Course 2021-2022 in ESG and Climate Risks
Lecture 6. Mathematical Methods, Technical Tools and Exercises

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- **Lecture 1: Introduction**
  - *Definition, Actors, the Market of ESG Investing* (42 slides)
- **Lecture 2: ESG Investing**
  - *ESG Scoring, ESG Ratings, Performance of ESG Investing, ESG Financing, ESG Premium* (132 slides)
- **Lecture 3: Other ESG Topics**
  - *Sustainable Financing Products, Impact Investing, Voting Policy & Engagement, ESG and Climate Accounting* (82 slides)
- **Lecture 4: Climate Risk**
  - *Definition, Global Warming, Economic Modeling, Risk Measures* (176 slides)
- **Lecture 5: Climate Investing**
  - *Portfolio Decarbonization, Net Zero Carbon Metrics, Portfolio Alignment* (164 slides)
- **Lecture 6: Mathematical Methods, Technical Tools and Exercises**
  - *Scoring System, Trend Modeling, Geolocation Data, Numerical Computations, Optimization* (150+ slides)

General information

1. **Overview**
   The objective of this course is to understand the concepts of sustainable finance from the viewpoint of asset owners and managers
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, ESG, Responsible Investing, Climate Change
5. **Hours**
   Lectures: 18h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `http://www.thierry-roncalli.com/SustainableFinance.html`

Class schedule

Course sessions

- Date 1 (6 hours, AM+PM)
- Date 2 (6 hours, AM+PM)
- Date 3 (6 hours, AM+PM)

Class times: Friday 9:00am-12:00pm, 1:00pm–4:00pm, Location: University of Evry

Additional materials

http://www.thierry-roncalli.com/SustainableFinance.html

- Slides
- Past examinations
- Exercises + Solutions
- LaTeX source of the slides + figures (in pdf format)
- Links to the references

Main references

Amundi publications on ESG Investing

1. Bennani *et al.* (2018), How ESG Investing Has Impacted the Asset Pricing in the Equity Market, DP-36-2018, 36 pages, November 2018
2. Drei *et al.* (2019), ESG Investing in Recent Years: New Insights from Old Challenges, DP-42-2019, 32 pages, December 2019
3. Ben Slimane *et al.* (2020), ESG Investing and Fixed Income: It's Time to Cross the Rubicon, DP-45-2019, 36 pages, January 2020
4. Roncalli, T. (2020), ESG & Factor Investing: A New Stage Has Been Reached, Amundi Viewpoint, May 2020

Available at `https://research-center.amundi.com` or `www.ssrn.com`

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Amundi publications on Climate Investing

1. Le Guenedal, T. (2019), Economic Modeling of Climate Risk, WP-83-2019, 92 pages, April 2019
2. Bouchet, V., and Le Guenedal, T. (2020), Credit Risk Sensitivity to Carbon Price, WP-95-2020, 48 pages, May 2020
3. Le Guenedal *et al.* (2020), Trajectory Monitoring in Portfolio Management and Issuer Intentionality Scoring, WP-97-2020, 54 pages, May 2020
4. Roncalli *et al.* (2020), Measuring and Managing Carbon Risk in Investment Portfolios, WP-99-2020, 67 pages, August 2020
5. Ben Slimane, M., Da Fonseca, D., and Mahtani, V. (2020), Facts and Fantasies about the Green Bond Premium, WP-102-2020, 52 pages, December 2020
6. Le Guenedal, Drobinski, P., and Tankov, P. (2021), Measuring and Pricing Cyclone-Related Physical Risk under Changing Climate, WP-111-2021, 42 pages, June 2021
7. Adenot *et al.* (2022), Cascading Effects of Carbon Price through the Value Chain and their Impacts on Firm's Valuation, WP-122-2022, 82 pages, February 2022
8. Le Guenedal *et al.* (2022), Net Zero Carbon Metrics, WP-123-2022, 82 pages, February 2022

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Amundi ESG Thema

1. Créhalet, E. (2021), Introduction to Net Zero, *Amundi ESG Thema #1*, https://research-center.amundi.com
2. Créhalet, E., Foll, J., Haustant, P., and Hessenberger, T. (2021), Carbon Offsetting: How Can It Contribute to the Net Zero Goal?, *Amundi ESG Thema #5*, https://research-center.amundi.com
3. Créhalet, E., and Talwar, S. (2021), Carbon-efficient Technologies in the Race to Net Zero, *Amundi ESG Thema #6*, https://research-center.amundi.com
4. Le Meaux, C., Le Berthe, T., Jaulin, T., Créhalet, E., Jouanneau, M., Pouget-Abadie, T., and Elbaz, J. (2021), How can Investors Contribute to Net Zero Efforts?, *Amundi ESG Thema #3*, https://research-center.amundi.com

Available at https://research-center.amundi.com or www.ssrn.com

Main references

Academic publications

1. Andersson, M., Bolton, P., and Samama, F. (2016), Hedging Climate Risk, *Financial Analysts Journal*, www.ssrn.com/abstract=2499628.
2. Ardia, D., Bluteau, K., Boudt, K., and Inghelbrecht, K. (2021), Climate Change Concerns and the Performance of Green versus Brown Stocks, *National Bank of Belgium, Working Paper*, www.ssrn.com/abstract=3717722.
3. Battiston, S., Mandel, A., Monasterolo, I., Schütze, F., and Visentin, G. (2017), A Climate Stress-test of the Financial System, *Nature Climate Change*, www.ssrn.com/abstract=2726076.
4. Berg, F. Koelbel, J.F., and Rigobon, R. (2019), Aggregate Confusion: The Divergence of ESG Ratings, *Working Paper*, www.ssrn.com/abstract=3438533
5. Berg, F., Fabisik, K., and Sautner, Z. (2021), Is History Repeating Itself? The (Un)predictable Past of ESG Ratings , *Working Paper*, www.ssrn.com/abstract=3722087
6. Bolton, P., and Kacperczyk, M. (2021), Do Investors Care about Carbon Risk?, *Journal of Financial Economics*, www.ssrn.com/abstract=3594189
7. Bolton, P., Kacperczyk, M., and Samama, F. (2021), Net-Zero Carbon Portfolio Alignment, *Working Paper*, www.ssrn.com/abstract=3922686
8. Coqueret, G. (2021), Perspectives in ESG Equity Investing, *Working Paper*, www.ssrn.com/abstract=3715753

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Academic publications

9. Crifo, P., Diaye, M.A., and Oueghlissi, R. (2015), Measuring the Effect of Government ESG Performance on Sovereign Borrowing Cost, *Quarterly Review of Economics and Finance*, hal.archives-ouvertes.fr/hal-00951304v3
10. Dennig, F., Budolfson, M.B., Fleurbaey, M., Siebert, A., and Socolow, R.H. (2015), Inequality, Climate Impacts on the Future Poor, and Carbon Prices, *Proceedings of the National Academy of Sciences*, www.pnas.org/content/112/52/15827
11. Engle, R.F., Giglio, S., Kelly, B., Lee, H., and Stroebel, J. (2020), Hedging Climate Change News, *Review of Financial Studies*, www.ssrn.com/abstract=3317570
12. Görgen, M., Jacob, A., Nerlinger, M., Riordan, R., Rohleder, M., and Wilkens, M. (2020), Carbon Risk, *Working Paper*, www.ssrn.com/abstract=2930897
13. Harris, J. (2015), The Carbon Risk Factor, *Working Paper*, www.ssrn.com/abstract=2666757
14. Karydas, C., and Xepapadeas, A. (2021), Climate Change Financial Risks: Implications for Asset Pricing and Interest Rates, *Working Paper*
15. Le Guenedal, T., and Roncalli, T. (2022), Portfolio Construction and Climate Risk Measures, *Climate Investing*, www.ssrn.com/abstract=3999971

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Academic publications

16. Martellini, L., and Vallée, L. (2021), Measuring and Managing ESG Risks in Sovereign Bond Portfolios and Implications for Sovereign Debt Investing, *Journal of Portfolio Management*, `www.risk.edhec.edu/measuring-and-managing-esg-risks-sovereign-bond`
17. Pedersen, L.H., Fitzgibbons, S., and Pomorski, L. (2021), Responsible Investing: The ESG-Efficient Frontier, *Journal of Financial Economics*, `www.ssrn.com/abstract=3466417`
18. Pástor, L., Stambaugh, R.F., and Taylor, L.A. (2021), Sustainable Investing in Equilibrium, *Journal of Financial Economics*, `www.ssrn.com/abstract=3498354`
19. Roncalli, T., Le Guenedal, T., Lepetit, F., Roncalli, T., and Sekine, T. (2021), The Market Measure of Carbon Risk and its Impact on the Minimum Variance Portfolio, *Journal of Portfolio Management*, `www.ssrn.com/abstract=3772707`
20. Van der Beck, P. (2021), Flow-driven ESG returns, *Working Paper*, `www.ssrn.com/abstract=3929359`

Computation of the carbon budget

- We consider the following computation:

$$\begin{aligned} \mathcal{CB}_i(t_0, t) &= \int_{t_0}^{t} \left(\mathcal{CE}_i(s) - \mathcal{CE}_i^\star\right) \mathrm{d}s \\ &= \mathcal{I}\left(\mathcal{CE}_i(s), \mathcal{CE}_i^\star; t_0, t\right) \end{aligned}$$

- In the case where $\mathcal{CE}_i^\star$ is not constant, we have:

$$\begin{aligned} \int_{t_0}^{t} \left(\mathcal{CE}_i(s) - \mathcal{CE}_i^\star(s)\right) \mathrm{d}s &= \int_{t_0}^{t} \mathcal{CE}_i(s) \, \mathrm{d}s - \int_{t_0}^{t} \mathcal{CE}_i^\star(s) \, \mathrm{d}s \\ &= \mathcal{I}\left(\mathcal{CE}_i(s), 0; t_0, t\right) - \mathcal{I}\left(\mathcal{CE}_i^\star(s), 0; t_0, t\right) \end{aligned}$$

- We only need a numerical approximation of $\mathcal{I}\left(\mathcal{CE}_i(s), \mathcal{CE}_i^\star; t_0, t\right)$

Numerical solution

- We consider the partition $\{[t_0, t_0 + \Delta t], \ldots, [t - \Delta t, t]\}$ of $[t_0, t]$
- We note:

$$m = \frac{t - t_0}{\Delta t}$$

- In the case of a yearly partition, we have $\Delta t = 1$
- We assume that $t_0 \leq t_{\mathcal{L}ast} \leq t$ where t_0 is the starting date, $t_{\mathcal{L}ast}$ is the last reporting date and t is the ending date

Numerical solution

- The right Riemann approximation is:

$$\begin{aligned}\mathcal{CB}_i(t_0, t) &= \int_{t_0}^{t} \left(\mathcal{CE}_i(s) - \mathcal{CE}_i^\star\right) \mathrm{d}s \\ &\approx \sum_{k=1}^{m} \left(\mathcal{CE}_i(t_0 + k\Delta t) - \mathcal{CE}_i^\star\right) \cdot \Delta t\end{aligned}$$

- The left Riemann sum is:

$$\mathcal{CB}_i(t_0, t) \approx \sum_{k=0}^{m-1} \left(\mathcal{CE}_i(t_0 + k\Delta t) - \mathcal{CE}_i^\star\right) \cdot \Delta t$$

- The midpoint rule is given by:

$$\mathcal{CB}_i(t_0, t) \approx \sum_{k=1}^{m} \left(\mathcal{CE}_i\left(t_0 + \frac{k}{2}\Delta t\right) - \mathcal{CE}_i^\star\right) \cdot \Delta t$$

Special cases
Constant linear reduction

We use a constant linear reduction rate:

$$\mathcal{R}_i\left(t_{\mathcal{L}ast}, t\right) = \mathcal{R}_i \cdot \left(t - t_{\mathcal{L}ast}\right)$$

We have:

$$\begin{aligned}
\int_{t_{\mathcal{L}ast}}^{t} \mathcal{R}_i\left(t_{\mathcal{L}ast}, s\right) \mathrm{d}s &= \mathcal{R}_i \int_{t_{\mathcal{L}ast}}^{t} \left(s - t_{\mathcal{L}ast}\right) \mathrm{d}s \\
&= \mathcal{R}_i \frac{\left(t - t_{\mathcal{L}ast}\right)^2}{2}
\end{aligned}$$

We obtain the following semi-analytical expression:

$$\begin{aligned}
\mathcal{CB}_i\left(t_0, t\right) &= \left(t - t_{\mathcal{L}ast}\right)\left(\mathcal{CE}_i\left(t_{\mathcal{L}ast}\right) - \mathcal{CE}_i^{\star}\right) - \left(t_{\mathcal{L}ast} - t_0\right)\mathcal{CE}_i^{\star} + \\
&\quad \int_{t_0}^{t_{\mathcal{L}ast}} \mathcal{CE}_i\left(s\right) \mathrm{d}s - \mathcal{R}_i \frac{\left(t - t_{\mathcal{L}ast}\right)^2}{2} \mathcal{CE}_i\left(t_{\mathcal{L}ast}\right)
\end{aligned}$$

Special cases

Constant compound reduction

We have:

$$\mathcal{CE}_i(t) = (1 - \mathcal{R}_i)^{(t - t_{\mathcal{L}ast})} \cdot \mathcal{CE}_i(t_{\mathcal{L}ast})$$

We deduce that:

$$\begin{aligned}
\int_{t_{\mathcal{L}ast}}^{t} \mathcal{CE}_i(s)\,\mathrm{d}s &= \mathcal{CE}_i(t_{\mathcal{L}ast}) \int_{t_{\mathcal{L}ast}}^{t} (1 - \mathcal{R}_i)^{(s - t_{\mathcal{L}ast})}\,\mathrm{d}s \\
&= \mathcal{CE}_i(t_{\mathcal{L}ast}) \left[\frac{(1 - \mathcal{R}_i)^{(s - t_{\mathcal{L}ast})}}{\ln(1 - \mathcal{R}_i)} \right]_{t_{\mathcal{L}ast}}^{t} \\
&= \frac{(1 - \mathcal{R}_i)^{(t - t_{\mathcal{L}ast})} - 1}{\ln(1 - \mathcal{R}_i)} \mathcal{CE}_i(t_{\mathcal{L}ast})
\end{aligned}$$

Special cases

Constant compound reduction

It follows that:

$$\begin{aligned}
\mathcal{CB}_i(t_0, t) &= -(t - t_0) \cdot \mathcal{CE}_i^{\star} + \int_{t_0}^{t} \mathcal{CE}_i(s)\,\mathrm{d}s \\
&= -(t - t_0) \cdot \mathcal{CE}_i^{\star} + \int_{t_0}^{t_{\mathcal{L}ast}} \mathcal{CE}_i(s)\,\mathrm{d}s + \\
&\quad \left(\frac{(1 - \mathcal{R}_i)^{(t - t_{\mathcal{L}ast})} - 1}{\ln(1 - \mathcal{R}_i)}\right) \mathcal{CE}_i(t_{\mathcal{L}ast})
\end{aligned}$$

Special cases

Linear function

We assume that:

$$\mathcal{CE}_i(t) = \beta_{i,0} + \beta_{i,1} t$$

It follows that:

$$\begin{aligned}
\mathcal{CB}_i(t_0, t) &= \int_{t_0}^{t} \left(\beta_{i,0} + \beta_{i,1} s - \mathcal{CE}_i^{\star}\right) \mathrm{d}s \\
&= \left[\frac{1}{2}\beta_{i,1} s^2 + \left(\beta_{i,0} - \mathcal{CE}_i^{\star}\right) s\right]_{t_0}^{t} \\
&= \frac{1}{2}\beta_{i,1}\left(t^2 - t_0^2\right) + \left(\beta_{i,0} - \mathcal{CE}_i^{\star}\right)(t - t_0)
\end{aligned}$$

Special cases

Piecewise linear function

We assume that $\mathcal{CE}_i(t)$ is known for $t \in \{t_0, t_1, \ldots, t_m\}$ and $\mathcal{CE}_i(t)$ is linear between two consecutive dates:

$$\mathcal{CE}_i(t) = \mathcal{CE}_i(t_{k-1}) + \frac{\mathcal{CE}_i(t_k) - \mathcal{CE}_i(t_{k-1})}{t_k - t_{k-1}}(t - t_{k-1}) \qquad \text{if } t \in [t_{k-1}, t_k]$$

We also have:

$$\begin{aligned}
\mathcal{CE}_i(t) &= \underbrace{\frac{t_k}{t_k - t_{k-1}}\mathcal{CE}_i(t_{k-1}) - \frac{t_{k-1}}{t_k - t_{k-1}}\mathcal{CE}_i(t_k)}_{\beta_{i,0,k}} + \\
&\quad \underbrace{\frac{\mathcal{CE}_i(t_k) - \mathcal{CE}_i(t_{k-1})}{t_k - t_{k-1}}}_{\beta_{i,1,k}} t \\
&= \beta_{i,0,k} + \beta_{i,1,k} \cdot t
\end{aligned}$$

Special cases

Piecewise linear function

We deduce that:

$$\begin{aligned}\mathcal{CB}_i(t_0,t) &= \sum_{k=1}^{k(t)} \int_{t_{k-1}}^{t_k} \left(\mathcal{CE}_i(s) - \mathcal{CE}_i^\star\right) \mathrm{d}s + \\ &\quad \int_{t_{k(t)}}^{t} \left(\mathcal{CE}_i(s) - \mathcal{CE}_i^\star\right) \mathrm{d}s\end{aligned}$$

where $k(t) = \{\max k : t_k \leq t\}$ and:

$$\begin{aligned}\mathcal{CB}_i(t_0,t) &= \frac{1}{2}\sum_{k=1}^{k(t)} \beta_{i,1,k}\left(t_k^2 - t_{k-1}^2\right) + \sum_{k=1}^{k(t)} \left(\beta_{i,0,k} - \mathcal{CE}_i^\star\right)\left(t_k - t_{k-1}\right) + \\ &\quad \frac{1}{2}\beta_{i,1,k(t)+1}\left(t^2 - t_{k(t)}^2\right) + \left(\beta_{i,0,k(t)+1} - \mathcal{CE}_i^\star\right)\left(t - t_{k(t)}\right)\end{aligned}$$

Special cases

Piecewise linear function

We can simplify the previous expression as follows:

$$
\begin{aligned}
\mathcal{CB}_i(t_0, t) = \; & \frac{1}{2}\sum_{k=1}^{k(t)} \left(\mathcal{CE}_i(t_k) - \mathcal{CE}_i(t_{k-1})\right)(t_k + t_{k-1}) + \\
& \sum_{k=1}^{k(t)} \left(\mathcal{CE}_i(t_{k-1}) - \mathcal{CE}_i^{\star}\right) t_k - \sum_{k=1}^{k(t)} \left(\mathcal{CE}_i(t_k) - \mathcal{CE}_i^{\star}\right) t_{k-1} + \\
& \frac{1}{2}\left(\mathcal{CE}_i(t) - \mathcal{CE}_i\left(t_{k(t)}\right)\right)\left(t + t_{k(t)}\right) + \\
& \left(\mathcal{CE}_i\left(t_{k(t)}\right) - \mathcal{CE}_i^{\star}\right) t - \sum_{k=1}^{k(t)} \left(\mathcal{CE}_i(t) - \mathcal{CE}_i^{\star}\right) t_{k(t)}
\end{aligned}
$$

Example

Historical data

We consider the following carbon emissions (expressed in $ktCO_2e$):

t	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
$\mathcal{CE}(t)$	30.3	31.2	34.5	37.5	34.5	38.5	32.0	28.5	23.0	20.0	19.9

The goal is to compute the carbon budget:

$$\mathcal{CB}(2010, 2020) = \int_{2010}^{2020} \mathcal{CE}(t)\, \mathrm{d}t$$

Example

Riemann sums

- The right Riemann sum is equal to:

$$\mathcal{CB}\left(2010, 2020\right) = \left(31.2 + 34.5 + \ldots + 20.0 + 19.9\right) \times 1 = 299.60$$

- The left Riemann sum is equal to:

$$\mathcal{CB}\left(2010, 2020\right) = \left(30.3 + 31.2 + \ldots + 23.0 + 20.0\right) \times 1 = 329.90$$

- The midpoint rule is equal to:

$$\mathcal{CB}\left(2010, 2020\right) = \left(30.75 + 32.85 + \ldots + 21.50 + 19.95\right) \times 1 = 304.80$$

Example
Linear reduction rate

- We have:

$$\mathcal{CE}(2020) = (1 - (2020 - 2010) \times \mathcal{R}) \times \mathcal{CE}(2010)$$

- We deduce that the linear reduction rate between 2010 and 2020 was equal to:

$$\mathcal{R} = \frac{1}{10} \times \left(1 - \frac{\mathcal{CE}(2020)}{\mathcal{CE}(2010)}\right) = 3.4323\%$$

- We obtain:

$$\begin{aligned}\mathcal{CB}(2010, 2020) &= (2020 - 2010) \times \mathcal{CE}(2010) - \\ &\quad \mathcal{R} \times \frac{(2020 - 2010)^2}{2} \times \mathcal{CE}(2010) \\ &= 251.00\end{aligned}$$

Example

Compound reduction rate

- We have:

$$\mathcal{CE}(2020) = \left(1 - \mathcal{R}^{(2020-2010)}\right) \times \mathcal{CE}(2010)$$

- We deduce that the compound reduction rate between 2010 and 2020 was equal to:

$$\mathcal{R} = 1 - \left(\frac{\mathcal{CE}(2020)}{\mathcal{CE}(2010)}\right)^{\frac{1}{(2020-2010)}} = 4.1171\%$$

- We obtain:

$$\begin{aligned}\mathcal{CB}(2010, 2020) &= \frac{(1-\mathcal{R})^{(2020-2010)} - 1}{\ln(1-\mathcal{R})} \times \mathcal{CE}(2010) \\ &= 247.37\end{aligned}$$

Example

Linear function

- We estimate the linear trend model:

$$\hat{\beta} = \left(X^\top X\right)^{-1} X^\top Y = \begin{pmatrix} 2810.6909 \\ -1.3800 \end{pmatrix}$$

- We deduce that:

$$\mathcal{CE}(t) = 2810.6909 - 1.3800 \times t$$

- It follows that:

$$\begin{aligned} \mathcal{CB}(2010, 2020) &= -\frac{1.38}{2} \times \left(2020^2 - 2010^2\right) + \\ & \quad 2810.6909 \times (2020 - 2010) \\ &= 299.91 \end{aligned}$$

Example

Piecewise linear function

- If we consider that $\mathcal{CE}(t)$ is a piecewise linear function, we obtain:

$$\mathcal{CB}(2010, 2020) = 304.80$$

- This is exactly the value obtained with the midpoint rule!

Trend modeling

Managing ESG and Climate Constraints in Portfolio Optimization

Impact of climate risk on credit risk

Modeling climate risks with jump processes

Geolocation and GPS positioning

Atmospheric measurement

Physical data

Probability distribution of an ESG score

Question 1

We consider an investment universe of 8 issuers with the following ESG scores:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
E	−2.80	−1.80	−1.75	0.60	0.75	1.30	1.90	2.70
S	−1.70	−1.90	0.75	−1.60	1.85	1.05	0.90	0.70
G	0.30	−0.70	−2.75	2.60	0.45	2.35	2.20	1.70

Probability distribution of an ESG score

Question 1.a

Calculate the ESG score of the issuers if we assume the following weighting scheme: 40% for **E**, 40% for **S** and 20% for **G**.

Probability distribution of an ESG score

- We have:

$$s_i^{(\mathrm{ESG})} = 0.4 \times s_i^{(\mathrm{E})} + 0.4 \times s_i^{(\mathrm{S})} + 0.2 \times s_i^{(\mathrm{G})}$$

- We obtain the following results:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$s_i^{(\mathrm{E})}$	−2.80	−1.80	−1.75	0.60	0.75	1.30	1.90	2.70
$s_i^{(\mathrm{S})}$	−1.70	−1.90	0.75	−1.60	1.85	1.05	0.90	0.70
$s_i^{(\mathrm{G})}$	0.30	−0.70	−2.75	2.60	0.45	2.35	2.20	1.70
$s_i^{(\mathrm{ESG})}$	−1.74	−1.62	−0.95	0.12	1.13	1.41	1.56	1.70

Probability distribution of an ESG score

Question 1.b

Calculate the ESG score of the equally-weighted portfolio x_{ew}.

Probability distribution of an ESG score

- We obtain:

$$
\begin{aligned}
\mathcal{S}^{(\mathrm{ESG})}\left(x_{\mathrm{ew}}\right) &= \sum_{i=1}^{8} x_{\mathrm{ew},i} \times \mathcal{S}_i^{(\mathrm{ESG})} \\
&= 0.2013
\end{aligned}
$$

Probability distribution of an ESG score

Question 2

We assume that the ESG scores are *iid* and follow a standard Gaussian distribution:

$$s_i \sim \mathcal{N}(0, 1)$$

Probability distribution of an ESG score

Question 2.a

We note $x_{\mathrm{ew}}^{(n)}$ the equally-weighted portfolio composed of n issuers. Calculate the distribution of the ESG score $\mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right)$ of the portfolio $x_{\mathrm{ew}}^{(n)}$.

Probability distribution of an ESG score

- We have:

$$\begin{aligned} \mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right) &= \sum_{i=1}^{n} x_{\mathrm{ew},i}^{(n)} \times \mathcal{S}_i \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathcal{S}_i \end{aligned}$$

We deduce that $\mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right)$ follows a Gaussian distribution.

Probability distribution of an ESG score

- Its mean is equal to:

$$\mathbb{E}\left[\mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\mathcal{S}_i\right] = 0$$

- Its standard deviation is equal to:

$$\begin{aligned}\sigma\left(\mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right)\right) &= \sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2\left(\mathcal{S}_i\right)} \\ &= \frac{1}{\sqrt{n}}\end{aligned}$$

- Finally, we obtain:

$$\mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right) \sim \mathcal{N}\left(0, \frac{1}{n}\right)$$

Probability distribution of an ESG score

Question 2.b

What is the ESG score of a well-diversified portfolio?

Probability distribution of an ESG score

- The behavior of a well-diversified portfolio is close to an equally-weighted portfolio with n sufficiently large. Therefore, the ESG score is close to zero because we have:

$$\lim_{n \rightarrow \infty} \mathcal{S}\left(x_{\mathrm{ew}}^{(n)}\right) = 0$$

Probability distribution of an ESG score

Question 2.c

We note $T \sim \mathbf{F}_\alpha$ where $\mathbf{F}_\alpha(t) = t^\alpha$, $t \in [0,1]$ and $\alpha \geq 0$. Draw the graph of the probability density function $f_\alpha(t)$ when α is respectively equal to 0.5, 1.5, 2.5 and 70. What do you notice?

Probability distribution of an ESG score

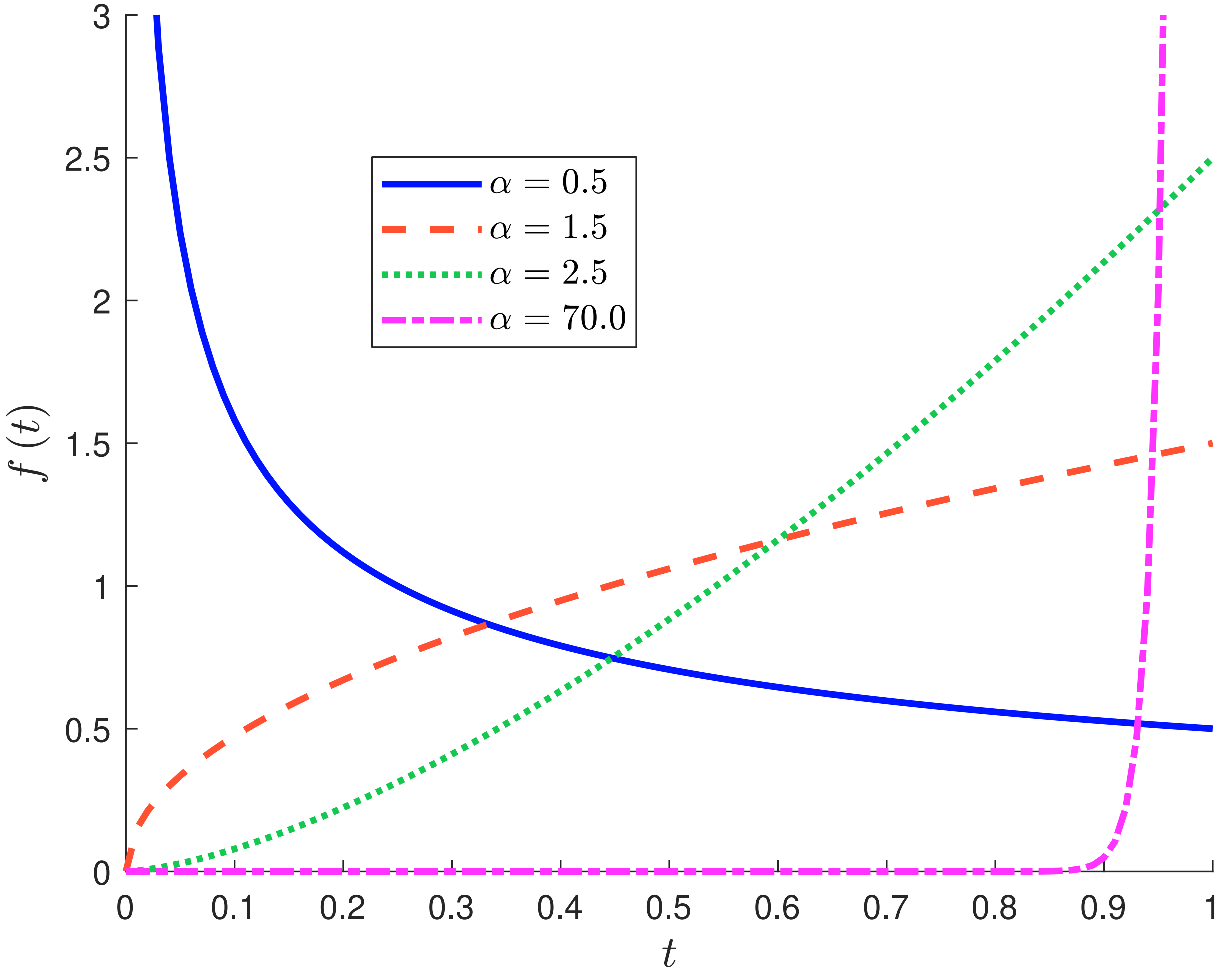


Figure 1: Probability density function $f_\alpha(t)$

Probability distribution of an ESG score

- We have:

$$f_{\alpha}(t) = \alpha t^{\alpha-1}$$

- We notice that the function $f_{\alpha}(t)$ tends to the dirac delta function when α tends to infinity:

$$\lim_{\alpha \to \infty} f_{\alpha}(t) = \delta_1(t) = \begin{cases} 0 & \text{if } t \neq 1 \\ +\infty & \text{if } t = 1 \end{cases}$$

Probability distribution of an ESG score

Question 2.d

We assume that the weights of the portfolio $x = (x_1, \ldots, x_n)$ follow a power-law distribution $\mathbf{F}_\alpha$:

$$x_i \sim c T_i$$

where $T_i \sim \mathbf{F}_\alpha$ are *iid* random variables and c is a normalization constant. Explain how to simulate the portfolio weights $x = (x_1, \ldots, x_n)$. Represent one simulation of the portfolio x for the previous values of α. Comment on these results. Deduce the relationship between the Herfindahl index $\mathcal{H}_\alpha(x)$ of the portfolio weights x and the parameter α.

Remark

We use $n = 50$ in the rest of the exercise.

Probability distribution of an ESG score

- To simulate T_i, we use the property of the probability integral transform:

$$U_i = \mathbf{F}_\alpha \left(T_i \right) \sim \mathcal{U}_{[0,1]}$$

We deduce that:

$$\begin{array}{rcl} T_i & = & \mathbf{F}_\alpha^{-1} \left(U_i \right) \\ & = & U_i^{1/\alpha} \end{array}$$

Probability distribution of an ESG score

The algorithm for simulating the portfolio x is then the following:

1. We simulate n independent uniform random numbers $(u_1, \ldots, u_n)$.
2. We compute the random variates $(t_1, \ldots, t_n)$ where:

$$t_i = u_i^{1/\alpha}$$

3. We calculate the normalization constant:

$$c = \left(\sum_{i=1}^{n} t_i \right)^{-1} = \left(\sum_{i=1}^{n} u_i^{1/\alpha} \right)^{-1}$$

4. We deduce the portfolio weights $x = (x_1, \ldots, x_n)$:

$$x_i = c \cdot t_i = c \cdot u_i^{1/\alpha} = \frac{u_i^{1/\alpha}}{\sum_{j=1}^{n} u_j^{1/\alpha}}$$

Probability distribution of an ESG score

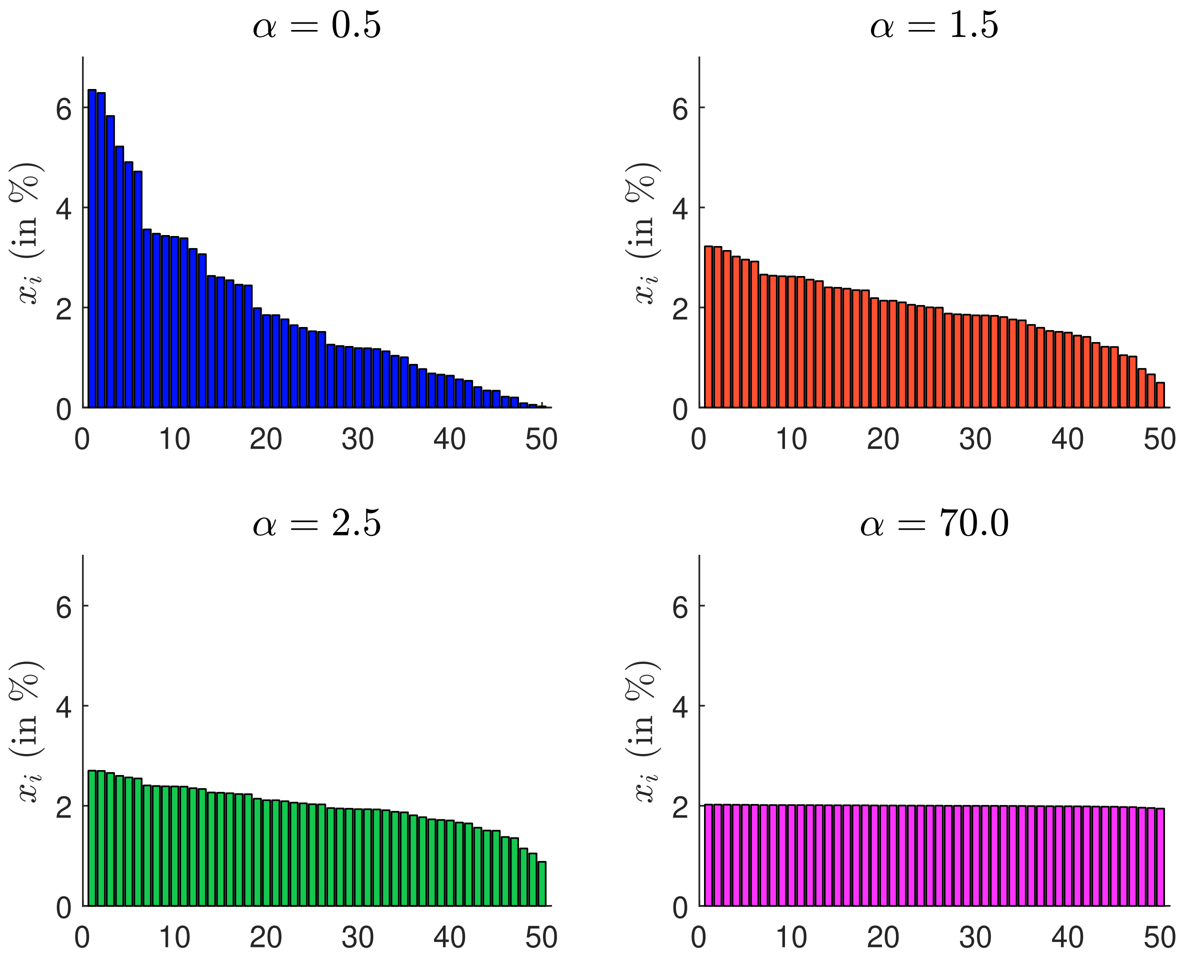


Figure 2: Repartition of the portfolio weights in descending order

Probability distribution of an ESG score

- In Figure 2, we have represented the composition of the portfolio x for the 4 values of α. The weights are ranked in descending order. We deduce that the portfolio x is uniform when $\alpha \to \infty$. The parameter α controls the concentration of the portfolio. Indeed, when α is small, the portfolio is highly concentrated. It follows that the Herfindahl index $\mathcal{H}_\alpha(x)$ of the portfolio weights is a decreasing function of the parameter α.

Probability distribution of an ESG score

Question 2.e

We assume that the weight x_i and the ESG score $\mathcal{S}_i$ of the issuer i are independent. How to simulate the portfolio ESG score $\mathcal{S}(x)$? Using 50 000 replications, estimate the probability distribution function of $\mathcal{S}(x)$ by the Monte Carlo method. Comment on these results.

Probability distribution of an ESG score

- We simulate $x = (x_1, \ldots, x_n)$ using the previous algorithm. The vector of ESG scores $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is generated with normally-distributed random variables since we have $\mathcal{S}_i \sim \mathcal{N}(0, 1)$. We deduce that the simulated value of the portfolio ESG score $\mathcal{S}(x)$ is equal to:

$$\mathcal{S}(x) = \sum_{i=1}^{n} x_i \cdot \mathcal{S}_i$$

- We replicate the simulation of $\mathcal{S}(x)$ 50 000 times and draw the corresponding histogram in Figure 3. We also report the fitted Gaussian distribution. We observe that the portfolio ESG score $\mathcal{S}(x)$ is equal to zero on average, and its variance is an increasing function of the portfolio concentration.

Probability distribution of an ESG score

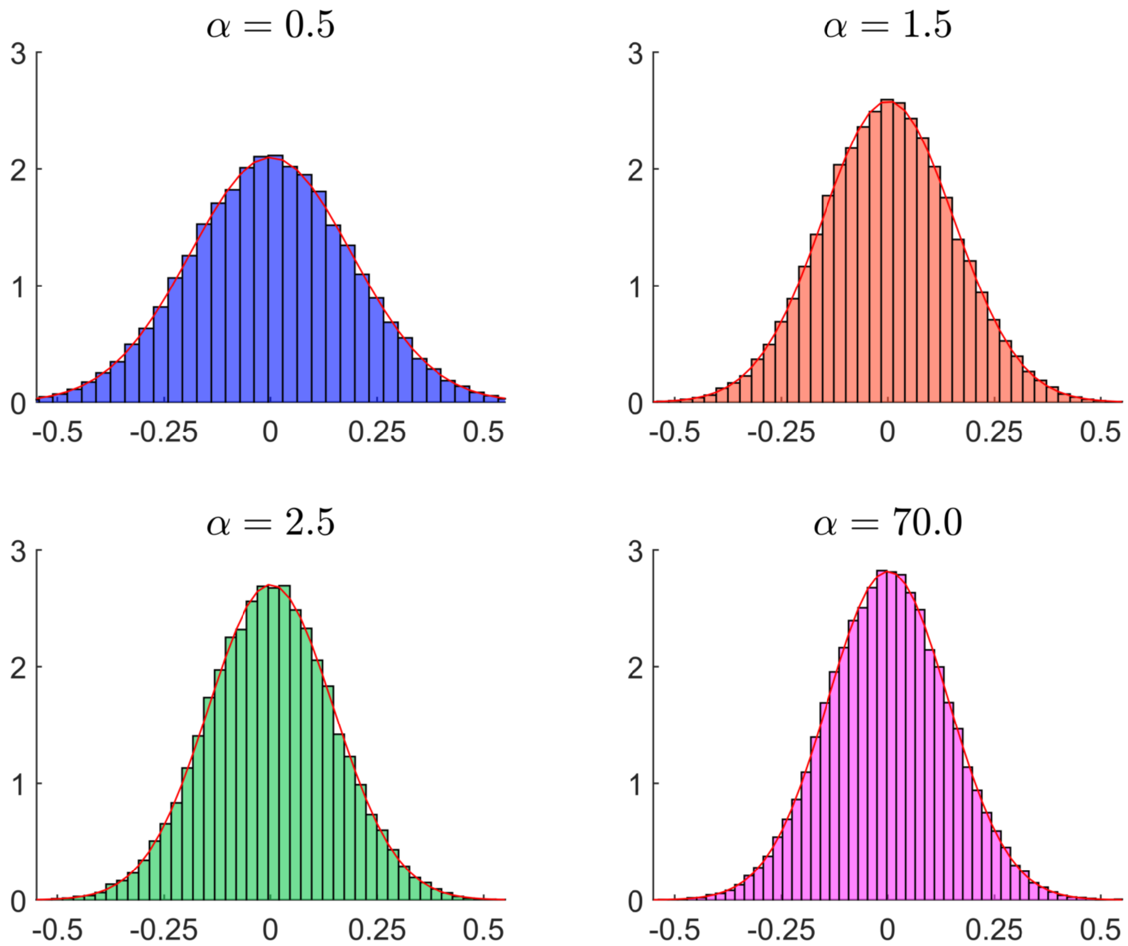


Figure 3: Histogram of the portfolio ESG score $\mathcal{S}(x)$

Probability distribution of an ESG score

Question 2.f

We now assume that the weight x_i and the ESG score $\mathcal{S}_i$ of the issuer i are positively correlated. More precisely, the dependence function between x_i and $\mathcal{S}_i$ is the Normal copula function with parameter ρ. Show that this is also the copula function between T_i and $\mathcal{S}_i$. Deduce an algorithm to simulate $\mathcal{S}(x)$.

Probability distribution of an ESG score

- Since $x_i \sim cT_i$, x_i is an increasing function of T_i. We deduce that the copula function of $(T_i, \mathcal{S}_i)$ is the same as the copula function of $(x_i, \mathcal{S}_i)$.
- To simulate the Normal copula function $\mathbf{C}(u, v)$, we use the transformation algorithm based on the Cholesky decomposition:

$$\left\{ \begin{array}{l} u_i = \Phi\left(g_i'\right) \\ v_i = \Phi\left(\rho g_i' + \sqrt{1-\rho^2} g_i''\right) \end{array} \right.$$

where g_i' and g_i'' are two independent random numbers from the probability distribution $\mathcal{N}(0, 1)$.

Probability distribution of an ESG score

Here is the algorithm to simulate the ESG portfolio score $\mathcal{S}(x)$:

1. We simulate n independent normally-distributed random numbers $g_i'$and g_i'' and we compute (u_i, v_i):

$$\begin{cases} u_i = \Phi\left(g_i'\right) \\ v_i = \Phi\left(\rho g_i' + \sqrt{1-\rho^2} g_i''\right) \end{cases}$$

2. We compute the random variates $(t_1, \ldots, t_n)$ where $t_i = u_i^{1/\alpha}$
3. We deduce the vector of weights $x = (x_1, \ldots, x_n)$:

$$x_i = t_i \bigg/ \sum_{j=1}^{n} t_j$$

4. We simulate the vector of scores $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$:

$$\mathcal{S}_i = \Phi^{-1}(v_i) = \rho g_i' + \sqrt{1-\rho^2} g_i''$$

5. We calculate the portfolio score:

$$\mathcal{S}(x) = \sum_{i=1}^{n} x_i \cdot \mathcal{S}_i$$

Probability distribution of an ESG score

Question 2.g

Using 50 000 replications, estimate the probability distribution function of $\mathcal{S}(x)$ by the Monte Carlo method when the correlation parameter ρ is set to 50%. Comment on these results.

Probability distribution of an ESG score

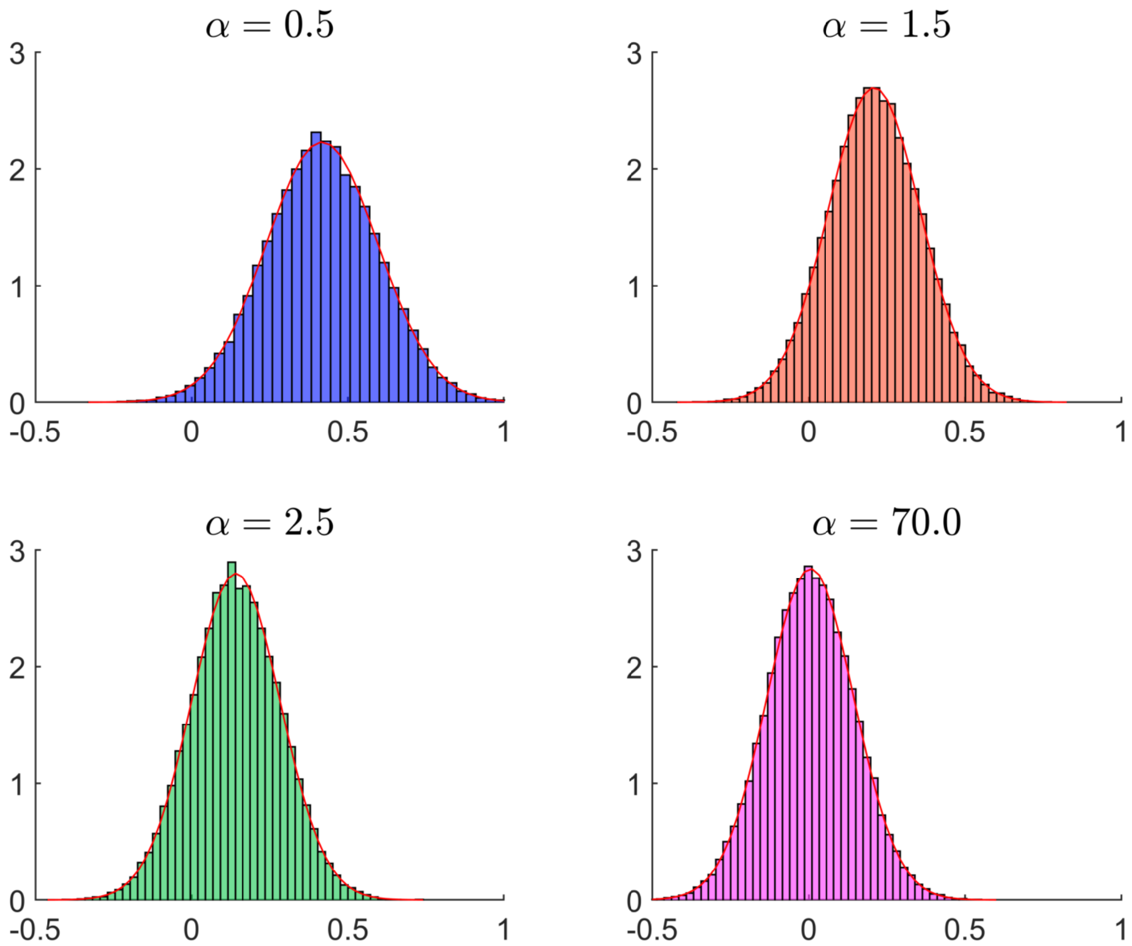


Figure 4: Histogram of the portfolio ESG score $\mathcal{S}(x)$ ($\rho = 50\%$)

Probability distribution of an ESG score

- In the independent case, we found that $\mathbb{E}\left[\mathcal{S}\left(x\right)\right] = 0$. In Figure 4, we notice that $\mathbb{E}\left[\mathcal{S}\left(x\right)\right] \neq 0$ when ρ is equal to 50%. Indeed, we obtain:

$$\mathbb{E}\left[\mathcal{S}\left(x\right)\right] = \begin{cases} 0.418 & \text{if } \alpha = 0.5 \\ 0.210 & \text{if } \alpha = 1.5 \\ 0.142 & \text{if } \alpha = 2.5 \\ 0.006 & \text{if } \alpha = 70.0 \end{cases}$$

Probability distribution of an ESG score

Question 2.h

Estimate the relationship between the correlation parameter ρ and the expected ESG score $\mathbb{E}[s(x)]$ of the portfolio x. Comment on these results.

Probability distribution of an ESG score

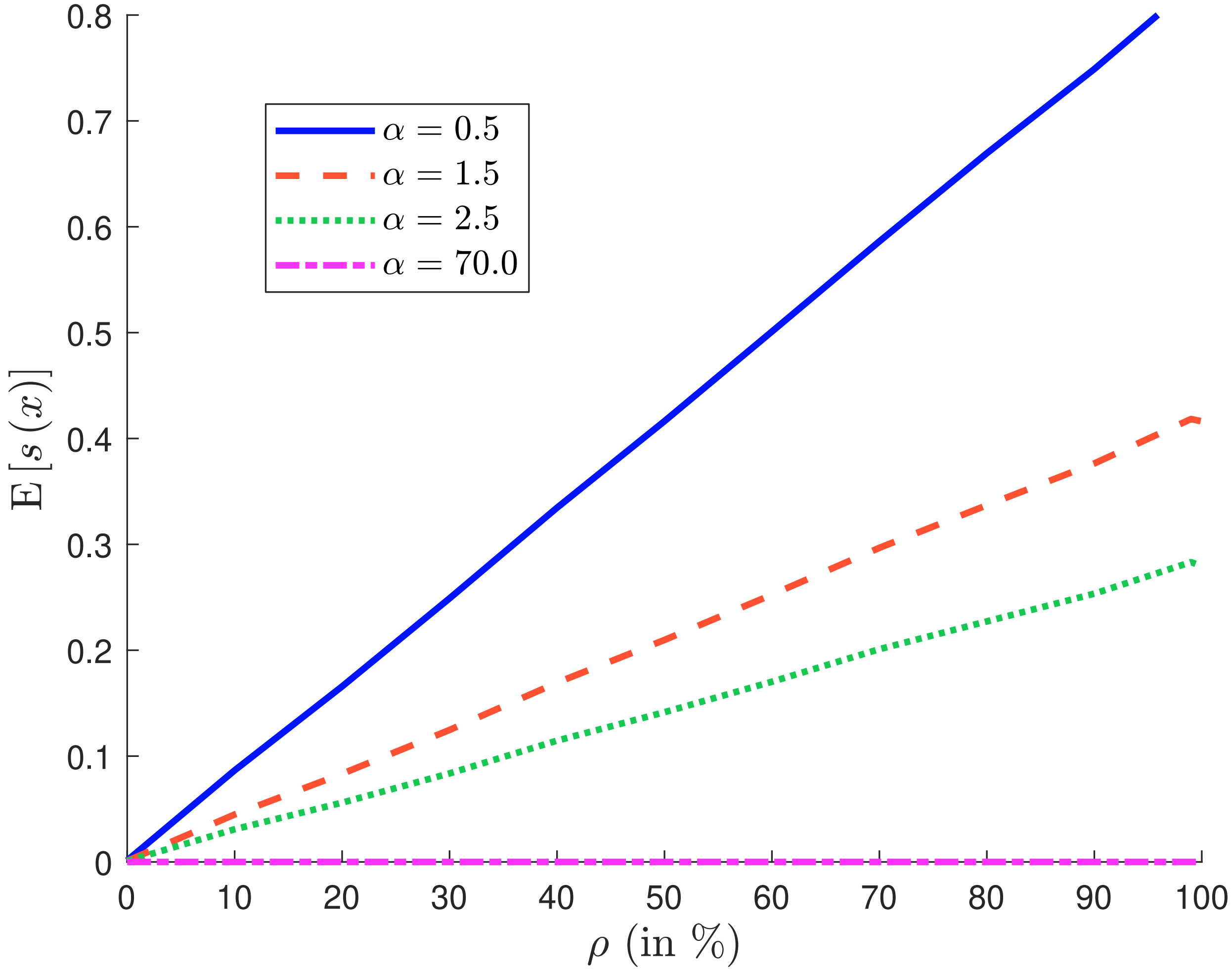


Figure 5: Relationship between ρ and $\mathbb{E}[s(x)]$

Probability distribution of an ESG score

- We notice that there is a positive relationship between ρ and $\mathbb{E}\left[\mathcal{S}\left(x\right)\right]$ and the slope increases with the concentration of the portfolio.

Probability distribution of an ESG score

Question 2.i

How are the previous results related to the size bias of ESG scoring?

Probability distribution of an ESG score

- Big cap companies have more (financial and human) resources to develop an ESG policy than small cap companies.
- Therefore, we observe a positive correlation between the market capitalization and the ESG score of an issuer.
- It follows that ESG portfolios have generally a size bias. For instance, we generally observe that cap-weighted indexes have an ESG score which is greater than the average of ESG scores.
- In the previous questions, we verify that $\mathbb{E}[\mathcal{S}(x)] \geq \mathbb{E}[\mathcal{S}]$ when the Herfindahl index of the portfolio x is high and the correlation between x_i and $\mathcal{S}_i$ is positive.

Probability distribution of an ESG score

Question 3

Let $\mathcal{s}$ be the ESG score of the issuer. We assume that the ESG score follows a standard Gaussian distribution:

$$\mathcal{s} \sim \mathcal{N}(0,1)$$

The ESG score $\mathcal{s}$ is also converted into an ESG rating $\mathcal{R}$, which can take the values **A**, **B**, **C** and **D** — **A** is the best rating and **D** is the worst rating.

Probability distribution of an ESG score

Question 3.a

We assume that the breakpoints of the rating system are -1.5, 0 and $+1.5$. Compute the frequencies of the ratings.

Probability distribution of an ESG score

- We have:

$$\begin{aligned}\Pr\{\mathcal{R} = \mathbf{A}\} &= \Pr\{\mathcal{s} \geq 1.5\} \\ &= 1 - \Phi(1.5) \\ &= 6.68\%\end{aligned}$$

and:

$$\begin{aligned}\Pr\{\mathcal{R} = \mathbf{B}\} &= \Pr\{0 \leq \mathcal{s} < 1.5\} \\ &= \Phi(1.5) - \Phi(0) \\ &= 43.32\%\end{aligned}$$

- Since the Gaussian distribution is symmetric around 0, we also have:

$$\Pr\{\mathcal{R} = \mathbf{C}\} = \Pr\{\mathcal{R} = \mathbf{B}\} = 43.32\%$$

and:

$$\Pr\{\mathcal{R} = \mathbf{D}\} = \Pr\{\mathcal{R} = \mathbf{A}\} = 6.68\%$$

Probability distribution of an ESG score

- The mapping function is:

$$\mathcal{M}_{\text{appring}}(\mathcal{s}) = \begin{cases} \textbf{A} & \text{if } \mathcal{s} < -1.5 \\ \textbf{B} & \text{if } -1.5 \leq \mathcal{s} < 0 \\ \textbf{C} & \text{if } 0 \leq \mathcal{s} < 1.5 \\ \textbf{D} & \text{if } \mathcal{s} \geq 1.5 \end{cases}$$

Probability distribution of an ESG score

Question 3.b

We would like to build a rating system such that each category has the same frequency. Find the mapping function.

Probability distribution of an ESG score

- We have:

  $$\Pr\{\mathcal{R}(t) = \mathbf{A}\} = \Pr\{\mathcal{R}(t) = \mathbf{B}\} = \Pr\{\mathcal{R}(t) = \mathbf{C}\} = \Pr\{\mathcal{R}(t) = \mathbf{D}\}$$

  and:

  $$\Pr\{\mathcal{R}(t) = \mathbf{A}\} + \Pr\{\mathcal{R}(t) = \mathbf{B}\} + \Pr\{\mathcal{R}(t) = \mathbf{C}\} + \Pr\{\mathcal{R}(t) = \mathbf{D}\} = 1$$

  We deduce that:

  $$\Pr\{\mathcal{R}(t) = \mathbf{A}\} = \frac{1}{4} = 25\%$$

  and $\Pr\{\mathcal{R}(t) = \mathbf{B}\} = \Pr\{\mathcal{R}(t) = \mathbf{C}\} = \Pr\{\mathcal{R}(t) = \mathbf{D}\} = 25\%$.

- We want to find the breakpoints (s_1, s_2, s_3) such that:

  $$\begin{cases} \Pr\{s < s_1\} = 25\% \\ \Pr\{s_1 \leq s < s_2\} = 25\% \\ \Pr\{s_2 \leq s < s_3\} = 25\% \\ \Pr\{s \geq s_3\} = 25\% \end{cases}$$

Probability distribution of an ESG score

- We deduce that:

$$\left\{\begin{array}{l} s_1 = \Phi^{-1}(0.25) = -0.6745 \\ s_2 = \Phi^{-1}(0.50) = 0 \\ s_3 = \Phi^{-1}(0.75) = +0.6745 \end{array}\right.$$

- The mapping function is:

$$\mathcal{M}_{\text{appring}}(s) = \left\{\begin{array}{ll} \textbf{A} & \text{if } s < -0.6745 \\ \textbf{B} & \text{if } -0.6745 \leq s < 0 \\ \textbf{C} & \text{if } 0 \leq s < 0.6745 \\ \textbf{D} & \text{if } s \geq 0.6745 \end{array}\right.$$

Probability distribution of an ESG score

Question 3.c

We would like to build a rating system such that the frequency of the median ratings **B** and **C** is 40% and the frequency of the extreme ratings **A** and **D** is 10%. Find the mapping function.

Probability distribution of an ESG score

- We have:

$$\begin{cases} s_1 = \Phi^{-1}(0.10) = -1.2816 \\ s_2 = \Phi^{-1}(0.50) = 0 \\ s_3 = \Phi^{-1}(0.90) = +1.2816 \end{cases}$$

- The mapping function is:

$$\mathcal{M}_{\text{appring}}(\mathcal{s}) = \begin{cases} \textbf{A} & \text{if } \mathcal{s} < -1.2816 \\ \textbf{B} & \text{if } -1.2816 \leq \mathcal{s} < 0 \\ \textbf{C} & \text{if } 0 \leq \mathcal{s} < 1.2816 \\ \textbf{D} & \text{if } \mathcal{s} \geq 1.2816 \end{cases}$$

Probability distribution of an ESG score

Question 4

Let $s(t)$ be the ESG score of the issuer at time t. The ESG scoring system is evaluated every month. The index time t corresponds to the current month, whereas the previous month is $t-1$. We assume that:

- The ESG score at time $t-1$ follows a standard Gaussian distribution:

$$s(t-1) \sim \mathcal{N}(0,1)$$

- The variation of the ESG score is Gaussian between two months:

$$\Delta s(t) = s(t) - s(t-1) \sim \mathcal{N}\left(0, \sigma^2\right)$$

- The ESG score $s(t-1)$ and the variation $\Delta s(t)$ are independent.

Probability distribution of an ESG score

Question 4

The ESG score $\mathcal{s}(t)$ is converted into an ESG rating $\mathcal{R}(t)$, which can take following grades:

$$\mathcal{R}_1 \prec \mathcal{R}_2 \prec \cdots \prec \mathcal{R}_k \prec \cdots \prec \mathcal{R}_{K-1} \prec \mathcal{R}_K$$

We assume that the breakpoints of the rating system are $(s_1, s_2, \ldots, s_{K-1})$. We also note $s_0 = -\infty$ and $s_K = +\infty$.

Probability distribution of an ESG score

Question 4.a

Compute the bivariate probability distribution of the random vector $(s(t-1), \Delta s(t))$.

Probability distribution of an ESG score

- The joint distribution of $(s(t-1), \Delta s(t))$ is:

$$\begin{pmatrix} s(t-1) \\ \Delta s(t) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right)$$

Probability distribution of an ESG score

Question 4.b

Compute the bivariate distribution of the random vector $(\mathcal{s}(t-1), \mathcal{s}(t))$.

Probability distribution of an ESG score

- Since we have:

$$s(t) = s(t-1) + \Delta s(t)$$

we deduce that:

$$\begin{pmatrix} s(t-1) \\ s(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s(t-1) \\ \Delta s(t) \end{pmatrix}$$

We conclude that $(s(t-1), s(t))$ is a Gaussian random vector.

Probability distribution of an ESG score

- We have:

$$\operatorname{var}\left(s\left(t\right)\right) = 1 + \sigma^2$$

and:

$$\begin{aligned}\operatorname{cov}\left(s\left(t-1\right), s\left(t\right)\right) &= \mathbb{E}\left[s\left(t-1\right) \cdot s\left(t\right)\right] \\ &= \mathbb{E}\left[s^2\left(t-1\right) + s\left(t-1\right) \cdot \Delta s\left(t\right)\right] \\ &= 1\end{aligned}$$

Probability distribution of an ESG score

- It follows that:

$$\begin{pmatrix} \mathcal{S}(t-1) \\ \mathcal{S}(t) \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}_2, \Sigma_\sigma\right)$$

where Σ_σ is the covariance matrix:

$$\Sigma_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1+\sigma^2 \end{pmatrix}$$

Probability distribution of an ESG score

Question 4.c

Compute the probability $p_k = \Pr\left\{\mathcal{R}\left(t-1\right) = \mathcal{R}_k\right\}$.

Probability distribution of an ESG score

- We have:

$$\begin{aligned}\Pr\left\{\mathcal{R}\left(t-1\right)=\mathcal{R}_k\right\} &= \Pr\left\{s_{k-1}\leq \mathcal{S}\left(t-1\right)<s_k\right\} \\ &= \Phi\left(s_k\right)-\Phi\left(s_{k-1}\right)\end{aligned}$$

Probability distribution of an ESG score

Question 4.d

Compute the joint probability $\Pr\left\{\mathcal{R}\left(t\right)=\mathcal{R}_k,\mathcal{R}\left(t-1\right)=\mathcal{R}_j\right\}$.

Probability distribution of an ESG score

- We have:

$$\begin{aligned}(*) &= \Pr\left\{\mathcal{R}(t) = \mathcal{R}_k, \mathcal{R}(t-1) = \mathcal{R}_j\right\} \\ &= \Pr\left\{s_{k-1} \leq \mathcal{s}(t) < s_k, s_{j-1} \leq \mathcal{s}(t-1) < s_j\right\} \\ &= \Phi_2\left(s_j, s_k; \Sigma_\sigma\right) - \Phi_2\left(s_{j-1}, s_k; \Sigma_\sigma\right) - \\ &\quad \Phi_2\left(s_j, s_{k-1}; \Sigma_\sigma\right) + \Phi_2\left(s_{j-1}, s_{k-1}; \Sigma_\sigma\right)\end{aligned}$$

where $\Phi_2(x, y; \Sigma_\sigma)$ is the bivariate Normal cdf with covariance matrix Σ_σ.

Probability distribution of an ESG score

Question 4.e

Compute the transition probability
$p_{j,k} = \Pr\{\mathcal{R}(t) = \mathcal{R}_k \mid \mathcal{R}(t-1) = \mathcal{R}_j\}$.

Probability distribution of an ESG score

- We have:

$$
\begin{aligned}
p_{j,k} &= \Pr\left\{\mathcal{R}(t) = \mathcal{R}_k \mid \mathcal{R}(t-1) = \mathcal{R}_j\right\} \\
&= \frac{\Pr\left\{\mathcal{R}(t) = \mathcal{R}_k, \mathcal{R}(t-1) = \mathcal{R}_j\right\}}{\Pr\left\{\mathcal{R}(t-1) = \mathcal{R}_j\right\}} \\
&= \frac{\Phi_2\left(s_j, s_k; \Sigma_\sigma\right) + \Phi_2\left(s_{j-1}, s_{k-1}; \Sigma_\sigma\right)}{\Phi\left(s_j\right) - \Phi\left(s_{j-1}\right)} - \\
&\quad\; \frac{\Phi_2\left(s_{j-1}, s_k; \Sigma_\sigma\right) + \Phi_2\left(s_j, s_{k-1}; \Sigma_\sigma\right)}{\Phi\left(s_j\right) - \Phi\left(s_{j-1}\right)}
\end{aligned}
$$

Probability distribution of an ESG score

Question 4.f

Compute the monthly turnover $\mathcal{T}\left(\mathcal{R}_k\right)$ of the ESG rating $\mathcal{R}_k$.

Probability distribution of an ESG score

- We have:

$$\begin{aligned}\mathcal{T}\left(\mathcal{R}_k\right) &= \Pr\left\{\mathcal{R}\left(t\right) \neq \mathcal{R}_k \mid \mathcal{R}\left(t-1\right) = \mathcal{R}_k\right\} \\ &= 1 - \Pr\left\{\mathcal{R}\left(t\right) = \mathcal{R}_k \mid \mathcal{R}\left(t-1\right) = \mathcal{R}_k\right\} \\ &= 1 - p_{k,k}\end{aligned}$$

Probability distribution of an ESG score

Question 4.g

Compute the monthly turnover $\mathcal{T}\left(\mathcal{R}_1, \ldots, \mathcal{R}_K\right)$ of the ESG rating system.

Probability distribution of an ESG score

- We have:

$$
\begin{aligned}
\mathcal{T}\left(\mathcal{R}_1,\ldots,\mathcal{R}_K\right) &= \sum_{k=1}^{K} \Pr\left\{\mathcal{R}\left(t-1\right)=\mathcal{R}_k\right\}\cdot\mathcal{T}\left(\mathcal{R}_k\right) \\
&= \sum_{k=1}^{K} \Pr\left\{\mathcal{R}\left(t\right)\neq\mathcal{R}_k,\mathcal{R}\left(t-1\right)=\mathcal{R}_k\right\}
\end{aligned}
$$

Probability distribution of an ESG score

Question 4.h

For each rating system given in Questions 3.a, 3.b and 3.c, determine the corresponding ESG migration matrix and the monthly turnover of the rating system if we assume that σ is equal to 10%. What is the best ESG rating system if we would like to control the turnover of ESG ratings?

Probability distribution of an ESG score

Table 1: ESG rating migration matrix (Question 3.a)

Rating	s_k	p_k	Transition probability $p_{j,k}$				$\mathcal{T}(\mathcal{R}_k)$
D		6.68%	92.96%	7.04%	0.00%	0.00%	7.04%
	-1.50						
C		43.32%	1.31%	95.03%	3.66%	0.00%	4.97%
	0.00						
B		43.32%	0.00%	3.66%	95.03%	1.31%	4.97%
	1.50						
A		6.68%	0.00%	0.00%	7.04%	92.96%	7.04%
$\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$							5.25%

Probability distribution of an ESG score

Table 2: ESG rating migration matrix (Question 3.b)

Rating	s_k	p_k	Transition probability $p_{j,k}$				$\mathcal{T}(\mathcal{R}_k)$
D		25.00%	95.15%	4.85%	0.00%	0.00%	4.85%
	-0.67						
C		25.00%	5.27%	88.38%	6.35%	0.00%	11.62%
	0.00						
B		25.00%	0.00%	6.35%	88.38%	5.27%	11.62%
	0.67						
A		25.00%	0.00%	0.00%	4.85%	95.15%	4.85%
$\mathcal{T}(\mathcal{R}_1,\ldots,\mathcal{R}_K)$							8.23%

Probability distribution of an ESG score

Table 3: ESG rating migration matrix (Question 3.c)

Rating	s_k	p_k	Transition probability $p_{j,k}$				$\mathcal{T}(\mathcal{R}_k)$
D		10.00%	93.54%	6.46%	0.00%	0.00%	6.46%
	-1.28						
C		40.00%	1.89%	94.14%	3.97%	0.00%	5.86%
	0.00						
B		40.00%	0.00%	3.97%	94.14%	1.89%	5.86%
	1.28						
A		10.00%	0.00%	0.00%	6.46%	93.54%	6.46%
$\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$							5.98%

Probability distribution of an ESG score

The ESG rating system defined in Question 3.a is the best rating system if we would like to reduce the monthly turnover of ESG ratings.

Probability distribution of an ESG score

Question 4.i

Draw the relationship between the parameter σ and the turnover $\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$ for the three ESG rating systems.

Probability distribution of an ESG score

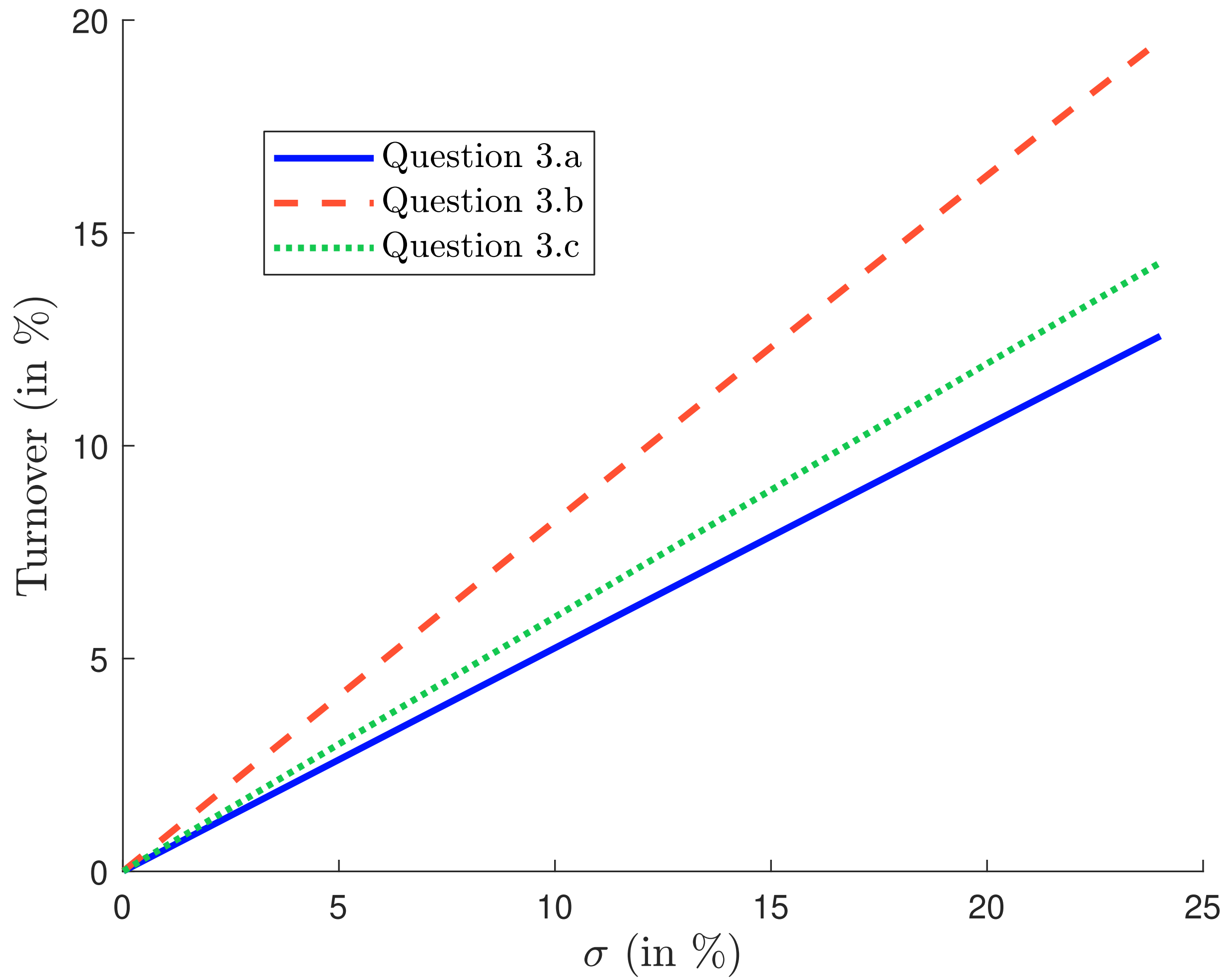


Figure 6: Relationship between σ and $\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$

Probability distribution of an ESG score

Question 4.j

We consider a uniform ESG rating system where:

$$\Pr\left\{\mathcal{R}\left(t-1\right)=\mathcal{R}_k\right\}=\frac{1}{K}$$

Draw the relationship between the number of notches K and the turnover $\mathcal{T}\left(\mathcal{R}_1,\ldots,\mathcal{R}_K\right)$ when the parameter σ takes the values 5%, 10% and 25%.

Probability distribution of an ESG score

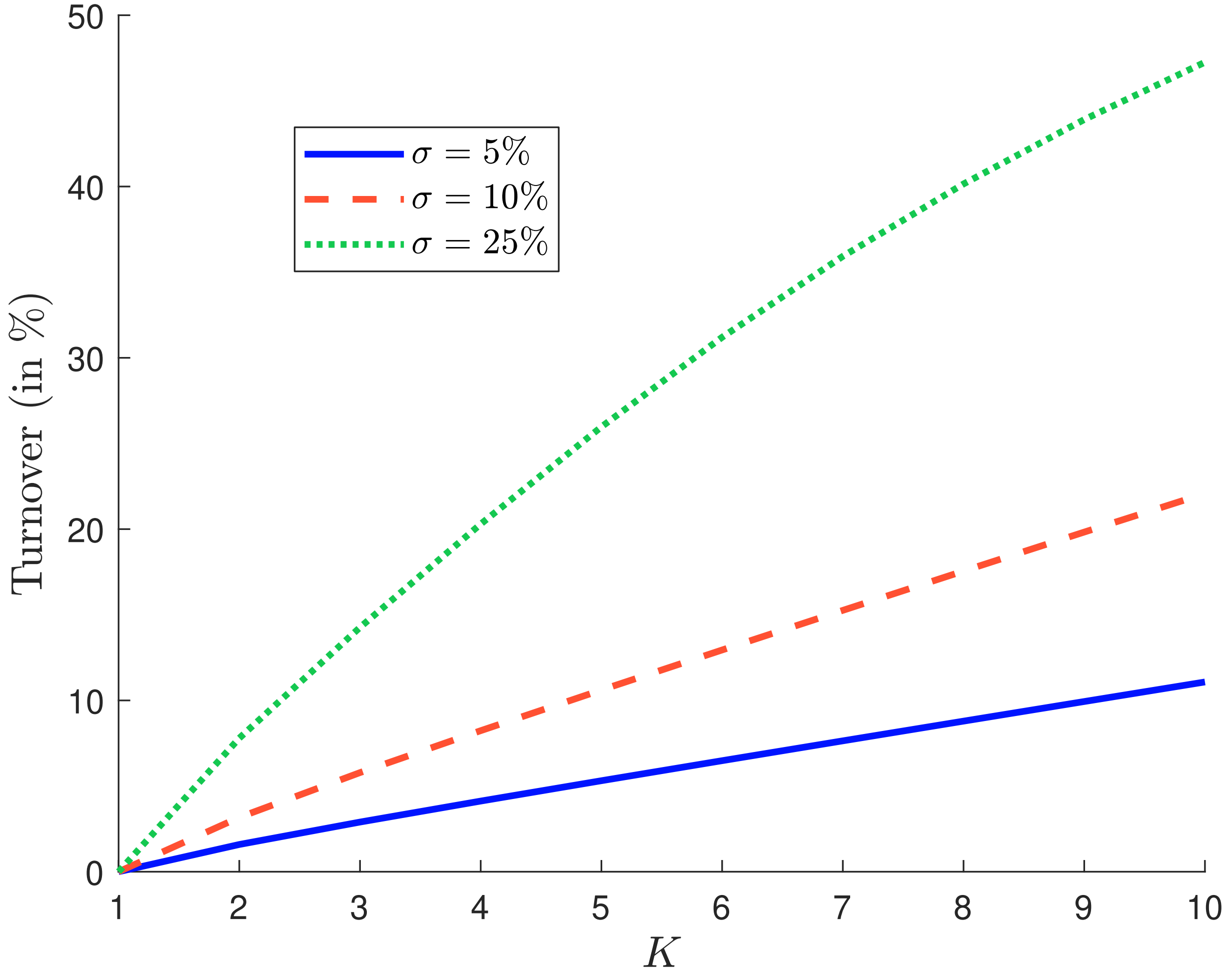


Figure 7: Relationship between K and $\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$

Probability distribution of an ESG score

Question 4.k

Why is an ESG rating system different than a credit rating system? What do you conclude from the previous analysis? What is the issue of ESG exclusion policy and negative screening?

Probability distribution of an ESG score

- An ESG rating system is mainly quantitative and highly depends on the mapping function. This is not the case of a credit rating system, which is mainly qualitative and discretionary.
- This explains that the turnover of an ESG rating system is higher than the turnover of a credit rating system.
- The stabilization of the ESG rating system implies to reduce the turnover $\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$, which depends on:
  1. The number of notches[2] K;
  2. The volatility σ of score changes
  3. The design of the ESG rating system $(s_1, \ldots, s_{K-1})$
- The turnover $\mathcal{T}(\mathcal{R}_1, \ldots, \mathcal{R}_K)$ has a big impact on an ESG exclusion (or negative screening) policy, because it creates noisy short-term entry/exit positions that do not necessarily correspond to a decrease or increase of the long-term ESG risks.

[2] This is why ESG rating systems have less notches than credit rating systems

Enhanced ESG score & tracking error control

Exercise

We consider a capitalization-weighted equity index, which is composed of 8 stocks. Their weights, volatilities and ESG scores are the following:

Stock	#1	#2	#3	#4	#5	#6	#7	#8
CW weight	0.23	0.19	0.17	0.13	0.09	0.08	0.06	0.05
Volatility	0.22	0.20	0.25	0.18	0.35	0.23	0.13	0.29
ESG score	−1.20	0.80	2.75	1.60	−2.75	−1.30	0.90	−1.70

The correlation matrix is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

Enhanced ESG score & tracking error control

Question 1

Calculate the ESG score of the benchmark.

Enhanced ESG score & tracking error control

- We note b_i and $\mathcal{S}_i$ the weight in the benchmark and the ESG score of Stock i
- The ESG score of the benchmark is equal to:

$$\mathcal{S}(b) = \sum_{i=1}^{8} b_i \cdot \mathcal{S}_i = 0.1690$$

Enhanced ESG score & tracking error control

Question 2

We consider the EW and ERC portfolios. Calculate the ESG score of these two portfolios. Define the ESG excess score with respect to the benchmark. Comment on these results.

Enhanced ESG score & tracking error control

- The composition of the EW portfolio is $x_i = 12.5\%$ and we have:

$$\mathcal{S}\left(x_{\mathrm{ew}}\right) = \sum_{i=1}^{8} \frac{\mathcal{S}_i}{8} = -0.1125$$

- The composition of the ERC portfolio is $x_1 = 12.42\%$, $x_2 = 14.03\%$, $x_3 = 10.17\%$, $x_4 = 13.79\%$, $x_5 = 7.59\%$, $x_6 = 12.34\%$, $x_7 = 20.61\%$ and $x_8 = 9.06\%$. We have:

$$\mathcal{S}\left(x_{\mathrm{erc}}\right) = \sum_{i=1}^{8} x_i \cdot \mathcal{S}_i = 0.1259$$

Enhanced ESG score & tracking error control

- The ESG excess score with respect to the benchmark is:

$$\mathcal{S}(x \mid b) = \mathcal{S}(x) - \mathcal{S}(b)$$

We have:

$$\begin{aligned} \mathcal{S}(x_{\mathrm{ew}} \mid b) &= -0.1125 - 0.1690 = -0.2815 \\ \mathcal{S}(x_{\mathrm{erc}} \mid b) &= 0.1259 - 0.1690 = -0.0431 \end{aligned}$$

- The two portfolios are riskier than the benchmark portfolio in terms of ESG risk

Enhanced ESG score & tracking error control

Question 3

Write the γ-problem of the ESG optimized portfolio when the goal is to improve the ESG score of the benchmark and control at the same time the tracking error volatility. Give the QP objective function.

Enhanced ESG score & tracking error control

- We have:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2}\sigma^2(x \mid b) - \gamma\mathcal{S}(x \mid b) \\ &\text{u.c.} \quad \begin{cases} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \\ x \in \Omega \end{cases} \end{aligned}$$

- Since $\sigma^2(x \mid b) = (x-b)^\top \Sigma (x-b)$ and $\mathcal{S}(x \mid b) = (x-b)^\top \mathcal{S}$, we deduce that the QP objective function is:

$$x^\star = \arg\min \frac{1}{2} x^\top \Sigma x - x^\top (\gamma \mathcal{S} + \Sigma b)$$

Enhanced ESG score & tracking error control

Question 4

Draw the efficient frontier between the tracking error volatility and the ESG excess score[a].

[a] We notice that $\gamma \in [0, 1.2\%]$ is sufficient for drawing the efficient frontier.

Enhanced ESG score & tracking error control

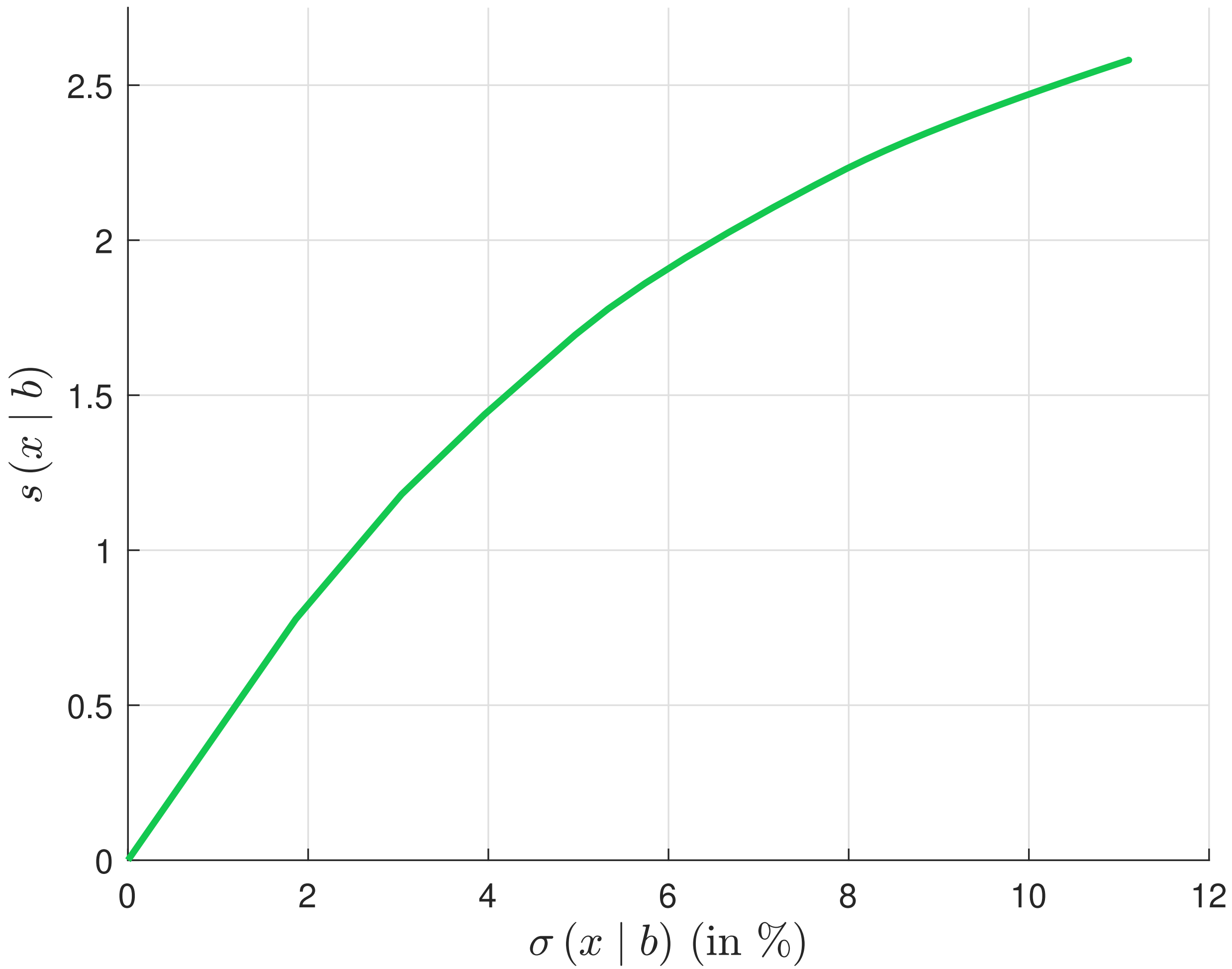


Figure 8: ESG efficient frontier

Enhanced ESG score & tracking error control

Question 5

Using the bisection algorithm, find the optimal portfolio if we would like to improve the ESG score of the benchmark by 0.5. Give the optimal value of γ. Compute the tracking error volatility $\sigma(x \mid b)$.

Enhanced ESG score & tracking error control

- The solution is equal to:

Stock	s_i	b_i	$x_i^\star$
#1	-1.200	23.000	25.029
#2	0.800	19.000	14.251
#3	2.750	17.000	21.947
#4	1.600	13.000	27.305
#5	-2.750	9.000	3.718
#6	-1.300	8.000	1.339
#7	0.900	6.000	1.675
#8	-1.700	5.000	4.736

- The optimal value of γ is 0.02768%
- The tracking error volatility is equal to 1.17636%

Enhanced ESG score & tracking error control

Question 6

Same question if we would like to improve the ESG score of the benchmark by 1.0.

Enhanced ESG score & tracking error control

- The solution is equal to:

Stock	s_i	b_i	$x_i^\star$
#1	−1.200	23.000	21.699
#2	0.800	19.000	12.443
#3	2.750	17.000	28.739
#4	1.600	13.000	33.555
#5	−2.750	9.000	0.002
#6	−1.300	8.000	0.000
#7	0.900	6.000	2.433
#8	−1.700	5.000	1.129

- The optimal value of γ is 0.07276%
- The tracking error volatility is equal to 2.48574%

Enhanced ESG score & tracking error control

Question 7

We impose that the portfolio weights can not be greater than 30%. Find the optimal portfolio if we would like to improve the ESG score of the benchmark by 1.0.

Enhanced ESG score & tracking error control

- The solution is equal to:

Stock	s_i	b_i	$x_i^\star$
#1	−1.200	23.000	20.116
#2	0.800	19.000	14.082
#3	2.750	17.000	29.481
#4	1.600	13.000	30.000
#5	−2.750	9.000	0.644
#6	−1.300	8.000	0.000
#7	0.900	6.000	4.662
#8	−1.700	5.000	1.015

- The optimal value of γ is 0.07355%
- The tracking error volatility is equal to 2.50317%

Enhanced ESG score & tracking error control

Question 8

Comment on these results.

Enhanced ESG score & tracking error control

- We notice that the evolution of the weights is not necessarily monotonous with respect to the ESG excess score $\mathcal{S}(x \mid b)$. For instance, if we target an improvement of 0.5, the weight of Stock #1 increases ($23\% \Rightarrow 25.029\%$). If we target an improvement of 1.0, the the weight of Stock #1 decreases ($25.029\% \Rightarrow 21.699\%$)
- Generally, the optimiser reduces the weight of stocks with low ESG scores and increases the weight of stocks with high ESG scores
- Nevertheless, the weight differences are not ranked in the same order than the ESG scores. For instance, if we target an improvement of 0.5, the largest variation is observed for Stock #4, which has an ESG score of 1.6. This is not the largest ESG score, since Stock #3 has an ESG score of 2.75
- This is due to the structure of the covariance matrix (Stock #3 is riskier than Stock #4)

Tilted portfolios with ESG and carbon intensity constraints

Exercise

We consider the CAPM model:

$$R_i - r = \beta_i \cdot (R_m - r) + \varepsilon_i$$

where R_i is the return of Asset i, R_m is the return of the market portfolio, r is the risk free asset, β_i is the beta of Asset i with respect to the market portfolio and ε_i is the idiosyncratic risk. We assume that $R_m \perp \varepsilon_i$ and $\varepsilon_i \perp \varepsilon_j$. We note σ_m the volatility of the market portfolio and $\tilde{\sigma}_i$ the idiosyncratic volatility. We consider a universe of 5 assets:

Asset i	1	2	3	4	5
β_i	0.30	0.50	0.90	1.30	2.00
$\tilde{\sigma}_i$	15%	16%	10%	11%	12%

and $\sigma_m = 20\%$. The risk free return is set to 1% and we assume that the expected return of the market portfolio is equal to $\mu_m = 6\%$.

Tilted portfolios with ESG and carbon intensity constraints

Question 1

We assume that the CAPM is valid.

Tilted portfolios with ESG and carbon intensity constraints

Question 1.a

Calculate the vector μ of expected returns.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\begin{aligned} \mu &= \mathbb{E}\left[R_i\right] \\ &= r + \beta_i \cdot (\mu_m - r) \\ &= \left(\begin{array}{r} 2.50 \\ 3.50 \\ 5.50 \\ 7.50 \\ 11.00 \end{array}\right) \quad \text{(in \%)} \end{aligned}$$

Tilted portfolios with ESG and carbon intensity constraints

Question 1.b

Compute the covariance matrix Σ.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\begin{aligned}
\Sigma &= \operatorname{cov}(R) \\
&= \beta\beta^\top \sigma_m^2 + \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_5^2\right) \\
&= \begin{pmatrix}
261.00 & 60.00 & 108.00 & 156.00 & 240.00 \\
60.00 & 356.00 & 180.00 & 260.00 & 400.00 \\
108.00 & 180.00 & 424.00 & 468.00 & 720.00 \\
156.00 & 260.00 & 468.00 & 797.00 & 1040.00 \\
240.00 & 400.00 & 720.00 & 1040.00 & 1744.00
\end{pmatrix} \times 10^{-4}
\end{aligned}$$

Tilted portfolios with ESG and carbon intensity constraints

Question 1.c

Deduce the volatility σ_i of the assets and find the correlation matrix $C = (\rho_{i,j})$ between asset returns.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\begin{aligned}
\sigma_i &= \sqrt{\Sigma_{i,i}} \\
&= \sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2} \\
&= \begin{pmatrix} 16.16 \\ 18.87 \\ 20.59 \\ 28.23 \\ 41.76 \end{pmatrix} \quad \text{(in \%)}
\end{aligned}$$

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\rho_{i,j} = \frac{\operatorname{cov}(R_i, R_j)}{\sigma(R_i)\cdot\sigma(R_j)} = \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}}\cdot\sqrt{\Sigma_{i,i}}} \frac{\beta_i\beta_j\sigma_m^2}{\sqrt{\beta_i^2\sigma_m^2+\tilde{\sigma}_i^2}\cdot\sqrt{\beta_j^2\sigma_m^2+\tilde{\sigma}_i^2}}$$

and:

$$C = \begin{pmatrix} 100.00 & 19.68 & 32.47 & 34.20 & 35.57 \\ 19.68 & 100.00 & 46.33 & 48.81 & 50.76 \\ 32.47 & 46.33 & 100.00 & 80.51 & 83.73 \\ 34.20 & 48.81 & 80.51 & 100.00 & 88.21 \\ 35.57 & 50.76 & 83.73 & 88.21 & 100.00 \end{pmatrix} \text{ (in \%)}$$

Tilted portfolios with ESG and carbon intensity constraints

Question 2

We assume that:

Asset i	1	2	3	4	5
μ_i	3%	4%	5%	7%	10%
$\mathcal{S}_i^{\mathrm{esg}}$	1.10	2.70	−0.90	−2.20	0.40
$\mathcal{CI}_i$	50	170	490	180	320
b_i	20%	20%	20%	20%	20%

where μ_i, $\mathcal{S}_i^{\mathrm{esg}}$, $\mathcal{CI}_i$ and b_i are respectively the expected return, the ESG score[a], the carbon intensity in $tCO_2e/\$$ mn and the benchmark weight of Asset i. The covariance matrix is given by the CAPM model and corresponds to the one calculated in Question 1.b. In what follows, we consider **long-only** portfolios.

[a] It corresponds to a z-score between -3 (worst score) and $+3$ (best score).

Tilted portfolios with ESG and carbon intensity constraints

Question 2.a

Compute the ESG score $\mathcal{S}^{\mathrm{esg}}(b)$ and the carbon intensity $\mathcal{CI}(b)$ of the benchmark b.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\begin{aligned}\mathcal{S}^{\mathrm{esg}}(b) &= \sum_{i=1}^{5} b_i \cdot \mathcal{S}_i^{\mathrm{esg}} \\ &= b^\top \mathcal{S}^{\mathrm{esg}} \\ &= 0.22\end{aligned}$$

- We have:

$$\begin{aligned}\mathcal{CI}(b) &= \sum_{i=1}^{5} b_i \cdot \mathcal{CI}_i \\ &= b^\top \mathcal{CI} \\ &= 242\end{aligned}$$

Tilted portfolios with ESG and carbon intensity constraints

Question 2.b

The current portfolio of the fund manager is equal to:

$$x = \begin{pmatrix} 10\% \\ 10\% \\ 30\% \\ 30\% \\ 20\% \end{pmatrix}$$

Compute the excess expected return $\mu(x \mid b)$, the tracking error volatility $\sigma(x \mid b)$, the ESG score $\mathcal{S}^{\text{esg}}(x)$ and the carbon intensity $\mathcal{CI}(x)$ of the portfolio x. Deduce its information ratio $\mathrm{IR}(x \mid b)$. Comment on these results.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\mu(x \mid b) = (x - b)^\top \mu = 50 \text{ bps}$$

and:

$$\sigma(x \mid b) = \sqrt{(x - b)^\top \Sigma (x - b)} = 3.85\%$$

- We deduce that:

$$\mathrm{IR}(x \mid b) = \frac{\mu(x \mid b)}{\sigma(x \mid b)} = \frac{0.50}{3.85} = 0.13$$

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\mathcal{S}^{\mathrm{esg}}(x) = \sum_{i=1}^{5} x_i \cdot \mathcal{S}_i^{\mathrm{esg}} = -0.47 \ll \mathcal{S}^{\mathrm{esg}}(b) = 0.22$$

and:

$$\mathcal{CI}(x) = \sum_{i=1}^{5} x_i \cdot \mathcal{CI}_i = 287 \gg \mathcal{CI}(b) = 242$$

- x is a good portfolio from the viewpoint of financial analysis since it has a positive information ratio. Nevertheless, it is a bad portfolio from the viewpoint of extra-financial analysis if we compare it with the benchmark. Indeed, it has a lower ESG score and a higher carbon intensity.

Tilted portfolios with ESG and carbon intensity constraints

Question 2.c

We would like to tilt the benchmark b in order to improve its expected return. Formulate the γ-problem of portfolio optimization in the presence of a benchmark. Find the corresponding QP problem. We note $x^\star(\gamma)$ the optimized portfolio. Draw the efficient frontier betwen the tracking error volatility $\sigma(x^\star(\gamma) \mid b)$ and the excess expected return $\mu(x^\star(\gamma) \mid b)$.

Question 2.d

Draw the relationship between $\sigma(x^\star(\gamma) \mid b)$ and $\mathcal{S}^{\mathrm{esg}}(x^\star(\gamma))$. Comment on these results.

Question 2.e

Draw the relationship between $\sigma(x^\star(\gamma) \mid b)$ and $\mathcal{CI}(x^\star(\gamma))$. Comment on these results.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$
\begin{aligned}
x^{\star}(\gamma) &= \arg\min \frac{1}{2}\sigma^{2}(x \mid b) - \gamma\mu(x \mid b) \\
& \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \\ x \in \Omega \end{array} \right.
\end{aligned}
$$

- Since $\sigma^{2}(x \mid b) = (x - b)^{\top} \Sigma (x - b)$ and $\mu(x \mid b) = (x - b)^{\top} \mu$, we deduce that the QP objective function is:

$$
x^{\star}(\gamma) = \arg\min \frac{1}{2} x^{\top} \Sigma x - x^{\top} (\gamma\mu + \Sigma b)
$$

Tilted portfolios with ESG and carbon intensity constraints

- We recall that the formulation of a standard QP problem is:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^\top Q x - x^\top R \\
&\text{u.c.} \quad \left\{
\begin{array}{l}
Ax = B \\
Cx \leq D \\
x^- \leq x \leq x^+
\end{array}
\right.
\end{aligned}
$$

- We have the following QP correspondences:

$$
\begin{aligned}
Q &= \Sigma \\
R &= \gamma\mu + \Sigma b \\
A &= \mathbf{1}_n^\top \\
B &= 1 \\
x^- &= \mathbf{0}_n \\
x^+ &= \mathbf{1}_n
\end{aligned}
$$

Tilted portfolios with ESG and carbon intensity constraints

- We compute $x^{\star}(\gamma)$ for several values of $\gamma \in [0, 10]$.
- For a given optimized portfolio $x^{\star}(\gamma)$, we compute:

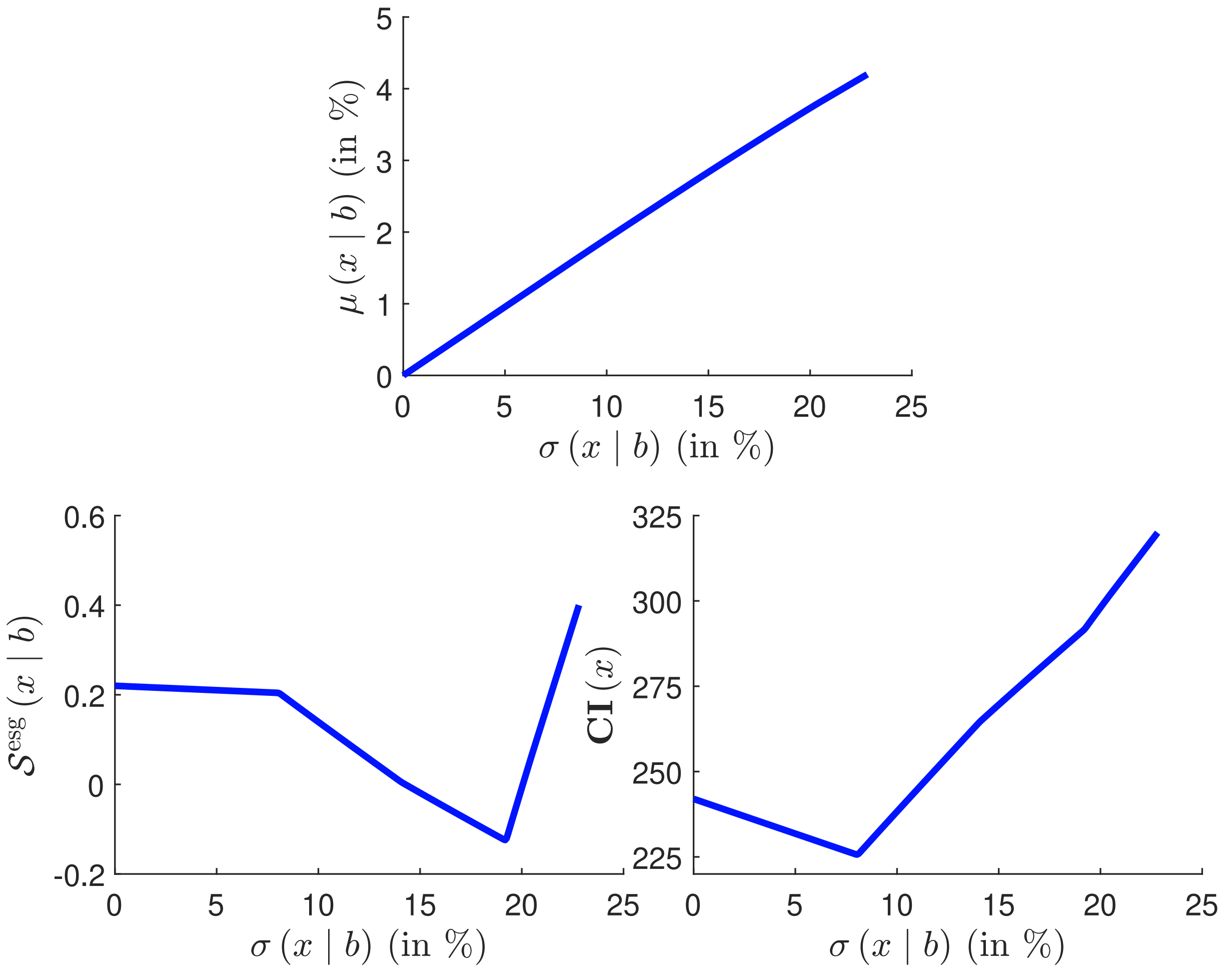
$$\mathcal{S}^{\mathrm{esg}}(x^{\star}(\gamma)) = \sum_{i=1}^{5} x_i^{\star}(\gamma) \cdot \mathcal{S}_i^{\mathrm{esg}}$$

and:

$$\mathcal{CI}(x^{\star}(\gamma)) = \sum_{i=1}^{5} x_i^{\star}(\gamma) \cdot \mathcal{CI}_i$$

Tilted portfolios with ESG and carbon intensity constraints

Figure 9: The efficient frontier of optimal portfolios



Tilted portfolios with ESG and carbon intensity constraints

- We do not observe a monotonous function between the tracking error volatility and the ESG score or the carbon intensity.
- When the tracking error volatility is low, the ESG score decreases weakly but we obtain a better carbon intensity.
- When the tracking error volatility is high, the ESG score is improved but the carbon intensity is degraded.

Tilted portfolios with ESG and carbon intensity constraints

Question 2.f

Find the optimal portfolio $x^\star$ if we target an ex-ante tracking error volatility of 5%. Give the optimal value of γ, the expected excess return $\mu\left(x^\star \mid b\right)$ and the information ratio $\mathrm{IR}\left(x^\star \mid b\right)$. Compute also the ESG score $\mathcal{S}^{\mathrm{esg}}\left(x^\star\right)$ and the carbon intensity $\mathcal{CI}\left(x^\star\right)$ of the optimal portfolio $x^\star$.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$x^{\star} = \begin{pmatrix} 14.66 \\ 18.66 \\ 7.60 \\ 23.96 \\ 35.13 \end{pmatrix} \quad \text{(in \%)}$$

- The optimal value of γ is equal to 26.16%.

Tilted portfolios with ESG and carbon intensity constraints

- We obtain $\mu(x^\star \mid b) = 96$ bps, $\sigma(x^\star \mid b) = 5\%$ and $\mathrm{IR}(x^\star \mid b) = 19\%$.
- We have $\mathcal{S}^{\mathrm{esg}}(x^\star) = 0.21$ and $\mathcal{S}^{\mathrm{esg}}(x^\star \mid b) = -0.01 < 0$. We obtain a lower ESG score, but it is close to zero.
- We have $\mathcal{CI}(x^\star) = 231.81$ and $\mathcal{CI}(x^\star \mid b) = -10.19 < 0$. We have improved the carbon intensity of the benchmark.

Tilted portfolios with ESG and carbon intensity constraints

Question 3

We now assume that $\mu = \mathbf{0}_5$.

Tilted portfolios with ESG and carbon intensity constraints

Question 3.a

We would like to reduce the carbon intensity of the benchmark portfolio by 20%. Give the QP formulation of the optimization problem. Compute the optimal portfolio $x^\star$ such that it has the lowest tracking error volatility $\sigma(x \mid b)$. Give the values of $\sigma(x^\star \mid b)$, $\mathcal{S}^{\mathrm{esg}}(x^\star)$, $\mathcal{S}^{\mathrm{esg}}(x^\star \mid b)$, $\mathcal{CI}(x^\star)$, $\mathcal{CI}(x^\star \mid b)$ and the reduction rate $\mathcal{R}(x^\star \mid b)$ of the carbon intensity. Comment on these results.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2}\sigma^2(x \mid b) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \\ \mathcal{CI}(x) \leq (1 - \mathcal{R}) \cdot \mathcal{CI}(b) \end{array} \right. \end{aligned}$$

where $\mathcal{R} = 20\%$.

Tilted portfolios with ESG and carbon intensity constraints

- We recall that the formulation of a standard QP problem is:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\ & \text{u.c.} \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

- We have the following QP correspondences:

$$\begin{array}{ll} Q = \Sigma & C = \mathcal{CI}^{\top} \\ R = \Sigma b & D = \mathcal{CI}^{+} = (1 - \mathcal{R}) \cdot \left(b^{\top} \mathcal{CI}\right) \\ A = \mathbf{1}_n^{\top} & x^{-} = \mathbf{0}_n \\ B = 1 & x^{+} = \mathbf{1}_n \end{array}$$

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$x^{\star} = \begin{pmatrix} 25.21 \\ 21.93 \\ 6.36 \\ 25.90 \\ 20.60 \end{pmatrix} \quad \text{(in \%)}$$

- We obtain $\sigma\left(x^{\star} \mid b\right) = 1.74\%$, $\mathcal{S}^{\mathrm{esg}}\left(x^{\star}\right) = 0.32$, $\mathcal{S}^{\mathrm{esg}}\left(x^{\star} \mid b\right) = 0.10$, $\mathcal{CI}\left(x^{\star}\right) = 193.60$, $\mathcal{CI}\left(x^{\star} \mid b\right) = -48.40$ and $\mathcal{R}\left(x^{\star} \mid b\right) = 20\%$.
- We obtain a better ESG score with an improvement of the carbon intensity.

Tilted portfolios with ESG and carbon intensity constraints

Question 3.b

We would like to improve the ESG score of the benchmark portfolio by $+0.50$. Give the QP formulation of the optimization problem. Compute the optimal portfolio $x^{\star}$ such that it has the lowest tracking error volatility $\sigma(x \mid b)$. Give the values of $\sigma(x^{\star} \mid b)$, $\mathcal{S}^{\mathrm{esg}}(x^{\star})$, $\mathcal{S}^{\mathrm{esg}}(x^{\star} \mid b)$, $\mathcal{CI}(x^{\star})$, $\mathcal{CI}(x^{\star} \mid b)$ and the reduction rate $\mathcal{R}(x^{\star} \mid b)$ of the carbon intensity. Comment on these results.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$
\begin{aligned}
x^{\star}(\gamma) &= \arg\min \frac{1}{2}\sigma^2(x \mid b) \\
\text{u.c.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_n^\top x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
\mathcal{S}^{\mathrm{esg}}(x) \geq \mathcal{S}^{\mathrm{esg}}(b) + 0.5
\end{array}
\right.
\end{aligned}
$$

Tilted portfolios with ESG and carbon intensity constraints

- We recall that the formulation of a standard QP problem is:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\ & \text{u.c.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

- We have the following QP correspondences:

$$\begin{array}{ll} Q = \Sigma & C = \left(-\mathcal{S}^{\mathrm{esg}}\right)^{\top} \\ R = \Sigma b & D = -\left(b^{\top} \mathcal{S}^{\mathrm{esg}} + 0.5\right) \\ A = \mathbf{1}_n^{\top} & x^{-} = \mathbf{0}_n \\ B = 1 & x^{+} = \mathbf{1}_n \end{array}$$

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$x^{\star} = \begin{pmatrix} 22.36 \\ 26.70 \\ 14.71 \\ 9.98 \\ 26.24 \end{pmatrix} \quad \text{(in \%)}$$

- We obtain $\sigma\left(x^{\star} \mid b\right) = 1.84\%$, $\mathcal{S}^{\mathrm{esg}}\left(x^{\star}\right) = 0.72$, $\mathcal{S}^{\mathrm{esg}}\left(x^{\star} \mid b\right) = 0.50$, $\mathcal{CI}\left(x^{\star}\right) = 230.61$, $\mathcal{CI}\left(x^{\star} \mid b\right) = -11.39$ and $\mathcal{R}\left(x^{\star} \mid b\right) = 4.71\%$.
- We obtain a better carbon intensity with an improvement of the ESG score. Nevertheless, the reduction of the carbon intensity is low and less than 5%.

Tilted portfolios with ESG and carbon intensity constraints

Question 3.c

Same question if we mix the two constraints.

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$
\begin{aligned}
x^{\star}(\gamma) &= \arg\min \frac{1}{2}\sigma^{2}(x \mid b) \\
\text{u.c.} & \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
\mathcal{S}^{\mathrm{esg}}(x) \geq \mathcal{S}^{\mathrm{esg}}(b) + 0.5 \\
\mathcal{CI}(x) \leq (1 - \mathcal{R}) \cdot \mathcal{CI}(b)
\end{array}
\right.
\end{aligned}
$$

Tilted portfolios with ESG and carbon intensity constraints

- We recall that the formulation of a standard QP problem is:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\
\text{u.c.} & \left\{
\begin{array}{l}
Ax = B \\
Cx \leq D \\
x^{-} \leq x \leq x^{+}
\end{array}
\right.
\end{aligned}
$$

- We have the following QP correspondences:

$$
\begin{array}{ll}
Q = \Sigma & C = \left[\begin{array}{c} \boldsymbol{\mathcal{CI}}^{\top} \\ \left(-\mathcal{S}^{\mathrm{esg}}\right)^{\top} \end{array} \right] \\
R = \Sigma b & D = \left[\begin{array}{c} (1 - 20\%) \cdot \left(b^{\top} \boldsymbol{\mathcal{CI}}\right) \\ -\left(b^{\top} \mathcal{S}^{\mathrm{esg}} + 0.5\right) \end{array} \right] \\
A = \mathbf{1}_n^{\top} & x^{-} = \mathbf{0}_n \\
B = 1 & x^{+} = \mathbf{1}_n
\end{array}
$$

Tilted portfolios with ESG and carbon intensity constraints

- We have:

$$x^{\star} = \begin{pmatrix} 26.16 \\ 27.13 \\ 4.64 \\ 16.41 \\ 25.67 \end{pmatrix} \quad \text{(in \%)}$$

- We obtain $\sigma(x^{\star} \mid b) = 2.29\%$, $\mathcal{S}^{\mathrm{esg}}(x^{\star}) = 0.72$, $\mathcal{S}^{\mathrm{esg}}(x^{\star} \mid b) = 0.50$, $\mathcal{CI}(x^{\star}) = 193.60$, $\mathcal{CI}(x^{\star} \mid b) = -48.40$ and $\mathcal{R}(x^{\star} \mid b) = 20\%$.
- It is possible to target the two objectives, but the tracking error volatility increases and is greater than 2%.

Net Zero Alignment Portfolio