ES}}(99.9\%), 2.00 \cdot \mathrm{SR}_t^+\right) \qquad \text{(RP \#1)}$$

$$c_t = \max\left(c_{\mathrm{VaR}}(99\%), 1.10 \cdot \mathrm{SR}_t^+\right) \qquad \text{(RP \#2)}$$

$$c_t = 1.10 \cdot \mathrm{SR}_t^+ \cdot \mathbb{1}\left\{\mathrm{SR}_t^+ > 0\right\} + \infty \cdot \mathbb{1}\left\{\mathrm{SR}_t^+ \leq 0\right\} \qquad \text{(RP \#3)}$$

1. Empirical covariance matrix (260 days)
2. Simple moving average based on the daily returns (260 days)

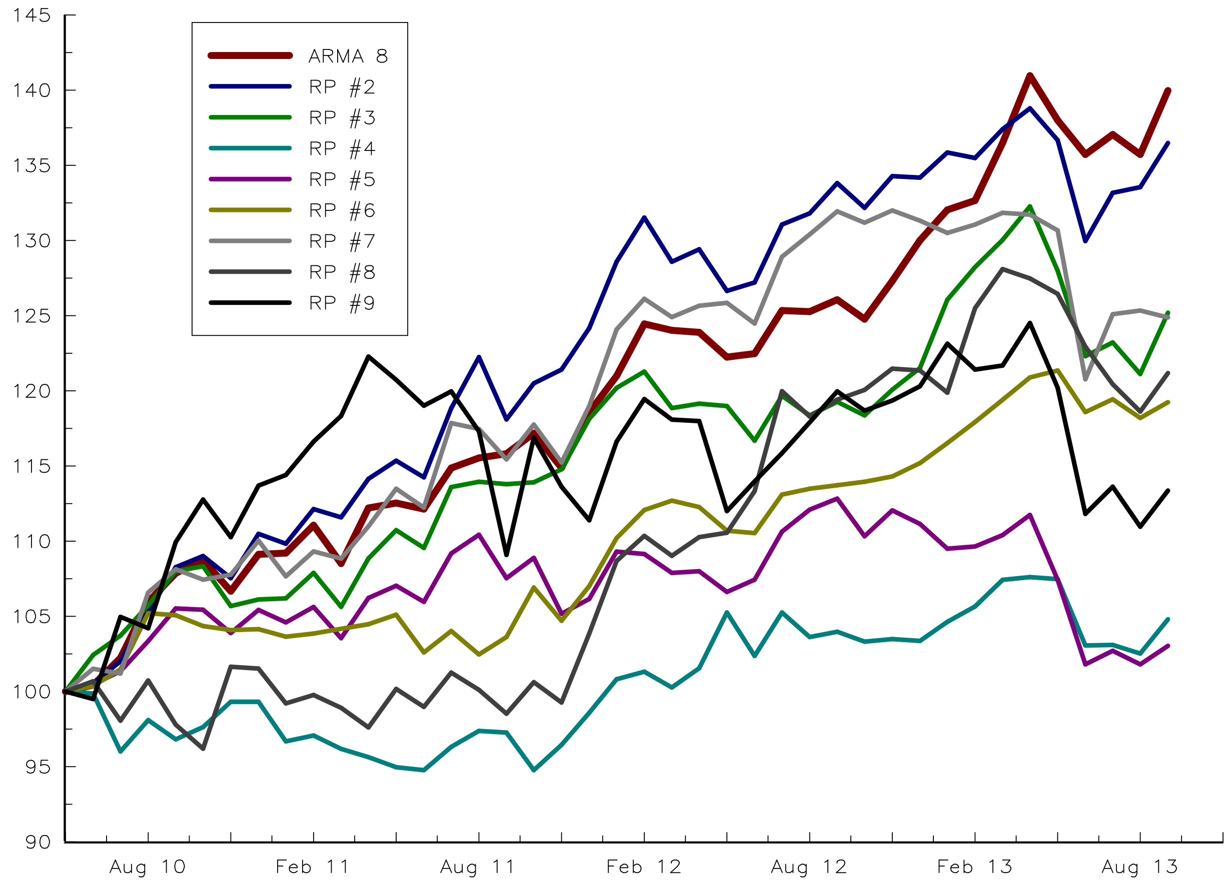
Table: Statistics of dynamic risk parity strategies

RP		$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2	τ
Static	#0	5.10	7.30	0.35	−21.39	0.07	2.68	0.30
	#1	5.68	7.25	0.44	−18.06	0.10	2.48	1.14
Active	#2	6.58	7.80	0.52	−12.78	0.05	2.80	2.98
	#3	7.41	8.00	0.61	−12.84	0.04	2.74	3.65

All risk parity funds are not alike

- Choice of the investment universe
- Choice of the risk budgets
- Choice of the TAA model
- Choice of the leverage implementation
- Choice of the rebalancing frequency
- etc.

Figure: Performance of RP funds



Conclusion

- Portfolio optimization leads to concentrated portfolios in terms of weights and risk.
- The use of weights constraints to diversify is equivalent to a discretionary shrinkage method.
- The risk parity approach is a better method to diversify portfolios and to capture risk premia (Risk parity = **risk premium** parity).
- It is a good candidate to define a neutral allocation.
- But it is not a magic allocation method:

  "It cannot free investors of their duty of making their own choices".

For Further Reading

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