Financial Risk Management
Tutorial Class — Session 4

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We consider a sample of $n$ individual losses $\{x_1, \ldots, x_n\}$. We assume that they can be described by different probability distributions:

(i) $X$ follows a log-normal distribution $\mathcal{LN}(\mu, \sigma^2)$.

(ii) $X$ follows a Pareto distribution $\mathcal{P}(\alpha, x^{-})$ defined by:

$$\Pr\{X \leq x\} = 1 - \left( \frac{x}{x^{-}} \right)^{-\alpha}$$

with $x \geq x^{-}$ and $\alpha > 0$.

(iii) $X$ follows a gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_{0}^{x} \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \, dt$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss $X$ follows a gamma distribution: $\ln X \sim \Gamma(\alpha; \beta)$.
Estimation of the loss severity distribution

Question 1
We consider the case (i).

(i) $X$ follows a log-normal distribution $\mathcal{LN}(\mu, \sigma^2)$. 
Question 1.a
Show that the probability density function is:

\[ f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2 \right) \]
Estimation of the loss severity distribution

The density of the Gaussian distribution $Y \sim \mathcal{N} (\mu, \sigma^2)$ is:

$$g (y) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( - \frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right)$$

Let $X \sim \mathcal{LN} (\mu, \sigma^2)$. We have $X = \exp Y$. It follows that:

$$f (x) = g (y) \left| \frac{dy}{dx} \right|$$

with $y = \ln x$. We deduce that:

$$f (x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( - \frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right) \times \frac{1}{x}$$

$$= \frac{1}{x\sigma \sqrt{2\pi}} \exp \left( - \frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2 \right)$$
Question 1.b

Calculate the two first moments of $X$. Deduce the orthogonal conditions of the generalized method of moments.
For $m \geq 1$, the non-centered moment is equal to:

$$E[X^m] = \int_{0}^{\infty} x^m \frac{1}{x\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2 \right) \, dx$$
Estimation of the loss severity distribution

By considering the change of variables \( y = \sigma^{-1} (\ln x - \mu) \) and \( z = y - m\sigma \), we obtain:

\[
\mathbb{E}[X^m] = \int_{-\infty}^{\infty} e^{m\mu + m\sigma y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy
\]

\[
= e^{m\mu} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2 + m\sigma y} \, dy
\]

\[
= e^{m\mu} \times e^{\frac{1}{2}m^2\sigma^2} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-m\sigma)^2} \, dy
\]

\[
= e^{m\mu + \frac{1}{2}m^2\sigma^2} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}z^2 \right) \, dz
\]

\[
= e^{m\mu + \frac{1}{2}m^2\sigma^2}
\]
Estimation of the loss severity distribution

We deduce that:

\[ \mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2} \]

and:

\[ \text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] \]

\[ = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \]

\[ = e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right) \]

We can estimate the parameters \( \mu \) and \( \sigma \) with the generalized method of moments by using the following empirical moments:

\[
\begin{align*}
    h_{i,1}(\mu, \sigma) &= x_i - e^{\mu + \frac{1}{2}\sigma^2} \\
    h_{i,2}(\mu, \sigma) &= (x_i - e^{\mu + \frac{1}{2}\sigma^2})^2 - e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right)
\end{align*}
\]
Estimation of the loss severity distribution

Question 1.c
Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$. 
The log-likelihood function of the sample \( \{x_1, \ldots, x_n\} \) is:

\[
\ell(\mu, \sigma) = \sum_{i=1}^{n} \ln f(x_i) = -\frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln 2\pi - \sum_{i=1}^{n} \ln x_i - \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\ln x_i - \mu}{\sigma} \right)^2
\]

To find the ML estimators \( \hat{\mu} \) and \( \hat{\sigma} \), we can proceed in two different ways.
Estimation of the loss severity distribution

#1 \( X \sim \mathcal{LN}(\mu, \sigma^2) \) implies that \( Y = \ln X \sim \mathcal{N}(\mu, \sigma^2) \). We know that the ML estimators \( \hat{\mu} \) and \( \hat{\sigma} \) associated to \( Y \) are:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} y_i
\]

\[
\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\mu})^2}
\]

We deduce that the ML estimators \( \hat{\mu} \) and \( \hat{\sigma} \) associated to the sample \( \{x_1, \ldots, x_n\} \) are:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \ln x_i
\]

\[
\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\ln x_i - \hat{\mu})^2}
\]
#2 We maximize the log-likelihood function. The first-order conditions are \( \partial_\mu \ell (\mu, \sigma) = 0 \) and \( \partial_\sigma \ell (\mu, \sigma) = 0 \). We deduce that:

\[
\partial_\mu \ell (\mu, \sigma) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (\ln x_i - \mu) = 0
\]

and:

\[
\partial_\sigma \ell (\mu, \sigma) = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(\ln x_i - \mu)^2}{\sigma^3} = 0
\]

We finally obtain:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \ln x_i
\]

and:

\[
\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\ln x_i - \hat{\mu})^2}
\]
Question 2

We consider the case (ii).

(ii) $X$ follows a Pareto distribution $\mathcal{P}(\alpha, x^-)$ defined by:

$$\Pr\{X \leq x\} = 1 - \left(\frac{x}{x^-}\right)^{-\alpha}$$

with $x \geq x^-$ and $\alpha > 0$. 
Estimation of the loss severity distribution

Question 2.a

Calculate the two first moments of $X$. Deduce the GMM conditions for estimating the parameter $\alpha$. 
Estimation of the loss severity distribution

The probability density function is:

\[
f(x) = \frac{\partial}{\partial x} \Pr\{X \leq x\} = \frac{\alpha x^{-(\alpha+1)}}{x^{-\alpha}}
\]

For \( m \geq 1 \), we have:

\[
\mathbb{E}[X^m] = \int_{x_-}^{\infty} x^m \frac{x^{-(\alpha+1)}}{x^{-\alpha}} \, dx
\]

\[
= \frac{\alpha}{x_-^{-\alpha}} \int_{x_-}^{\infty} x^{m-\alpha-1} \, dx
\]

\[
= \frac{\alpha}{x_-^{-\alpha}} \left[ \frac{x^{m-\alpha}}{m-\alpha} \right]_{x_-}^{\infty}
\]

\[
= \frac{\alpha}{\alpha - m} x_-^m
\]
Estimation of the loss severity distribution

We deduce that:

$$\mathbb{E}[X] = \frac{\alpha}{\alpha - 1} x_-$$

and:

$$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

$$= \frac{\alpha}{\alpha - 2} x_-^2 - \left(\frac{\alpha}{\alpha - 1} x_-\right)^2$$

$$= \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)} x_-^2$$
We can then estimate the parameter $\alpha$ by considering the following empirical moments:

$$h_{i,1}(\alpha) = x_i - \frac{\alpha}{\alpha - 1}x_-$$

$$h_{i,2}(\alpha) = \left( x_i - \frac{\alpha}{\alpha - 1}x_-' \right)^2 - \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)}x_-^2$$

The generalized method of moments can consider either the first moment $h_{i,1}(\alpha)$, the second moment $h_{i,2}(\alpha)$ or the joint moments $(h_{i,1}(\alpha), h_{i,2}(\alpha))$. In the first case, the estimator is:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i - nx_-}$$
Question 2.b
Find the maximum likelihood estimator $\hat{\alpha}$. 
Operational Risk
Asset Liability Management Risk
Estimation of the loss severity distribution
Estimation of the loss frequency distribution

Estimation of the loss severity distribution

The log-likelihood function is:

$$
\ell(\alpha) = \sum_{i=1}^{n} \ln f(x_i) = n \ln \alpha - (\alpha + 1) \sum_{i=1}^{n} \ln x_i + n\alpha \ln x_-
$$

The first-order condition is:

$$
\frac{\partial \ell(\alpha)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln x_- = 0
$$

We deduce that:

$$
n = \alpha \sum_{i=1}^{n} \ln \frac{x_i}{x_-}
$$

The ML estimator is then:

$$
\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} (\ln x_i - \ln x_-)}
$$
Question 3

We consider the case (iii). Write the log-likelihood function associated to the sample of individual losses \( \{x_1, \ldots, x_n\} \). Deduce the first-order conditions of the maximum likelihood estimators \( \hat{\alpha} \) and \( \hat{\beta} \).

(iii) \( X \) follows a gamma distribution \( \Gamma (\alpha, \beta) \) defined by:

\[
\Pr \{ X \leq x \} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma (\alpha)} \, dt
\]

with \( x \geq 0, \alpha > 0 \) and \( \beta > 0 \).
Estimation of the loss severity distribution

The probability density function of (iii) is:

\[ f(x) = \frac{\partial \Pr \{ X \leq x \}}{\partial x} = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \]

It follows that the log-likelihood function is:

\[ \ell(\alpha, \beta) = \sum_{i=1}^{n} \ln f(x_i) = -n \ln \Gamma(\alpha) + n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} x_i \]

The first-order conditions \( \partial_\alpha \ell(\alpha, \beta) = 0 \) and \( \partial_\beta \ell(\alpha, \beta) = 0 \) imply that:

\[ n \left( \ln \beta - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right) + \sum_{i=1}^{n} \ln x_i = 0 \]

and:

\[ n\frac{\alpha}{\beta} - \sum_{i=1}^{n} x_i = 0 \]
Question 4

We consider the case (iv). Show that the probability density function of $X$ is:

$$f(x) = \frac{\beta^\alpha (\ln x)^{\alpha-1}}{\Gamma(\alpha) x^{\beta+1}}$$

What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$.

(iv) The natural logarithm of the loss $X$ follows a gamma distribution:

$$\ln X \sim \Gamma(\alpha; \beta).$$
Let $Y \sim \Gamma(\alpha, \beta)$ and $X = \exp Y$. We have:

$$f_X(x) \, |dx| = f_Y(y) \, |dy|$$

where $f_X$ and $f_Y$ are the probability density functions of $X$ and $Y$. We deduce that:

$$f_X(x) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} \times \frac{1}{e^y}$$

$$= \frac{\beta^\alpha (\ln x)^{\alpha-1} e^{-\beta \ln x}}{x \Gamma(\alpha)}$$

$$= \frac{\beta^\alpha (\ln x)^{\alpha-1}}{\Gamma(\alpha) x^{\beta+1}}$$

The support of this probability density function is $[0, +\infty)$. 

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Financial Risk Management (Tutorial Class — Session 4)
The log-likelihood function associated to the sample of individual losses \( \{x_1, \ldots, x_n\} \) is:

\[
\ell(\alpha, \beta) = \sum_{i=1}^{n} \ln f(x_i)
\]

\[
= -n \ln \Gamma(\alpha) + n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^{n} \ln (\ln x_i) - (\beta + 1) \sum_{i=1}^{n} \ln x_i
\]
Question 5

We now assume that the losses \( \{x_1, \ldots, x_n\} \) have been collected beyond a threshold \( H \) meaning that \( X \geq H \).
Question 5.a

What becomes the generalized method of moments in the case (i).

(i) $X$ follows a log-normal distribution $\mathcal{LN}(\mu, \sigma^2)$. 
Using Bayes’ formula, we have:

\[
Pr \{ X \leq x \mid X \geq H \} = \frac{Pr \{ H \leq X \leq x \}}{Pr \{ X \geq H \}}
\]

\[
= \frac{F(x) - F(H)}{1 - F(H)}
\]

where \( F \) is the cdf of \( X \). We deduce that the conditional probability density function is:

\[
f(x \mid X \geq H) = \frac{\partial}{\partial x} \Pr \{ X \leq x \mid X \geq H \}
\]

\[
= \frac{f(x)}{1 - F(H)} \times 1 \{ x \geq H \}
\]
Estimation of the loss severity distribution

For the log-normal probability distribution, we obtain:

\[
f(x \mid X \geq H) = \frac{1}{1 - \Phi \left( \frac{\ln H - \mu}{\sigma} \right)} \times \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2} dx
\]

\[
= \varphi \times \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2} dx
\]
We note $\mathcal{M}_m(\mu, \sigma)$ the conditional moment $\mathbb{E}[X^m | X \geq H]$. We have:

$$
\mathcal{M}_m(\mu, \sigma) = \varphi \times \int_H^\infty \frac{x^{m-1}}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2} \, dx
$$

$$
= \varphi \times \int_{\ln H}^\infty \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 + mx} \, dx
$$

$$
= \varphi \times e^{m\mu + \frac{1}{2} m^2 \sigma^2} \times \int_{\ln H}^\infty \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-(\mu+m\sigma^2)}{\sigma^2} \right)^2} \, dx
$$

$$
= 1 - \Phi \left( \frac{\ln H - \mu - m\sigma^2}{\sigma} \right) e^{m\mu + \frac{1}{2} m^2 \sigma^2}
$$

$$
= \frac{1 - \Phi \left( \frac{\ln H - \mu}{\sigma} \right)}{1 - \Phi \left( \frac{\ln H - \mu - m\sigma^2}{\sigma} \right)} e^{m\mu + \frac{1}{2} m^2 \sigma^2}
$$
The first two moments of $X \mid X \geq H$ are then:

$$M_1 (\mu, \sigma) = \mathbb{E} [X \mid X \geq H] = \frac{1 - \Phi \left( \frac{\ln H - \mu - \sigma^2}{\sigma} \right)}{1 - \Phi \left( \frac{\ln H - \mu}{\sigma} \right)} e^{\mu + \frac{1}{2} \sigma^2}$$

and:

$$M_2 (\mu, \sigma) = \mathbb{E} [X^2 \mid X \geq H] = \frac{1 - \Phi \left( \frac{\ln H - \mu - 2\sigma^2}{\sigma} \right)}{1 - \Phi \left( \frac{\ln H - \mu}{\sigma} \right)} e^{2\mu + 2\sigma^2}$$
We can therefore estimate $\mu$ and $\sigma$ by considering the following empirical moments:

$$
\begin{align*}
    h_{i,1}(\mu, \sigma) &= x_i - M_1(\mu, \sigma) \\
    h_{i,2}(\mu, \sigma) &= (x_i - M_1(\mu, \sigma))^2 - (M_2(\mu, \sigma) - M_1^2(\mu, \sigma))
\end{align*}
$$
Question 5.b

Calculate the maximum likelihood estimator $\hat{\alpha}$ in the case (ii).

(ii) $X$ follows a Pareto distribution $\mathcal{P}(\alpha, x^{-})$ defined by:

$$
\Pr \{X \leq x\} = 1 - \left(\frac{x}{x^{-}}\right)^{-\alpha}
$$

with $x \geq x^{-}$ and $\alpha > 0$. 
Estimation of the loss severity distribution

We have:

\[ f(x \mid X \geq H) = \frac{f(x)}{1 - F(H)} \times 1 \{x \geq H\} \]

\[ = \left( \frac{x^{-(\alpha+1)}}{\alpha x^{-\alpha}} \right) \bigg/ \left( \frac{H^{-\alpha}}{x^{-\alpha}} \right) \]

\[ = \alpha \frac{x^{-(\alpha+1)}}{H^{-\alpha}} \]

The conditional probability function is then a Pareto distribution with the same parameter \( \alpha \) but with a new threshold \( x_- = H \). We can then deduce that the ML estimator \( \hat{\alpha} \) is:

\[ \hat{\alpha} = \frac{n}{\left( \sum_{i=1}^{n} \ln x_i \right) - n \ln H} \]
Question 5.c

Write the log-likelihood function in the case (iii).

(iii) $X$ follows a gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)} \, dt$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$. 
Estimation of the loss severity distribution

The conditional probability density function is:

\[ f(x \mid X \geq H) = \frac{f(x)}{1 - F(H)} \times 1 \{x \geq H\} = \left( \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \right) \left/ \int_H^\infty \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \, dt \right. \]

\[ = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\int_H^\infty \beta^\alpha t^{\alpha-1} e^{-\beta t} \, dt} \]

We deduce that the log-likelihood function is:

\[ \ell(\alpha, \beta) = n\alpha \ln \beta - n \ln \left( \int_H^\infty \beta^\alpha t^{\alpha-1} e^{-\beta t} \, dt \right) + (\alpha - 1) \sum_{i=1}^n \ln x_i - \beta \sum_{i=1}^n x_i \]
Exercise

We consider a dataset of individual losses \( \{x_1, \ldots, x_n\} \) corresponding to a sample of \( T \) annual loss numbers \( \{N_{Y_1}, \ldots, N_{Y_T}\} \). This implies that:

\[
\sum_{t=1}^{T} N_{Y_t} = n
\]

If we measure the number of losses per quarter \( \{N_{Q_1}, \ldots, N_{Q_{4T}}\} \), we use the notation:

\[
\sum_{t=1}^{4T} N_{Q_t} = n
\]
Question 1

We assume that the annual number of losses follows a Poisson distribution \( P(\lambda_Y) \). Calculate the maximum likelihood estimator \( \hat{\lambda}_Y \) associated to the sample \( \{N_{Y_1}, \ldots, N_{Y_T}\} \).
Estimation of the loss frequency distribution

We have:

$$\Pr \{N = n\} = e^{-\lambda_Y} \frac{\lambda_Y^n}{n!}$$

We deduce that the expression of the log-likelihood function is:

$$\ell (\lambda_Y) = \sum_{t=1}^{T} \ln \Pr \{N = N_{Y_t}\} = -\lambda_Y T + \left( \sum_{t=1}^{T} N_{Y_t} \right) \ln \lambda_Y - \sum_{t=1}^{T} \ln (N_{Y_t}!)$$

The first-order condition is:

$$\frac{\partial \ell (\lambda_Y)}{\partial \lambda_Y} = -T + \frac{1}{\lambda_Y} \left( \sum_{t=1}^{T} N_{Y_t} \right) = 0$$

We deduce that the ML estimator is:

$$\hat{\lambda}_Y = \frac{1}{T} \sum_{t=1}^{T} N_{Y_t} = \frac{n}{T}$$
Question 2

We assume that the quarterly number of losses follows a Poisson distribution \( P(\lambda_Q) \). Calculate the maximum likelihood estimator \( \hat{\lambda}_Q \) associated to the sample \( \{N_{Q1}, \ldots, N_{Q4T}\} \).
Using the same arguments, we obtain:

$$\hat{\lambda}_Q = \frac{1}{4T} \sum_{t=1}^{4T} N_{Qt} = \frac{n}{4T} = \frac{\hat{\lambda}_Y}{4}$$
Question 3

What is the impact of considering a quarterly or annual basis on the computation of the capital charge?
Considering a quarterly or annual basis has no impact on the capital charge. Indeed, the capital charge is computed with a one-year time horizon. If we use a quarterly basis, we have to find the distribution of the annual loss number. In this case, the annual loss number is the sum of the four quarterly loss numbers:

\[ N_Y = N_{Q1} + N_{Q2} + N_{Q3} + N_{Q4} \]

We know that each quarterly loss number follows a Poisson distribution \( P(\hat{\lambda}_Q) \) and that they are independent. Because the Poisson distribution is infinitely divisible, we obtain:

\[ N_{Q1} + N_{Q2} + N_{Q3} + N_{Q4} \sim P\left(4\hat{\lambda}_Q\right) \]

We deduce that the annual loss number follows a Poisson distribution \( P(\hat{\lambda}_Y) \) in both cases.
Question 4

What does this result become if we consider a method of moments based on the first moment?
Since we have $\mathbb{E}[\mathcal{P}(\lambda)] = \lambda$, the MM estimator in the case of annual loss numbers is:

$$\hat{\lambda}_Y = \frac{1}{T} \sum_{t=1}^{T} N_{Y_t} = \frac{n}{T}$$

The MM estimator is exactly the ML estimator.
Question 5

Same question if we consider a method of moments based on the second moment.
Since we have $\text{var}(\mathcal{P}(\lambda)) = \lambda$, the MM estimator in the case of annual loss numbers is:

$$\hat{\lambda}_Y = \frac{1}{T} \sum_{t=1}^{T} N_{Y_t} - \frac{n^2}{T^2}$$

If we use a quarterly basis, we obtain:

$$\hat{\lambda}_Q = \frac{1}{4} \left( \frac{1}{T} \sum_{t=1}^{4T} N_{Q_t}^2 - \frac{n^2}{4T^2} \right)$$

$$\neq \frac{\hat{\lambda}_Y}{4}$$

There is no reason that $\hat{\lambda}_Y = 4\hat{\lambda}_Q$ meaning that the capital charge will not be the same.
Exercise

In what follows, we consider a debt instrument, whose remaining maturity is equal to $m$. We note $t$ the current date and $T = t + m$ the maturity date.
Computation of the amortization functions

Question 1

We consider a bullet repayment debt. Define its amortization function $S(t, u)$. Calculate the survival function $S^*(t, u)$ of the stock. Show that:

$$S^*(t, u) = 1 \{ t \leq u < t + m \} \cdot \left( 1 - \frac{u - t}{m} \right)$$

in the case where the new production is constant. Comment on this result.
By definition, we have:

\[ S(t, u) = \mathbb{1} \{ t \leq u < t + m \} = \begin{cases} 
1 & \text{if } u \in [t, t + m] \\
0 & \text{otherwise}
\end{cases} \]

This means that the survival function is equal to one when \( u \) is between the current date \( t \) and the maturity date \( T = t + m \). When \( u \) reaches \( T \), the outstanding amount is repaid, implying that \( S(t, T) \) is equal to zero.

It follows that:

\[ S^*(t, u) = \frac{\int_{-\infty}^{t} NP(s) S(s, u) \, ds}{\int_{-\infty}^{t} NP(s) S(s, t) \, ds} = \frac{\int_{-\infty}^{t} NP(s) \cdot \mathbb{1} \{ s \leq u < s + m \} \, ds}{\int_{-\infty}^{t} NP(s) \cdot \mathbb{1} \{ s \leq t < s + m \} \, ds} \]
For the numerator, we have:

\[ 1 \{ s \leq u < s + m \} = 1 \quad \Rightarrow \quad u < s + m \]

\[ \Leftrightarrow \quad s > u - m \]

and:

\[ \int_{-\infty}^{t} \text{NP}(s) \cdot 1 \{ s \leq u < s + m \} \, ds = \int_{u-m}^{t} \text{NP}(s) \, ds \]
Computations of the amortization functions

For the denominator, we have:

\[ 1 \{ s \leq t < s + m \} = 1 \implies t < s + m \]
\[ \iff s > t - m \]

and:

\[
\int_{-\infty}^{t} \text{NP}(s) \cdot 1 \{ s \leq t < s + m \} \, ds = \int_{t-m}^{t} \text{NP}(s) \, ds
\]

We deduce that:

\[
S^*(t, u) = 1 \{ t \leq u < t + m \} \cdot \frac{\int_{u-m}^{t} \text{NP}(s) \, ds}{\int_{t-m}^{t} \text{NP}(s) \, ds}
\]
Computation of the amortization functions

In the case where the new production is a constant, we have $NP(s) = c$ and:

$$S^*(t, u) = 1 \{ t \leq u < t + m \} \cdot \frac{\int_{u-m}^{t} ds}{\int_{t-m}^{t} ds}$$

$$= 1 \{ t \leq u < t + m \} \cdot \frac{s_{u-m}^{t}}{s_{t-m}^{t}}$$

$$= 1 \{ t \leq u < t + m \} \cdot \left( \frac{t-u+m}{t-t+m} \right)$$

$$= 1 \{ t \leq u < t + m \} \cdot \left( 1 - \frac{u-t}{m} \right)$$

The survival function $S^*(t, u)$ corresponds to the case of a linear amortization.
Computation of the amortization functions

**Question 2**

Same question if we consider a debt instrument, whose amortization rate is constant.
Computation of the amortization functions

If the amortization is linear, we have:

\[ S(t, u) = \mathbb{1} \{ t \leq u < t + m \} \cdot \left( 1 - \frac{u - t}{m} \right) \]

We deduce that:

\[ S^*(t, u) = \mathbb{1} \{ t \leq u < t + m \} \cdot \frac{\int_{u-m}^{t} \text{NP}(s) \left( 1 - \frac{u - s}{m} \right) \, ds}{\int_{t-m}^{t} \text{NP}(s) \left( 1 - \frac{t - s}{m} \right) \, ds} \]

In the case where the new production is a constant, we obtain:

\[ S^*(t, u) = \mathbb{1} \{ t \leq u < t + m \} \cdot \frac{\int_{u-m}^{t} \left( 1 - \frac{u - s}{m} \right) \, ds}{\int_{t-m}^{t} \left( 1 - \frac{t - s}{m} \right) \, ds} \]
Computation of the amortization functions

For the numerator, we have:

\[ \int_{u-m}^{t} \left(1 - \frac{u-s}{m}\right) \, ds = \left[ s - \frac{su}{m} + \frac{s^2}{2m}\right]_{u-m}^{t} \]

\[ = \left( t - \frac{tu}{m} + \frac{t^2}{2m}\right) - \left( u - m - \frac{u^2 - mu}{m} + \frac{(u-m)^2}{2m}\right) \]

\[ = \left( t - \frac{tu}{m} + \frac{t^2}{2m}\right) - \left( u - m - \frac{u^2}{2} - \frac{u^2}{2m}\right) \]

\[ = \frac{m^2 + u^2 + t^2 + 2mt - 2mu - 2tu}{2m} \]

\[ = \frac{(m - u + t)^2}{2m} \]
For the denominator, we use the previous result and we set $u = t$:

\[
\int_{t-m}^{t} \left(1 - \frac{t-s}{m} \right) \, ds = \frac{(m-t+t)^2}{2m} = \frac{m}{2}
\]
We deduce that:

\[ S^*(t, u) = \mathbb{1}_{t \leq u < t + m} \cdot \frac{(m - u + t)^2}{m^2} \]

\[ = \mathbb{1}_{t \leq u < t + m} \cdot \left(1 - \frac{u - t}{m}\right)^2 \]

The survival function \( S^*(t, u) \) corresponds to the case of a parabolic amortization.
Computation of the amortization functions

Question 3

Same question if we assume\(^a\) that the amortization function is exponential with parameter \(\lambda\).

\(^a\)By definition of the exponential amortization, we have \(m = +\infty\).
Computation of the amortization functions

If the amortization is exponential, we have:

$$S(t, u) = e^{-\int_{t}^{u} \lambda \, ds} = e^{-\lambda(u-t)}$$

It follows that:

$$S^*(t, u) = \frac{\int_{-\infty}^{t} NP(s) e^{-\lambda(u-s)} \, ds}{\int_{-\infty}^{t} NP(s) e^{-\lambda(t-s)} \, ds}$$

In the case where the new production is a constant, we obtain:

$$S^*(t, u) = \frac{\int_{-\infty}^{t} e^{-\lambda(u-s)} \, ds}{\int_{-\infty}^{t} e^{-\lambda(t-s)} \, ds} = \frac{\left[\frac{1}{\lambda} e^{-\lambda(u-s)}\right]_{-\infty}^{t}}{\left[\frac{1}{\lambda} e^{-\lambda(t-s)}\right]_{-\infty}^{t}} = e^{-\lambda(u-t)} = S(t, u)$$

The stock amortization function is equal to the flow amortization function.
Computation of the amortization functions

**Question 4**

Find the expression of $\mathcal{D}^*(t)$ when the new production is constant.
Computation of the amortization functions

We recall that the liquidity duration is equal to:

\[ \mathcal{D}(t) = \int_t^\infty (u - t) f(t, u) \, du \]

where \( f(t, u) \) is the density function associated to the survival function \( S(t, u) \). For the stock, we have:

\[ \mathcal{D}^*(t) = \int_t^\infty (u - t) f^*(t, u) \, du \]

where \( f^*(t, u) \) is the density function associated to the survival function \( S^*(t, u) \):

\[ f^*(t, u) = \frac{\int_t^s \text{NP}(s) f(s, u) \, ds}{\int_t^\infty \text{NP}(s) S(s, t) \, ds} \]
In the case where the new production is constant, we obtain:

\[
D^* (t) = \frac{\int_{t}^{\infty} (u - t) \int_{-\infty}^{t} f(s, u) \, ds \, du}{\int_{-\infty}^{t} S(s, t) \, ds}
\]

Since we have \(\int_{-\infty}^{t} f(s, u) \, ds = S(t, u)\), we deduce that:

\[
D^* (t) = \frac{\int_{t}^{\infty} (u - t) S(t, u) \, du}{\int_{-\infty}^{t} S(s, t) \, ds}
\]
Computation of the amortization functions

Question 5

Calculate the durations $D(t)$ and $D^*(t)$ for the three previous cases.
In the case of the bullet repayment debt, we have:

\[ D(t) = m \]

and:

\[ D^*(t) = \frac{\int_t^{t+m} (u - t) \, du}{\int_t^{t-m} ds} \]

\[ = \frac{\left[ \frac{1}{2} (u - t)^2 \right]_{t}^{t+m}}{\left[ s \right]_t^{t-m}} \]

\[ = \frac{m}{2} \]
Computation of the amortization functions

In the case of the linear amortization, we have:

\[ f(t, u) = 1 \{ t \leq u < t + m \} \cdot \frac{1}{m} \]

and:

\[ D(t) = \int_{t}^{t+m} \frac{(u-t)}{m} \, du \]

\[ = \frac{1}{m} \left[ \frac{1}{2} (u-t)^2 \right]_{t}^{t+m} \]

\[ = \frac{m}{2} \]
For the stock duration, we deduce that

\[ D^*(t) = \frac{\int_{t}^{t+m} (u - t) \left( 1 - \frac{u - t}{m} \right) \, du}{\int_{t}^{t} \left( 1 - \frac{t - s}{m} \right) \, ds \int_{t-m}^{t+m} \left( 1 - \frac{t - s}{m} \right) \, ds} \]

\[ = \int_{t}^{t+m} \left( u - t - \frac{u^2}{m} + 2\frac{tu}{m} - \frac{t^2}{m} \right) \, du \]

\[ = \frac{\int_{t}^{t} \left( 1 - \frac{t}{m} + \frac{s}{m} \right) \, ds \int_{t-m}^{t-m} \left( 1 - \frac{t}{m} + \frac{s}{m} \right) \, ds}{\left[ \frac{u^2}{2} - tu - \frac{u^3}{3m} + \frac{tu^2}{m} - \frac{t^2 u}{m} \right]_{t}^{t+m}} \]

\[ = \left[ s - \frac{st}{m} + \frac{s^2}{2m} \right]_{t-m}^{t} \]
The numerator is equal to:

\[ (*) = \left[ \frac{u^2}{2} - tu - \frac{u^3}{3m} + \frac{tu^2}{m} - \frac{t^2u}{m} \right]^{t+m} \]

\[ = \frac{1}{6m} \left[ 3mu^2 - 6mtu - 2u^3 + 6tu^2 - 6t^2u \right]^{t+m} \]

\[ = \frac{1}{6m} \left( m^3 - 3mt^2 - 2t^3 \right) + \frac{1}{6m} \left( 3mt^2 + 2t^3 \right) \]

\[ = \frac{m^2}{6} \]
Computation of the amortization functions

The denominator is equal to:

\[(\star) \quad = \quad \left[ s - \frac{st}{m} + \frac{s^2}{2m} \right]_t^m \]

\[= \quad \frac{1}{2m} \left[ s^2 - 2s(t - m) \right]_t^m \]

\[= \quad \frac{1}{2m} \left( t^2 - 2t(t - m) - (t - m)^2 + 2(t - m)^2 \right) \]

\[= \quad \frac{1}{2m} \left( t^2 - 2t^2 + 2mt + t^2 - 2mt + m^2 \right) \]

\[= \quad \frac{m^2}{2} \]
We deduce that:

\[ D^*(t) = \frac{m}{3} \]
For the exponential amortization, we have:

\[ f(t, u) = \lambda e^{-\lambda(u-t)} \]

and\(^1\):

\[ D(t) = \int_t^\infty (u - t) \lambda e^{-\lambda(u-t)} \, du = \int_0^\infty v \lambda e^{-\lambda v} \, dv = \frac{1}{\lambda} \]

For the stock duration, we deduce that:

\[ D^*(t) = \frac{\int_t^\infty (u - t) e^{-\lambda(u-t)} \, du}{\int_{-\infty}^t e^{-\lambda(t-s)} \, ds} = \frac{\int_0^\infty ve^{-\lambda v} \, dv}{\int_0^\infty e^{-\lambda v} \, dv} = \frac{1}{\lambda} \]

We verify that \( D(t) = D^*(t) \) since we have demonstrated that \( S^*(t, u) = S(t, u) \).

\(^1\)We use the change of variable \( v = u - t \).
Computation of the amortization functions

Question 6
Calculate the corresponding dynamics $dN(t)$. 
In the case of the bullet repayment debt, we have:

$$dN(t) = (\text{NP}(t) - \text{NP}(t - m)) \, dt$$
Computation of the amortization functions

In the case of the linear amortization, we have:

\[ f(s, t) = \frac{\mathbf{1}\{s \leq t < s + m\}}{m} \]

It follows that:

\[ \int_{-\infty}^{t} NP(s) f(s, t) \, ds = \frac{1}{m} \int_{-\infty}^{t} \mathbf{1}\{s \leq t < s + m\} \cdot NP(s) \, ds \]

\[ = \frac{1}{m} \int_{t-m}^{t} NP(s) \, ds \]

We deduce that:

\[ dN(t) = \left( NP(t) - \frac{1}{m} \int_{t-m}^{t} NP(s) \, ds \right) \, dt \]
Computation of the amortization functions

For the exponential amortization, we have:

\[ f(s, t) = \lambda e^{-\lambda(t-s)} \]

and:

\[
\int_{-\infty}^{t} NP(s) f(s, t) \, ds = \int_{-\infty}^{t} NP(s) \lambda e^{-\lambda(t-s)} \, ds
\]

\[
= \lambda \int_{-\infty}^{t} NP(s) e^{-\lambda(t-s)} \, ds
\]

\[
= \lambda N(t)
\]

We deduce that:

\[ dN(t) = (NP(t) - \lambda N(t)) \, dt \]
Impact of prepayment

Exercise

We recall that the outstanding balance of a CAM (constant amortization mortgage) at time $t$ is given by:

$$N(t) = 1 \{ t < m \} \cdot N_0 \cdot \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}$$

where $N_0$ is the notional, $i$ is the interest rate and $m$ is the maturity.
Impact of prepayment

Question 1

Find the dynamics $dN(t)$. 

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Impact of prepayment

We deduce that the dynamics of $N(t)$ is equal to:

$$dN(t) = \mathbb{1} \{ t < m \} \cdot N_0 \frac{-ie^{-i(m-t)}}{1 - e^{-im}} \, dt$$

$$= -ie^{-i(m-t)} \left( \mathbb{1} \{ t < m \} \cdot N_0 \frac{1}{1 - e^{-im}} \right) \, dt$$

$$= -\frac{ie^{-i(m-t)}}{1 - e^{-i(m-t)}} N(t) \, dt$$
Impact of prepayment

Question 2

We note $\tilde{N}(t)$ the modified outstanding balance that takes into account the prepayment risk. Let $\lambda_p(t)$ be the prepayment rate at time $t$. Write the dynamics of $\tilde{N}(t)$. 
Impact of prepayment

The prepayment rate has a negative impact on $dN(t)$ because it reduces the outstanding amount $N(t)$:

$$d\tilde{N}(t) = - \frac{ie^{-i(m-t)}}{1 - e^{-i(m-t)}}\tilde{N}(t) \, dt - \lambda_p(t)\tilde{N}(t) \, dt$$
Impact of prepayment

Question 3

Show that $\tilde{N}(t) = N(t) S_p(t)$ where $S_p(t)$ is the prepayment-based survival function.
Impact of prepayment

It follows that:

$$d \ln \tilde{N}(t) = - \left( \frac{ie^{-imt}}{1 - e^{-imt}} + \lambda_p(t) \right) dt$$

and:

$$\ln \tilde{N}(t) - \ln \tilde{N}(0) = \int_0^t \frac{-ie^{-imt}}{1 - e^{-imt}} ds - \int_0^t \lambda_p(s) ds$$

$$= \left[ \ln \left( 1 - e^{-imt} \right) \right]_0^t - \int_0^t \lambda_p(s) ds$$

$$= \ln \left( \frac{1 - e^{-imt}}{1 - e^{-im}} \right) - \int_0^t \lambda_p(s) ds$$
Impact of prepayment

We deduce that:

\[
\tilde{N}(t) = \left( N_0 \frac{1 - e^{-i(m-t)}}{1 - e^{-im}} \right) e^{-\int_0^t \lambda_p(s) \, ds} = N(t) S_p(t)
\]

where \( S_p(t) \) is the survival function associated to the hazard rate \( \lambda_p(t) \).
Impact of prepayment

**Question 4**

Calculate the liquidity duration $\tilde{D}(t)$ associated to the outstanding balance $\tilde{N}(t)$ when the hazard rate of prepayments is constant and equal to $\lambda_p$. 
Impact of prepayment

We have:

\[ \tilde{N}(t, u) = 1 \{ t \leq u < t + m \} \cdot N(t) \frac{1 - e^{-i(t+m-u)}}{1 - e^{-im}} e^{-\lambda_p(u-t)} \]

this implies that:

\[ \tilde{S}(t, u) = 1 \{ t \leq u < t + m \} \cdot \frac{\lambda_p e^{-\lambda_p(u-t)} + (i - \lambda_p) e^{-im + (i - \lambda_p)(u-t)}}{1 - e^{-im}} \]

and:

\[ \tilde{f}(t, u) = 1 \{ t \leq u < t + m \} \cdot \frac{\lambda_p e^{-\lambda_p(u-t)} + (i - \lambda_p) e^{-im + (i - \lambda_p)(u-t)}}{1 - e^{-im}} \]
Impact of prepayment

It follows that:

\[
\tilde{D}(t) = \frac{\lambda_p}{1 - e^{-im}} \int_{t}^{t+m} (u - t) e^{-\lambda_p (u-t)} \, du + \frac{(i - \lambda_p) e^{-im}}{1 - e^{-im}} \int_{t}^{t+m} (u - t) e^{(i-\lambda_p)(u-t)} \, du
\]

\[
= \frac{\lambda_p}{1 - e^{-im}} \int_{0}^{m} ve^{-\lambda_p v} \, dv + \frac{(i - \lambda_p) e^{-im}}{1 - e^{-im}} \int_{0}^{m} ve^{(i-\lambda_p)v} \, dv
\]

\[
= \frac{\lambda_p}{1 - e^{-im}} \left( \frac{me^{-\lambda_p m}}{-\lambda_p} - \frac{e^{-\lambda_p m} - 1}{\lambda_p^2} \right) + \frac{(i - \lambda_p) e^{-im}}{1 - e^{-im}} \left( \frac{me^{(i-\lambda_p)m}}{(i - \lambda_p)} - \frac{e^{(i-\lambda_p)m} - 1}{(i - \lambda_p)^2} \right)
\]

\[
= \frac{1}{1 - e^{-im}} \left( \frac{e^{-im} - e^{-\lambda_p m}}{i - \lambda_p} + \frac{1 - e^{-\lambda_p m}}{\lambda_p} \right)
\]
because we have:

\[
\int_0^m v e^{\alpha v} \, dv = \left[ \frac{v e^{\alpha v}}{\alpha} \right]_0^m - \int_0^m \frac{e^{\alpha v}}{\alpha} \, dv \\
= \left[ \frac{v e^{\alpha v}}{\alpha} \right]_0^m - \left[ \frac{e^{\alpha v}}{\alpha^2} \right]_0^m \\
= \frac{m e^{\alpha m}}{\alpha} - \frac{e^{\alpha m} - 1}{\alpha^2}
\]