Asset Management
Lecture 2. Risk Budgeting

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General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `http://www.thierry-roncalli.com/RiskBasedAM.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 8 (6 hours, AM+PM)
- January 15 (6 hours, AM+PM)
- January 22 (6 hours, AM+PM)
- January 29 (6 hours, AM+PM)

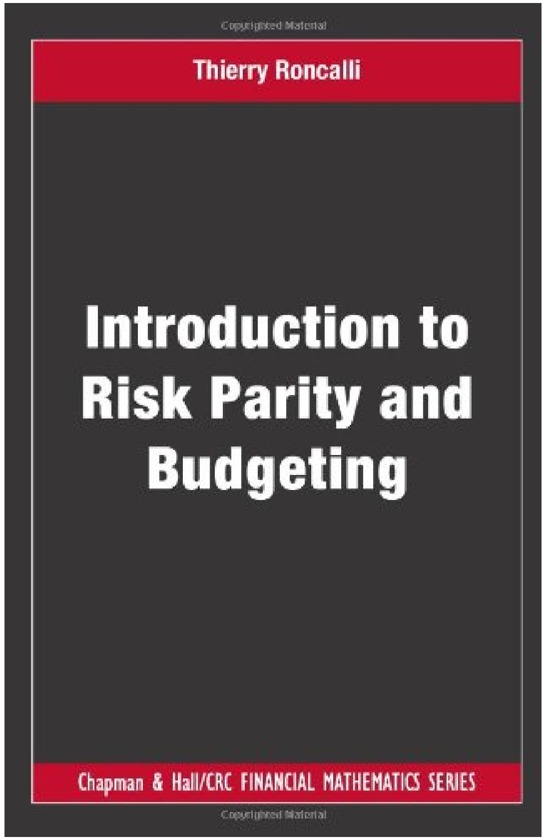
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Textbook

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskBasedAM.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Agenda

- Lecture 1: Portfolio Optimization
- **Lecture 2: Risk Budgeting**
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Portfolio optimization & portfolio diversification

Example 1

- We consider an investment universe of 5 assets
- (μ_i, σ_i) are respectively equal to $(8\%, 12\%)$, $(7\%, 10\%)$, $(7.5\%, 11\%)$, $(8.5\%, 13\%)$ and $(8\%, 12\%)$
- The correlation matrix is $\mathcal{C}_5(\rho)$ with $\rho = 60\%$

The optimal portfolio $x^\star$ such that $\sigma(x^\star) = 10\%$ is equal to:

$$x^\star = \begin{pmatrix} 23.97\% \\ 6.42\% \\ 16.91\% \\ 28.73\% \\ 23.97\% \end{pmatrix}$$

Portfolio optimization & portfolio diversification

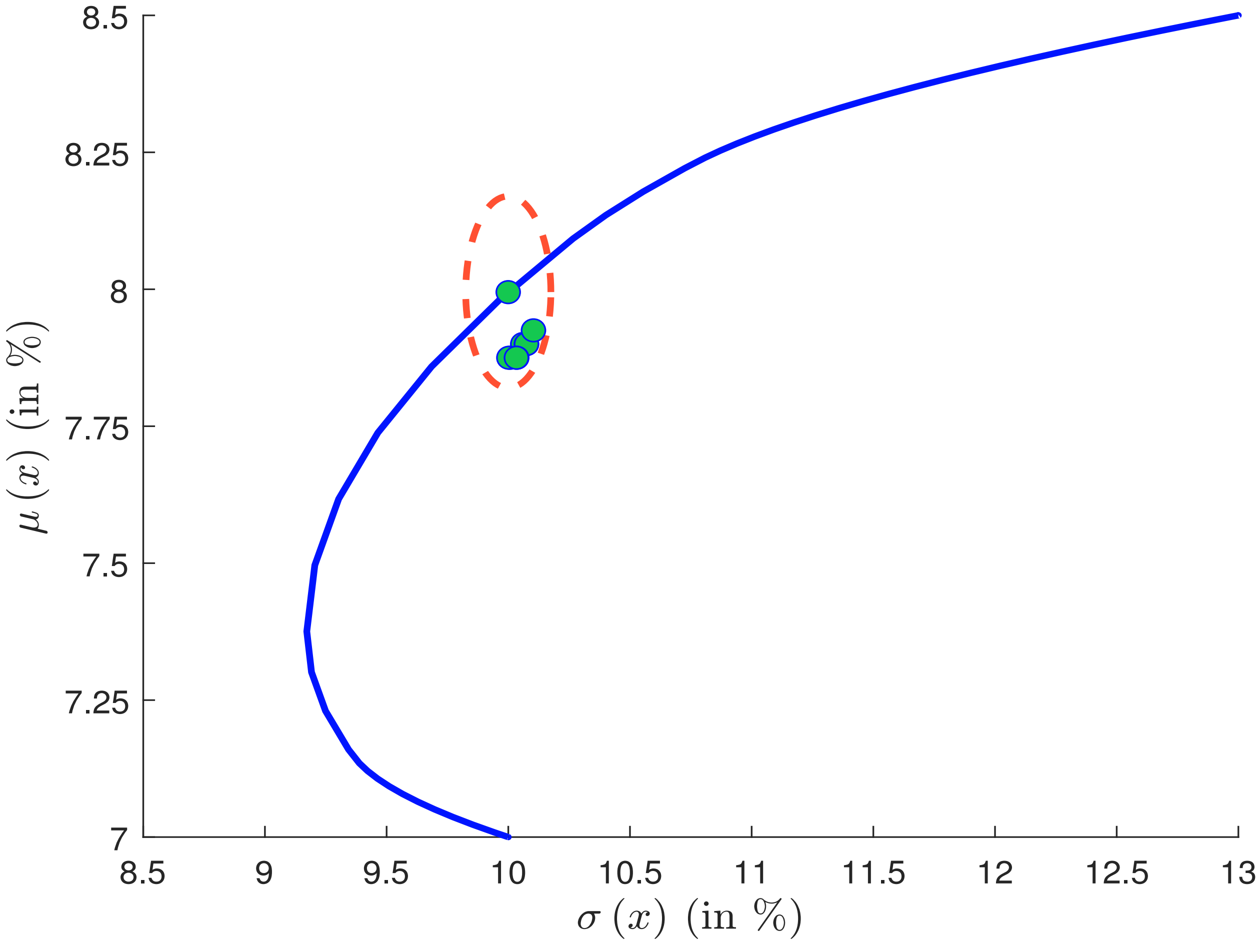


Figure 1: Optimized portfolios versus optimal diversified portfolios

Portfolio optimization & portfolio diversification

Table 1: Some equivalent mean-variance portfolios

x_1	23.97		5	5	35	35	50	5	5	10
x_2	6.42	25		25	10	25	10	30		25
x_3	16.91	5	40		10	5	15		45	10
x_4	28.73	35	20	30	5	35	10	35	20	45
x_5	23.97	35	35	40	40		15	30	30	10
$\mu(x)$	7.99	7.90	7.90	7.90	7.88	7.90	7.88	7.88	7.88	7.93
$\sigma(x)$	10.00	10.07	10.06	10.07	10.01	10.07	10.03	10.00	10.03	10.10

$\Rightarrow$ These portfolios have very different compositions, but lead to very close mean-variance features

Some of these portfolios appear more balanced and more diversified than the optimized portfolio

Other methods to build a portfolio

1. Weight budgeting (WB)
2. Risk budgeting (RB)
3. Performance budgeting (PB)

Ex-ante analysis
$\neq$
Ex-post analysis

Important result

RB = PB

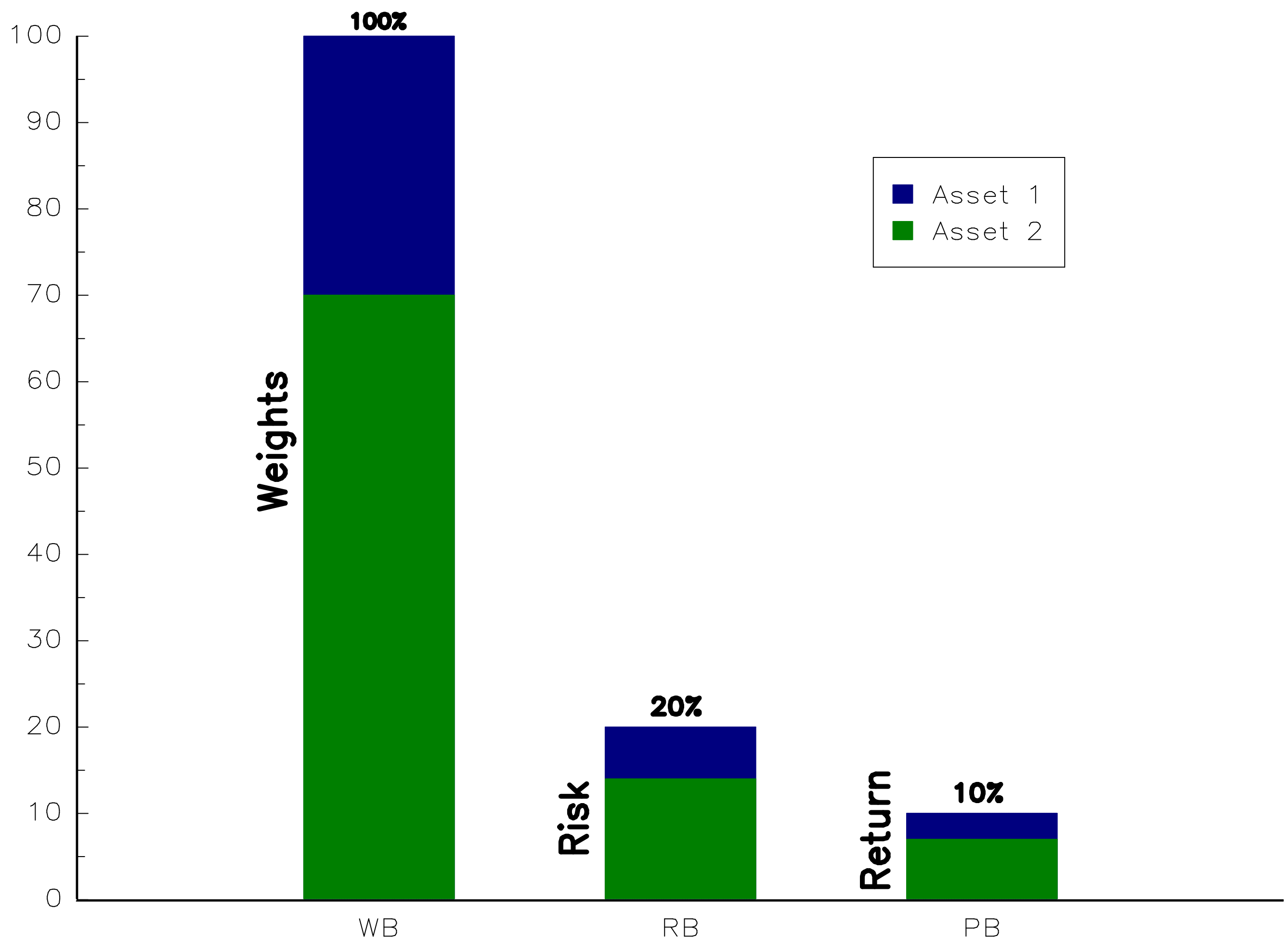


Figure 2: The 30/70 rule

Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathcal{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned}\mathcal{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial\, \mathcal{R}(x_1, \ldots, x_n)}{\partial\, x_i} \\ &= \sum_{i=1}^{n} \mathcal{RC}_i(x_1, \ldots, x_n)\end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

2. Risk budgeting (RB)

$$\mathcal{RC}_i = b_i \cdot \mathcal{R}(x_1, \ldots, x_n)$$

Importance of the coherency and convexity properties

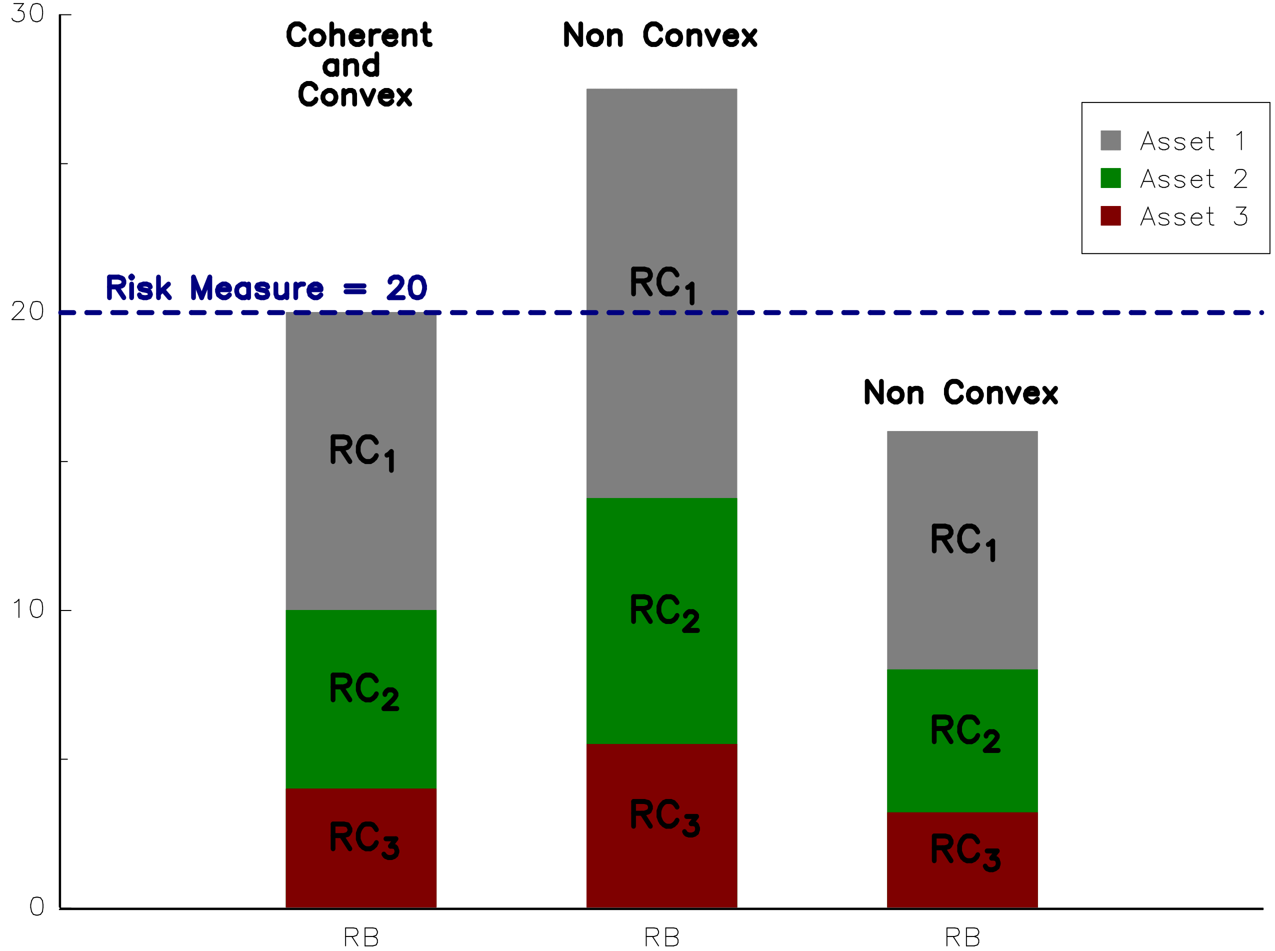


Figure 3: Risk Measure = 20 with a 50/30/20 budget rule

Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We note x the vector of the portfolio's weights:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

It follows that the portfolio volatility is equal to:

$$\sigma(x) = \sqrt{x^\top \Sigma x}$$

Computation of the marginal volatilities

The vector of marginal volatilities is equal to:

$$\begin{aligned}
\frac{\partial\,\sigma\left(x\right)}{\partial\,x} &= \begin{pmatrix} \dfrac{\partial\,\sigma\left(x\right)}{\partial\,x_1} \\ \vdots \\ \dfrac{\partial\,\sigma\left(x\right)}{\partial\,x_n} \end{pmatrix} \\
&= \frac{\partial}{\partial\,x}\left(x^\top \Sigma x\right)^{1/2} \\
&= \frac{1}{2}\left(x^\top \Sigma x\right)^{1/2-1}\left(2\Sigma x\right) \\
&= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}
\end{aligned}$$

It follows that the marginal volatility of Asset i is given by:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_i} = \frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}} = \sum_{j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j x_j}{\sigma\left(x\right)} = \sigma_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma\left(x\right)}$$

Computation of the risk contributions

We deduce that the risk contribution of the i^{th} asset is then:

$$
\begin{aligned}
\mathcal{RC}_i &= x_i \cdot \frac{\partial \, \sigma(x)}{\partial \, x_i} \\
&= \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
&= \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j} \sigma_j}{\sigma(x)}
\end{aligned}
$$

The Euler allocation principle

We verify that the volatility satisfies the full allocation property:

$$\begin{aligned}\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \rho_{i,j}\sigma_i\sigma_j \\ &= \frac{\sigma^2(x)}{\sigma(x)} = \sigma(x)\end{aligned}$$

An alternative proof uses the definition of the dot product:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a^\top b$$

Indeed, we have:

$$\sum_{i=1}^{n} \mathcal{RC}_i = \sum_{i=1}^{n} \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{1}{\sqrt{x^\top \Sigma x}} \sum_{i=1}^{n} x_i \cdot (\Sigma x)_i = \frac{1}{\sqrt{x^\top \Sigma x}} x^\top \Sigma x = \sigma(x)$$

Definition of the risk contribution

Definition

The marginal risk contribution of Asset i is:

$$\mathcal{MR}_i = \frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

The absolute risk contribution of Asset i is:

$$\mathcal{RC}_i = x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

The relative risk contribution of Asset i is:

$$\mathcal{RC}_i^\star = \frac{\mathcal{RC}_i}{\sigma(x)} = \frac{x_i \cdot (\Sigma x)_i}{x^\top \Sigma x}$$

The Euler allocation principle

Remark

We have $\sum_{i=1}^{n} \mathcal{RC}_i = \sigma(x)$ *and* $\sum_{i=1}^{n} \mathcal{RC}_i^{\star} = 100\%$.

Application

Example 2

We consider three assets. We assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

We consider the portfolio x, which is given by:

$$x = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix}$$

Application

Using the relationship $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, we deduce that the covariance matrix is[1]:

$$\Sigma = \begin{pmatrix} 9.00 & 4.80 & 2.25 \\ 4.80 & 4.00 & 0.90 \\ 2.25 & 0.90 & 2.25 \end{pmatrix} \times 10^{-2}$$

It follows that the variance of the portfolio is:

$$\begin{aligned} \sigma^2(x) &= 0.50^2 \times 0.09 + 0.20^2 \times 0.04 + 0.30^2 \times 0.0225 + \\ & \quad 2 \times 0.50 \times 0.20 \times 0.0480 + 2 \times 0.50 \times 0.30 \times 0.0225 + \\ & \quad 2 \times 0.20 \times 0.30 \times 0.0090 \\ &= 4.3555\% \end{aligned}$$

The volatility is then $\sigma(x) = \sqrt{4.3555\%} = 20.8698\%$.

[1] The covariance term between assets 1 and 2 is equal to $\Sigma_{1,2} = 80\% \times 30\% \times 20\%$ or $\Sigma_{1,2} = 4.80\%$

Application

The computation of the marginal volatilities gives:

$$\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \frac{1}{20.8698\%} \begin{pmatrix} 6.1350\% \\ 3.4700\% \\ 1.9800\% \end{pmatrix} = \begin{pmatrix} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{pmatrix}$$

Application

Finally, we obtain the risk contributions by multiplying the weights by the marginal volatilities:

$$x \circ \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix} \circ \begin{pmatrix} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{pmatrix} = \begin{pmatrix} 14.6982\% \\ 3.3254\% \\ 2.8462\% \end{pmatrix}$$

We verify that the sum of risk contributions is equal to the volatility:

$$\sum_{i=1}^{3} \mathcal{RC}_i = 14.6982\% + 3.3254\% + 2.8462\% = 20.8698\%$$

Application

Table 2: Risk decomposition of the portfolio's volatility (Example 2)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	50.00	29.40	14.70	70.43
2	20.00	16.63	3.33	15.93
3	30.00	9.49	2.85	13.64
$\sigma(x)$			20.87	

The ERC portfolio

Definition

- Let Σ be the covariance matrix of asset returns
- The risk measure corresponds to the volatility risk measure
- The ERC portfolio is the **unique** portfolio x such that the risk contributions are equal:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{x_j \cdot (\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

ERC = Equal Risk Contribution

The concept of risk budgeting

Example 3

- 3 assets
- Volatilities are respectively equal to 20%, 30% and 15%
- Correlations are set to 60% between the 1st asset and the 2nd asset and 10% between the first two assets and the 3rd asset
- Budgets are set to 50%, 25% and 25%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional approach)

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	17.99%	9.00%	54.40%
2	**25.00%**	25.17%	6.29%	38.06%
3	**25.00%**	4.99%	1.25%	7.54%
Volatility			16.54%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	41.62%	16.84%	7.01%	**50.00%**
2	15.79%	22.19%	3.51%	**25.00%**
3	42.58%	8.23%	3.51%	**25.00%**
Volatility			14.02%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	30.41%	15.15%	4.61%	**33.33%**
2	20.28%	22.73%	4.61%	**33.33%**
3	49.31%	9.35%	4.61%	**33.33%**
Volatility			13.82%	

The concept of risk budgeting

We have:

$$\sigma\left(50\%, 25\%, 25\%\right) = 16.54\%$$

The marginal risk for the first asset is:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = \lim_{\varepsilon \to 0} \frac{\sigma\left(x_1 + \varepsilon, x_2, x_3\right) - \sigma\left(x_1, x_2, x_3\right)}{\left(x_1 + \varepsilon\right) - x_1}$$

If $\varepsilon = 1\%$, we have:

$$\sigma\left(0.51, 0.25, 0.25\right) = 16.72\%$$

We deduce that:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} \simeq \frac{16.72\% - 16.54\%}{1\%} = 18.01\%$$

whereas the true value is equal to:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = 17.99\%$$

The concept of risk budgeting

Example 4

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

The concept of risk budgeting

Question

We assume that the portfolio's wealth is set to \$1 000. Calculate the nominal volatility of the previous WB, RB and ERC portfolios.

The concept of risk budgeting

We have:

$$\begin{aligned}
\sigma(x_{\mathrm{wb}}) &= 10^3 \times 20.87\% = \$208.7 \\
\sigma(x_{\mathrm{rb}}) &= 10^3 \times 17.49\% = \$174.9 \\
\sigma(x_{\mathrm{erc}}) &= 10^3 \times 16.13\% = \$161.3
\end{aligned}$$

The concept of risk budgeting

Question

We increase the exposure of the 3 portfolios by \$10 as follows:

$$\Delta x = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix} = \begin{pmatrix} \$1 \\ \$5 \\ \$4 \end{pmatrix}$$

Calculate the nominal volatility of these new portfolios.

The concept of risk budgeting

By assuming that $\Delta x \simeq 0$, we have:

$$
\begin{array}{rcl}
\sigma\left(x_{\mathrm{wb}} + \Delta x\right) & \approx & (\$500 + \$1) \times 0.2940 + \\
& & (\$200 + \$5) \times 0.1663 + \\
& & (\$300 + \$4) \times 0.0949 \\
& \approx & \$210.2
\end{array}
$$

$\sigma\left(x_{\mathrm{rb}} + \Delta x\right) \approx \$176.4$ and $\sigma\left(x_{\mathrm{erc}} + \Delta x\right) \approx \$162.9$.

Uniform correlation

- We assume a constant correlation matrix $\mathcal{C}_n(\rho)$, meaning that $\rho_{i,j} = \rho$ for all $i \neq j$
- We have:

$$
\begin{aligned}
(\Sigma x)_i &= \sum_{k=1}^{n} \rho_{i,k} \sigma_i \sigma_k x_k \\
&= \sigma_i^2 x_i + \rho \sigma_i \sum_{k \neq i} \sigma_k x_k \\
&= \sigma_i^2 x_i + \rho \sigma_i \sum_{k=1}^{n} \sigma_k x_k - \rho \sigma_i^2 x_i \\
&= (1-\rho) x_i \sigma_i^2 + \rho \sigma_i \sum_{k=1}^{n} x_k \sigma_k \\
&= \sigma_i \left((1-\rho) x_i \sigma_i + \rho \sum_{k=1}^{n} x_k \sigma_k \right)
\end{aligned}
$$

Uniform correlation

- Since we have:

$$\mathcal{RC}_i = \frac{x_i \left(\Sigma x\right)_i}{\sigma\left(x\right)}$$

we deduce that $\mathcal{RC}_i = \mathcal{RC}_j$ is equivalent to:

$$x_i \sigma_i \left(\left(1-\rho\right) x_i \sigma_i + \rho \sum_{k=1}^{n} x_k \sigma_k\right) = x_j \sigma_j \left(\left(1-\rho\right) x_j \sigma_j + \rho \sum_{k=1}^{n} x_k \sigma_k\right)$$

It follows that $x_i \sigma_i = x_j \sigma_j$. Because $\sum_{i=1}^{n} x_i = 1$, we deduce that:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

Result

The weight allocated to Asset i is inversely proportional to its volatility and does not depend on the value of the correlation

Minimum uniform correlation

- The global minimum variance portfolio is equal to:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1}\mathbf{1}_n}$$

- Let $\Sigma = \sigma\sigma^\top \circ \mathcal{C}_n(\rho)$ be the covariance matrix with $\mathcal{C}_n(\rho)$ the constant correlation matrix
- We have:

$$\Sigma^{-1} = \Gamma \circ \mathcal{C}_n^{-1}(\rho)$$

with $\Gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}$ and:

$$\mathcal{C}_n^{-1}(\rho) = \frac{\rho \mathbf{1}_n \mathbf{1}_n^\top - ((n-1)\rho + 1) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

Minimum uniform correlation

- We deduce that the expression of the GMV weights are:

$$x_{\mathrm{gmv},i} = \frac{-\left(\left(n-1\right)\rho+1\right)\sigma_i^{-2} + \rho \sum_{j=1}^n \left(\sigma_i \sigma_j\right)^{-1}}{\sum_{k=1}^n \left(-\left(\left(n-1\right)\rho+1\right)\sigma_k^{-2} + \rho \sum_{j=1}^n \left(\sigma_k \sigma_j\right)^{-1}\right)}$$

- The lower bound of $\mathcal{C}_n\left(\rho\right)$ is achieved for $\rho = -\left(n-1\right)^{-1}$
- In this case, the solution becomes:

$$x_{\mathrm{gmv},i} = \frac{\sum_{j=1}^n \left(\sigma_i \sigma_j\right)^{-1}}{\sum_{k=1}^n \sum_{j=1}^n \left(\sigma_k \sigma_j\right)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^n \sigma_k^{-1}}$$

Result

The ERC portfolio is equal to the GMV portfolio when the correlation is at its lowest possible value:

$$\lim_{\rho \rightarrow -\left(n-1\right)^{-1}} x_{\mathrm{gmv}} = x_{\mathrm{erc}}$$

Uniform volatility

- If all volatilities are equal, i.e. $\sigma_i = \sigma$ for all i, the risk contribution becomes:

$$\mathcal{RC}_i = \frac{\left(\sum_{k=1}^{n} x_i x_k \rho_{i,k}\right) \sigma^2}{\sigma(x)}$$

- The ERC portfolio verifies then:

$$x_i \left(\sum_{k=1}^{n} x_k \rho_{i,k}\right) = x_j \left(\sum_{k=1}^{n} x_k \rho_{j,k}\right)$$

- We deduce that:

$$x_i = \frac{\left(\sum_{k=1}^{n} x_k \rho_{i,k}\right)^{-1}}{\sum_{j=1}^{n} \left(\sum_{k=1}^{n} x_k \rho_{j,k}\right)^{-1}}$$

Uniform volatility

Result

The weight of asset i is inversely proportional to the weighted average of correlations of Asset i

Remark

Contrary to the previous case, this solution is endogenous since x_i is a function of itself directly

General case

- In the general case, we have:

$$\beta_i = \beta(\mathbf{e}_i \mid x) = \frac{\mathbf{e}_i^\top \Sigma x}{x^\top \Sigma x} = \frac{(\Sigma x)_i}{\sigma^2(x)}$$

and:

$$\mathcal{RC}_i = \frac{x_i (\Sigma x)_i}{\sigma(x)} = \sigma(x) x_i \beta_i$$

- We deduce that $\mathcal{RC}_i = \mathcal{RC}_j$ is equivalent to:

$$x_i \beta_i = x_j \beta_j$$

- It follows that:

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^n \beta_j^{-1}}$$

General case

- We notice that:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \frac{\mathcal{RC}_i}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n} \mathcal{RC}_i = 1$$

and:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} \right) = 1$$

It follows that:

$$\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{1}{n}$$

- We finally obtain:

$$x_i = \frac{1}{n \beta_i}$$

General case

Result

The weight of Asset i is proportional to the inverse of its beta:

$$x_i \propto \beta_i^{-1}$$

Remark

This solution is endogenous since x_i is a function of itself because $\beta_i = \beta\left(\mathbf{e}_i \mid x\right)$.

General case

Example 5

We consider an investment universe of four assets with $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 30\%$ and $\sigma_4 = 10\%$. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.00 & 0.20 & 1.00 & \\ -0.10 & 0.40 & 0.70 & 1.00 \end{pmatrix}$$

General case

Table 3: Composition of the ERC portfolio (Example 5)

Asset	x_i	$\mathcal{MR}_i$	β_i	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	31.34%	8.52%	0.80	2.67%	25.00%
2	17.49%	15.27%	1.43	2.67%	25.00%
3	13.05%	20.46%	1.92	2.67%	25.00%
4	38.12%	7.00%	0.66	2.67%	25.00%
Volatility				10.68%	

We verify that:

$$x_1 = \frac{1}{(4 \times 0.7978)} = 31.34\%$$

Existence and uniqueness

We consider the following optimization problem:

$$\begin{aligned} y^{\star}(c) &= \arg\min \frac{1}{2} y^{\top} \Sigma y \\ &\text{u.c.} \quad \sum_{i=1}^{n} \ln y_i \geq c \end{aligned}$$

The Lagrange function is equal to:

$$\mathcal{L}(y; \lambda_c) = \frac{1}{2} y^{\top} \Sigma y - \lambda_c \left(\sum_{i=1}^{n} \ln y_i - c \right)$$

At the optimum, we have:

$$\frac{\partial \mathcal{L}(y; \lambda_c, \lambda)}{\partial y} = \mathbf{0}_n \Leftrightarrow (\Sigma y)_i - \frac{\lambda_c}{y_i} = 0$$

Existence and uniqueness

It follows that:

$$y_i \left(\Sigma y\right)_i = \lambda_c$$

or equivalently:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

Since we minimize a convex function subject to a lower convex bound, the solution $y^{\star}(c)$ exists and is unique

Existence and uniqueness

Question

What is the difference between $y^{\star}(c)$ and $y^{\star}(c')$?

Let $y' = \alpha y^{\star}(c)$. The first-order conditions are:

$$y_i^{\star}(c)\left(\Sigma y^{\star}(c)\right)_i = \lambda_c$$

and:

$$y_i'\left(\Sigma y'\right)_i = \alpha^2 \lambda_c = \lambda_{c'}$$

Since $\lambda_c \neq 0$, the Kuhn-Tucker condition becomes:

$$\min\left(\lambda_c, \sum_{i=1}^{n} \ln y_i^{\star}(c) - c\right) = 0 \Leftrightarrow \sum_{i=1}^{n} \ln y_i^{\star}(c) - c = 0$$

Existence and uniqueness

It follows that:

$$\sum_{i=1}^{n} \ln \frac{y_i'(c)}{\alpha} = c$$

or:

$$\sum_{i=1}^{n} \ln y_i'(c) = c + n \ln \alpha = c'$$

We deduce that:

$$\alpha = \exp\left(\frac{c' - c}{n}\right)$$

$y^\star(c')$ is a scaled solution of $y^\star(c)$:

$$y^\star(c') = \exp\left(\frac{c' - c}{n}\right) y^\star(c)$$

Existence and uniqueness

The ERC portfolio is the solution $y^{\star}(c)$ such that $\sum_{i=1}^{n} y_i^{\star}(c) = 1$:

$$x_{\mathrm{erc}} = \frac{y^{\star}(c)}{\sum_{i=1}^{n} y_i^{\star}(c)}$$

and corresponds to the following value of the logarithmic barrier:

$$c_{\mathrm{erc}} = c - n \ln \sum_{i=1}^{n} y_i^{\star}(c)$$

Existence and uniqueness

Theorem

Because of the previous results, x_{erc} exists and is unique. This is the solution of the following optimization problem[a]:

$$\begin{array}{rcl} x_{\mathrm{erc}} & = & \arg\min \frac{1}{2} x^{\top} \Sigma x \\ & \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} \ln x_i \geq c_{\mathrm{erc}} \\ \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{array}$$

[a] We can add the last two constraints because they do not change the solution

Location of the ERC portfolio

The global global minimum variance portfolio is defined by:

$$\begin{aligned} x_{\mathrm{gmv}} &= \arg\min \sigma(x) \\ \text{u.c.} &\quad \mathbf{1}_n^\top x = 1 \end{aligned}$$

We have:

$$\mathcal{L}(x;\lambda_0) = \sigma(x) - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right)$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0)}{\partial\,x} = \mathbf{0}_n \Leftrightarrow \frac{\partial\,\sigma(x)}{\partial\,x} - \lambda_0 \mathbf{1}_n = \mathbf{0}_n$$

Location of the ERC portfolio

Theorem

The global minimum variance portfolio satisfies:

$$\frac{\partial \, \sigma(x)}{\partial \, x_i} = \frac{\partial \, \sigma(x)}{\partial \, x_j}$$

The marginal volatilities are then the same.

Location of the ERC portfolio

The equally-weighted portfolio is defined by:

$$x_i = \frac{1}{n}$$

We deduce that:

$$x_i = x_j$$

Location of the ERC portfolio

We have:

$$x_i = x_j \quad \text{(EW)}$$

$$\frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{\partial\, \sigma(x)}{\partial\, x_j} \quad \text{(GMV)}$$

$$x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = x_j \frac{\partial\, \sigma(x)}{\partial\, x_j} \quad \text{(ERC)}$$

The ERC portfolio is a combination of GMV and EW portfolios

Volatility of the ERC portfolio

We consider the following optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\ & \text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{cases} \end{aligned}$$

- We know that there exists a scalar c_{erc} such that:

$$x^{\star}(c_{\mathrm{erc}}) = x_{\mathrm{erc}}$$

- If $c = -\infty$, the logarithmic barrier constraint vanishes and we have:

$$x^{\star}(-\infty) = x_{\mathrm{mv}}$$

  where x_{mv} is the long-only minimum variance portfolio

Volatility of the ERC portfolio

- We notice that the function $f(x) = \sum_{i=1}^{n} \ln x_i$ such that $\mathbf{1}_n^\top x = 1$ reaches its maximum when:

$$\frac{1}{x_i} = \lambda_0$$

implying that $x_i = x_j = n^{-1}$. In this case, we have:

$$c_{\max} = \sum_{i=1}^{n} \ln \frac{1}{n} = -n \ln n$$

- If $c = -n \ln n$, we have:

$$x^{\star}(-n \ln n) = x_{\mathrm{ew}}$$

- Because we have a convex minimization problem and a lower convex bound, we deduce that:

$$c_2 \geq c_1 \Leftrightarrow \sigma(x^{\star}(c_2)) \geq \sigma(x^{\star}(c_1))$$

Volatility of the ERC portfolio

Theorem

We obtain the following inequality:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

The ERC portfolio may be viewed as a portfolio "between" the MV portfolio and the EW portfolio.

Remark

The ERC portfolio is a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of weights

Relationship with naive diversification $(1/n)$

Optimality of the ERC portfolio

Let us consider the tangency (or maximum Sharpe ratio) portfolio defined by:

$$x_{\mathrm{msr}} = \arg\max \frac{\mu(x) - r}{\sigma(x)}$$

where $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. We recall that the portfolio is MSR if and only if:

$$\frac{\partial_{x_i} \mu(x) - r}{\partial_{x_i} \sigma(x)} = \frac{\mu(x) - r}{\sigma(x)}$$

Therefore, the MSR portfolio x_{msr} verifies the following relationship:

$$\begin{aligned} \mu - r\mathbf{1}_n &= \left(\frac{\mu(x_{\mathrm{msr}}) - r}{\sigma^2(x_{\mathrm{msr}})}\right) \Sigma x_{\mathrm{msr}} \\ &= \mathrm{SR}(x_{\mathrm{msr}} \mid r) \frac{\Sigma x_{\mathrm{msr}}}{\sigma(x_{\mathrm{msr}})} \end{aligned}$$

Optimality of the ERC portfolio

- If we assume a constant correlation matrix, the ERC portfolio is defined by:

$$x_i = \frac{c}{\sigma_i}$$

where $c = \left(\sum_{j=1}^{n} \sigma_j^{-1} \right)^{-1}$

- We have:

$$(\Sigma x)_i = \sum_{j=1}^{n} \rho_{i,j} \sigma_i \sigma_j x_j = c\sigma_i \sum_{j=1}^{n} \rho_{i,j} = c\sigma_i \left(1 + \rho \left(n - 1\right)\right)$$

- We deduce that:

$$\frac{\partial \, \sigma\left(x\right)}{\partial \, x_i} = c \frac{\sigma_i \left(\left(1 - \rho\right) + \rho n\right)}{\sigma\left(x\right)}$$

Optimality of the ERC portfolio

- The portfolio volatility is equal to:

$$
\begin{aligned}
\sigma^2(x) &= \sigma(x) \sum\nolimits_{i=1}^n x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} \\
&= \sigma(x) \sum\nolimits_{i=1}^n \frac{c}{\sigma_i} \cdot c \frac{\sigma_i ((1-\rho)+\rho n)}{\sigma(x)} \\
&= nc^2 ((1-\rho)+\rho n)
\end{aligned}
$$

- The ERC portfolio is the MSR portfolio if and only if:

$$
\begin{aligned}
\mu_i - r &= \left(\frac{\sum_{j=1}^n (\mu_j - r) x_j}{\sigma^2(x)} \right) (\Sigma x)_i \\
&= \left(\frac{\sum_{j=1}^n (\mu_j - r) c\sigma_j^{-1}}{nc^2 ((1-\rho)+\rho n)} \right) c\sigma_i (1 + \rho (n-1)) \\
&= \left(\frac{1}{n} \sum_{j=1}^n \frac{\mu_j - r}{\sigma_j} \right) \sigma_i
\end{aligned}
$$

Optimality of the ERC portfolio

- We can write this condition as follows:

$$\mu_i = r + \mathrm{SR} \cdot \sigma_i$$

where:

$$\mathrm{SR} = \frac{1}{n} \sum_{j=1}^{n} \frac{\mu_j - r}{\sigma_j}$$

Theorem

The ERC portfolio is the tangency or MSR portfolio if and only if the correlation is uniform and the Sharpe ratio is the same for all the assets

Optimality of the ERC portfolio

Example 6

We consider an investment universe of five assets. The volatilities are respectively equal to 5%, 7%, 9%, 10% and 15%. The risk-free rate is equal to 2%. The correlation is uniform.

Optimality of the ERC portfolio

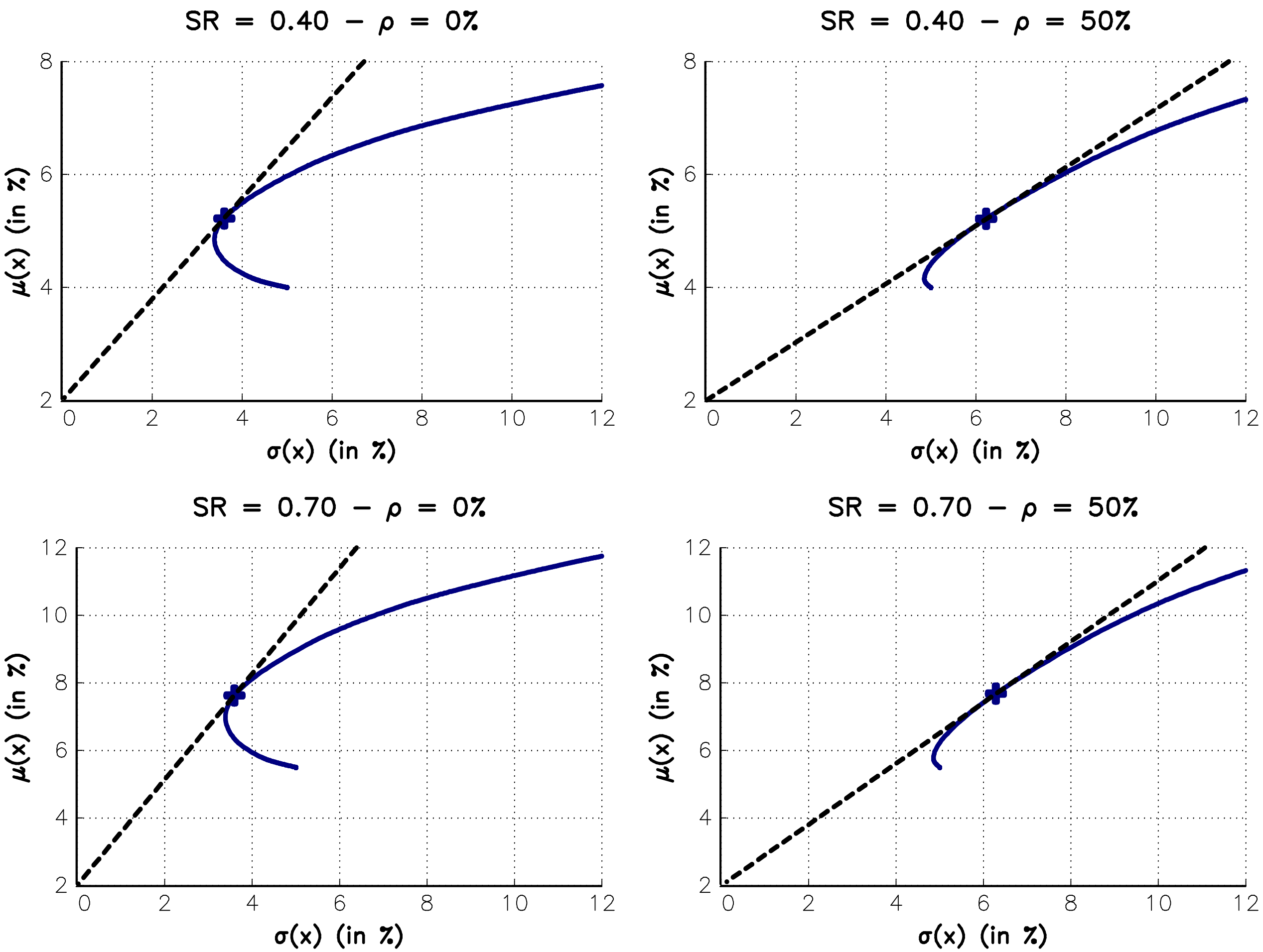


Figure 4: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are the same (Example 6)

Optimality of the ERC portfolio

Example 7

We consider an investment universe of five assets. The volatilities are respectively equal to 5%, 7%, 9%, 10% and 15%. The correlation matrix is equal to:

$$\rho = \begin{pmatrix} 1.00 & & & & \\ 0.50 & 1.00 & & & \\ 0.25 & 0.25 & 1.00 & & \\ 0.00 & 0.00 & 0.00 & 1.00 & \\ -0.25 & -0.25 & -0.25 & 0.00 & 1.00 \end{pmatrix}$$

Optimality of the ERC portfolio

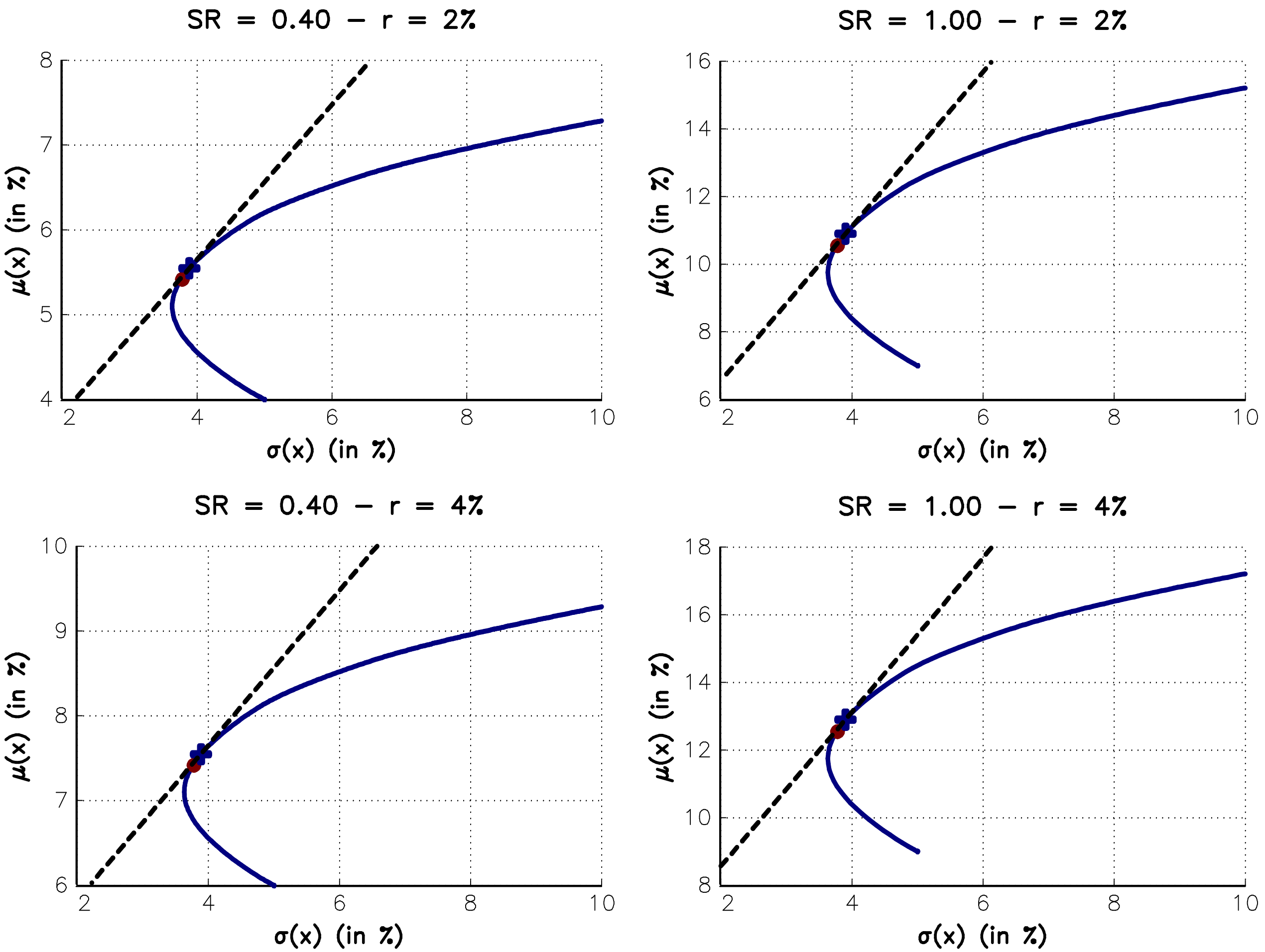


Figure 5: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are the same (Example 7)

The SQP approach

- The ERC portfolio satisfies:

$$x_i \cdot (\Sigma x)_i = x_j \cdot (\Sigma x)_j$$

or:

$$x_i \cdot (\Sigma x)_i = \frac{x^\top \Sigma x}{n}$$

- We deduce that:

$$\begin{aligned} x_{\mathrm{erc}} &= \arg\min f(x) \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

and $f(x_{\mathrm{erc}}) = 0$

Remark

The optimization problem is solved using the sequential quadratic programming (or SQP) algorithm

The SQP approach

- We can choose:

$$f(x) = \sum_{i=1}^{n} \left(x_i \cdot (\Sigma x)_i - \frac{1}{n} x^\top \Sigma x \right)^2$$

or:

$$f(x; b) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i \cdot (\Sigma x)_i - x_j \cdot (\Sigma x)_j \right)^2$$

The Jacobi approach

- We have:

$$\beta_i(x) = \frac{(\Sigma x)_i}{x^\top \Sigma x}$$

- The ERC portfolio satisfies:

$$x_i = \frac{\beta_i^{-1}(x)}{\sum_{j=1}^n \beta_j^{-1}(x)}$$

or:

$$x_i \propto \frac{1}{(\Sigma x)_i}$$

The Jacobi approach

The Jacobi algorithm consists in finding the fixed point by considering the following iterations:

1. We set $k \leftarrow 0$ and we note $x^{(0)}$ the vector of starting values[2]
2. At iteration $k+1$, we compute:

$$y_i^{(k+1)} \propto \frac{1}{\beta_i\left(x^{(k)}\right)} = \frac{1}{\left(\Sigma x^{(k)}\right)_i}$$

and:

$$x_i^{(k+1)} = \frac{y_i^{(k+1)}}{\sum_{j=1}^n y_j^{(k+1)}}$$

3. We iterate Step 2 until convergence

[2]For instance, we can use the following rule:

$$x_i^{(0)} = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

The Newton-Raphson approach

We consider the following optimization problem:

$$x^* = \arg\min f(x)$$

The Newton-Raphson iteration is defined by:

$$x^{(k+1)} = x^{(k)} - \Delta x^{(k)}$$

where $\Delta x^{(k)}$ is the inverse of the Hessian matrix of $f\left(x^{(k)}\right)$ times the gradient vector of $f\left(x^{(k)}\right)$:

$$\Delta x^{(k)} = \left[\partial_x^2 f\left(x^{(k)}\right)\right]^{-1} \partial_x f\left(x^{(k)}\right)$$

The Newton-Raphson approach

- We consider the Lagrange function:

$$f(y) = \frac{1}{2} y^\top \Sigma y - \lambda_c \sum_{i=1}^{n} \ln y_i$$

- We choose a value of λ_c (e.g. $\lambda_c = 1$)
- We note y^{-m} the vector $n \times 1$ matrix with elements $\left(y_1^{-m}, \ldots, y_n^{-m}\right)$ and $\operatorname{diag}\left(y^{-m}\right)$ the $n \times n$ diagonal matrix with elements $\left(y_1^{-m}, \ldots, y_n^{-m}\right)$:

$$\operatorname{diag}\left(y^{-m}\right) = \begin{pmatrix} y_1^{-m} & 0 & & 0 \\ 0 & y_2^{-m} & & \\ & & \ddots & 0 \\ 0 & & 0 & y_n^{-m} \end{pmatrix}$$

The Newton-Raphson approach

- We apply the Newton-Raphson algorithm with:

$$\partial_y f(y) = \Sigma y - \lambda_c y^{-1}$$

and:

$$\partial_y^2 f(y) = \Sigma + \lambda_c \operatorname{diag}\left(y^{-2}\right)$$

- The solution is given by:

$$x_{\mathrm{erc}} = \frac{y^\star}{\sum_{i=1}^n y_i^\star}$$

The Newton-Raphson approach

- For the starting value $y_i^{(0)}$, we can assume that the correlations are uniform:

$$y_i^{(0)} = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

- At the optimum, we recall that $\lambda_c = y_i^\star \cdot (\Sigma y^\star)_i$. We deduce that:

$$\lambda_c = \frac{1}{n} \sum_{i=1}^n y_i^\star \cdot (\Sigma y^\star)_i = \frac{\sigma^2 (y^\star)}{n}$$

Therefore, we can choose:

$$\lambda_c = \frac{\sigma^2 \left(y^{(0)}\right)}{n}$$

The Newton-Raphson approach

- From a numerical point of view, it may be important to control the magnitude order α of $y^{\star}$ (e.g. $\alpha = 10\%$, $\alpha = 1$ or $\alpha = 10$). For instance, we don't want that the magnitude order is 10^{-5} or 10^{5}. In this case, we can use the following rule:

$$\lambda_c = n\alpha^2\sigma^2\left(x_{\mathrm{erc}}\right)$$

- For example, if $n = 10$ and $\alpha = 5$, and we guess that the volatility of the ERC portfolio is around 10%, we set:

$$\lambda_c = 10 \times 5^2 \times 0.10^2 = 2.5$$

The CCD approach

Table 4: Cyclical coordinate descent algorithm

The goal is to find the solution $x^{\star} = \arg\min f(x)$
We initialize the vector $x^{(0)}$
Set $k \leftarrow 0$
repeat
  for $i = 1 : n$ **do**
    $x_i^{(k+1)} = \arg\min_{\varkappa} f\left(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, \varkappa, x_{i+1}^{(k)}, \ldots, x_n^{(k)}\right)$
  end for
  $k \leftarrow k + 1$
until convergence
return $x^{\star} \leftarrow x^{(k)}$

The CCD approach

We have:

$$\mathcal{L}(y;\lambda_c) = \arg\min \frac{1}{2} y^\top \Sigma y - \lambda_c \sum_{i=1}^{n} \ln y_i$$

The first-order condition is equal to:

$$\frac{\partial\, \mathcal{L}(y;\lambda)}{\partial\, y_i} = (\Sigma y)_i - \frac{\lambda_c}{y_i} = 0$$

or:

$$y_i \cdot (\Sigma y)_i - \lambda_c = 0$$

It follows that:

$$\sigma_i^2 y_i^2 + \left(\sigma_i \sum_{j \neq i} \rho_{i,j} \sigma_j y_j \right) y_i - \lambda_c = 0$$

The CCD approach

We recognize a second-degree equation:

$$\alpha_i y_i^2 + \beta_i y_i + \gamma_i = 0$$

1. The polynomial function is convex because we have $\alpha_i = \sigma_i^2 > 0$
2. The product of the roots is negative:

$$y_i' y_i'' = \frac{\gamma_i}{\alpha_i} = -\frac{\lambda_c}{\sigma_i^2} < 0$$

3. The discriminant is positive:

$$\Delta = \beta_i^2 - 4\alpha_i\gamma_i = \left(\sigma_i \sum_{j \neq i} \rho_{i,j}\sigma_j y_j\right)^2 + 4\sigma_i^2\lambda_c > 0$$

We always have two solutions with opposite signs. We deduce that the solution is the positive root of the second-degree equation:

$$y_i^\star = y_i'' = \frac{-\beta_i + \sqrt{\beta_i^2 - 4\alpha_i\gamma_i}}{2\alpha_i}$$

The CCD approach

The CCD algorithm consists in iterating the following formula:

$$y_i^{(k+1)} = \frac{-\beta_i^{(k+1)} + \sqrt{\left(\beta_i^{(k+1)}\right)^2 - 4\alpha_i^{(k+1)}\gamma_i^{(k+1)}}}{2\alpha_i^{(k+1)}}$$

where:

$$\begin{array}{rcl}
\alpha_i^{(k+1)} & = & \sigma_i^2 \\
\beta_i^{(k+1)} & = & \sigma_i \left(\sum_{j<i} \rho_{i,j}\sigma_j y_j^{(k+1)} + \sum_{j>i} \rho_{i,j}\sigma_j y_j^{(k)}\right) \\
\gamma_i^{(k+1)} & = & -\lambda_c
\end{array}$$

The ERC portfolio is the scaled solution $y^\star$:

$$x_{\mathrm{erc}} = \frac{y^\star}{\sum_{i=1}^n y_i^\star}$$

Efficiency of the algorithms

$$\text{CCD} \succ \text{NR} \succ \text{SQP} \succ \text{Jacobi}$$

Definition of RB portfolios

Definition

A risk budgeting (RB) portfolio x satisfies the following conditions:

$$\left\{\begin{array}{l} \mathcal{RC}_1 = b_1 \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_i = b_i \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_n = b_n \mathcal{R}(x) \end{array}\right.$$

where $\mathcal{R}(x)$ is a coherent and convex risk measure and $b = (b_1, \ldots, b_n)$ is a vector of risk budgets such that $b_i \geq 0$ and $\sum_{i=1}^n b_i = 1$

Definition of RB portfolios

Remark

The ERC portfolio is a particular case of RB portfolios when $\mathcal{R}(x) = \sigma(x)$ and $b_i = \frac{1}{n}$

Coherent risk measure

1. Subadditivity
$$\mathcal{R}(x_1 + x_2) \leq \mathcal{R}(x_1) + \mathcal{R}(x_2)$$
2. Homogeneity
$$\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x) \quad \text{if } \lambda \geq 0$$
3. Monotonicity
$$\text{if } x_1 \prec x_2, \text{ then } \mathcal{R}(x_1) \geq \mathcal{R}(x_2)$$
4. Translation invariance
$$\text{if } m \in \mathbb{R}, \text{ then } \mathcal{R}(x + m) = \mathcal{R}(x) - m$$

Convex risk measure

The convexity property is defined as follows:

$$\mathcal{R}\left(\lambda x_1 + (1-\lambda) x_2\right) \leq \lambda \mathcal{R}\left(x_1\right) + (1-\lambda)\mathcal{R}\left(x_2\right)$$

This condition means that diversification should not increase the risk

Euler allocation principle

This property is necessary for the Euler allocation principle:

$$\mathcal{R}\left(x\right) = \sum_{i=1}^{n} x_i \frac{\partial\, \mathcal{R}\left(x\right)}{\partial\, x_i}$$

Some risk measures

The portfolio loss is $L(x) = -R(x)$ where $R(x)$ is the portfolio return. We consider then different risk measures:

- Volatility of the loss

$$\mathcal{R}(x) = \sigma(L(x)) = \sigma(x)$$

- Standard deviation-based risk measure

$$\mathcal{R}(x) = \mathrm{SD}_c(x) = \mathbb{E}[L(x)] + c \cdot \sigma(L(x)) = -\mu(x) + c \cdot \sigma(x)$$

- Value-at-risk

$$\mathcal{R}(x) = \mathrm{VaR}_\alpha(x) = \inf\{\ell : \Pr\{L(x) \leq \ell\} \geq \alpha\}$$

- Expected shortfall

$$\mathcal{R}(x) = \mathrm{ES}_\alpha(x) = \mathbb{E}[L(x) \mid L(x) \geq \mathrm{VaR}_\alpha(x)] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(x)\,\mathrm{d}u$$

Gaussian risk measures

We assume that the asset returns are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$

We have:

$$\begin{aligned}
\sigma(x) &= \sqrt{x^\top \Sigma x} \\
\mathrm{SD}_c(x) &= -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} \\
\mathrm{VaR}_\alpha(x) &= -x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x} \\
\mathrm{ES}_\alpha(x) &= -x^\top \mu + \frac{\sqrt{x^\top \Sigma x}}{(1-\alpha)} \phi\left(\Phi^{-1}(\alpha)\right)
\end{aligned}$$

Gaussian risk contributions

- Volatility $\sigma(x)$

$$\mathcal{RC}_i = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

- Standard deviation-based risk measure $\mathrm{SD}_c(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + c \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \right)$$

- Value-at-risk $\mathrm{VaR}_\alpha(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + \Phi^{-1}(\alpha) \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \right)$$

- Expected shortfall $\mathrm{ES}_\alpha(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + \frac{(\Sigma x)_i}{(1-\alpha)\sqrt{x^\top \Sigma x}} \phi\left(\Phi^{-1}(\alpha)\right) \right)$$

Gaussian risk contributions

Example 8

We consider three assets. We assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The portfolio is equal to $(50\%, 20\%, 30\%)$.

Gaussian risk contributions

Table 5: Risk decomposition of the portfolio (Example 8)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Volatility	1	50.00	29.40	14.70	70.43
	2	20.00	16.63	3.33	15.93
	3	30.00	9.49	2.85	13.64
	$\sigma(x)$			20.87	
Value-at-risk	1	50.00	68.39	34.19	70.43
	2	20.00	38.68	7.74	15.93
	3	30.00	22.07	6.62	13.64
	$\mathrm{VaR}_{99\%}(x)$			48.55	
Expected shortfall	1	50.00	78.35	39.17	70.43
	2	20.00	44.31	8.86	15.93
	3	30.00	25.29	7.59	13.64
	$\mathrm{ES}_{99\%}(x)$			55.62	

Gaussian risk contributions

Example 9

We consider three assets. We assume that their expected returns are equal to 10%, 5% and 8% whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The portfolio is equal to $(50\%, 20\%, 30\%)$.

Gaussian risk contributions

Table 6: Risk decomposition of the portfolio (Example 9)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Volatility	1	50.00	29.40	14.70	70.43
	2	20.00	16.63	3.33	15.93
	3	30.00	9.49	2.85	13.64
	$\sigma(x)$			20.87	
Value-at-risk	1	50.00	58.39	29.19	72.71
	2	20.00	33.68	6.74	16.78
	3	30.00	14.07	4.22	10.51
	$\mathrm{VaR}_{99\%}(x)$			40.15	
Expected shortfall	1	50.00	68.35	34.17	72.37
	2	20.00	39.31	7.86	16.65
	3	30.00	17.29	5.19	10.98
	$\mathrm{ES}_{99\%}(x)$			47.22	

Non-Gaussian risk contributions

They are not frequently used in asset management and portfolio allocation, except in the case of skewed assets (Bruder *et al.*, 2016; Lezmi *et al.*, 2018)

Non-parametric risk contributions are given in Chapter 2 in Roncalli (2013)

Gaussian RB portfolios

Example 10

We consider three assets. We assume that their expected returns are equal to 10%, 5% and 8% whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The risk budgets are equal to $(50\%, 20\%, 30\%)$.

Gaussian RB portfolios

Table 7: Risk budgeting portfolios (Example 10)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Volatility	1	31.14	28.08	8.74	50.00
	2	21.90	15.97	3.50	20.00
	3	46.96	11.17	5.25	30.00
	$\sigma(x)$			17.49	
Value-at-risk	1	29.18	54.47	15.90	50.00
	2	20.31	31.30	6.36	20.00
	3	50.50	18.89	9.54	30.00
	$\mathrm{VaR}_{99\%}(x)$			31.79	
Expected shortfall	1	29.48	64.02	18.87	50.00
	2	20.54	36.74	7.55	20.00
	3	49.98	22.65	11.32	30.00
	$\mathrm{ES}_{99\%}(x)$			37.74	

Special cases

- The case of uniform correlation[3] $\rho_{i,j} = \rho$
  1. Minimum correlation
  $$x_i \left(-\frac{1}{n-1}\right) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$
  2. Zero correlation
  $$x_i (0) = \frac{\sqrt{b_i}\sigma_i^{-1}}{\sum_{j=1}^n \sqrt{b_j}\sigma_j^{-1}}$$
  3. Maximum correlation
  $$x_i (1) = \frac{b_i\sigma_i^{-1}}{\sum_{j=1}^n b_j\sigma_j^{-1}}$$
- The general case
  $$x_i = \frac{b_i\beta_i^{-1}}{\sum_{j=1}^n b_j\beta_j^{-1}}$$
  where β_i is the beta of Asset i with respect to the RB portfolio

[3] The solution is noted $x_i (\rho)$.

Existence and uniqueness

We have:

$$\frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma(x)}$$

Suppose that the risk budget b_k is equal to zero. This means that:

$$x_k \left(x_k \sigma_k^2 + \sigma_k \sum_{j \neq k} x_j \rho_{k,j} \sigma_j \right) = 0$$

We obtain two solutions:

1. The first one is:
$$x'_k = 0$$
2. The second one verifies:
$$x''_k = - \frac{\sum_{j \neq k} x_j \rho_{k,j} \sigma_j}{\sigma_k}$$

Existence and uniqueness

- If $\rho_{k,j} \geq 0$ for all j, we have $\sum_{j \neq k} x_j \rho_{k,j} \sigma_j \geq 0$ because $x_j \geq 0$ and $\sigma_j > 0$. This implies that $x_k'' \leq 0$ meaning that $x_k' = 0$ is the unique positive solution
- The only way to have $x_k'' > 0$ is to have some negative correlations $\rho_{k,j}$. In this case, this implies that:

$$\sum_{j \neq k} x_j \rho_{k,j} \sigma_j < 0$$

- If we consider a universe of three assets, this constraint is verified for $k = 3$ and a covariance matrix such that $\rho_{1,3} < 0$ and $\rho_{2,3} < 0$

Existence and uniqueness

Example 11

We have $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, $\sigma_3 = 5\%$, $\rho_{1,2} = 50\%$, $\rho_{1,3} = -25\%$ and $\rho_{2,3} = -25\%$

If the risk budgets are equal to $(50\%, 50\%, 0\%)$, the two solutions are:

$$(33.33\%, 66.67\%, 0\%)$$

and:

$$(20\%, 40\%, 40\%)$$

Two questions

1. How many solutions do we have in the general case?
2. Which solution is the best?

Existence and uniqueness

Table 8: First solution (Example 11)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.33	17.32	5.77	50.00
2	66.67	8.66	5.77	50.00
3	0.00	−1.44	0.00	0.00
Volatility			11.55	

Table 9: Second solution (Example 11)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	16.58	3.32	50.00
2	40.00	8.29	3.32	50.00
3	40.00	0.00	0.00	0.00
Volatility			6.63	

Existence and uniqueness

The case with strictly positive risk budgets

- We consider the following optimization problem:

$$\begin{aligned} y^{\star} &= \arg\min \mathcal{R}(y) \\ \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} b_i \ln y_i \geq c \\ y \geq \mathbf{0}_n \end{array} \right. \end{aligned}$$

  where c is an arbitrary constant

- The associated Lagrange function is:

$$\mathcal{L}(y; \lambda, \lambda_c) = \mathcal{R}(y) - \lambda^{\top} y - \lambda_c \left(\sum_{i=1}^{n} b_i \ln y_i - c \right)$$

  where $\lambda \in \mathbb{R}^n$ and $\lambda_c \in \mathbb{R}$

Existence and uniqueness

The case with strictly positive risk budgets

- The solution $y^{\star}$ verifies the following first-order condition:

$$\frac{\partial\,\mathcal{L}\left(y;\lambda,\lambda_c\right)}{\partial\,y_i}=\frac{\partial\,\mathcal{R}\left(y\right)}{\partial\,y_i}-\lambda_i-\lambda_c\frac{b_i}{y_i}=0$$

- The Kuhn-Tucker conditions are:

$$\left\{\begin{array}{l}\min\left(\lambda_i,y_i\right)=0\\ \min\left(\lambda_c,\sum_{i=1}^n b_i\ln y_i-c\right)=0\end{array}\right.$$

Existence and uniqueness

The case with strictly positive risk budgets

- Because $\ln y_i$ is not defined for $y_i = 0$, it follows that $y_i > 0$ and $\lambda_i = 0$
- We note that the constraint $\sum_{i=1}^{n} b_i \ln y_i = c$ is necessarily reached (because the solution cannot be $y^\star = \mathbf{0}_n$), then $\lambda_c > 0$ and we have:

$$y_i \frac{\partial \mathcal{R}(y)}{\partial y_i} = \lambda_c b_i$$

- We verify that the risk contributions are proportional to the risk budgets:

$$\mathcal{RC}_i = \lambda_c b_i$$

Existence and uniqueness

The case with strictly positive risk budgets

Theorem

The optimization program has a unique solution and the RB portfolio is equal to:

$$x_{\mathrm{rb}} = \frac{y^{\star}}{\sum_{i=1}^{n} y_i^{\star}}$$

Remark

We note that the convexity property of the risk measure is essential to the existence and uniqueness of the RB portfolio. If $\mathcal{R}(x)$ is not convex, the preceding analysis becomes invalid.

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

- Let $\mathcal{N}$ be the set of assets such that $b_i = 0$
- The Lagrange function becomes:

$$\mathcal{L}(y; \lambda, \lambda_c) = \mathcal{R}(y) - \lambda^\top y - \lambda_c \left(\sum_{i \notin \mathcal{N}} b_i \ln y_i - c \right)$$

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

- The solution $y^{\star}$ verifies the following first-order conditions:

$$\frac{\partial \mathcal{L}(y;\lambda,\lambda_c)}{\partial y_i} = \begin{cases} \partial_{y_i} \mathcal{R}(y) - \lambda_i - \lambda_c b_i y_i^{-1} = 0 & \text{if} \quad i \notin \mathcal{N} \\ \partial_{y_i} \mathcal{R}(y) - \lambda_i = 0 & \text{if} \quad i \in \mathcal{N} \end{cases}$$

  - If $i \notin \mathcal{N}$, the previous analysis is valid and we verify that risk contributions are proportional to the risk budgets:

$$y_i \frac{\partial \mathcal{R}(y)}{\partial y_i} = \lambda_c b_i$$

  - If $i \in \mathcal{N}$, we must distinguish two cases:
    1. If $y_i = 0$, it implies that $\lambda_i > 0$ and $\partial_{y_i} \mathcal{R}(y) > 0$
    2. In the other case, if $y_i > 0$, it implies that $\lambda_i = 0$ and $\partial_{y_i} \mathcal{R}(y) = 0$
  - The solution $y_i = 0$ or $y_i > 0$ if $i \in \mathcal{N}$ will then depend on the structure of the covariance matrix Σ (in the case of a Gaussian risk measure)

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

Theorem

We conclude that the solution $y^\star$ of the optimization problem exists and is unique even if some risk budgets are set to zero. As previously, we deduce the normalized RB portfolio x_{rb} by scaling $y^\star$. This solution, noted $\mathcal{S}_1$, satisfies the following relationships:

$$\left\{ \begin{array}{ll} \mathcal{RC}_i = x_i \cdot \partial_{x_i} \mathcal{R}(x) = b_i & \text{if} \quad i \notin \mathcal{N} \\ \left\{ \begin{array}{ll} x_i = 0 \text{ and } \partial_{x_i} \mathcal{R}(x) > 0 & (i) \\ \text{or} & \\ x_i > 0 \text{ and } \partial_{x_i} \mathcal{R}(x) = 0 & (ii) \end{array} \right. & \text{if} \quad i \in \mathcal{N} \end{array} \right.$$

The conditions (i) and (ii) are mutually exclusive for one asset $i \in \mathcal{N}$, but not necessarily for all the assets $i \in \mathcal{N}$.

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

The previous analysis implies that there may be several solutions to the following non-linear system when $b_i = 0$ for $i \in \mathcal{N}$:

$$\begin{cases} \mathcal{RC}_1 = b_1 \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_i = b_i \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_n = b_n \mathcal{R}(x) \end{cases}$$

- Let $\mathcal{N} = \mathcal{N}_1 \bigsqcup \mathcal{N}_2$ where $\mathcal{N}_1$ is the set of assets verifying the condition (i) and $\mathcal{N}_2$ is the set of assets verifying the condition (ii)
- The number of solutions is equal to 2^m where $m = |\mathcal{N}_2|$ is the cardinality of $\mathcal{N}_2$

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

We note $\mathcal{S}_2$ the solution with $x_i = 0$ for all assets such that $b_i = 0$. Even if $\mathcal{S}_2$ is the solution expected by the investor, the only acceptable solution is $\mathcal{S}_1$. Indeed, if we impose $b_i = \varepsilon_i$ where $\varepsilon_i > 0$ is a small number for $i \in \mathcal{N}$, we obtain:

$$\lim_{\varepsilon_i \to 0} \mathcal{S} = \mathcal{S}_1$$

The solution converges to $\mathcal{S}_1$, and not to $\mathcal{S}_2$ or the other solutions

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

Remark

The non-linear system is not well-defined, whereas the optimization problem is the right approach to define a RB portfolio

Definition

A RB portfolio is a minimum risk portfolio subject to a diversification constraint, which is defined by the logarithmic barrier function

Existence and uniqueness

Example 12

We consider a universe of three assets with $\sigma_1 = 20\%$, $\sigma_2 = 10\%$ and $\sigma_3 = 5\%$. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ \rho_{1,3} & \rho_{2,3} & 1.00 \end{pmatrix}$$

We would like to build a RB portfolio such that the risk budgets with respect to the volatility risk measure are $(50\%, 50\%, 0\%)$. Moreover, we assume that $\rho_{1,3} = \rho_{2,3}$.

Existence and uniqueness

Table 10: RB solutions when the risk budget b_3 is equal to 0 (Example 12)

$\rho_{1,3} = \rho_{2,3}$	Solution		1	2	3	$\sigma(x)$
-25%	$\mathcal{S}_1$	x_i	20.00%	40.00%	40.00%	6.63%
		$\mathcal{MR}_i$	16.58%	8.29%	0.00%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	
	$\mathcal{S}_2$	x_i	33.33%	66.67%	0.00%	11.55%
		$\mathcal{MR}_i$	17.32%	8.66%	-1.44%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	
	$\mathcal{S}'_1$	x_i	19.23%	38.46%	42.31%	6.38%
		$\mathcal{MR}_i$	16.42%	8.21%	0.15%	
		$\mathcal{RC}_i$	49.50%	49.50%	1.00%	
25%	$\mathcal{S}_1$	x_i	33.33%	66.67%	0.00%	11.55%
		$\mathcal{MR}_i$	17.32%	8.66%	1.44%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	

Existence and uniqueness

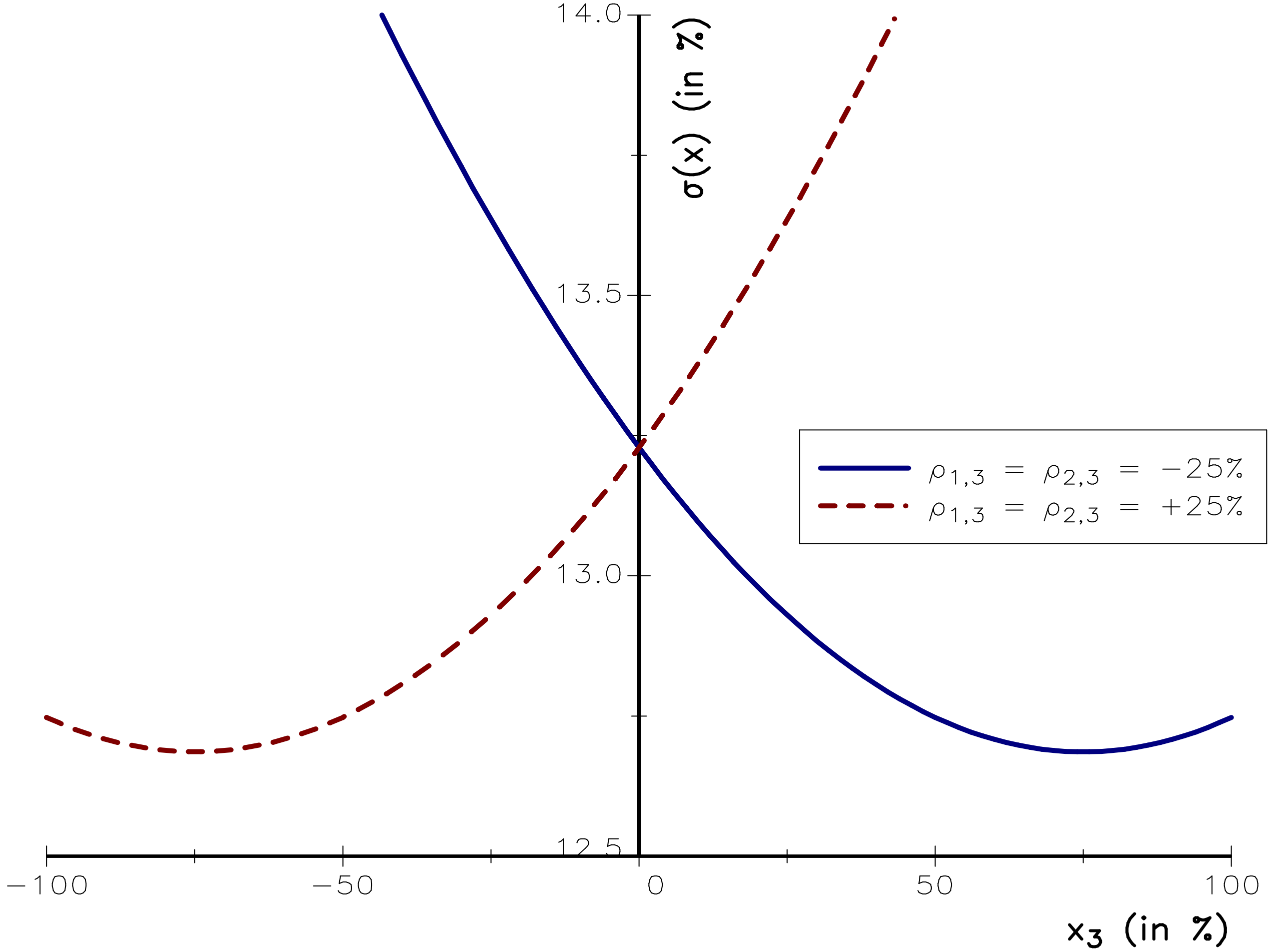


Figure 6: Evolution of the portfolio's volatility with respect to x_3

Location of the RB portfolio

We have:

$$\frac{x_i}{b_i} = \frac{x_j}{b_j} \qquad \text{(WB)}$$

$$\frac{\partial\,\mathcal{R}(x)}{\partial\,x_i} = \frac{\partial\,\mathcal{R}(x)}{\partial\,x_j} \qquad \text{(MR)}$$

$$\frac{1}{b_i}\left(x_i \frac{\partial\,\mathcal{R}(x)}{\partial\,x_i}\right) = \frac{1}{b_j}\left(x_j \frac{\partial\,\mathcal{R}(x)}{\partial\,x_j}\right) \qquad \text{(ERC)}$$

The RB portfolio is a combination of MR (long-only minimum risk) and WB (weight budgeting) portfolios

Risk of the RB portfolio

Theorem

We obtain the following inequality:

$$\mathcal{R}\left(x_{\mathrm{mr}}\right) \leq \mathcal{R}\left(x_{\mathrm{rb}}\right) \leq \mathcal{R}\left(x_{\mathrm{wb}}\right)$$

The RB portfolio may be viewed as a portfolio "between" the MR portfolio and the WB portfolio

Diversification index

Definition

The diversification index is equal to:

$$\begin{aligned}\mathcal{D}(x) &= \frac{\mathcal{R}\left(\sum_{i=1}^{n} L_i\right)}{\sum_{i=1}^{n}\mathcal{R}(L_i)} \\ &= \frac{\mathcal{R}(x)}{\sum_{i=1}^{n} x_i\mathcal{R}(\mathbf{e}_i)}\end{aligned}$$

Diversification index

- The diversification index is the ratio between the risk measure of portfolio x and the weighted risk measure of the assets
- If $\mathcal{R}$ is a coherent risk measure, we have $\mathcal{D}(x) \leq 1$
- If $\mathcal{D}(x) = 1$, it implies that the losses are comonotonic
- If $\mathcal{R}$ is the volatility risk measure, we obtain:

$$\mathcal{D}(x) = \frac{\sqrt{x^\top \Sigma x}}{\sum_{i=1}^n x_i \sigma_i}$$

It takes the value one if the asset returns are perfectly correlated meaning that the correlation matrix is $\mathcal{C}_n(1)$

Concentration index

- Let $\pi \in \mathbb{R}_{+}^{n}$ such that $\mathbf{1}_n^\top \pi = 1 \Rightarrow \pi$ is a probability distribution
- The probability distribution π^{+} is perfectly concentrated if there exists one observation i_0 such that $\pi_{i_0}^{+} = 1$ and $\pi_i^{+} = 0$ if $i \neq i_0$
- When n tends to $+\infty$, the limit distribution is noted π_{∞}^{+}
- On the opposite, the probability distribution π^{-} such that $\pi_i^{-} = 1/n$ for all $i = 1, \ldots, n$ has no concentration

Concentration index

Definition

A concentration index is a mapping function $\mathcal{C}(\pi)$ such that $\mathcal{C}(\pi)$ increases with concentration and verifies:

$$\mathcal{C}(\pi^-) \leq \mathcal{C}(\pi) \leq \mathcal{C}(\pi^+)$$

- For instance, if π represents the weights of the portfolio, $\mathcal{C}(\pi)$ measures then the weight concentration
- By construction, $\mathcal{C}(\pi)$ reaches the minimum value if the portfolio is equally weighted
- To measure the risk concentration of the portfolio, we define π as the distribution of the risk contributions. In this case, the portfolio corresponding to the lower bound $\mathcal{C}(\pi^-) = 0$ is the ERC portfolio

Herfindahl index

Definition

The Herfindahl index associated with π is defined as:

$$\mathcal{H}(\pi) = \sum_{i=1}^{n} \pi_i^2$$

- This index takes the value 1 for the probability distribution π^+ and $1/n$ for the distribution with uniform probabilities π^-
- To scale the statistics onto $[0, 1]$, we consider the normalized index $\mathcal{H}^\star(\pi)$ defined as follows:

$$\mathcal{H}^\star(\pi) = \frac{n\mathcal{H}(\pi) - 1}{n - 1}$$

Gini index

- The Gini index is based on the Lorenz curve of inequality
- Let X and Y be two random variables. The Lorenz curve $y = \mathbb{L}(x)$ is defined by the following parameterization:

$$\left\{ \begin{array}{l} x = \Pr\{X \leq x\} \\ y = \Pr\{Y \leq y \mid X \leq x\} \end{array} \right.$$

- The Lorenz curve admits two limit cases
  1. If the portfolio is perfectly concentrated, the distribution of the weights corresponds to π_{∞}^{+}
  2. On the opposite, the least concentrated portfolio is the equally weighted portfolio and the Lorenz curve is the bisecting line $y = x$

Gini index

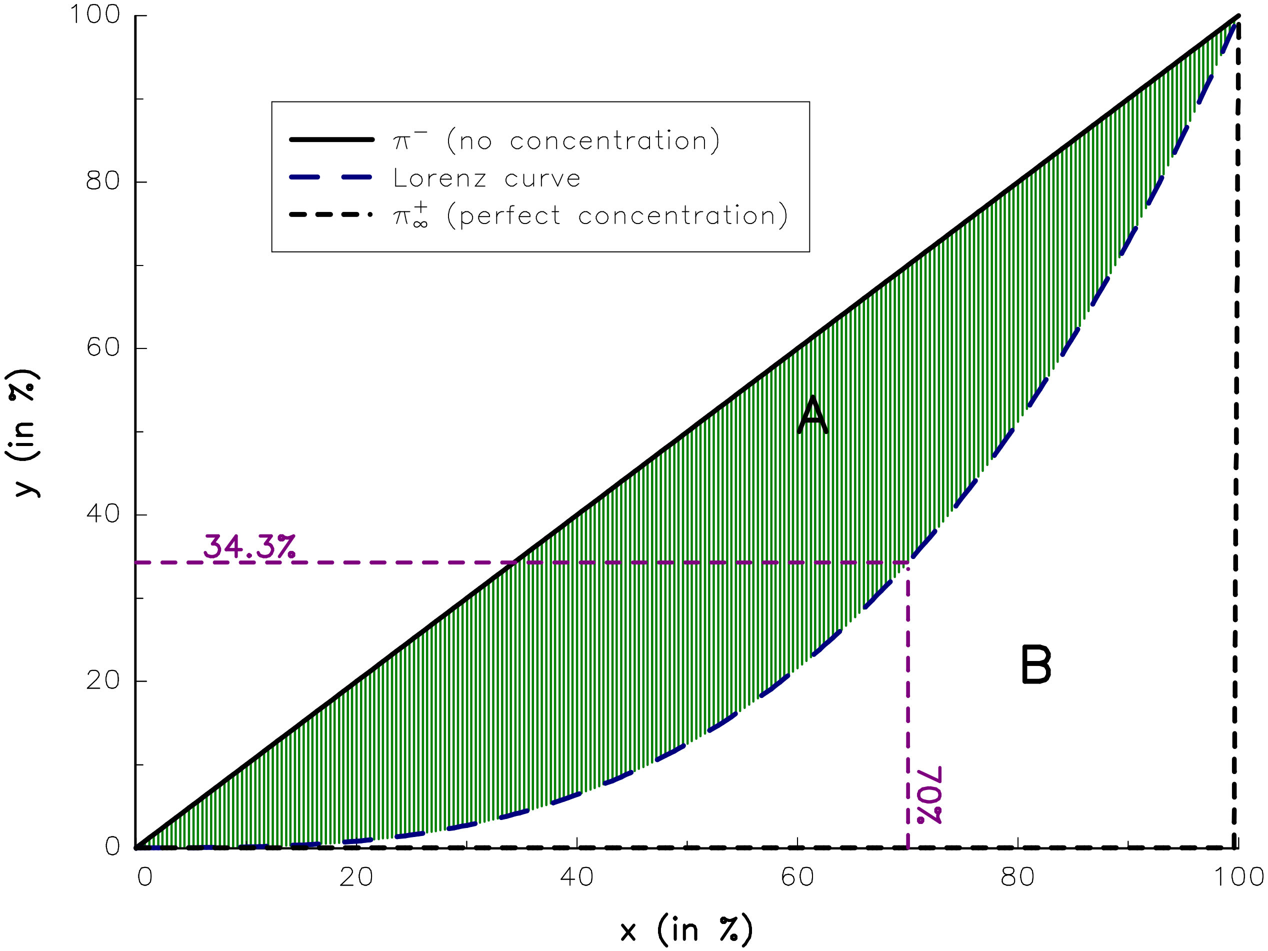


Figure 7: Geometry of the Lorenz curve

Gini index

Definition

The Gini index is then defined as:

$$\mathcal{G}(\pi) = \frac{A}{A+B}$$

with A the area between $\mathbb{L}(\pi^-)$ and $\mathbb{L}(\pi)$, and B the area between $\mathbb{L}(\pi)$ and $\mathbb{L}(\pi_\infty^+)$

Gini index

- By construction, we have $\mathcal{G}(\pi^-) = 0$, $\mathcal{G}(\pi^+_\infty) = 1$ and:

$$\begin{aligned} \mathcal{G}(\pi) &= \frac{(A+B)-B}{A+B} \\ &= 1 - \frac{1}{A+B} B \\ &= 1 - 2\int_0^1 \mathbb{L}(x)\,\mathrm{d}x \end{aligned}$$

In the case when π is a discrete probability distribution, we obtain:

$$\mathcal{G}(\pi) = \frac{2\sum_{i=1}^n i\pi_{i:n}}{n\sum_{i=1}^n \pi_{i:n}} - \frac{n+1}{n}$$

where $\{\pi_{1:n}, \ldots, \pi_{n:n}\}$ are the ordered statistics of $\{\pi_1, \ldots, \pi_n\}$.

Shannon entropy

Definition

The Shannon entropy is equal to:

$$\mathcal{I}(\pi) = -\sum_{i=1}^{n} \pi_i \ln \pi_i$$

- The diversity index corresponds to the statistic:

$$\mathcal{I}^{\star}(\pi) = \exp(\mathcal{I}(\pi))$$

- We have $\mathcal{I}^{\star}(\pi^{-}) = n$ and $\mathcal{I}^{\star}(\pi^{+}) = 1$

Impact of the reparametrization on the asset universe

- We consider a set of m primary assets $(\mathcal{A}'_1, \ldots, \mathcal{A}'_m)$ with a covariance matrix Ω
- We define n synthetic assets $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ which are composed of the primary assets
- We denote $W = (w_{i,j})$ the weight matrix such that $w_{i,j}$ is the weight of the primary asset $\mathcal{A}'_j$ in the synthetic asset $\mathcal{A}_i$ (we have $\sum_{j=1}^{m} w_{i,j} = 1$)
- The covariance matrix of the synthetic assets Σ is equal to $W \Omega W^\top$
- The synthetic assets can be interpreted as portfolios of the primary assets
- For example, $\mathcal{A}'_j$ may represent a stock whereas $\mathcal{A}_i$ may be an index

Impact of the reparametrization on the asset universe

1. We consider a portfolio $x = (x_1, \ldots, x_n)$ defined with respect to the synthetic assets. We have:

$$\mathcal{RC}_i = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

2. We also define the portfolio with respect to the primary assets. In this case, the composition is $y = (y_1, \ldots, y_m)$ where $y_j = \sum_{i=1}^n x_i w_{i,j}$ (or $y = W^\top x$). We have:

$$\mathcal{RC}_j = y_j \cdot \frac{(\Omega y)_j}{\sqrt{y^\top \Omega y}}$$

Impact of the reparametrization on the asset universe

Example 13

We have six primary assets. The volatility of these assets is respectively 20%, 30%, 25%, 15%, 10% and 30%. We assume that the assets are not correlated. We consider two equally weighted synthetic assets with:

$$W = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & & \\ & & 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

Impact of the reparametrization on the asset universe

Table 11: Risk decomposition of Portfolio #1 with respect to the synthetic assets (Example 13)

Asset i	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
$\mathcal{A}_1$	36.00	9.44	3.40	33.33
$\mathcal{A}_2$	38.00	8.90	3.38	33.17
$\mathcal{A}_3$	26.00	13.13	3.41	33.50

Table 12: Risk decomposition of Portfolio #1 with respect to the primary assets (Example 13)

Asset j	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{A}'_1$	9.00	3.53	0.32	3.12
$\mathcal{A}'_2$	9.00	7.95	0.72	7.02
$\mathcal{A}'_3$	31.50	19.31	6.08	59.69
$\mathcal{A}'_4$	31.50	6.95	2.19	21.49
$\mathcal{A}'_5$	9.50	0.93	0.09	0.87
$\mathcal{A}'_6$	9.50	8.39	0.80	7.82

Impact of the reparametrization on the asset universe

Table 13: Risk decomposition of Portfolio #2 with respect to the synthetic assets (Example 13)

Asset i	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
$\mathcal{A}_1$	48.00	9.84	4.73	49.91
$\mathcal{A}_2$	50.00	9.03	4.51	47.67
$\mathcal{A}_3$	2.00	11.45	0.23	2.42

Table 14: Risk decomposition of Portfolio #2 with respect to the primary assets (Example 13)

Asset j	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{A}_1'$	12.00	5.07	0.61	6.43
$\mathcal{A}_2'$	12.00	11.41	1.37	14.46
$\mathcal{A}_3'$	25.50	16.84	4.29	45.35
$\mathcal{A}_4'$	25.50	6.06	1.55	16.33
$\mathcal{A}_5'$	12.50	1.32	0.17	1.74
$\mathcal{A}_6'$	12.50	11.88	1.49	15.69

Impact of the reparametrization on the asset universe

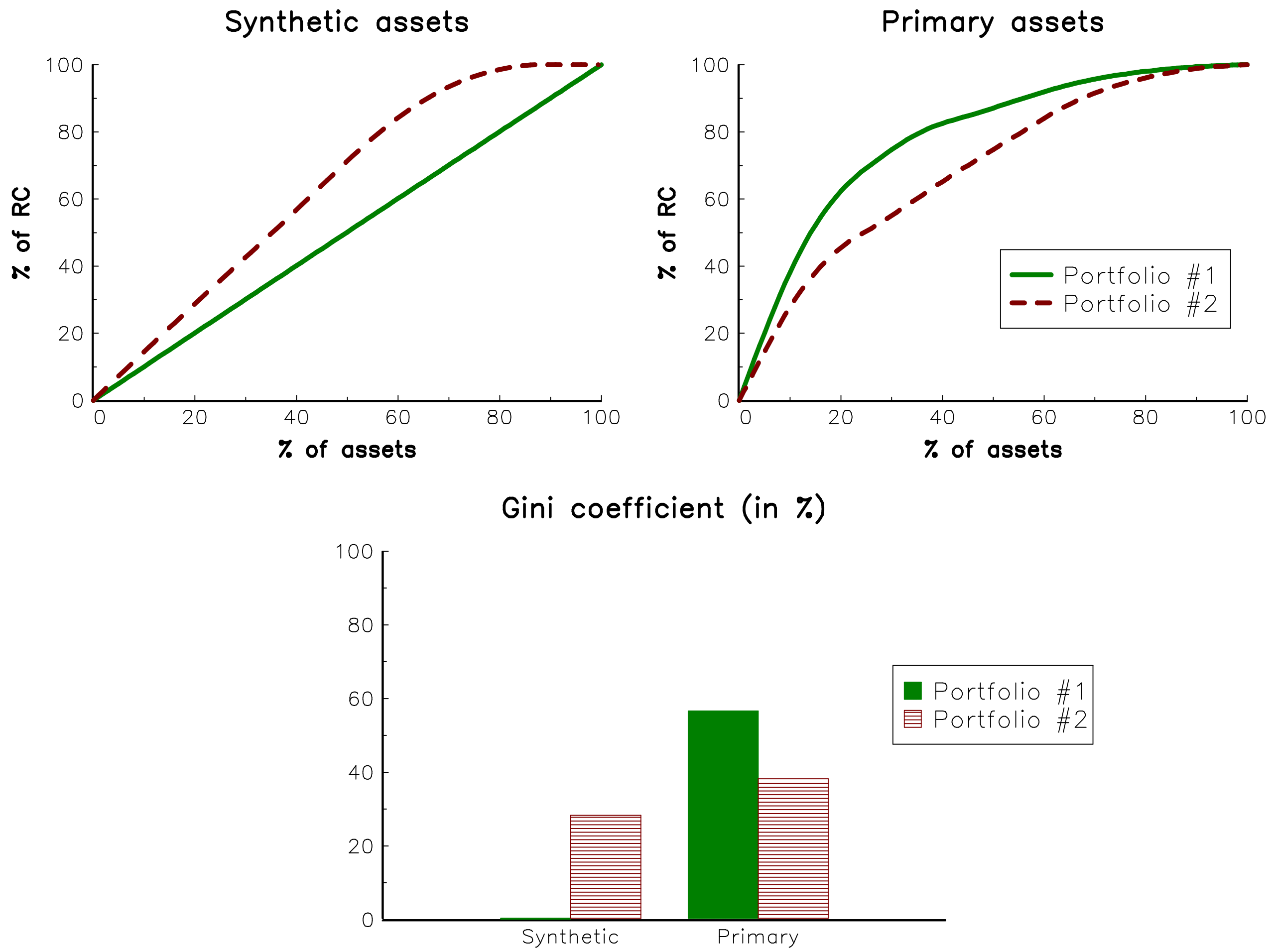


Figure 8: Lorenz curve of risk contributions (Example 13)

Risk decomposition with respect to the risk factors

- We consider a set of n assets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and a set of m risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$
- R_t is the $(n \times 1)$ vector of asset returns at time t
- Σ is the covariance matrix of asset returns
- $\mathcal{F}_t$ is the $(m \times 1)$ vector of factor returns at time t
- Ω is the covariance matrix of factor returns

Risk decomposition with respect to the risk factors

Linear factor model

We consider the linear factor model:

$$R_t = A\mathcal{F}_t + \varepsilon_t$$

where $\mathcal{F}_t$ and ε_t are two uncorrelated random vectors, ε_t is a centered random vector $(n \times 1)$ of covariance D and A is the $(n \times m)$ loadings matrix

We have the following relationship:

$$\Sigma = A\Omega A^\top + D$$

Risk decomposition with respect to the risk factors

We decompose the portfolio's asset exposures x by the portfolio's risk factors exposures y in the following way:

$$x = B^{+} y + \tilde{B}^{+} \tilde{y}$$

where:

- B^{+} is the Moore-Penrose inverse of $A^{\top}$
- $\tilde{B}^{+}$ is any $n \times (n - m)$ matrix that spans the left nullspace of B^{+}
- $\tilde{y}$ corresponds to $n - m$ residual (or additional) factors that have no economic interpretation

It follows that:

$$\left\{ \begin{array}{l} y = A^{\top} x \\ \tilde{y} = \tilde{B} x \end{array} \right.$$

where $\tilde{B} = \ker \left(A^{\top} \right)^{\top}$

Risk decomposition with respect to the risk factors

Risk decomposition I

- We can show that the marginal risk of the j^{th} factor exposure is given by:

$$\mathcal{MR}\left(\mathcal{F}_j\right) = \frac{\partial\,\sigma\left(x\right)}{\partial\,y_j} = \frac{\left(A^{+}\Sigma x\right)_j}{\sigma\left(x\right)}$$

whereas its risk contribution is equal to:

$$\mathcal{RC}\left(\mathcal{F}_j\right) = y_j \frac{\partial\,\sigma\left(x\right)}{\partial\,y_j} = \frac{\left(A^{\top} x\right)_j \cdot \left(A^{+}\Sigma x\right)_j}{\sigma\left(x\right)}$$

Risk decomposition with respect to the risk factors

Risk decomposition II

- For the residual factors, we have:

$$\mathcal{MR}\left(\tilde{\mathcal{F}}_j\right) = \frac{\partial\,\sigma\left(x\right)}{\partial\,\tilde{y}_j} = \frac{\left(\tilde{B}\Sigma x\right)_j}{\sigma\left(x\right)}$$

and:

$$\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \tilde{y}_j \frac{\partial\,\sigma\left(x\right)}{\partial\,\tilde{y}_j} = \frac{\left(\tilde{B}x\right)_j \cdot \left(\tilde{B}\Sigma x\right)_j}{\sigma\left(x\right)}$$

Risk decomposition with respect to the risk factors

Remark

We can show that these risk contributions satisfy the allocation principle:

$$\sigma(x) = \sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) + \sum_{j=1}^{n-m} \mathcal{RC}(\tilde{\mathcal{F}}_j)$$

Risk decomposition with respect to the risk factors

Let $\mathrm{pinv}(C)$ and $\mathrm{null}(C)$ be the Moore-Penrose pseudo-inverse and the orthonormal basis for the right null space of C

1. Computation of A^{+}

$$A^{+} = \mathrm{pinv}(A) = \left(A^{\top}A\right)^{-1} A^{\top}$$

2. Computation of B

$$B = A^{\top}$$

3. Computation of B^{+}

$$B^{+} = \mathrm{pinv}(B) = B^{\top}\left(BB^{\top}\right)^{-1}$$

4. Computation of $\tilde{B}$

$$\tilde{B} = \mathrm{pinv}\left(\mathrm{null}\left({B^{+}}^{\top}\right)\right) \cdot \left(I_n - B^{+}A^{\top}\right)$$

Risk decomposition with respect to the risk factors

Remark

The previous results can be extended to other coherent and convex risk measures (Roncalli and Weisang, 2016)

Risk decomposition with respect to the risk factors

Example 14

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Risk decomposition with respect to the risk factors

The correlation matrix of asset returns is (in %):

$$\rho = \begin{pmatrix} 100.0 & & & \\ 69.0 & 100.0 & & \\ 79.5 & 76.4 & 100.0 & \\ 66.2 & 57.2 & 66.3 & 100.0 \end{pmatrix}$$

and their volatilities are respectively equal to 21.19%, 27.09%, 26.25% and 23.04%.

Risk decomposition with respect to the risk factors

We obtain that:

$$A^{+} = \begin{pmatrix} 1.260 & -0.383 & 1.037 & -1.196 \\ -3.253 & 2.435 & -1.657 & 2.797 \\ -0.835 & 0.208 & -1.130 & 2.348 \end{pmatrix}$$

and:

$$\tilde{B} = \begin{pmatrix} 0.533 & 0.452 & -0.692 & -0.183 \end{pmatrix}$$

Risk decomposition with respect to the risk factors

Table 15: Risk decomposition of the EW portfolio with respect to the assets (Example 14)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	25.00	18.81	4.70	21.97
2	25.00	23.72	5.93	27.71
3	25.00	24.24	6.06	28.32
4	25.00	18.83	4.71	22.00
Volatility			21.40	

Table 16: Risk decomposition of the EW portfolio with respect to the risk factors (Example 14)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	100.00	17.22	17.22	80.49
$\mathcal{F}_2$	22.50	9.07	2.04	9.53
$\mathcal{F}_3$	35.00	6.06	2.12	9.91
$\tilde{\mathcal{F}}_1$	2.75	0.52	0.01	0.07
Volatility			21.40	

Risk factor parity (or RFP) portfolios

RFP portfolios are defined by:

$$\mathcal{RC}\left(\mathcal{F}_j\right) = b_j \mathcal{R}\left(x\right)$$

They are computed using the following optimization problem:

$$\begin{aligned} \left(y^{\star}, \breve{y}^{\star}\right) & = & \arg\min \sum_{j=1}^{m} \left(\mathcal{RC}\left(\mathcal{F}_j\right) - b_j \mathcal{R}\left(x\right)\right)^2 \\ & & \text{u.c.} \quad \mathbf{1}_n^{\top} \left(B^{+} y + \tilde{B}^{+} \tilde{y}\right) = 1 \end{aligned}$$

Risk factor parity (or RFP) portfolios

Example 15

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%. We consider the following factor risk budgets:

$$b = (49\%, 25\%, 25\%)$$

Risk factor parity (or RFP) portfolios

Table 17: Risk decomposition of the RFP portfolio with respect to the risk factors (Example 15)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	93.38	11.16	10.42	49.00
$\mathcal{F}_2$	24.02	22.14	5.32	25.00
$\mathcal{F}_3$	39.67	13.41	5.32	25.00
$\tilde{\mathcal{F}}_1$	16.39	1.30	0.21	1.00
Volatility			21.27	

Table 18: Risk decomposition of the RFP portfolio with respect to the assets (Example 15)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	15.08	17.44	2.63	12.36
2	38.38	23.94	9.19	43.18
3	0.89	21.82	0.20	0.92
4	45.65	20.29	9.26	43.54
Volatility			21.27	

Minimizing the risk concentration between the risk factors

We now consider the following problem:

$$\mathcal{RC}\left(\mathcal{F}_j\right) \simeq \mathcal{RC}\left(\mathcal{F}_k\right)$$

$\Rightarrow$ The portfolios are computed by minimizing the risk concentration between the risk factors

Remark

We can use the Herfindahl index, the Gini index or the Shanon entropy

Minimizing the risk concentration between the risk factors

Example 16

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Minimizing the risk concentration between the risk factors

Table 19: Risk decomposition of the balanced RFP portfolio with respect to the risk factors (Example 16)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	91.97	7.91	7.28	33.26
$\mathcal{F}_2$	25.78	28.23	7.28	33.26
$\mathcal{F}_3$	42.22	17.24	7.28	33.26
$\tilde{\mathcal{F}}_1$	6.74	0.70	0.05	0.21
Volatility			21.88	

Table 20: Risk decomposition of the balanced RFP portfolio with respect to the assets (Example 16)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	0.30	16.11	0.05	0.22
2	39.37	23.13	9.11	41.63
3	0.31	20.93	0.07	0.30
4	60.01	21.09	12.66	57.85
Volatility			21.88	

Minimizing the risk concentration between the risk factors

We have $\mathcal{H}^{\star} = 0$, $\mathcal{G} = 0$ and $\mathcal{I}^{\star} = 3$

Minimizing the risk concentration between the risk factors

Table 21: Balanced RFP portfolios with $x_i \geq 10\%$ (Example 16)

Criterion	$\mathcal{H}(x)$	$\mathcal{G}(x)$	$\mathcal{I}(x)$
x_1	10.00	10.00	10.00
x_2	22.08	18.24	24.91
x_3	10.00	10.00	10.00
x_4	57.92	61.76	55.09
$\mathcal{H}^{\star}$	0.0436	0.0490	0.0453
$\mathcal{G}$	0.1570	0.1476	0.1639
$\mathcal{I}^{\star}$	2.8636	2.8416	2.8643

Justification of diversified funds

Investor Profiles

1. Conservative (low risk)
2. Moderate (medium risk)
3. Aggressive (high risk)

Fund Profiles

1. Defensive (20% equities and 80% bonds)
2. Balanced (50% equities and 50% bonds)
3. Dynamic (80% equities and 20% bonds)

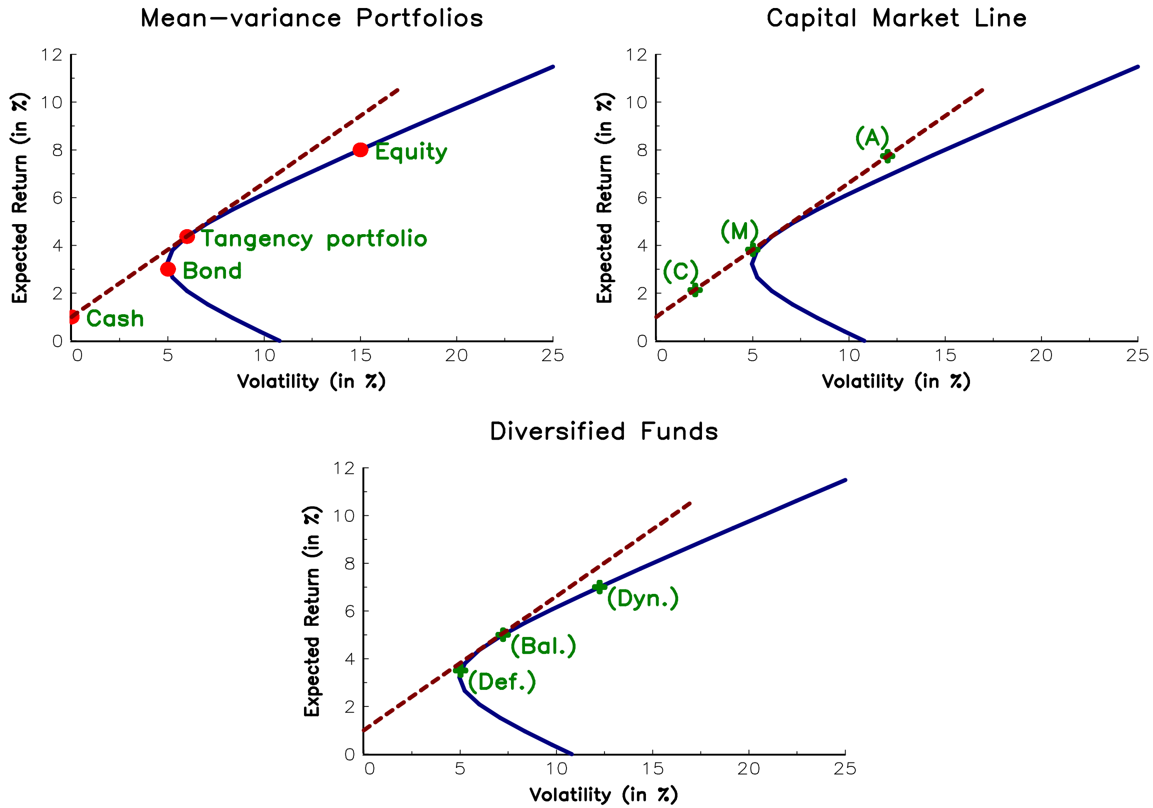


Figure 9: The asset allocation puzzle

What type of diversification is offered by diversified funds?

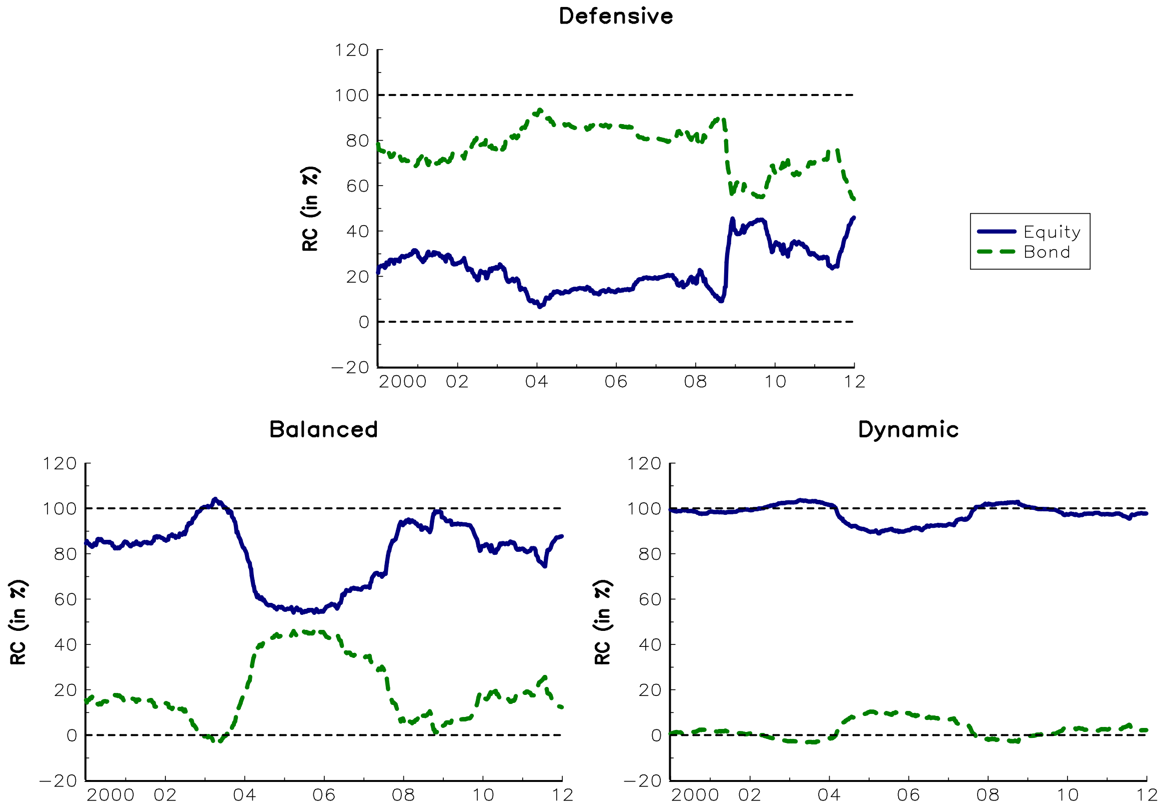


Figure 10: Equity (MSCI World) and bond (WGBI) risk contributions

Diversified funds
=
Marketing idea?

- Contrarian constant-mix strategy
- Deleverage of an equity exposure
- Low risk diversification
- No mapping between fund profiles and investor profiles
- Static weights
- Dynamic risk contributions

Risk-balanced allocation

- Multi-dimensional target volatility strategy
- Trend-following portfolio (if negative correlation between return and risk)
- Dynamic weights
- Static risk contributions (risk budgeting)
- High diversification

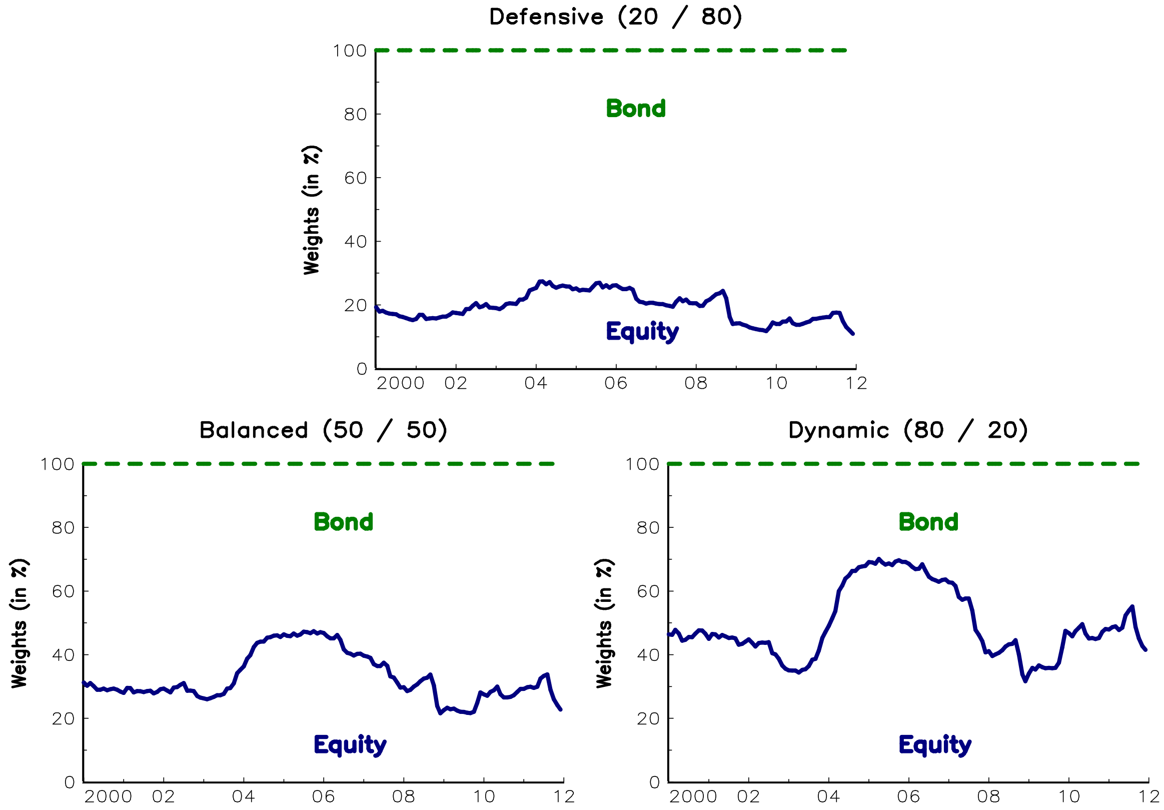


Figure 11: Equity and bond allocation

Characterization of the stock/bond market portfolio

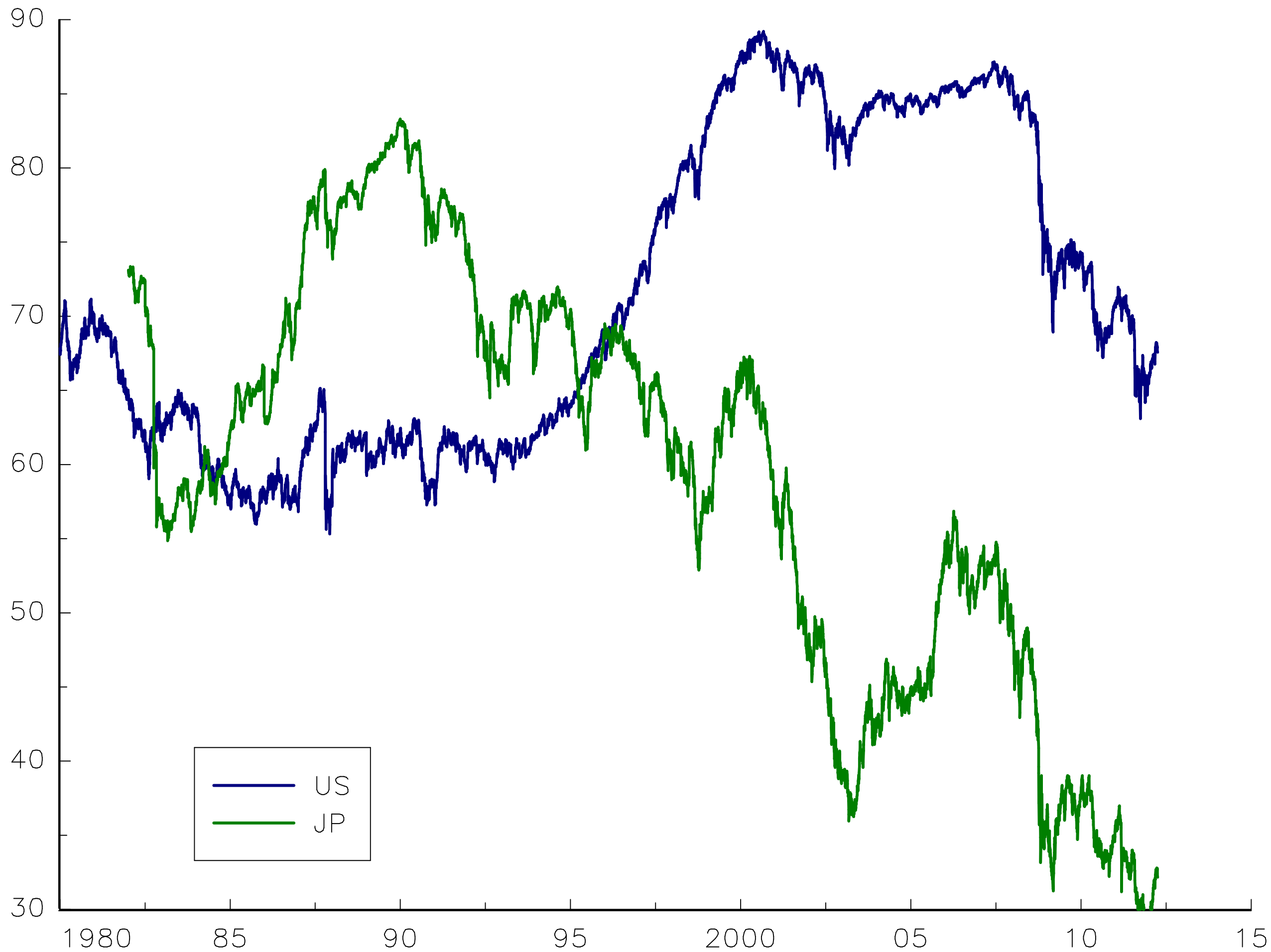


Figure 12: Evolution of the equity weight for United States and Japan

Characterization of the stock/bond market portfolio

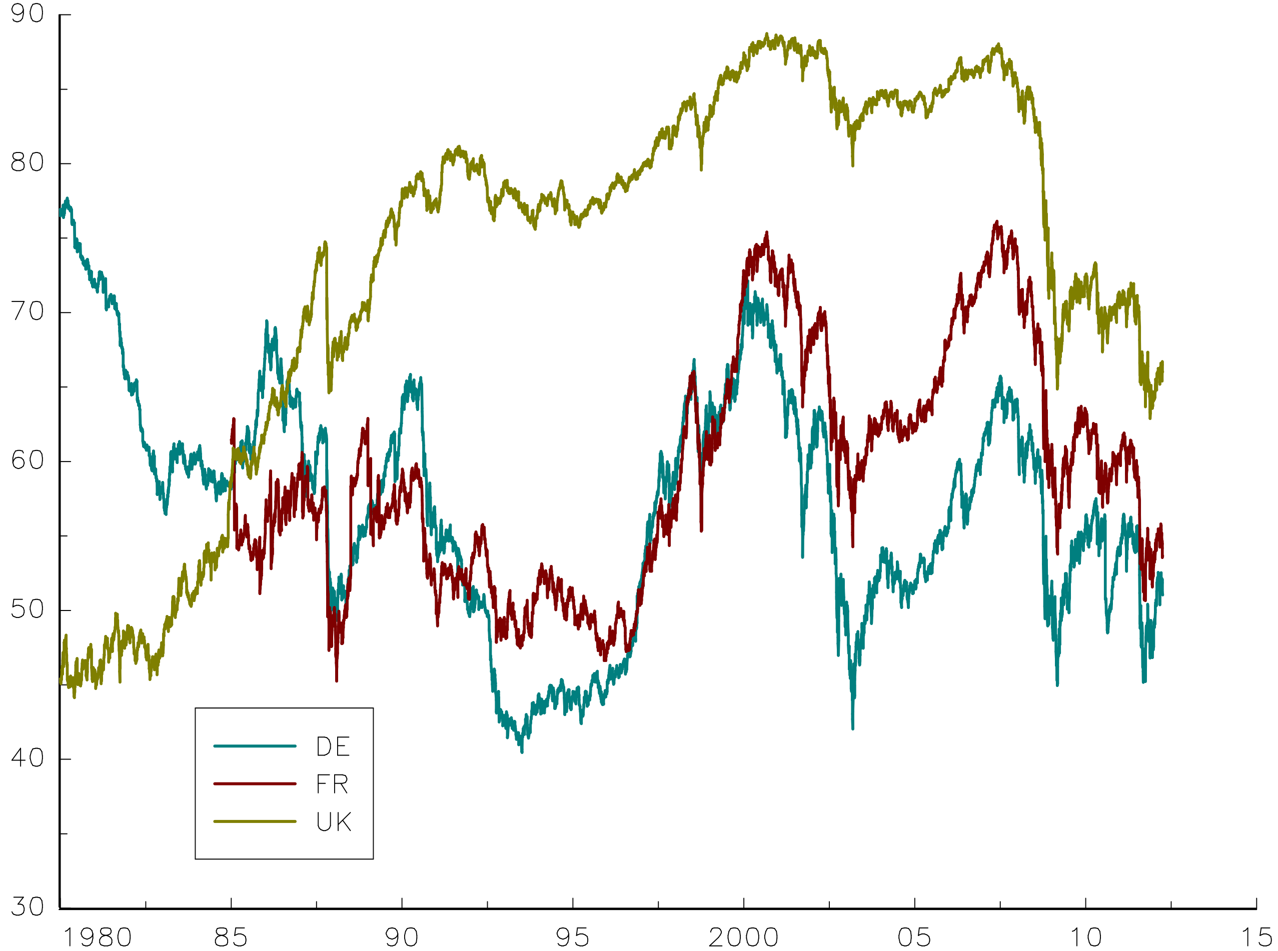


Figure 13: Evolution of the equity weight for Germany, France and UK

Link between risk premium and risk contribution

Let π_i and π_M be the risk premium of Asset i and the market risk premium. We have:

$$\begin{aligned}\pi_i &= \beta_i \cdot \pi_M \\ &= \frac{\operatorname{cov}(R_i, R_M)}{\sigma(R_M)} \cdot \frac{\pi_M}{\sigma(R_M)} \\ &= \frac{\partial\, \sigma(x_M)}{\partial\, x_i} \cdot \operatorname{SR}(x_M)\end{aligned}$$

The risk premium of Asset i is then proportional to the marginal volatility of Asset i with respect to the market portfolio

Foundation of the risk budgeting approach

For the tangency portfolio, we have:

performance contribution = risk contribution

Link between risk premium and risk contribution

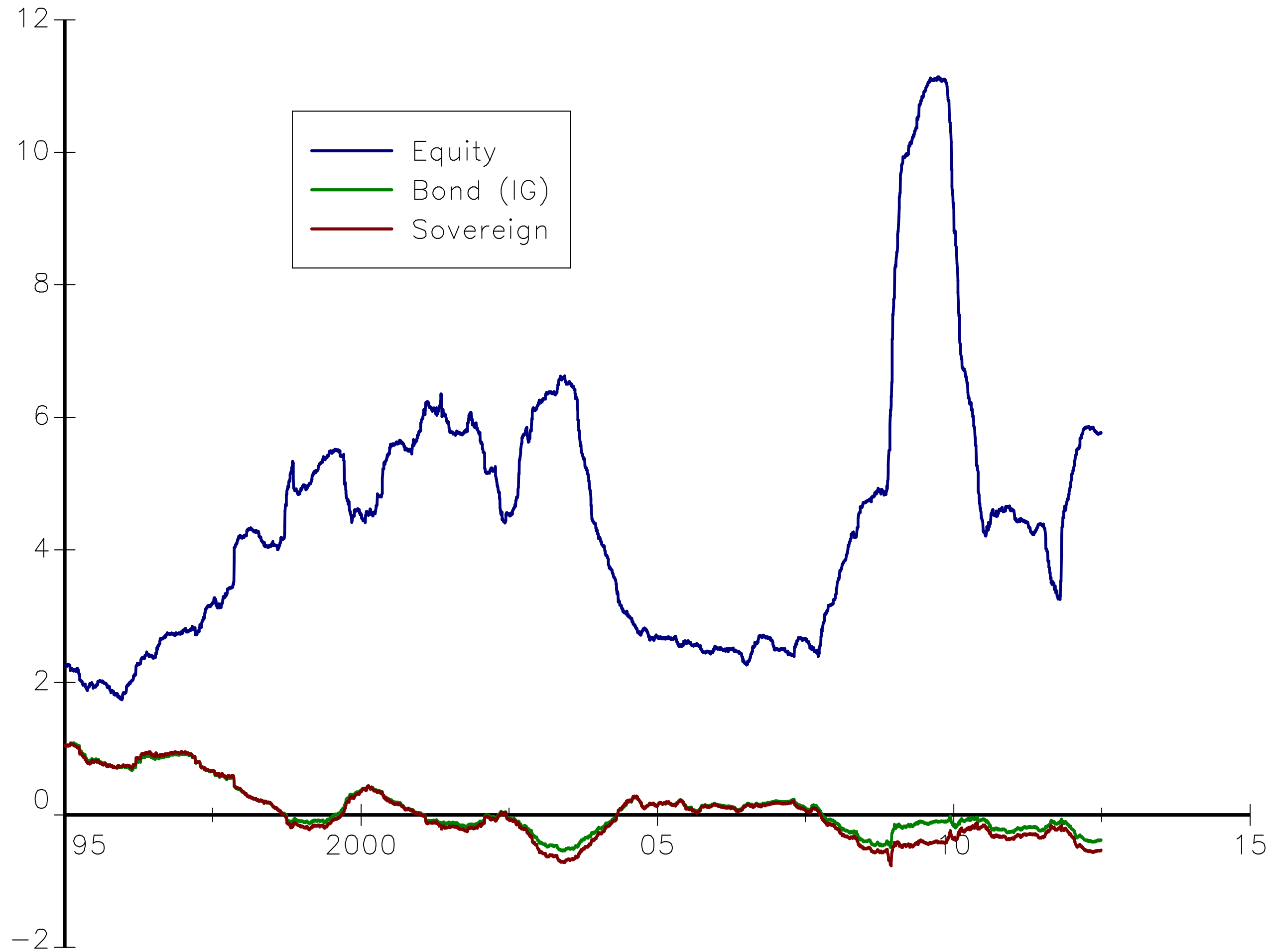


Figure 14: Risk premia (in %) for the US market portfolio ($\mathrm{SR}(x_M) = 25\%$)

Link between risk premium and risk contribution

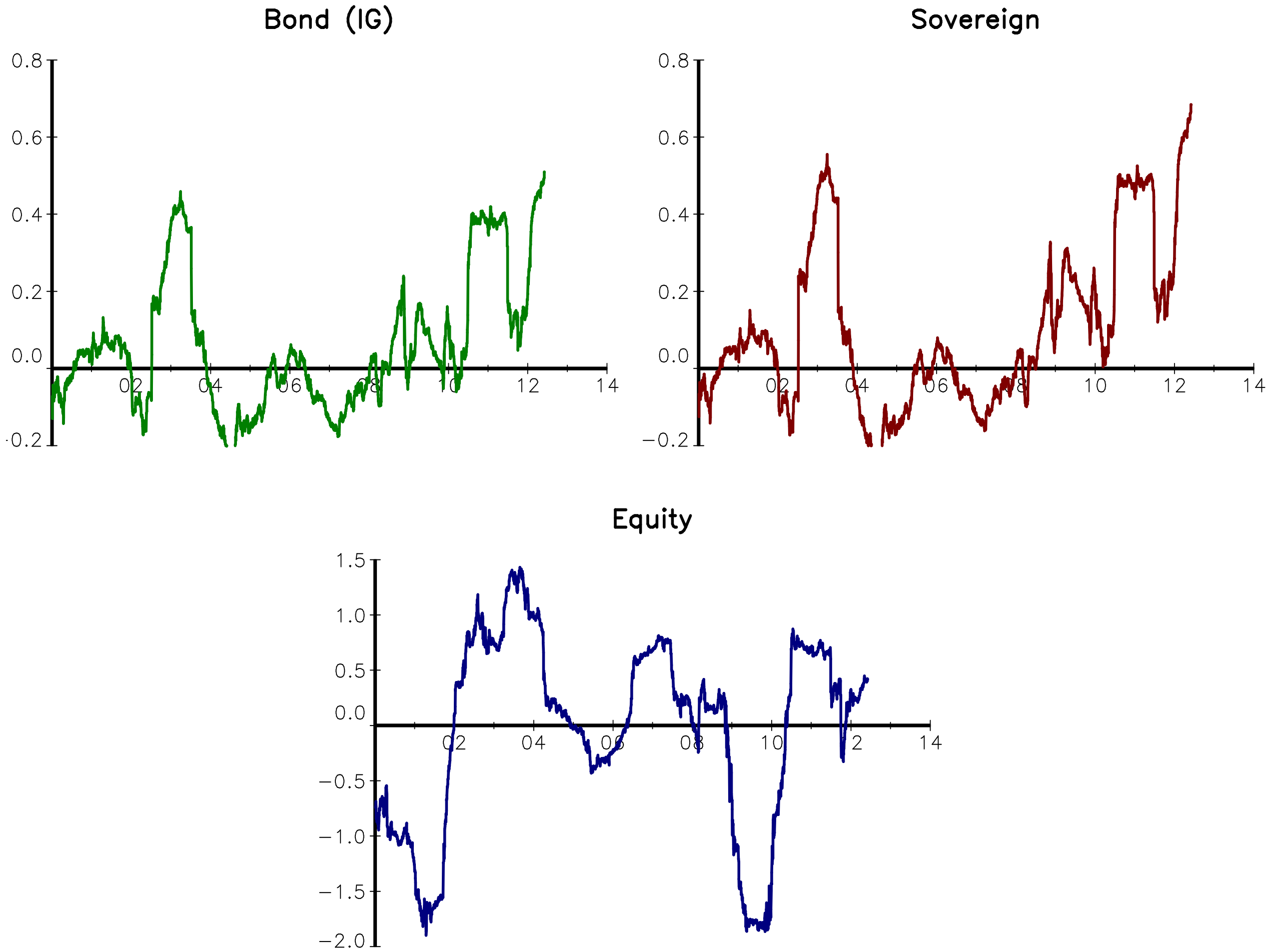


Figure 15: Difference (in %) between EURO and US risk premia $(\mathrm{SR}(x_M) = 25\%)$

Sharpe theory of risk premia

The one-factor risk model

We deduce that:

$$R_i - R_f = \alpha_i + \underbrace{\beta_i \cdot (R_M - R_f)}_{\text{Systematic Risk}} + \underbrace{\varepsilon_i}_{\text{Specific Risk}}$$

We necessarily have:

1. $\alpha_i = 0$
2. $\mathbb{E}[\varepsilon_i] = 0$

$\Rightarrow$ On average, only the systematic risk is rewarded, not the idiosyncratic risk

Sharpe theory of risk premia

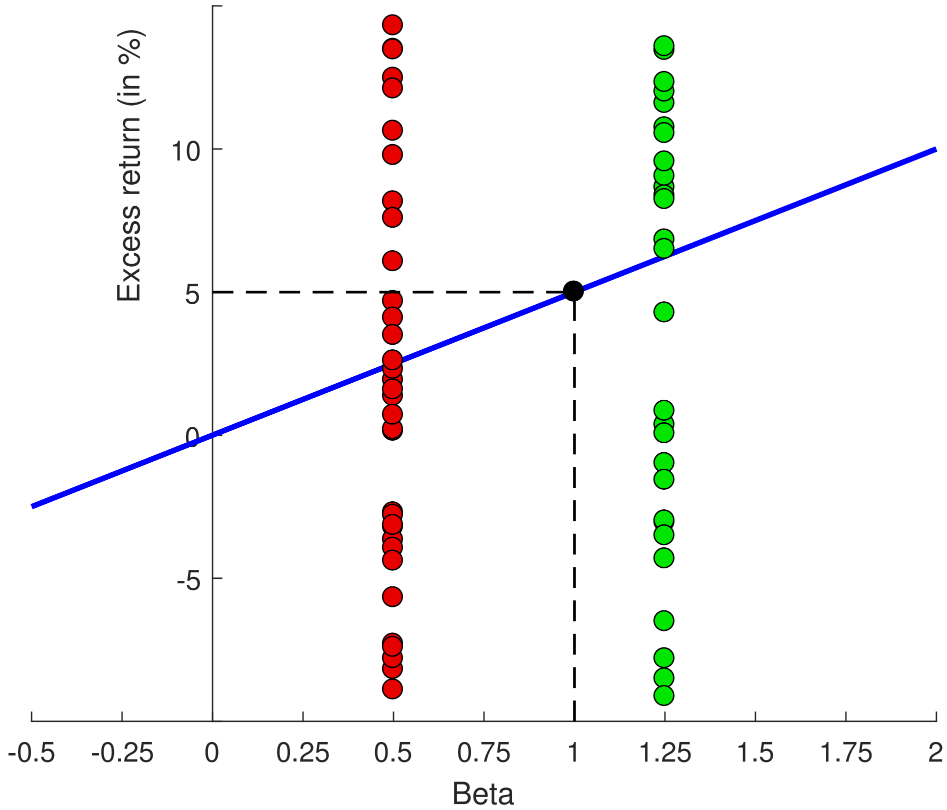


Figure 16: The security market line (SML)

- Risk premium is an increasing function of the systematic risk
- Risk premium may be negative (meaning that some assets can have a return lower than the risk-free asset!)
- More risk $\neq$ more return

Black-Litterman theory of risk premia

In the Black-Litterman model, the expected (or ex-ante/implied) risk premia are equal to:

$$\tilde{\pi} = \tilde{\mu} - r = \mathrm{SR}\left(x \mid r\right) \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the expected Sharpe ratio of the portfolio.

Black-Litterman theory of risk premia

Example 17

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

We also assume that the return of the risk-free asset is equal to 1.5%.

Black-Litterman theory of risk premia

Table 22: Black-Litterman risk premia (Example 17)

Asset	CAPM		Black-Litterman			
	π_i	$x_i^\star$	x_i	$\tilde{\pi}_i$	x_i	$\tilde{\pi}_i$
#1	3.50%	63.63%	25.00%	2.91%	40.00%	3.33%
#2	4.50%	19.27%	25.00%	4.71%	30.00%	4.97%
#3	6.50%	50.28%	25.00%	7.96%	20.00%	7.69%
#4	4.50%	−33.17%	25.00%	9.07%	10.00%	8.18%
$\mu(x)$	6.37%		6.25%		6.00%	
$\sigma(x)$	14.43%		18.27%		15.35%	
$\tilde{\mu}(x)$	6.37%		7.66%		6.68%	

Black-Litterman theory of risk premia

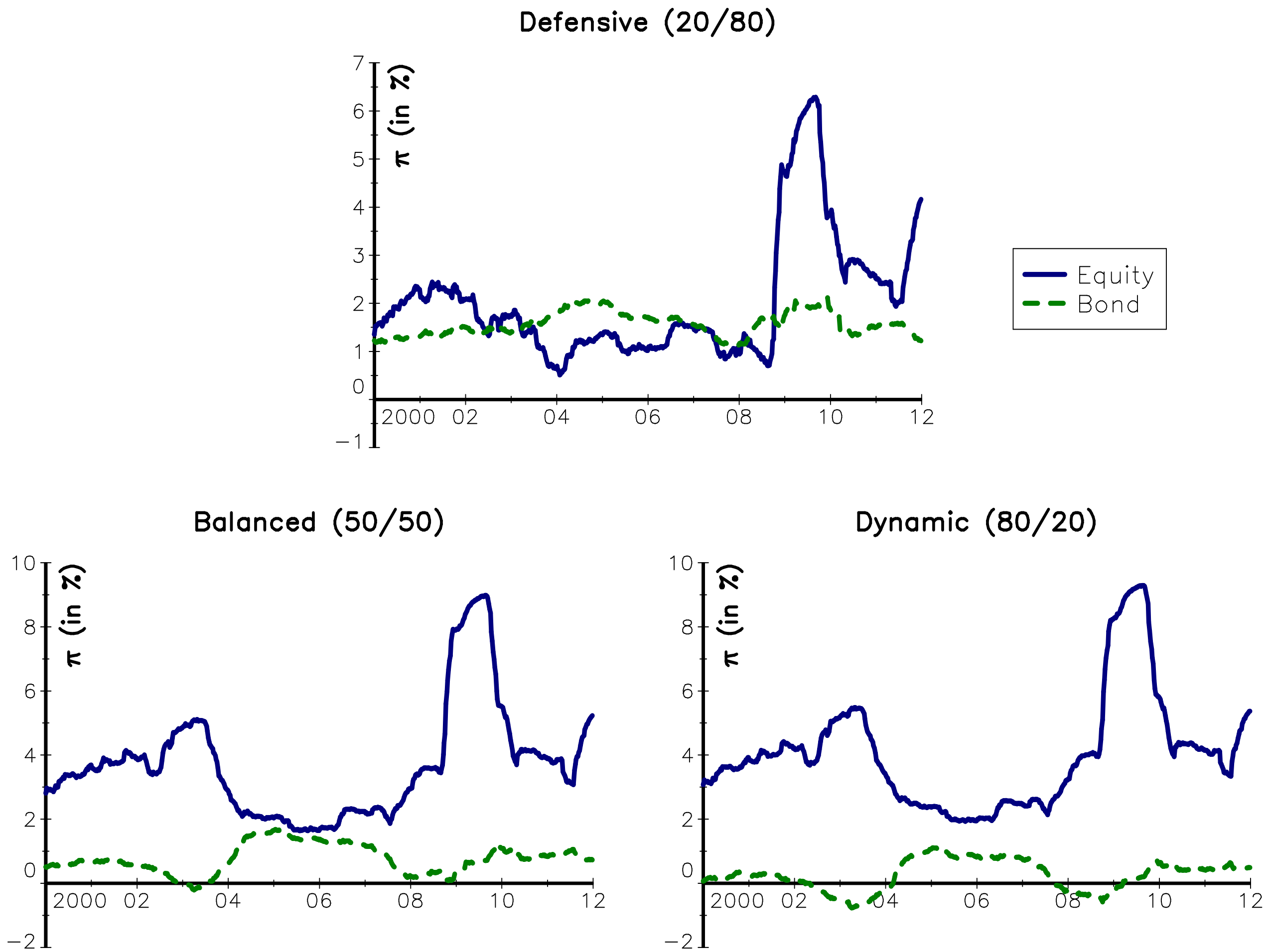


Figure 17: Equity and bond implied risk premia for diversified funds

Performance assets versus hedging assets

- We recall that:

$$\tilde{\pi} = \mathrm{SR}\left(x \mid r\right) \frac{\partial\, \sigma\left(x\right)}{\partial\, x}$$

  where $\sigma\left(x\right)$ is the volatility of portfolio x

- We have:

$$\begin{aligned}\frac{\partial\, \sigma\left(x\right)}{\partial\, x_i} &= \frac{\left(\Sigma x\right)_i}{\sigma\left(x\right)} \\ &= \frac{\left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right)}{\sigma\left(x\right)}\end{aligned}$$

- We deduce that

$$\tilde{\pi}_i = \mathrm{SR}\left(x \mid r\right) \frac{\left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right)}{\sigma\left(x\right)}$$

Performance assets versus hedging assets

In the two-asset case, we obtain:

$$\tilde{\pi}_1 = c(x) \left(\underbrace{x_1 \sigma_1^2}_{\text{variance}} + \underbrace{\rho \sigma_1 \sigma_2 (1 - x_1)}_{\text{covariance}} \right)$$

and:

$$\tilde{\pi}_2 = c(x) \left(\underbrace{x_2 \sigma_2^2}_{\text{variance}} + \underbrace{\rho \sigma_1 \sigma_2 (1 - x_2)}_{\text{covariance}} \right)$$

where $c(x)$ is equal to $\mathrm{SR}(x \mid r) / \sigma(x)$ and ρ is the cross-correlation between the two asset returns

Performance assets versus hedging assets

In the two-asset case, the implied risk premium becomes:

$$\tilde{\pi}_i = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma\left(x\right)} \left(\underbrace{x_i \cdot \sigma_i^2}_{\text{variance}} + \underbrace{\left(1 - x_i\right) \cdot \rho \sigma_i \sigma_j}_{\text{covariance}} \right)$$

There are two components in the risk premium:

- a variance risk component, which is an increasing function of the volatility and the weight of the asset
- a (positive or negative) covariance risk component, which depends on the correlation between asset returns

Performance asset versus hedging asset

- When $\tilde{\pi}_i > 0$, the asset i is a performance asset for Portfolio x
- When $\tilde{\pi}_i < 0$, the asset i is a hedging asset for Portfolio x

Performance assets versus hedging assets

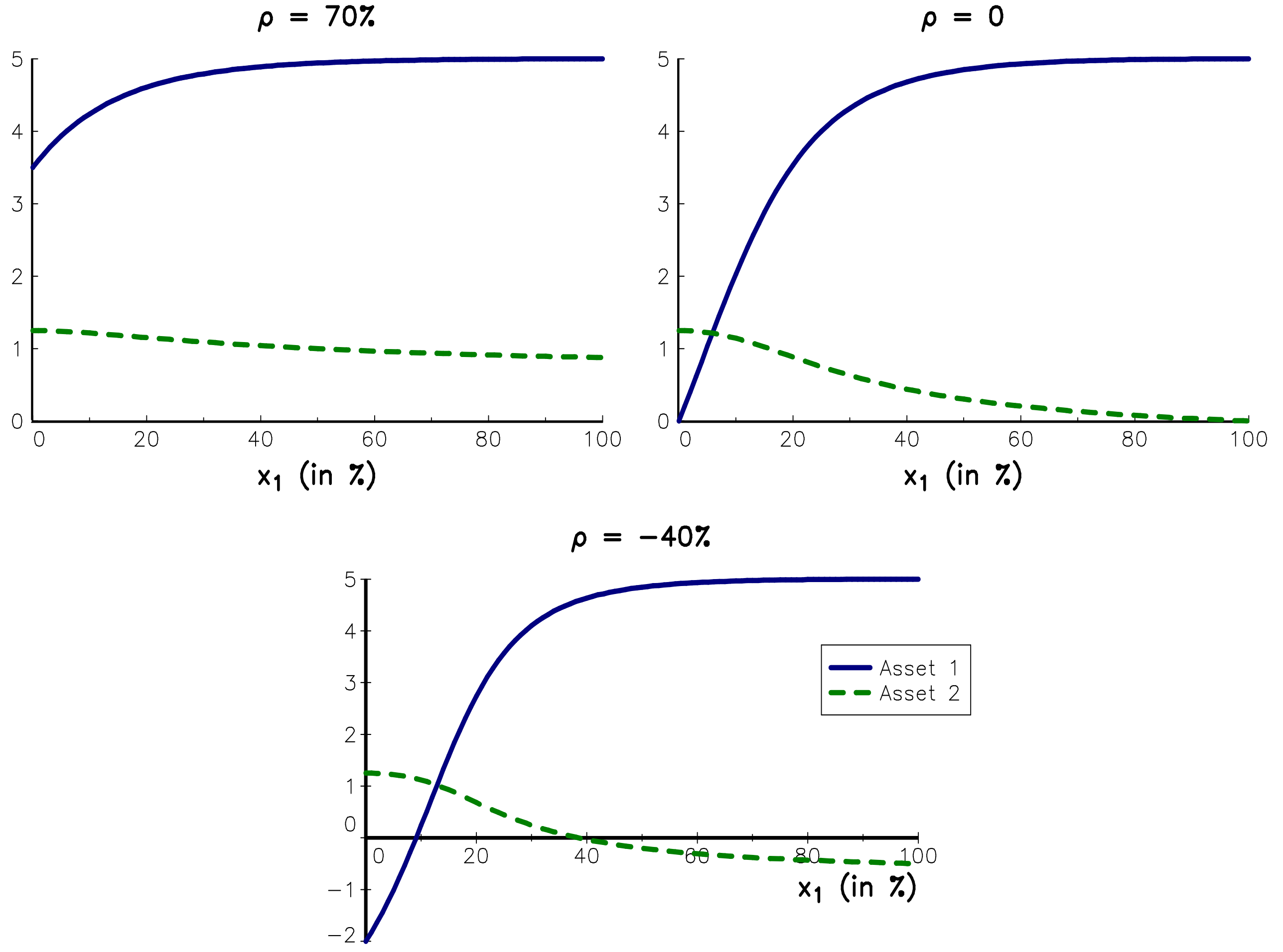


Figure 18: Impact of the correlation on the expected risk premium ($\sigma_1 = 20\%$, $\sigma_2 = 5\%$ and $\mathrm{SR}(x) = 0.25$)

Are bonds performance or hedging assets?

- Stocks are always considered as performance assets, while bonds may be performance or hedging assets, depending on the region and the period[4]
- 1990-2008: (Sovereign) bonds were perceived as performance assets
- The 2008 GFC has strengthened the fly-to-quality characteristic of bonds
- 2013-2017: Bonds are now more and more perceived as hedging assets[5]

Diversified stock-bond portfolios ⇒ Deleveraged equity portfolios

[4] For instance bonds were hedging assets in 2008 and performance assets in 2011

[5] This is particular true in the US and Europe, where the implied risk premium is negative. In Japan, the implied risk premium continue to be positive

Diversified versus risk parity funds

Table 23: Statistics of diversified and risk parity portfolios (2000-2012)

Portfolio	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2
Defensive	5.41	6.89	0.42	−17.23	0.19	2.67
Balanced	3.68	9.64	0.12	−33.18	−0.13	3.87
Dynamic	1.70	14.48	−0.06	−48.90	−0.18	5.96
Risk parity	5.12	7.29	0.36	−21.22	0.08	2.65
Static	4.71	7.64	0.29	−23.96	0.03	2.59
Leveraged RP	6.67	9.26	0.45	−23.74	0.01	0.78

- The 60/40 constant mix strategy is not the right benchmark
- Results depend on the investment universe (number/granularity of asset classes)
- What is the impact of rising interest rates?

Optimality of the RB portfolio

We consider the utility function:

$$\mathcal{U}(x) = (\mu(x) - r) - \phi \mathcal{R}(x)$$

Portfolio x is optimal if the vector of expected risk premia satisfies this relationship:

$$\tilde{\pi} = \phi \frac{\partial \mathcal{R}(x)}{\partial x}$$

If the RB portfolio is optimal, we deduce that the (excess) performance contribution $\mathcal{PC}_i$ of asset i is proportional to its risk budget:

$$\begin{aligned} \mathcal{PC}_i &= x_i \tilde{\pi}_i \\ &= \phi \cdot \mathcal{RC}_i \\ &\propto b_i \end{aligned}$$

Optimality of the RB portfolio

In the Black-Litterman approach of risk premia, we have:

$$\tilde{\pi}_i = \tilde{\mu}_i - r = \mathrm{SR}\left(x \mid r\right) \frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}}$$

This implies that the (excess) performance contribution is equal to:

$$\begin{aligned} \mathcal{PC}_i &= \mathrm{SR}\left(x \mid r\right) \frac{x_i \cdot \left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}} \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{RC}_i \end{aligned}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the expected Sharpe ratio of the RB portfolio

Optimality of the RB portfolio

Remark

From an ex-ante point of view, performance budgeting and risk budgeting are equivalent

Optimality of the RB portfolio

Example 18

We consider a universe of four assets. The volatilities are respectively 10%, 20%, 30% and 40%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.20 & 0.20 & 1.00 & \\ 0.20 & 0.20 & 0.50 & 1.00 \end{pmatrix}$$

The risk-free rate is equal to zero

Optimality of the RB portfolio

Table 24: Implied risk premia when $b = (20\%, 25\%, 40\%, 15\%)$ (Example 18)

Asset	x_i	$\mathcal{MR}_i$	$\tilde{\mu}_i$	$\mathcal{PC}_i$	$\mathcal{PC}_i^\star$
1	40.91	7.10	3.55	1.45	20.00
2	25.12	14.46	7.23	1.82	25.00
3	25.26	23.01	11.50	2.91	40.00
4	8.71	25.04	12.52	1.09	15.00
Expected return				7.27	

Table 25: Implied risk premia when $b = (10\%, 10\%, 10\%, 70\%)$ (Example 18)

Asset	x_i	$\mathcal{MR}_i$	$\tilde{\mu}_i$	$\mathcal{PC}_i$	$\mathcal{PC}_i^\star$
1	35.88	5.27	2.63	0.94	10.00
2	17.94	10.53	5.27	0.94	10.00
3	10.18	18.56	9.28	0.94	10.00
4	35.99	36.75	18.37	6.61	70.00
Expected return				9.45	

Main result

There is no neutral allocation. Every portfolio corresponds to an active bet.

Variation on the ERC portfolio

Question 1

We note Σ the covariance matrix of asset returns.

Variation on the ERC portfolio

Question 1.a

What is the risk contribution $\mathcal{RC}_i$ of asset i with respect to portfolio x?

Variation on the ERC portfolio

Let $\mathcal{R}(x)$ be a risk measure of the portfolio x. If this risk measure satisfies the Euler principle, we have (TR-RPB, page 78):

$$\mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial\, \mathcal{R}(x)}{\partial\, x_i}$$

We can then decompose the risk measure as a sum of asset contributions. This is why we define the risk contribution $\mathcal{RC}_i$ of asset i as the product of the weight by the marginal risk:

$$\mathcal{RC}_i = x_i \frac{\partial\, \mathcal{R}(x)}{\partial\, x_i}$$

When the risk measure is the volatility $\sigma(x)$, it follows that:

$$\begin{aligned} \mathcal{RC}_i &= x_i \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\ &= \frac{x_i \left(\sum_{k=1}^{n} \rho_{i,k}\sigma_i\sigma_k x_k\right)}{\sigma(x)} \end{aligned}$$

Variation on the ERC portfolio

Question 1.b

Define the ERC portfolio.

Variation on the ERC portfolio

The ERC portfolio corresponds to the risk budgeting portfolio when the risk measure is the return volatility $\sigma(x)$ and when the risk budgets are the same for all the assets (TR-RPB, page 119). It means that $\mathcal{RC}_i = \mathcal{RC}_j$, that is:

$$x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = x_j \frac{\partial\, \sigma(x)}{\partial\, x_j}$$

Variation on the ERC portfolio

Question 1.c

Calculate the variance of the risk contributions. Define an optimization program to compute the ERC portfolio. Find an equivalent maximization program based on the $\mathcal{L}^2$ norm.

Variation on the ERC portfolio

We have:

$$\begin{aligned}\overline{\mathcal{RC}} &= \frac{1}{n}\sum_{i=1}^{n}\mathcal{RC}_i \\ &= \frac{1}{n}\sigma(x)\end{aligned}$$

It follows that:

$$\begin{aligned}\operatorname{var}(\mathcal{RC}) &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{RC}_i - \overline{\mathcal{RC}}\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{RC}_i - \frac{1}{n}\sigma(x)\right)^2 \\ &= \frac{1}{n^2\sigma(x)}\sum_{i=1}^{n}\left(nx_i(\Sigma x)_i - \sigma^2(x)\right)^2\end{aligned}$$

Variation on the ERC portfolio

To compute the ERC portfolio, we may consider the following optimization program:

$$x^{\star} = \arg\min \sum_{i=1}^{n} \left(n x_i \left(\Sigma x \right)_i - \sigma^2 \left(x \right) \right)^2$$

Because we know that the ERC portfolio always exists (TR-RPB, page 108), the objective function at the optimum $x^{\star}$ is necessarily equal to 0. Another equivalent optimization program is to consider the L^2 norm. In this case, we have (TR-RPB, page 102):

$$x^{\star} = \arg\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i \cdot \left(\Sigma x \right)_i - x_j \cdot \left(\Sigma x \right)_j \right)^2$$

Variation on the ERC portfolio

Question 1.d

Let $\beta_i(x)$ be the beta of asset i with respect to portfolio x. Show that we have the following relationship in the ERC portfolio:

$$x_i \beta_i(x) = x_j \beta_j(x)$$

Propose a numerical algorithm to find the ERC portfolio.

Variation on the ERC portfolio

We have:

$$\begin{aligned}\beta_i(x) &= \frac{(\Sigma x)_i}{x^\top \Sigma x} \\ &= \frac{\mathcal{MR}_i}{\sigma(x)}\end{aligned}$$

We deduce that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \mathcal{MR}_i \\ &= x_i \beta_i(x)\,\sigma(x)\end{aligned}$$

The relationship $\mathcal{RC}_i = \mathcal{RC}_j$ becomes:

$$x_i \beta_i(x) = x_j \beta_j(x)$$

It means that the weight is inversely proportional to the beta:

$$x_i \propto \frac{1}{\beta_i(x)}$$

Variation on the ERC portfolio

We can use the Jacobi power algorithm (TR-RPB, page 308). Let $x^{(k)}$ be the portfolio at iteration k. We define the portfolio $x^{(k+1)}$ as follows:

$$x^{(k+1)} = \frac{\beta_i^{-1}\left(x^{(k)}\right)}{\sum_{j=1}^{n} \beta_j^{-1}\left(x^{(k)}\right)}$$

Starting from an initial portfolio $x^{(0)}$, the limit portfolio is the ERC portfolio if the algorithm converges:

$$\lim_{k \to \infty} x^{(k)} = x_{\mathrm{erc}}$$

Variation on the ERC portfolio

Question 1.e

We suppose that the volatilities are 15%, 20% and 25% and that the correlation matrix is:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 40\% & 30\% & 100\% \end{pmatrix}$$

Compute the ERC portfolio using the beta algorithm.

Variation on the ERC portfolio

Starting from the EW portfolio, we obtain for the first five iterations:

k	0	1	2	3	4	5
$x_1^{(k)}$ (in %)	33.3333	43.1487	40.4122	41.2314	40.9771	41.0617
$x_2^{(k)}$ (in %)	33.3333	32.3615	31.9164	32.3529	32.1104	32.2274
$x_3^{(k)}$ (in %)	33.3333	24.4898	27.6714	26.4157	26.9125	26.7109
$\beta_1\left(x^{(k)}\right)$	0.7326	0.8341	0.8046	0.8147	0.8113	0.8126
$\beta_2\left(x^{(k)}\right)$	0.9767	1.0561	1.0255	1.0397	1.0337	1.0363
$\beta_3\left(x^{(k)}\right)$	1.2907	1.2181	1.2559	1.2405	1.2472	1.2444

Variation on the ERC portfolio

The next iterations give the following results:

k	6	7	8	9	10	11
$x_1^{(k)}$ (in %)	41.0321	41.0430	41.0388	41.0405	41.0398	41.0401
$x_2^{(k)}$ (in %)	32.1746	32.1977	32.1878	32.1920	32.1902	32.1909
$x_3^{(k)}$ (in %)	26.7933	26.7593	26.7734	26.7676	26.7700	26.7690
$\beta_1\left(x^{(k)}\right)$	0.8121	0.8123	0.8122	0.8122	0.8122	0.8122
$\beta_2\left(x^{(k)}\right)$	1.0352	1.0356	1.0354	1.0355	1.0355	1.0355
$\beta_3\left(x^{(k)}\right)$	1.2456	1.2451	1.2453	1.2452	1.2452	1.2452

Variation on the ERC portfolio

Finally, the algorithm converges after 14 iterations with the following stopping criteria:

$$\sup_i \left| x_i^{(k+1)} - x_i^{(k)} \right| \leq 10^{-6}$$

and we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	41.04%	12.12%	4.97%	33.33%
2	32.19%	15.45%	4.97%	33.33%
3	26.77%	18.58%	4.97%	33.33%

Variation on the ERC portfolio

Question 2

We now suppose that the return of asset i satisfies the CAPM model:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the idiosyncratic risk. We note $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\mathrm{var}(R_m) = \sigma_m^2$ and $\mathrm{cov}(\varepsilon) = D = \mathrm{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$.

Variation on the ERC portfolio

Question 2.a

Calculate the risk contribution $\mathcal{RC}_i$.

Variation on the ERC portfolio

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$$

We deduce that:

$$\begin{aligned} \mathcal{RC}_i &= \frac{x_i \left(\sum_{k=1}^n \beta_i \beta_k \sigma_m^2 x_k + \tilde{\sigma}_i^2 x_i\right)}{\tilde{\sigma}\left(x\right)} \\ &= \frac{x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2}{\sigma\left(x\right)} \end{aligned}$$

with:

$$B = \sum_{k=1}^n x_k \beta_k \sigma_m^2$$

Variation on the ERC portfolio

Question 2.b

We assume that $\beta_i = \beta_j$. Show that the ERC weight x_i is a decreasing function of the idiosyncratic volatility $\tilde{\sigma}_i$.

Variation on the ERC portfolio

Using Equation 2.a, we deduce that the ERC portfolio satisfies:

$$x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2 = x_j \beta_j B + x_j^2 \tilde{\sigma}_j^2$$

or:

$$\left(x_i \beta_i - x_j \beta_j\right) B = \left(x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2\right)$$

Variation on the ERC portfolio

If $\beta_i = \beta_j = \beta$, we have:

$$(x_i - x_j)\,\beta B = \left(x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2\right)$$

Because $\beta > 0$, we deduce that:

$$\begin{array}{rcl} x_i > x_j & \Leftrightarrow & x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2 > 0 \\ & \Leftrightarrow & x_j \tilde{\sigma}_j > x_i \tilde{\sigma}_i \\ & \Leftrightarrow & \tilde{\sigma}_i < \tilde{\sigma}_j \end{array}$$

We conclude that the weight x_i is a decreasing function of the specific volatility $\tilde{\sigma}_i$.

Variation on the ERC portfolio

Question 2.c

We assume that $\tilde{\sigma}_i = \tilde{\sigma}_j$. Show that the ERC weight x_i is a decreasing function of the sensitivity β_i to the common factor.

Variation on the ERC portfolio

If $\tilde{\sigma}_i = \tilde{\sigma}_j = \tilde{\sigma}$, we have:

$$\left(x_i \beta_i - x_j \beta_j\right) B = \left(x_j^2 - x_i^2\right) \tilde{\sigma}^2$$

We deduce that:

$$\begin{array}{rcl} x_i > x_j & \Leftrightarrow & \left(x_i \beta_i - x_j \beta_j\right) B < 0 \\ & \Leftrightarrow & x_i \beta_i < x_j \beta_j \\ & \Leftrightarrow & \beta_i < \beta_j \end{array}$$

We conclude that the weight x_i is a decreasing function of the sensitivity β_i.

Variation on the ERC portfolio

Question 2.d

We consider the numerical application: $\beta_1 = 1$, $\beta_2 = 0.9$, $\beta_3 = 0.8$, $\beta_4 = 0.7$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$, $\tilde{\sigma}_3 = 10\%$, $\tilde{\sigma}_4 = 10\%$, and $\sigma_m = 20\%$. Find the ERC portfolio.

Variation on the ERC portfolio

We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	21.92%	19.73%	4.32%	25.00%
2	24.26%	17.83%	4.32%	25.00%
3	25.43%	17.00%	4.32%	25.00%
4	28.39%	15.23%	4.32%	25.00%

Weight concentration of a portfolio

Question 1

We consider the Lorenz curve defined by:

$$\begin{array}{rcl} [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & \mathbb{L}(x) \end{array}$$

We assume that $\mathbb{L}$ is an increasing function and $\mathbb{L}(x) > x$.

Weight concentration of a portfolio

Question 1.a

Represent graphically the function $\mathbb{L}$ and define the Gini coefficient $\mathcal{G}$ associated with $\mathbb{L}$.

Weight concentration of a portfolio

We have represented the function $y = \mathcal{L}(x)$ in Figure 19. It verifies $\mathcal{L}(x) \geq x$ and $\mathcal{L}(x) \leq 1$. The Gini coefficient is defined as follows (TR-RPB, page 127):

$$
\begin{aligned}
G &= \frac{A}{A+B} \\
&= \left(\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - \frac{1}{2} \right) \Big/ \frac{1}{2} \\
&= 2 \int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1
\end{aligned}
$$

Weight concentration of a portfolio

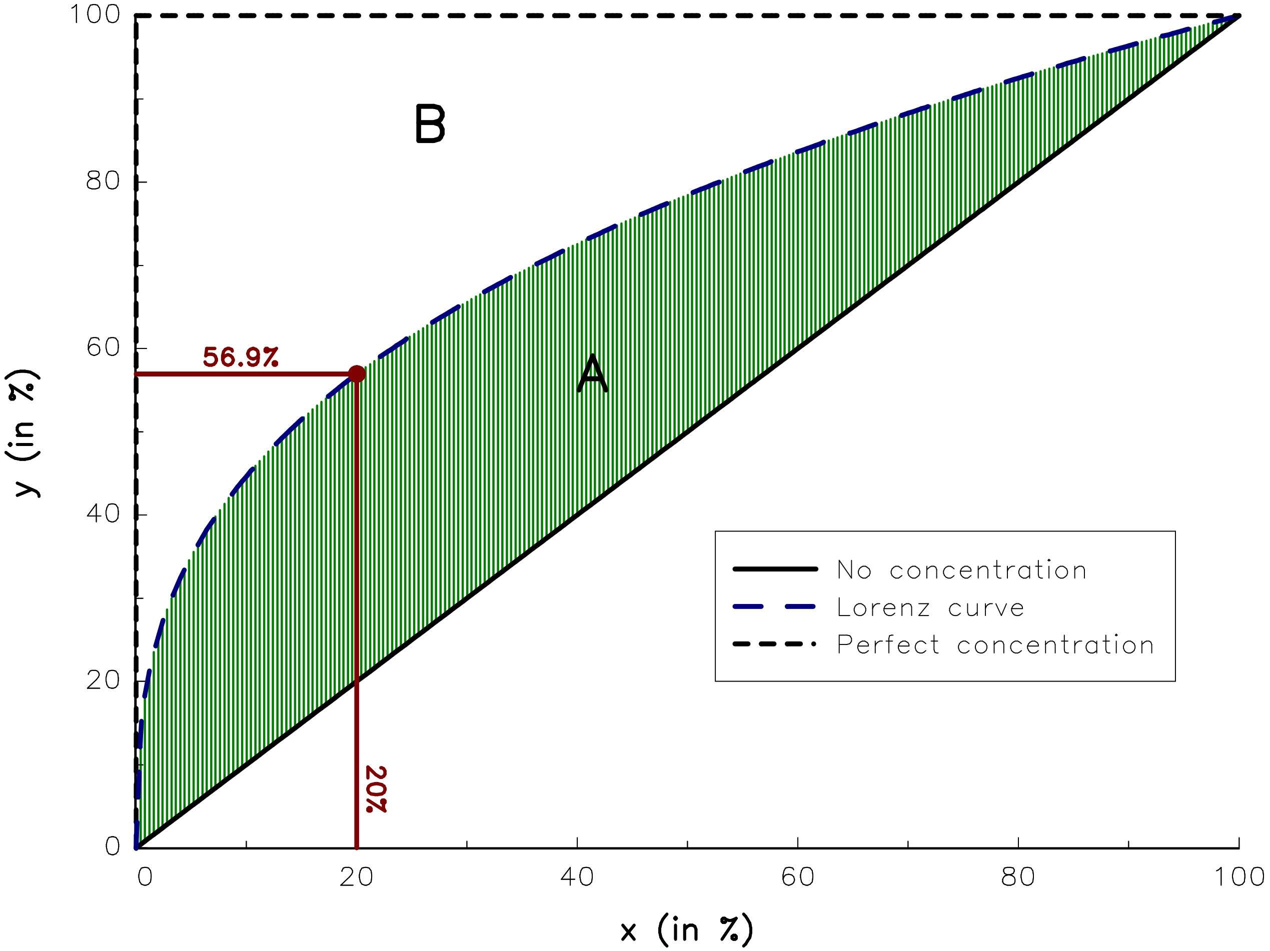


Figure 19: Lorenz curve

Weight concentration of a portfolio

Question 1.b

We set $\mathbb{L}_\alpha(x) = x^\alpha$ with $\alpha \geq 0$. Is the function $\mathbb{L}_\alpha$ a Lorenz curve? Calculate the Gini coefficient $\mathcal{G}(\alpha)$ with respect to α. Deduce $\mathcal{G}(0)$, $\mathcal{G}\left(\frac{1}{2}\right)$ and $\mathcal{G}(1)$.

Weight concentration of a portfolio

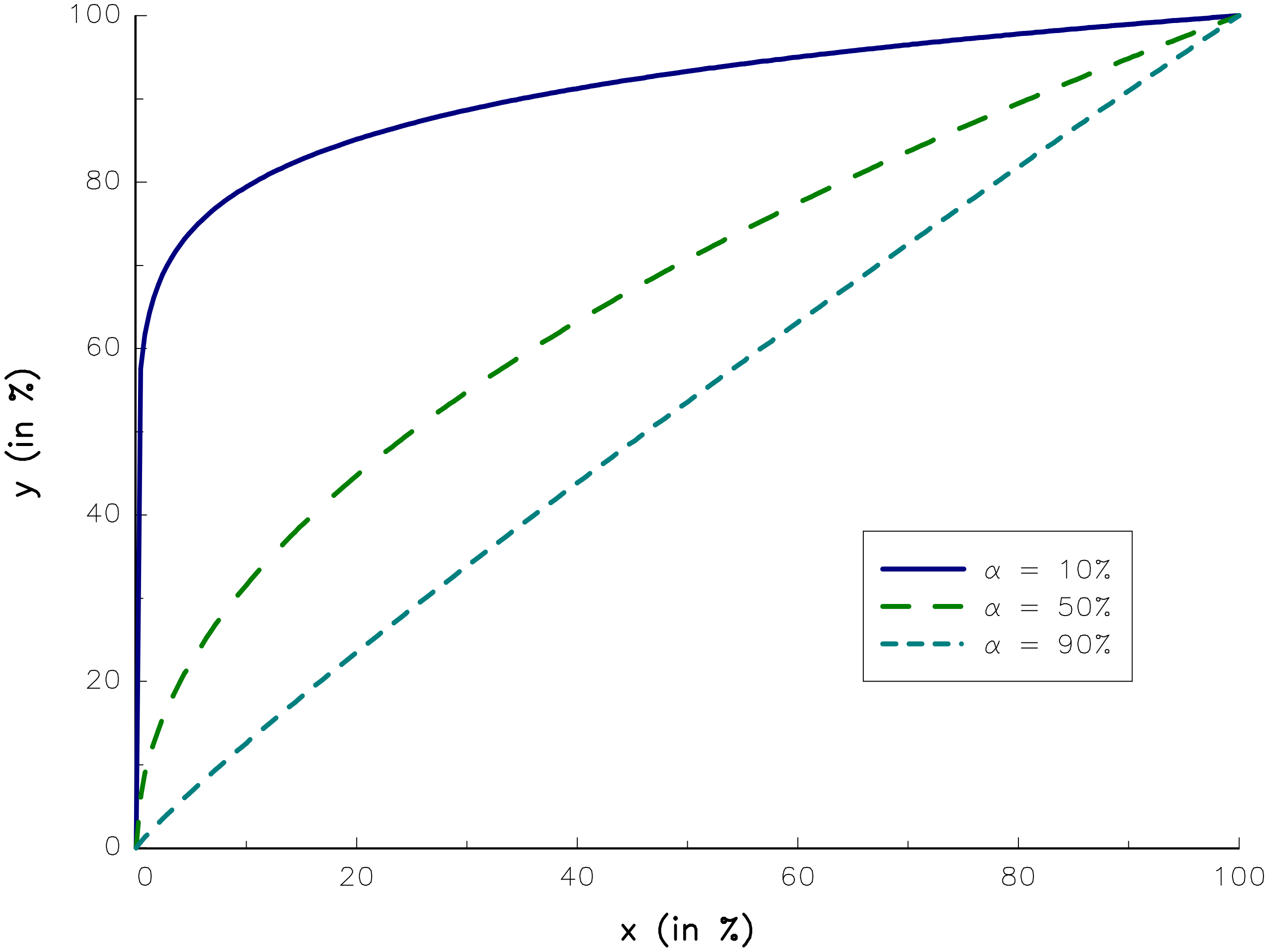


Figure 20: Function $y = x^{\alpha}$

Weight concentration of a portfolio

If $\alpha \geq 0$, the function $\mathcal{L}_\alpha(x) = x^\alpha$ is increasing. We have $\mathcal{L}_\alpha(1) = 1$, $\mathcal{L}_\alpha(x) \leq 1$ and $\mathcal{L}_\alpha(x) \geq x$. We deduce that $\mathcal{L}_\alpha$ is a Lorenz curve. For the Gini index, we have:

$$\begin{aligned} \mathcal{G}(\alpha) &= 2\int_0^1 x^\alpha \,\mathrm{d}x - 1 \\ &= 2\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 - 1 \\ &= \frac{1-\alpha}{1+\alpha} \end{aligned}$$

We deduce that $\mathcal{G}(0) = 1$, $\mathcal{G}\left(\frac{1}{2}\right) = 1/3$ et $\mathcal{G}(1) = 0$. $\alpha = 0$ corresponds to the perfect concentration whereas $\alpha = 1$ corresponds to the perfect equality.

Weight concentration of a portfolio

Question 2

Let w be a portfolio of n assets. We suppose that the weights are sorted in a descending order: $w_1 \geq w_2 \geq \ldots \geq w_n$.

Weight concentration of a portfolio

Question 2.a

We define $\mathbb{L}_w(x)$ as follows:

$$\mathbb{L}_w(x) = \sum_{j=1}^{i} w_j \quad \text{if} \quad \frac{i}{n} \leq x < \frac{i+1}{n}$$

with $\mathbb{L}_w(0) = 0$. Is the function $\mathbb{L}_w$ a Lorenz curve ? Calculate the Gini coefficient with respect to the weights w_i. In which cases does $\mathcal{G}$ take the values 0 and 1?

Weight concentration of a portfolio

We have $\mathcal{L}_w(0) = 0$ and $\mathcal{L}_w(1) = \sum_{j=1}^{n} w_j = 1$. If $x_2 \geq x_1$, we have $\mathcal{L}_w(x_2) \geq \mathcal{L}_w(x_2)$. $\mathcal{L}_w$ is then a Lorenz curve. The Gini coefficient is equal to:

$$\begin{aligned}\mathcal{G} &= 2\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1 \\ &= \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} w_j - 1\end{aligned}$$

If $w_j = n^{-1}$, we have:

$$\begin{aligned}\lim_{n\to\infty}\mathcal{G} &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\frac{i}{n} - 1 \\ &= \lim_{n\to\infty}\frac{2}{n}\cdot\frac{n(n+1)}{2n} - 1 \\ &= \lim_{n\to\infty}\frac{1}{n} = 0\end{aligned}$$

Weight concentration of a portfolio

If $w_1 = 1$, we have:

$$\begin{aligned} \lim_{n \to \infty} \mathcal{G} &= \lim_{n \to \infty} 1 - \frac{1}{n} \\ &= 1 \end{aligned}$$

We note that the perfect equality does not correspond to the case $\mathcal{G} = 0$ except in the asymptotic case. This is why we may slightly modify the definition of $\mathcal{L}_w(x)$:

$$\mathcal{L}_w(x) = \begin{cases} \sum_{j=1}^{i} w_j & \text{if} \quad x = n^{-1} i \\ \sum_{j=1}^{i} w_j + w_{i+1}(nx - i) & \text{if} \quad n^{-1} i < x < n^{-1}(i+1) \end{cases}$$

While the previous definition corresponds to a constant piecewise function, this one defines an affine piecewise function. In this case, the computation of the Gini index is done using a trapezoidal integration:

$$\mathcal{G} = \frac{2}{n} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{i} w_j + \frac{1}{2} \right) - 1$$

Weight concentration of a portfolio

Question 2.b

The definition of the Herfindahl index is:

$$\mathcal{H} = \sum_{i=1}^{n} w_i^2$$

In which cases does $\mathcal{H}$ take the value 1? Show that $\mathcal{H}$ reaches its maximum when $w_i = n^{-1}$. What is the interpretation of this result?

Weight concentration of a portfolio

The Herfindahl index is equal to 1 if the portfolio is concentrated in only one asset. We seek to minimize $\mathcal{H} = \sum_{i=1}^n w_i^2$ under the constraint $\sum_{i=1}^n w_i = 1$. The Lagrange function is then:

$$f\left(w_1, \ldots, w_n; \lambda\right) = \sum_{i=1}^n w_i^2 - \lambda \left(\sum_{i=1}^n w_i - 1\right)$$

The first-order conditions are $2w_i - \lambda = 0$. We deduce that $w_i = w_j$. $\mathcal{H}$ reaches its minimum when $w_i = n^{-1}$. It corresponds to the equally weighted portfolio. In this case, we have:

$$\mathcal{H} = \frac{1}{n}$$

Weight concentration of a portfolio

Question 2.c

We set $\mathcal{N} = \mathcal{H}^{-1}$. What does the statistic $\mathcal{N}$ mean?

Weight concentration of a portfolio

The statistic $\mathcal{N}$ is the degree of freedom or the equivalent number of equally weighted assets. For instance, if $\mathcal{H} = 0.5$, then $\mathcal{N} = 2$. It is a portfolio equivalent to two equally weighted assets.

Weight concentration of a portfolio

Question 3

We consider an investment universe of five assets. We assume that their asset returns are not correlated. The volatilities are given in the table below:

σ_i	2%	5%	10%	20%	30%
$w_i^{(1)}$		10%	20%	30%	40%
$w_i^{(2)}$	40%	20%		30%	10%
$w_i^{(3)}$	20%	15%	25%	35%	5%

Weight concentration of a portfolio

Question 3.a

Find the minimum variance portfolio $w^{(4)}$.

Weight concentration of a portfolio

The minimum variance portfolio is equal to:

$$w^{(4)} = \begin{pmatrix} 82.342\% \\ 13.175\% \\ 3.294\% \\ 0.823\% \\ 0.366\% \end{pmatrix}$$

Weight concentration of a portfolio

Question 3.b

Calculate the Gini and Herfindahl indices and the statistic $\mathcal{N}$ for the four portfolios $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ and $w^{(4)}$.

Weight concentration of a portfolio

For each portfolio, we sort the weights in descending order. For the portfolio $w^{(1)}$, we have $w_1^{(1)} = 40\%$, $w_2^{(1)} = 30\%$, $w_3^{(1)} = 20\%$, $w_4^{(1)} = 10\%$ and $w_5^{(1)} = 0\%$. It follows that:

$$\begin{aligned}\mathcal{H}\left(w^{(1)}\right) &= \sum_{i=1}^{5}\left(w_i^{(1)}\right)^2 \\ &= 0.10^2 + 0.20^2 + 0.30^2 + 0.40^2 \\ &= 0.30\end{aligned}$$

We also have:

$$\begin{aligned}\mathcal{G}\left(w^{(1)}\right) &= \frac{2}{5}\left(\sum_{i=1}^{4}\sum_{j=1}^{i}\tilde{w}_j^{(1)} + \frac{1}{2}\right) - 1 \\ &= \frac{2}{5}\left(0.40 + 0.70 + 0.90 + 1.00 + \frac{1}{2}\right) - 1 \\ &= 0.40\end{aligned}$$

Weight concentration of a portfolio

For the portfolios $w^{(2)}$, $w^{(3)}$ and $w^{(4)}$, we obtain $\mathcal{H}\left(w^{(2)}\right) = 0.30$, $\mathcal{H}\left(w^{(3)}\right) = 0.25$, $\mathcal{H}\left(w^{(4)}\right) = 0.70$, $\mathcal{G}\left(w^{(2)}\right) = 0.40$, $\mathcal{G}\left(w^{(3)}\right) = 0.28$ and $\mathcal{G}\left(w^{(4)}\right) = 0.71$. We have $\mathcal{N}\left(w^{(2)}\right) = \mathcal{N}\left(w^{(1)}\right) = 3.33$, $\mathcal{N}\left(w^{(3)}\right) = 4.00$ and $\mathcal{N}\left(w^{(4)}\right) = 1.44$.

Weight concentration of a portfolio

Question 3.c

Comment on these results. What differences do you make between portfolio concentration and portfolio diversification?

Weight concentration of a portfolio

All the statistics show that the least concentrated portfolio is $w^{(3)}$. The most concentrated portfolio is paradoxically the minimum variance portfolio $w^{(4)}$. We generally assimilate variance optimization to diversification optimization. We show in this example that diversifying in the Markowitz sense does not permit to minimize the concentration.

The optimization problem of the ERC portfolio

Question 1

We consider four assets. Their volatilities are equal to 10%, 15%, 20% and 25% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 100\% & & & \\ 60\% & 100\% & & \\ 40\% & 40\% & 100\% & \\ 30\% & 30\% & 20\% & 100\% \end{pmatrix}$$

The optimization problem of the ERC portfolio

Question 1.a

Find the long-only minimum variance, ERC and equally weighted portfolios.

The optimization problem of the ERC portfolio

The weights of the three portfolios are:

Asset	MV	ERC	EW
1	87.51%	37.01%	25.00%
2	4.05%	24.68%	25.00%
3	4.81%	20.65%	25.00%
4	3.64%	17.66%	25.00%

The optimization problem of the ERC portfolio

Question 1.b

We consider the following portfolio optimization problem:

$$\begin{array}{rcl} x^{\star}(c) & = & \arg\min \sqrt{x^{\top}\Sigma x} \\ & \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}_n^{\top} x = 1 \\ x \geq \mathbf{0}_n \end{array} \right. \end{array} \tag{1}$$

with Σ the covariance matrix of asset returns. We note λ_c and λ_0 the Lagrange coefficients associated with the constraints $\sum_{i=1}^{n} \ln x_i \geq c$ and $\mathbf{1}_n^{\top} x = 1$. Write the Lagrange function of the optimization problem. Deduce then an equivalent optimization problem that is easier to solve than Problem (1).

The optimization problem of the ERC portfolio

The Lagrange function is:

$$\begin{aligned}\mathcal{L}(x;\lambda,\lambda_0,\lambda_c) &= \sqrt{x^\top \Sigma x} - \lambda^\top x - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right) - \lambda_c \left(\sum_{i=1}^n \ln x_i - c\right) \\ &= \left(\sqrt{x^\top \Sigma x} - \lambda_c \sum_{i=1}^n \ln x_i\right) - \lambda^\top x - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right) + \lambda_c c\end{aligned}$$

We deduce that an equivalent optimization problem is:

$$\begin{aligned}\tilde{x}^\star(\lambda_c) &= \arg\min \sqrt{\tilde{x}^\top \Sigma \tilde{x}} - \lambda_c \sum_{i=1}^n \ln \tilde{x}_i \\ &\text{u.c.} \quad \left\{\begin{array}{l}\mathbf{1}_n^\top \tilde{x} = 1 \\ \tilde{x} \geq \mathbf{0}_n\end{array}\right.\end{aligned}$$

The optimization problem of the ERC portfolio

We notice a strong difference between the two problems because they don't use the same control variable. However, the control variable c of the first problem may be deduced from the solution of the second problem:

$$c = \sum_{i=1}^{n} \ln \tilde{x}_i^\star \left(\lambda_c\right)$$

We also know that (TR-RPB, page 131):

$$c_- \leq \sum_{i=1}^{n} \ln x_i \leq c_+$$

where $c_- = \sum_{i=1}^{n} \ln \left(x_{\mathrm{mv}}\right)_i$ and $c_+ = -n \ln n$. It follows that:

$$\left\{ \begin{array}{ll} x^\star \left(c\right) = \tilde{x}^\star \left(0\right) & \text{if } c \leq c_- \\ x^\star \left(c\right) = \tilde{x}^\star \left(\infty\right) & \text{if } c \geq c_+ \end{array} \right.$$

If $c \in \left]c_-, c_+\right[$, there exists a scalar $\lambda_c > 0$ such that:

$$x^\star \left(c\right) = \tilde{x}^\star \left(\lambda_c\right)$$

The optimization problem of the ERC portfolio

Question 1.c

Represent the relationship between λ_c and $\sigma(x^\star(c))$, c and $\sigma(x^\star(c))$ and $\mathcal{I}^\star(x^\star(c))$ and $\sigma(x^\star(c))$ where $\mathcal{I}^\star(x)$ is the diversity index of the weights.

The optimization problem of the ERC portfolio

For a given value $\lambda_c \in [0, +\infty[$, we solve numerically the second problem and find the optimized portfolio $\tilde{x}^\star(\lambda_c)$. Then, we calculate $c = \sum_{i=1}^n \ln \tilde{x}_i^\star(\lambda_c)$ and deduce that $x^\star(c) = \tilde{x}^\star(\lambda_c)$. We finally obtain $\sigma(x^\star(c)) = \sigma(\tilde{x}^\star(\lambda_c))$ and $\mathcal{I}^\star(x^\star(c)) = \mathcal{I}^\star(\tilde{x}^\star(\lambda_c))$. The relationships between λ_c, c, $\mathcal{I}^\star(x^\star(c))$ and $\sigma(x^\star(c))$ are reported in Figure 21.

The optimization problem of the ERC portfolio

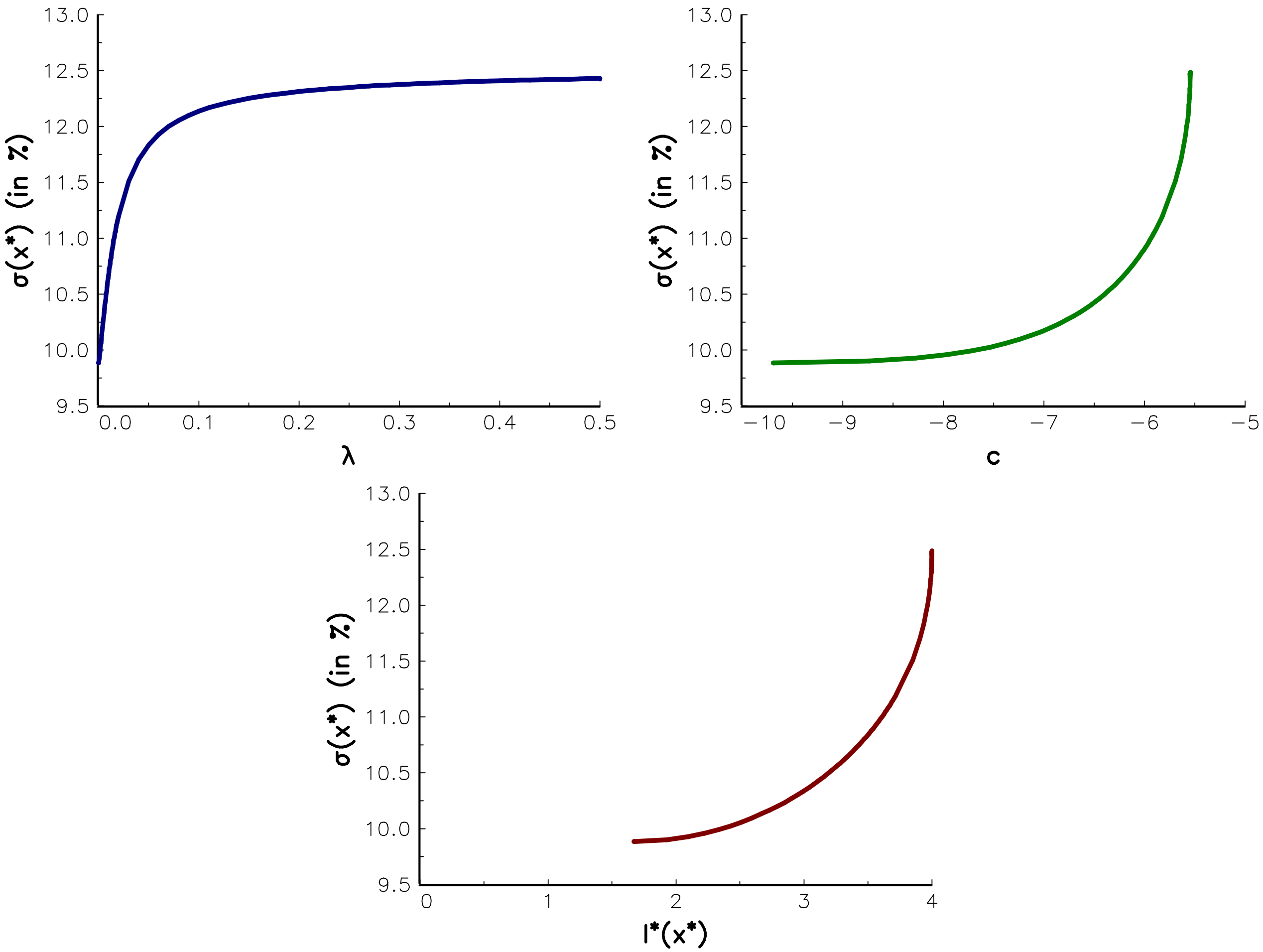


Figure 21: Relationship between λ_c, c, $\mathcal{I}^\star\left(x^\star\left(c\right)\right)$ and $\sigma\left(x^\star\left(c\right)\right)$

The optimization problem of the ERC portfolio

Question 1.d

Represent the relationship between λ_c and $\mathcal{I}^\star(\mathcal{RC})$, c and $\mathcal{I}^\star(\mathcal{RC})$ and $\mathcal{I}^\star(x^\star(c))$ and $\mathcal{I}^\star(\mathcal{RC})$ where $\mathcal{I}^\star(\mathcal{RC})$ is the diversity index of the risk contributions.

The optimization problem of the ERC portfolio

If we consider $\mathcal{I}^{\star}(\mathcal{RC})$ in place of $\sigma(x^{\star}(c))$, we obtain Figure 22.

The optimization problem of the ERC portfolio

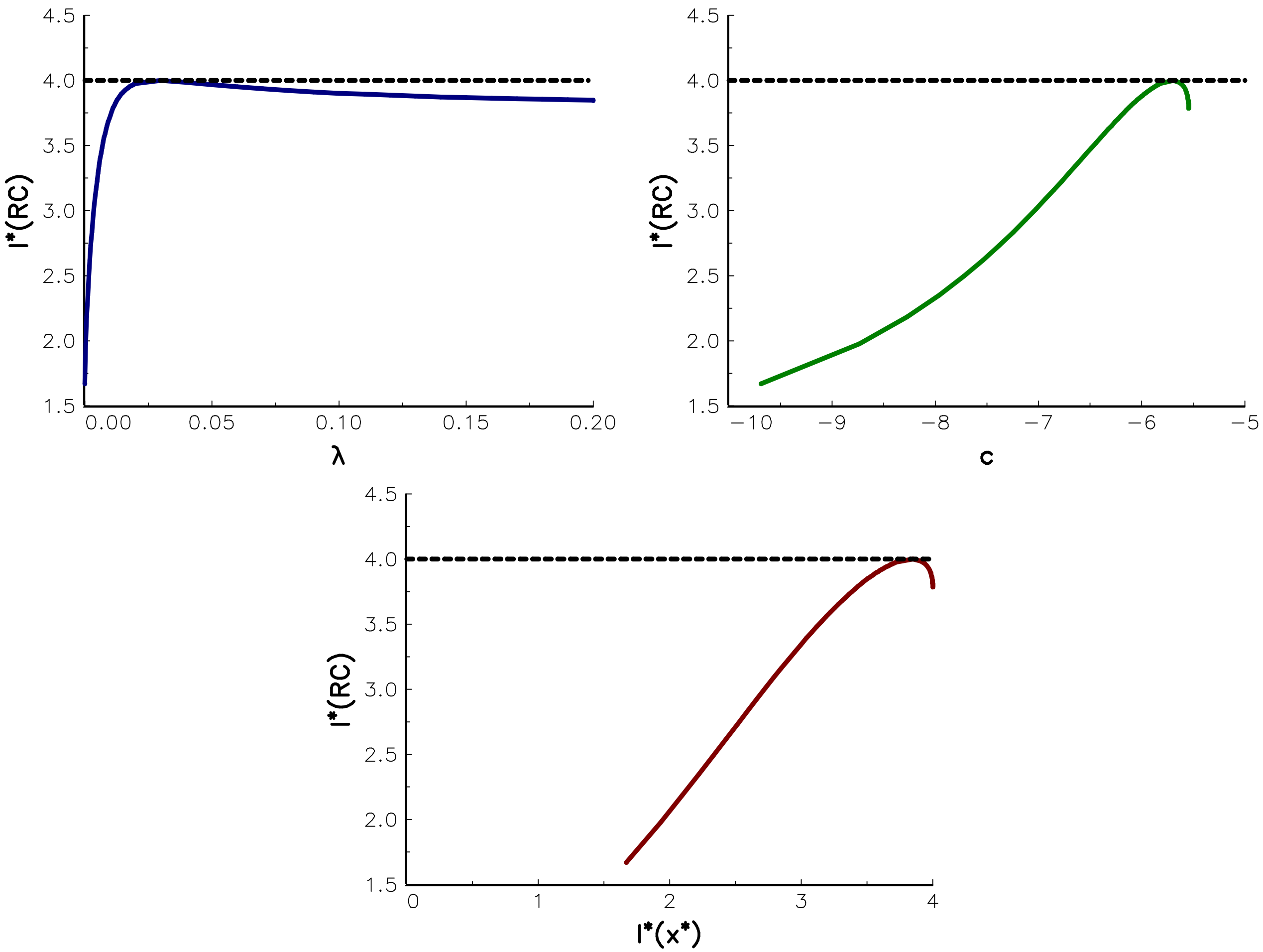


Figure 22: Relationship between λ_c, c, $\mathcal{I}^\star(x^\star(c))$ and $\mathcal{I}^\star(\mathcal{RC})$

The optimization problem of the ERC portfolio

Question 1.e

Draw the relationship between $\sigma\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$. Identify the ERC portfolio.

The optimization problem of the ERC portfolio

In Figure 23, we have reported the relationship between $\sigma\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$. The ERC portfolio satisfies the equation $\mathcal{I}^{\star}\left(\mathcal{RC}\right) = n$.

The optimization problem of the ERC portfolio

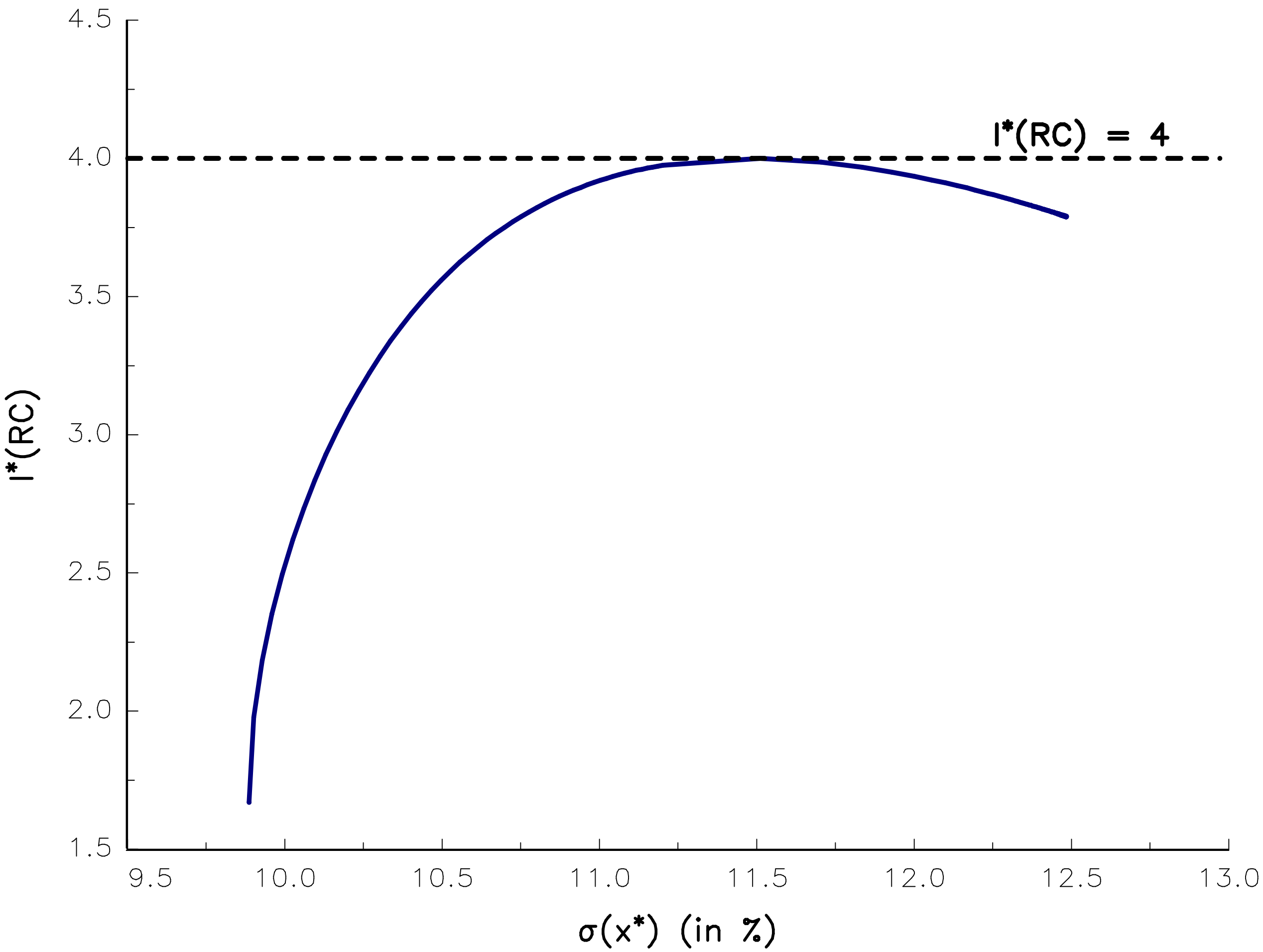


Figure 23: Relationship between $\sigma\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$

The optimization problem of the ERC portfolio

Question 2

We now consider a slight modification of the previous optimization problem:

$$
\begin{aligned}
x^{\star}(c) &= \arg\min \sqrt{x^{\top}\Sigma x} \\
\text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} \ln x_i \geq c \\ x \geq \mathbf{0}_n \end{array} \right.
\end{aligned}
\tag{2}
$$

The optimization problem of the ERC portfolio

Question 2.a

Why does the optimization problem (1) not define the ERC portfolio?

The optimization problem of the ERC portfolio

Let us consider the optimization problem when we impose the constraint $\mathbf{1}_n^\top x = 1$. The first-order condition is:

$$\frac{\partial\,\sigma(x)}{\partial\,x_i} - \lambda_i - \lambda_0 - \frac{\lambda_c}{x_i} = 0$$

Because $x_i > 0$, we deduce that $\lambda_i = 0$ and:

$$x_i \frac{\partial\,\sigma(x)}{\partial\,x_i} = \lambda_0 x_i + \lambda_c$$

If this solution corresponds to the ERC portfolio, we obtain:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \lambda_0 x_i + \lambda_c = \lambda_0 x_j + \lambda_c$$

If $\lambda_0 \neq 0$, we deduce that:

$$x_i = x_j$$

It corresponds to the EW portfolio meaning that the assumption $\mathcal{RC}_i = \mathcal{RC}_j$ is false.

The optimization problem of the ERC portfolio

Question 2.b

Find the optimized portfolio of the optimization problem (2) when c is equal to -10. Calculate the corresponding risk allocation.

The optimization problem of the ERC portfolio

If c is equal to -10, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	12.65%	7.75%	0.98%	25.00%
2	8.43%	11.63%	0.98%	25.00%
3	7.06%	13.89%	0.98%	25.00%
4	6.03%	16.25%	0.98%	25.00%
$\sigma(x)$			3.92%	

The optimization problem of the ERC portfolio

Question 2.c

Same question if $c = 0$.

The optimization problem of the ERC portfolio

If c is equal to 0, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	154.07%	7.75%	11.94%	25.00%
2	102.72%	11.63%	11.94%	25.00%
3	85.97%	13.89%	11.94%	25.00%
4	73.50%	16.25%	11.94%	25.00%
$\sigma(x)$			47.78%	

The optimization problem of the ERC portfolio

Question 2.d

Demonstrate then that the solution to the second optimization problem is:

$$x^{\star}(c) = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}$$

where $c_{\mathrm{erc}} = \sum_{i=1}^{n} \ln x_{\mathrm{erc},i}$. Comment on this result.

The optimization problem of the ERC portfolio

In this case, the first-order condition is:

$$\frac{\partial\,\sigma(x)}{\partial\,x_i} - \lambda_i - \frac{\lambda_c}{x_i} = 0$$

As previously, $\lambda_i = 0$ because $x_i > 0$ and we obtain:

$$x_i \frac{\partial\,\sigma(x)}{\partial\,x_i} = \lambda_c$$

The solution of the second optimization problem is then a non-normalized ERC portfolio because $\sum_{i=1}^n x_i$ is not necessarily equal to 1. If we note $c_{\mathrm{erc}} = \sum_{i=1}^n \ln(x_{\mathrm{erc}})_i$, we deduce that:

$$\begin{aligned} x_{\mathrm{erc}} &= \arg\min \sqrt{x^\top \Sigma x} \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n \ln x_i \geq c_{\mathrm{erc}} \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned}$$

The optimization problem of the ERC portfolio

Let $x^\star(c)$ be the portfolio defined by:

$$x^\star(c) = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}$$

We have $x^\star(c) > \mathbf{0}_n$,

$$\sqrt{x^\star(c)^\top \Sigma x^\star(c)} = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) \sqrt{x_{\mathrm{erc}}^\top \Sigma x_{\mathrm{erc}}}$$

and:

$$\begin{aligned}
\sum_{i=1}^n \ln x_i^\star(c) &= \sum_{i=1}^n \ln\left(\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}\right)_i \\
&= c - c_{\mathrm{erc}} + \sum_{i=1}^n \ln\left(x_{\mathrm{erc}}\right)_i \\
&= c
\end{aligned}$$

We conclude that $x^\star(c)$ is the solution of the optimization problem.

The optimization problem of the ERC portfolio

$x^{\star}(c)$ is then a leveraged ERC portfolio if $c > c_{\mathrm{erc}}$ and a deleveraged ERC portfolio if $c < c_{\mathrm{erc}}$.

In our example, c_{erc} is equal to -5.7046. If $c = -10$, we have:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 34.17\%$$

We verify that the solution of Question 2.b is such that $\sum_{i=1}^{n} x_i = 34.17\%$ and $\mathcal{RC}_i^{\star} = \mathcal{RC}_j^{\star}$.

If $c = 0$, we obtain:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 416.26\%$$

In this case, the solution is a leveraged ERC portfolio.

The optimization problem of the ERC portfolio

Question 2.e

Show that there exists a scalar c such that the Lagrange coefficient λ_0 of the optimization problem (1) is equal to zero. Deduce then that the volatility of the ERC portfolio is between the volatility of the long-only minimum variance portfolio and the volatility of the equally weighted portfolio:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

The optimization problem of the ERC portfolio

From the previous question, we know that the ERC optimization portfolio is the solution of the second optimization problem if we use c_{erc} for the control variable. In this case, we have $\sum_{i=1}^{n} x_i^\star \left(c_{\mathrm{erc}}\right) = 1$ meaning that x_{erc} is also the solution of the first optimization problem. We deduce that $\lambda_0 = 0$ if $c = c_{\mathrm{erc}}$. The first optimization problem is a convex problem with a convex inequality constraint. The objective function is then an increasing function of the control variable c:

$$c_1 \leq c_2 \Rightarrow \sigma\left(x^\star\left(c_1\right)\right) \geq \sigma\left(x^\star\left(c_2\right)\right)$$

The optimization problem of the ERC portfolio

We have seen that the minimum variance portfolio corresponds to $c = -\infty$, that the EW portfolio is obtained with $c = -n \ln n$ and that the ERC portfolio is the solution of the optimization problem when c is equal to c_{erc}. Moreover, we have $-\infty \leq c_{\mathrm{erc}} \leq -n \ln n$. We deduce that the volatility of the ERC portfolio is between the volatility of the long-only minimum variance portfolio and the volatility of the equally weighted portfolio:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

Risk parity funds

Question 1

We consider a universe of three asset classes[a] which are stocks (S), bonds (B) and commodities (C). We have computed the one-year historical covariance matrix of asset returns for different dates and we obtain the following results (all the numbers are expressed in %):

	31/12/1999			31/12/2002			30/12/2005		
σ_i	12.40	5.61	12.72	20.69	7.36	13.59	7.97	7.01	16.93
$\rho_{i,j}$	100.00			100.00			100.00		
	−5.89	100.00		−36.98	100.00		29.25	100.00	
	−4.09	−7.13	100.00	22.74	−13.12	100.00	15.75	15.05	100.00
	31/12/2007			31/12/2008			31/12/2010		
σ_i	12.94	5.50	14.54	33.03	9.73	29.00	16.73	6.88	16.93
$\rho_{i,j}$	100.00	−25.76		100.00			100.00		
	−25.76	100.00		−16.26	100.00		15.31	100.00	
	31.91	6.87	100.00	47.31	9.13	100.00	64.13	15.46	100.00

[a] In fact, we use the MSCI World index, the Citigroup WGBI index and the DJ UBS Commodity index to represent these asset classes.

Risk parity funds

Question 1.a

Compute the weights and the volatility of the risk parity[a] (RP portfolio) portfolios for the different dates.

[a]Here, risk parity refers to the ERC portfolio when we do not take into account the correlations.

Risk parity funds

The RP portfolio is defined as follows:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.89%	18.75%	38.35%	23.57%	18.07%	22.63%
B	52.81%	52.71%	43.60%	55.45%	61.35%	55.02%
C	23.29%	28.54%	18.05%	20.98%	20.58%	22.36%
$\sigma(x)$	4.83%	6.08%	6.26%	5.51%	11.64%	8.38%

Risk parity funds

Question 1.b

Same question by considering the ERC portfolio.

Risk parity funds

In the ERC portfolio, the risk contributions are equal for all the assets:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

with:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \tag{3}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.66%	18.18%	37.85%	23.28%	17.06%	20.33%
B	53.12%	58.64%	43.18%	59.93%	66.39%	59.61%
C	23.22%	23.18%	18.97%	16.79%	16.54%	20.07%
$\sigma(x)$	4.82%	5.70%	6.32%	5.16%	10.77%	7.96%

Risk parity funds

Question 1.c

What do you notice about the volatility of RP and ERC portfolios? Explain these results.

Risk parity funds

We notice that $\sigma(x_{\mathrm{erc}}) \leq \sigma(x_{\mathrm{rp}})$ except for the year 2005. This date corresponds to positive correlations between assets. Moreover, the correlation between stocks and bonds is the highest. Starting from the RP portfolio, it is then possible to approach the ERC portfolio by reducing the weights of stocks and bonds and increasing the weight of commodities. At the end, we find an ERC portfolio that has a slightly higher volatility.

Risk parity funds

Question 1.d

Find the analytical expression of the volatility $\sigma(x)$, the marginal risk $\mathcal{MR}_i$, the risk contribution $\mathcal{RC}_i$ and the normalized risk contribution $\mathcal{RC}_i^{\star}$ in the case of RP portfolios.

Risk parity funds

The volatility of the RP portfolio is:

$$
\begin{aligned}
\sigma(x) &= \frac{1}{\sum_{j=1}^{n} \sigma_j^{-1}} \sqrt{\left(\sigma^{-1}\right)^\top \Sigma \sigma^{-1}} \\
&= \frac{1}{\sum_{j=1}^{n} \sigma_j^{-1}} \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sigma_i \sigma_j} \rho_{i,j} \sigma_i \sigma_j} \\
&= \frac{1}{\sum_{j=1}^{n} \sigma_j^{-1}} \sqrt{n + 2 \sum_{i>j} \rho_{i,j}} \\
&= \frac{1}{\sum_{j=1}^{n} \sigma_j^{-1}} \sqrt{n \left(1 + (n-1)\bar{\rho}\right)}
\end{aligned}
$$

where $\bar{\rho}$ is the average correlation between asset returns.

Risk parity funds

For the marginal risk, we obtain:

$$
\begin{aligned}
\mathcal{MR}_i &= \frac{\left(\Sigma\sigma^{-1}\right)_i}{\sigma(x)\sum_{j=1}^n \sigma_j^{-1}} \\
&= \frac{1}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j\frac{1}{\sigma_j} \\
&= \frac{\sigma_i}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j} \\
&= \frac{\sigma_i\bar{\rho}_i\sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}}
\end{aligned}
$$

where $\bar{\rho}_i$ is the average correlation of asset i with the other assets (including itself).

Risk parity funds

The expression of the risk contribution is then:

$$\begin{aligned}\mathcal{RC}_i &= \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}} \frac{\sigma_i \bar{\rho}_i \sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}} \\ &= \frac{\bar{\rho}_i \sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}\sum_{j=1}^n \sigma_j^{-1}}\end{aligned}$$

We deduce that the normalized risk contribution is:

$$\begin{aligned}\mathcal{RC}_i^\star &= \frac{\bar{\rho}_i \sqrt{n}}{\sigma(x)\sqrt{1+(n-1)\bar{\rho}}\sum_{j=1}^n \sigma_j^{-1}} \\ &= \frac{\bar{\rho}_i}{1+(n-1)\bar{\rho}}\end{aligned}$$

Risk parity funds

Question 1.e

Compute the normalized risk contributions of the previous RP portfolios. Comment on these results.

Risk parity funds

We obtain the following normalized risk contributions:

Date	1999	2002	2005	2007	2008	2010
S	33.87%	34.96%	34.52%	32.56%	34.45%	36.64%
B	32.73%	20.34%	34.35%	24.88%	24.42%	26.70%
C	33.40%	44.69%	31.14%	42.57%	41.13%	36.67%

We notice that the risk contributions are not exactly equal for all the assets. Generally, the risk contribution of bonds is lower than the risk contribution of equities, which is itself lower than the risk contribution of commodities.

Risk parity funds

Question 2

We consider four parameter sets of risk budgets:

Set	b_1	b_2	b_3
#1	45%	45%	10%
#2	70%	10%	20%
#3	20%	70%	10%
#4	25%	25%	50%

Risk parity funds

Question 2.a

Compute the RB portfolios for the different dates.

Risk parity funds

We obtain the following RB portfolios:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	26.83%	22.14%	42.83%	27.20%	20.63%	25.92%
B	45%	59.78%	66.10%	48.77%	66.15%	73.35%	67.03%
C	10%	13.39%	11.76%	8.40%	6.65%	6.02%	7.05%
S	70%	40.39%	29.32%	65.53%	39.37%	33.47%	46.26%
B	10%	37.63%	51.48%	19.55%	47.18%	52.89%	37.76%
C	20%	21.98%	19.20%	14.93%	13.45%	13.64%	15.98%
S	20%	17.55%	16.02%	25.20%	18.78%	12.94%	13.87%
B	70%	69.67%	71.70%	66.18%	74.33%	80.81%	78.58%
C	10%	12.78%	12.28%	8.62%	6.89%	6.24%	7.55%
S	25%	21.69%	15.76%	34.47%	20.55%	14.59%	16.65%
B	25%	48.99%	54.03%	39.38%	55.44%	61.18%	53.98%
C	50%	29.33%	30.21%	26.15%	24.01%	24.22%	29.37%

Risk parity funds

Question 2.b

Compute the implied risk premium $\tilde{\pi}_i$ of the assets for these portfolios if we assume a Sharpe ratio equal to 0.40.

Risk parity funds

To compute the implied risk premium $\tilde{\pi}_i$, we use the following formula (TR-RPB, page 274):

$$\begin{aligned}\tilde{\pi}_i &= \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{MR}_i \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \frac{\left(\Sigma x\right)_i}{\sigma\left(x\right)}\end{aligned}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the Sharpe ratio of the portfolio.

Risk parity funds

We obtain the following results:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	3.19%	4.60%	2.49%	3.15%	8.64%	5.20%
B	45%	1.43%	1.54%	2.19%	1.29%	2.43%	2.01%
C	10%	1.42%	1.92%	2.82%	2.86%	6.58%	4.24%
S	70%	4.05%	6.45%	2.86%	4.31%	11.56%	6.32%
B	10%	0.62%	0.52%	1.37%	0.51%	1.04%	1.11%
C	20%	2.13%	2.81%	3.59%	3.61%	8.11%	5.23%
S	20%	2.06%	2.68%	1.91%	1.93%	5.61%	3.91%
B	70%	1.82%	2.10%	2.54%	1.71%	3.14%	2.42%
C	10%	1.42%	1.75%	2.79%	2.64%	5.82%	3.60%
S	25%	2.33%	3.78%	1.98%	2.74%	8.06%	5.13%
B	25%	1.03%	1.10%	1.74%	1.02%	1.92%	1.58%
C	50%	3.45%	3.95%	5.23%	4.69%	9.71%	5.82%

Risk parity funds

Question 2.c

Comment on these results.

Risk parity funds

We have:

$$x_i \tilde{\pi}_i = \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{RC}_i$$

We deduce that:

$$\tilde{\pi}_i \propto \frac{b_i}{x_i}$$

x_i is generally an increasing function of b_i. As a consequence, the relationship between the risk budgets b_i and the risk premiums $\tilde{\pi}_i$ is not necessarily increasing. However, we notice that the bigger the risk budget, the higher the risk premium. This is easily explained. If an investor allocates more risk budget to one asset class than another investor, he thinks that the risk premium of this asset class is higher than the other investor.

Risk parity funds

However, we must be careful. This interpretation is valid if we compare two sets of risk budgets. It is false if we compare the risk budgets among themselves. For instance, if we consider the third parameter set, the risk budget of bonds is 70% whereas the risk budget of stocks is 20%. It does not mean that the risk premium of bonds is higher than the risk premium of equities. In fact, we observe the contrary. If we would like to compare risk budgets among themselves, the right measure is the implied Sharpe ratio, which is equal to:

$$\begin{aligned} \mathrm{SR}_i &= \frac{\tilde{\pi}_i}{\sigma_i} \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \frac{\mathcal{MR}_i}{\sigma_i} \end{aligned}$$

For instance, if we consider the most diversified portfolio, the marginal risk is proportional to the volatility and we retrieve the result that Sharpe ratios are equal if the MDP is optimal.

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