Portfolio Allocation with Skewness Risk

Benjamin Bruder, Nazar Kostyuchyk and Thierry Roncalli

*Lyxor Asset Management\textsuperscript{1}, France

EDHEC Business School

Nice, October 6, 2016

\textsuperscript{1}The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Lyxor Asset Management.
The skewness puzzle

- Alternative risk premia = extension of equity factor investing to other asset classes (in a long/short format)

- Alternative risk premia encompasses two different types of risk factor:
  - Skewness risk premia (= pure risk premia)
  - Market anomalies (≠ risk premia)

- ARP (in particular skewness risk premia) are not all-weather strategies:
  - Extreme risks of ARP are high and may be correlated
  - Aggregation of skewness is not straightforward
  - Skewness diversification ≠ volatility diversification

\[
\sigma(X + Y) \leq \sigma(X) + \sigma(X) \quad \gamma_1(X + Y) \neq \gamma_1(X) + \gamma_1(X)
\]
The skewness puzzle

Figure: Skewness aggregation of L/S alternative risk premia

Worst skewness coefficients $\gamma^*_i$

Largest drawdown ratios $DD^*$

MSCI World index
The skewness puzzle

Figure: Skewness aggregation in the case of the bivariate log-normal distribution

\[ \gamma_1(X) = -1.8, \gamma_1(Y) = -1.8 \]

\[ \gamma_1(X) = -1.8, \gamma_1(Y) = -6.2 \]

\[ \gamma_1(X) = -0.8, \gamma_1(Y) = -0.3 \]

\[ \gamma_1(X) = -0.6, \gamma_1(Y) = -0.6 \]
Summary I

Recent trends in asset management

- Risk parity
- Equity factor investing
- Alternative risk premia

⇒ These 3 topics are related to the concept of diversification.

What is the issue?

- Risk parity = volatility risk measure
- Equity factor investing = distressed risk (default/liquidity)
- Alternative risk premia = skewness risk premia + market anomalies

⇒ Skewness risk?

Skewness diversification ≠ volatility diversification
Figure: Cumulative performance of US bonds, US equities and US short volatility.
Summary III

Figure: Volatility-based ERC portfolio
Figure: Skewness-based ERC portfolio
Figure: Comparison of the carry allocation
Summary VI

- **Factor investing**
  - The allocation in size and value risk factors is generally overestimated.

- **Alternative Risk premia**
  - The turnover issue
  - How to allocate between skewness risk premia and market anomalies?
  - Relevance of the trend-following strategy
  - Momentum crashes?

- **Skewness hedging vs volatility hedging**
  - It is difficult to diversify the skewness risk
  - Volatility optimization leads to non-optimal portfolios
  - Sizing the risk exposure is the only solution
  - Mixing skewness risk premia and market anomalies
  - The case of CTA strategies
The jump-diffusion representation

- $n$ risky assets represented by the vector of prices $S_t = (S_{1,t}, \ldots, S_{n,t})$

with:
\[
\begin{align*}
    dS_t &= \text{diag}(S_t) \, dL_t \\
    dL_t &= \mu \, dt + \Sigma^{1/2} \, dW_t + dZ_t
\end{align*}
\]

where $Z_t$ is a pure $n$-dimensional jump process.

- We assume that the jump process $Z_t$ is a compound Poisson process:
\[
Z_t = \sum_{i=1}^{N_t} Z_i
\]

where $N_t \sim \mathcal{P}(\lambda)$ and $Z_i \sim \mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$.

The characteristic function of asset returns $R_t = (R_{1,t}, \ldots, R_{n,t})$ for the holding period $dt$ may be approximated by:
\[
\mathbb{E} \left[ e^{-iu \cdot R_t} \right] \approx (1 - \lambda dt) \cdot e^{(iu^T \mu - \frac{1}{2} u^T \Sigma u) \, dt} + (\lambda dt) \cdot e^{iu^T (\mu dt + \tilde{\mu}) - \frac{1}{2} u^T (\Sigma dt + \tilde{\Sigma}) u}
\]
The Gaussian mixture representation

We consider a Gaussian mixture model with two regimes to define $R_t$:

1. The continuous component, which has the probability $(1 - \lambda dt)$ to occur, is driven by the Gaussian distribution $\mathcal{N}(\mu dt, \Sigma dt)$;
2. The jump component, which has the probability $\lambda dt$ to occur, is driven by the Gaussian distribution $\mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$.

The multivariate density function of $R_t$ is:

$$f(y) = \frac{1 - \lambda dt}{(2\pi)^{n/2} |\Sigma dt|^{1/2}} e^{-\frac{1}{2} (y - \mu dt)^\top (\Sigma dt)^{-1} (y - \mu dt) + \lambda dt} e^{-\frac{1}{2} (y - (\mu dt + \tilde{\mu}))^\top (\Sigma dt + \tilde{\Sigma})^{-1} (y - (\mu dt + \tilde{\mu}))}$$

The characteristic function of $R_t$ is equal to:

$$\mathbb{E} \left[ e^{-iu \cdot R_t} \right] = (1 - \lambda dt) \cdot e^{(iu^\top \mu - \frac{1}{2} u^\top \Sigma u) dt} + (\lambda dt) \cdot e^{iu^\top (\mu dt + \tilde{\mu}) - \frac{1}{2} u^\top (\Sigma dt + \tilde{\Sigma}) u}$$
Distribution function of the portfolio’s return

Let \( x = (x_1, \ldots, x_n) \) be the vector of weights in the portfolio. We have:

\[
R(x) = Y = B_1 \cdot Y_1 + B_2 \cdot Y_2
\]

where:

- \( B_1 \sim B(\pi_1), \ B_2 = 1 - B_1 \sim B(\pi_2), \ \pi_1 = 1 - \lambda \) and \( \pi_2 = \lambda \) \( (\mathcal{H} : \mathrm{d}t = 1) \);
- \( Y_1 \sim \mathcal{N}(\mu_1(x), \sigma_1^2(x)), \ \mu_1(x) = x^\top \mu \) and \( \sigma_1^2(x) = x^\top \Sigma x \);
- \( Y_2 \sim \mathcal{N}(\mu_2(x), \sigma_2^2(x)), \ \mu_2(x) = x^\top (\mu + \tilde{\mu}) \) and \( \sigma_2^2(x) = x^\top (\Sigma + \tilde{\Sigma}) x \).

⇒ The portfolio’s return \( R(x) \) has the following density function:

\[
f(y) = \pi_1 f_1(y) + \pi_2 f_2(y)
= (1 - \lambda) \frac{1}{\sigma_1(x)} \phi \left( \frac{y - \mu_1(x)}{\sigma_1(x)} \right) + \lambda \frac{1}{\sigma_2(x)} \phi \left( \frac{y - \mu_2(x)}{\sigma_2(x)} \right)
\]
Parameters: \( \mu_1 = 5\% \), \( \sigma_1 = 10\% \), \( \tilde{\mu}_1 = -20\% \), \( \tilde{\sigma}_1 = 5\% \), \( \mu_2 = 10\% \), \( \sigma_2 = 20\% \),
\( \tilde{\mu}_2 = -40\% \), \( \tilde{\sigma}_2 = 5\% \), \( \rho = 50\% \), \( \tilde{\rho} = 60\% \) and \( \lambda = 0.20 \).
The skewness of $R(x)$ is equal to:

$$
\gamma_1 = \frac{(\lambda - \lambda^2) \left( (1 - 2\lambda) (x^T \tilde{\mu})^3 + 3 (x^T \tilde{\mu}) (x^T \tilde{\Sigma} x) \right)}{\left( x^T \Sigma x + \lambda x^T \tilde{\Sigma} x + (\lambda - \lambda^2) (x^T \tilde{\mu})^2 \right)^{3/2}}
$$

The portfolio exhibits skewness, except for some limit cases:

$$
\gamma_1 = 0 \Leftrightarrow x^T \tilde{\mu} = 0 \text{ or } \lambda = 0 \text{ or } \lambda = 1
$$

We have:

- If $x^T \tilde{\mu} > 0$, then $\gamma_1 > 0$;
- If $x^T \tilde{\mu} < 0$, then $\gamma_1 < 0$ in most cases.

$\Rightarrow$ We retrieve the result of Hamdan et al. (2016):

**Skewness risk is maximum when volatility risk is minimum**
A Skewness Model of Asset Returns
Risk Parity Portfolios with Jumps
The Equity/Bond/Volatility Asset Mix Policy
Factor Investing, Alternative Risk Premia and Skewness Hedging

The Jump Model
The Mixture Model
Relationship between jump risk and skewness risk
Estimation of the parameters

Relationship between jump risk and skewness risk

Parameters: $\sigma = 20\%$, $\tilde{\mu} = -40\%$, $\tilde{\sigma} = 20\%$ and $\lambda = 25\%$. 
Estimation of the parameters

With the notations $\mu_1 = \mu$, $\Sigma_1 = \Sigma$, $\mu_2 = \mu + \tilde{\mu}$, $\Sigma_2 = \Sigma + \tilde{\Sigma}$, the log-likelihood function becomes:

$$
\ell(\theta) = \sum_{t=1}^{T} \ln \sum_{j=1}^{2} \pi_j \phi_n (R_t; \mu_j, \Sigma_j)
$$

⇒ The method of maximum likelihood is not suitable (weak identification)
⇒ Parameters are estimated using the EM algorithm

Posterior probability to have a jump at time $t$

We have:

$$
\pi_{2,t} = \Pr \{ B_2 = 1 \mid R_t \} = \frac{\pi_2 \phi_n (R_t; \mu_2, \Sigma_2)}{\sum_{s=1}^{2} \pi_s \phi_n (R_t; \mu_s, \Sigma_s)}
$$
The expected shortfall risk measure

Definition of the expected shortfall

\[ ES_\alpha (x) = \mathbb{E}[L(x) \mid L(x) \geq \text{VaR}_\alpha (x)] \]

where \( L(x) = -R(x) \) is the portfolio’s loss.

We obtain:

\[ ES_\alpha (x) = (1 - \lambda) \cdot \varphi (\text{VaR}_\alpha (x), \mu_1 (x), \sigma_1 (x)) + \lambda \cdot \varphi (\text{VaR}_\alpha (x), \mu_2 (x), \sigma_2 (x)) \]

where the function \( \varphi (a, b, c) \) is defined by:

\[ \varphi (a, b, c) = \frac{c}{1 - \alpha} \phi \left( \frac{a + b}{c} \right) - \frac{b}{1 - \alpha} \Phi \left( -\frac{a + b}{c} \right) \]

Here, the value-at-risk \( \text{VaR}_\alpha (x) \) is the root of the following equation:

\[ (1 - \lambda) \cdot \Phi \left( \frac{\text{VaR}_\alpha (x) + \mu_1 (x)}{\sigma_1 (x)} \right) + \lambda \cdot \Phi \left( \frac{\text{VaR}_\alpha (x) + \mu_2 (x)}{\sigma_2 (x)} \right) = \alpha \]
Analytical expression of risk contributions

We obtain a complicated expression of the risk contribution:

$$\mathcal{RC}_i(x) = x_i \frac{\partial \text{ES}_\alpha(x)}{\partial x_i} = \ldots$$

But it is an analytical formula!

⇒ No numerical issues for implementing the model

Euler decomposition

We have:

$$\sum_{i=1}^{n} \mathcal{RC}_i(x) = \text{ES}_\alpha(x)$$

⇒ Comparison with the value-at-risk based on the Cornish-Fisher expansion
Risk decomposition

Example

We consider three assets, whose expected returns are equal to 10%, 15% and 20%. Their volatilities are equal to 20%, 25% and 30% while the correlation matrix of asset returns is provided by the following matrix:

\[
\rho = \begin{pmatrix}
1.00 & 0.50 & 0.20 \\
0.50 & 1.00 & 0.40 \\
0.20 & 0.40 & 1.00
\end{pmatrix}
\]

For the jumps, we assume that \( \tilde{\mu}_i = -10\% \), \( \tilde{\sigma}_i = 20\% \) and \( \tilde{\rho}_{i,j} = 50\% \). Moreover, the intensity \( \lambda \) of jumps is equal to 0.25, meaning that we observe a jump every four years on average.
Risk decomposition

Table: Expected shortfall decomposition (without jumps)

<table>
<thead>
<tr>
<th>Asset</th>
<th>$x_i$</th>
<th>$MR_i$</th>
<th>$RC_i$</th>
<th>$RC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.00</td>
<td>8.90</td>
<td>1.78</td>
<td>6.26</td>
</tr>
<tr>
<td>2</td>
<td>20.00</td>
<td>18.22</td>
<td>3.64</td>
<td>12.80</td>
</tr>
<tr>
<td>3</td>
<td>60.00</td>
<td>38.40</td>
<td>23.04</td>
<td>80.94</td>
</tr>
<tr>
<td>ES$_\alpha$ ($x$)</td>
<td></td>
<td></td>
<td>28.46</td>
<td></td>
</tr>
</tbody>
</table>

Table: Expected shortfall decomposition (with jumps)

<table>
<thead>
<tr>
<th>Asset</th>
<th>$x_i$</th>
<th>$MR_i$</th>
<th>$RC_i$</th>
<th>$RC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20.00</td>
<td>20.39</td>
<td>4.08</td>
<td>10.96</td>
</tr>
<tr>
<td>2</td>
<td>20.00</td>
<td>27.31</td>
<td>5.46</td>
<td>14.67</td>
</tr>
<tr>
<td>3</td>
<td>60.00</td>
<td>46.13</td>
<td>27.68</td>
<td>74.37</td>
</tr>
<tr>
<td>ES$_\alpha$ ($x$)</td>
<td></td>
<td></td>
<td>37.22</td>
<td></td>
</tr>
</tbody>
</table>
The RB portfolio is defined by the following non-linear system:

\[
\begin{align*}
RC_i(x) &= b_i R(x) \\
0 &< b_i \\
x_i &> 0 \\
\sum_{i=1}^{n} b_i &= 1 \\
\sum_{i=1}^{n} x_i &= 1
\end{align*}
\]

where \( b_i \) is the ex-ante risk budget of asset \( i \) expressed in relative terms.

**Numerical solution of the RB portfolio**

\[
y^* = \arg\min_{y} ES_{\alpha}(y) - \sum_{i=1}^{n} b_i \ln y_i \quad \text{u.c.} \quad y \geq 0
\]

The RB portfolio corresponds to the normalized portfolio:

\[
x_i^* = \frac{y_i^*}{\sum_{j=1}^{n} y_j^*}
\]
Existence and uniqueness

We have to impose the following restriction:

\[ R(x) = ES_\alpha(x) \geq 0 \]

The issue comes from the homogeneity property \( R(\delta x) = \delta R(x) \) where \( \delta \) is a positive scalar.

**Theorem**

If \( \alpha \geq \max (\alpha^-, \lambda) \), the RB portfolio exists and is unique, where \( \alpha^- \) be the root of the equation below:

\[
\frac{1 - \lambda}{1 - \alpha^-} \Phi \left( \Phi^{-1} \left( \frac{\alpha^- - \lambda}{1 - \lambda} \right) \right) + \lambda \Phi^{-1} \left( \frac{\alpha^- - \lambda}{1 - \lambda} \right) = (1 + \lambda) SR_1^+
\]

and \( SR_1^+ \) is the maximum Sharpe ratio under the first regime.
Figure: Relationship between $\lambda$, $SR_1^+$ and $\alpha^-$
Example

We have $\mu_1 = 3\%$, $\mu_2 = 8\%$, $\mu_3 = 12\%$, $\sigma_1 = 8\%$, $\sigma_2 = 20\%$, $\sigma_3 = 30\%$ and:

$$
\rho = \begin{pmatrix}
1.00 \\
0.50 & 1.00 \\
0.20 & 0.40 & 1.00
\end{pmatrix}
$$

For the jumps, we have $\tilde{\mu}_1 = -15\%$, $\tilde{\mu}_2 = -40\%$, $\tilde{\mu}_3 = 0\%$, $\tilde{\sigma}_1 = 15\%$, $\tilde{\sigma}_2 = 20\%$, $\tilde{\sigma}_3 = 10\%$ and:

$$
\tilde{\rho} = \begin{pmatrix}
1.00 \\
0.50 & 1.00 \\
0.00 & 0.00 & 1.00
\end{pmatrix}
$$

The intensity $\lambda$ of jumps is equal to 0.25.
## ERC portfolio

**Table:** ERC portfolio (Gaussian risk measure)

<table>
<thead>
<tr>
<th>Asset</th>
<th>volatility risk measure</th>
<th>95% expected shortfall</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_i )</td>
<td>( MR_i )</td>
</tr>
<tr>
<td>1</td>
<td>60.94</td>
<td>5.96</td>
</tr>
<tr>
<td>2</td>
<td>22.20</td>
<td>16.35</td>
</tr>
<tr>
<td>3</td>
<td>16.87</td>
<td>21.52</td>
</tr>
<tr>
<td>( \sigma(x) )</td>
<td>10.89</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** ERC portfolio (95% expected shortfall with jumps)

<table>
<thead>
<tr>
<th>Asset</th>
<th>( x_i )</th>
<th>( MR_i )</th>
<th>( RC_i )</th>
<th>( RC^*_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.70</td>
<td>24.70</td>
<td>11.04</td>
<td>33.33</td>
</tr>
<tr>
<td>2</td>
<td>19.87</td>
<td>55.56</td>
<td>11.04</td>
<td>33.33</td>
</tr>
<tr>
<td>3</td>
<td>35.42</td>
<td>31.17</td>
<td>11.04</td>
<td>33.33</td>
</tr>
<tr>
<td>( ES_\alpha(x) )</td>
<td></td>
<td></td>
<td></td>
<td>33.12</td>
</tr>
</tbody>
</table>
Figure: Cumulative performance of US bonds, US equities and US short volatility
Statistics

Table: Worst returns (in %)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Daily</th>
<th>Weekly</th>
<th>Monthly</th>
<th>Annually</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>-1.67</td>
<td>-2.81</td>
<td>-4.40</td>
<td>-3.41</td>
<td>-6.03</td>
</tr>
<tr>
<td>Equities</td>
<td>-9.03</td>
<td>-18.29</td>
<td>-29.67</td>
<td>-49.69</td>
<td>-55.25</td>
</tr>
<tr>
<td>Carry</td>
<td>-6.82</td>
<td>-11.04</td>
<td>-23.43</td>
<td>-23.37</td>
<td>-27.30</td>
</tr>
</tbody>
</table>

Table: Skewness coefficients

<table>
<thead>
<tr>
<th>Asset</th>
<th>Daily</th>
<th>Weekly</th>
<th>Monthly</th>
<th>Annually</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>-0.12</td>
<td>-0.17</td>
<td>0.07</td>
<td>0.22</td>
<td>4.17</td>
</tr>
<tr>
<td>Equities</td>
<td>0.01</td>
<td>-0.44</td>
<td>-0.81</td>
<td>-0.57</td>
<td>18.38</td>
</tr>
<tr>
<td>Carry</td>
<td>-7.24</td>
<td>-5.77</td>
<td>-6.32</td>
<td>-2.23</td>
<td>5.50</td>
</tr>
</tbody>
</table>
A Skewness Model of Asset Returns
Risk Parity Portfolios with Jumps
The Equity/Bond/Volatility Asset Mix Policy
Factor Investing, Alternative Risk Premia and Skewness Hedging

EM estimates of the mixture model

\[ f(y) = (1 - \pi) \phi_1 (y; \mu_1 dt, \sigma_1^2 dt) + \pi \phi_1 (y; \mu_2 dt, \sigma_2^2 dt) \]

Figure: Bonds

Figure: Equities
**Figure: Carry**

**Main results:**
- We obtain a two-volatility regime!
- Daily and annually lag periods are not adapted.
How to obtain a mixture model with jumps?

The probability density function is:

\[ f(y) = (1 - \pi) \phi_3(y; \mu dt, \Sigma dt) + \pi \phi_3(y; \mu dt + \tilde{\mu}, \Sigma dt + \tilde{\Sigma}) \]

⇒ \( \pi \) can not be estimated by EM.

The two-step estimation procedure

1. We calibrate the probability \( \pi \) according to the expected drawdown:

\[ \mathbb{E}[DD_i(\tau)] = \tilde{\mu}_i + \Phi^{-1}\left(\frac{dt}{\tau}\right)\tilde{\sigma}_i \]

where \( \tau \) is the given return period.

2. We calibrate the parameters \( \mu, \sigma, \tilde{\mu} \) and \( \tilde{\Sigma} \) by CML.
Calibration of the jump probability

Figure: Expected weekly drawdown (in %) of the carry risk premium
Calibration of the other parameters

<table>
<thead>
<tr>
<th>Regime</th>
<th>Asset</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
<th>$\rho_{i,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>Bonds</td>
<td>5.38</td>
<td>4.17</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>Equities</td>
<td>7.89</td>
<td>15.64</td>
<td>-36.80</td>
</tr>
<tr>
<td></td>
<td>Carry</td>
<td>10.10</td>
<td>2.91</td>
<td>-25.17</td>
</tr>
<tr>
<td>Jump</td>
<td>Bonds</td>
<td>$\tilde{\mu}_i$</td>
<td>$\tilde{\sigma}_i$</td>
<td>$\tilde{\rho}_{i,j}$</td>
</tr>
<tr>
<td></td>
<td>Equities</td>
<td>-1.20</td>
<td>6.76</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>Carry</td>
<td>-2.23</td>
<td>2.57</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table: Estimation of the constrained mixture model when $\pi = 0.5\%$ (weekly model)
Figure: PDF of asset returns (weekly model)

- Bonds
- Equities
- Carry
### Comparing in-sample ERC portfolios

**Table:** Weights (in %) of the ERC portfolio

<table>
<thead>
<tr>
<th>Asset</th>
<th>Weekly model</th>
<th></th>
<th>Monthly model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gaussian</td>
<td>Jump model</td>
<td>Gaussian</td>
<td>Jump model</td>
</tr>
<tr>
<td></td>
<td>Normal</td>
<td>Mixture</td>
<td>Normal</td>
<td>Mixture</td>
</tr>
<tr>
<td>Bonds</td>
<td>59.17</td>
<td>46.46</td>
<td>52.71</td>
<td>61.36</td>
</tr>
<tr>
<td>Equities</td>
<td>10.43</td>
<td>9.18</td>
<td>10.36</td>
<td>12.09</td>
</tr>
<tr>
<td>carry</td>
<td>30.40</td>
<td>44.36</td>
<td>36.93</td>
<td>26.55</td>
</tr>
</tbody>
</table>

Benjamin Bruder, Nazar Kostyuchyk and Thierry Roncalli
How explaining these results?

Figure: One-year rolling volatility (in %) of the carry risk premium (weekly model)
Figure: Dynamics of the ERC weights (Gaussian model)
How to produce an out-of-sample skewness-based risk parity portfolio?

**Main assumption**

We assume that the parameters $\pi$, $\tilde{\mu}$ and $\tilde{\Sigma}$ are given once and for all, and are set to the previous estimates.

Alternative approach: we can calibrate $\pi$, $\tilde{\mu}$ and $\tilde{\Sigma}$ using stress scenarios.

**Bad solution**

We apply the ML method with the data of the rolling window:

$$\left(\hat{\mu}_t, \hat{\Sigma}_t\right) = \arg\max_{\left(\mu_t, \Sigma_t\right)} \sum_{s=1}^{n_{rw}} \ln \left( (1-\pi) \phi_3 \left( R_{t-s} - \mu_t \, dt, \Sigma_t \, dt \right) + \pi \phi_3 \left( R_{t-s} - (\mu_t \, dt + \tilde{\mu}), \Sigma_t \, dt + \tilde{\Sigma} \right) \right)$$

where $n_{rw}$ is the length of the rolling window.
Out-of-sample skewness-based ERC portfolio

Figure: Dynamics of the ERC weights (mixture model, ML method)
Introducing the thresholding approach

Ait-Sahalia and Jacod, 2012

We observe a jump at time $t$ when the absolute return of $R_t$ is larger than a given level $r^*$:

$$J_t = 1 \iff |R_t - v| \geq r^*$$

Using a sample of asset returns, we can then create two sub-samples:

- a sample of asset returns without jump that satisfy $|R_t - v| < r^*$;
- a sample of asset returns with jumps that satisfy $|R_t - v| \geq r^*$;
Introducing the filtering approach

Filtering approach to detect jump

- We calculate the posterior probability of the jump regime:

\[ \hat{\pi}_t = \frac{\pi \phi_n \left( R_t, \mu dt + \tilde{\mu}, \Sigma dt + \tilde{\Sigma} \right)}{(1 - \pi) \phi_n \left( R_t, \mu dt, \Sigma dt \right) + \pi \phi_n \left( R_t, \mu dt + \tilde{\mu}, \Sigma dt + \tilde{\Sigma} \right)} \]

- We will say that we observe a jump at time \( t \) when the posterior probability is larger than a threshold \( \pi^* \):

\[ J_t = 1 \iff \hat{\pi}_t \geq \pi^* \]
Equivalence between the two approaches \((n = 1)\)

\[ \hat{\pi}_t \geq \pi^* \iff R_t \leq R^- \text{ or } R_t \geq R^+ \]

**Figure:** Detecting the jumps of the carry risk premium (weekly model)
Non-equivalence between the two approaches \((n > 1)\)

\[
\hat{\pi}_t \geq \pi^* \iff R_i, t \leq R_i^- \text{ and } R_i, t \geq R_i^+
\]

\[
\hat{\pi}_t \geq \pi^* \iff (R_t - \nu)^\top Q (R_t - \nu) \geq r^*
\]
The out-of-sample filtering approach

Filtering algorithm

- Given $\hat{\mu}_{t-1}$ and $\hat{\Sigma}_{t-1}$, we estimate the jump probability for each dates of the rolling window which ends at time $t$;
- Given the previous jump probabilities, we estimate the parameters $\hat{\mu}_t$ and $\hat{\Sigma}_t$ by deleting the dates of the rolling window that correspond to a jump;
- We iterate the algorithm until today.

\[
\begin{align*}
0 & \quad t - n_{TW} & \quad t - j & \quad t \\
\checkmark & \checkmark & \times & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark
\end{align*}
\]

Rolling window

- ✓ if $\hat{\pi}_{t-j} < \pi^*$
- ✗ if $\hat{\pi}_{t-j} \geq \pi^*$
Estimating out-of-sample jump probabilities

**Figure:** Jump probabilities (in %)

- **In-sample jump probability**
  - Weekly model
  - Monthly model

- **Out-of-sample jump probability**
  - Weekly model
  - Monthly model
Using the filtering approach to estimate the volatility

**Figure:** Estimated volatility (in %) of the carry risk premium (weekly model)
Figure: Dynamics of the ERC weights (mixture model, filtering method)
Analysis of the results

- The turnover of the skewness-based ERC allocation is 40% lower than the turnover of the volatility-based ERC allocation.

- In terms of historical performance, volatility and Sharpe ratio, the two portfolios are equivalent, but:
  
  Before 2008 $\neq$ After 2008
Analysis of the results

Figure: Comparison of the carry allocation (weekly model)
• Size: distressed risk due to the liquidity risk of small cap stocks
• Value: distressed risk due to the default risk of value stocks
• Low beta: no skewness risk
• Momentum: skewness risk
• Quality: limited skewness risk, but correlated with skewness risk of Momentum

⇒ Value (and Size) allocation is generally overestimated
Skewness risk premia

- The equity/bond/volatility asset mix policy = an emblematic illustration
- Other issues:
  - Portfolio of cross-asset carry risk premia
  - Introduction of less liquid strategies
- The case of equity cross-section momentum risk premium
  - Momentum crashes (Daniel and Moskowitz, 2016)
  - Volatility-based risk parity strategies are not adapted
  - Winners-minus-losers strategy ≠ winners-minus-market strategy
### Table: Volatility and skewness risks of risk-based portfolios (weekly model)

<table>
<thead>
<tr>
<th>Portfolio Model</th>
<th>MV Gaussian (full sample)</th>
<th>MV Normal Jump model</th>
<th>ERC Mixture</th>
<th>MES Mixture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>63.26%</td>
<td>36.05%</td>
<td>52.71%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Equities</td>
<td>2.23%</td>
<td>0.00%</td>
<td>10.36%</td>
<td>0.00%</td>
</tr>
<tr>
<td>Carry</td>
<td>34.51%</td>
<td>63.95%</td>
<td>36.93%</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\sigma(x)$</td>
<td>2.62%</td>
<td>2.33%</td>
<td>2.75%</td>
<td>4.17%</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>-2.75</td>
<td>-19.81</td>
<td>-0.17</td>
<td>0.00</td>
</tr>
</tbody>
</table>

### The arithmetics of skewness

$$-(36.05\% \times 0.17 + 0\% \times 0.44 + 63.95\% \times 5.77) = -19.81$$
The case of CTA strategies

Belief / Misconceptions

- CTA is the right strategy for hedging the skewness risk
- Equity risk = skewness risk

CTA strategy and the equity market

- Equity risk = volatility risk, not skewness risk
- CTA is a good strategy for hedging the volatility risk of the equity market (e.g. 2008)
- It is not obvious that CTA is the right strategy for hedging skewness risks (e.g. 2011, January 2015, etc.)