Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 6. Equity and Bond Portfolio Optimization with Green Preferences (Exercise)

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `www.thierry-roncalli.com/AssetManagementCourse.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 12 (6 hours, AM+PM)
- January 19 (6 hours, AM+PM)
- January 26 (6 hours, AM+PM)
- February 2 (6 hours, AM+PM)

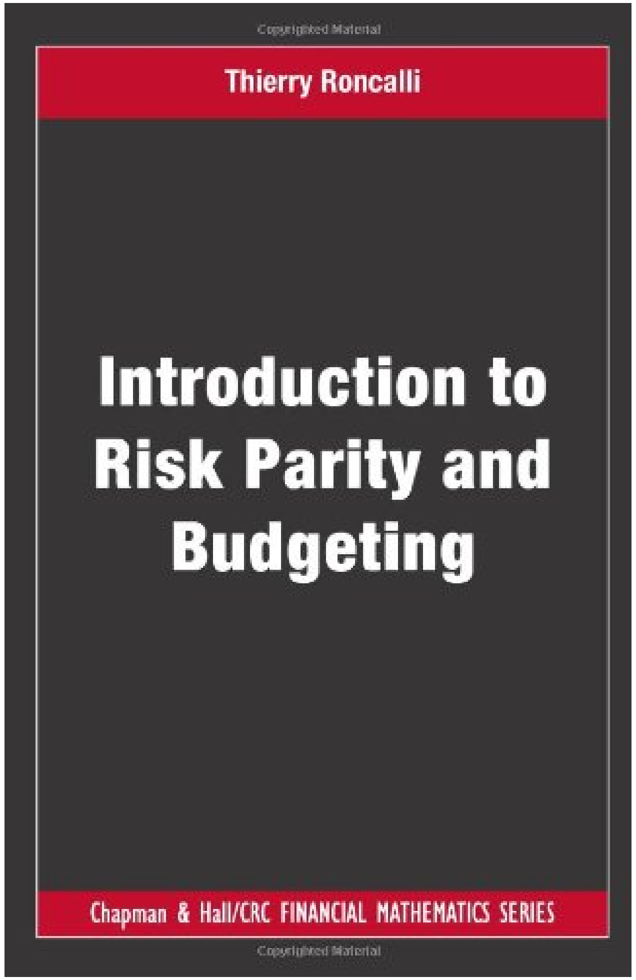
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

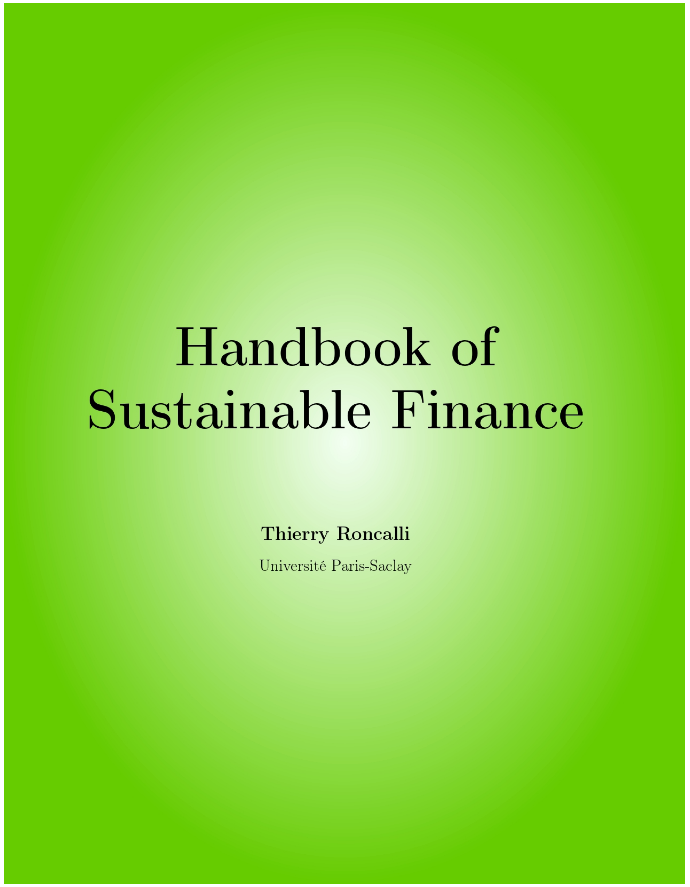
Textbook (Asset Management)

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Textbook (Sustainable Finance)

- Roncalli, T. (2024), *Handbook of Sustainable Finance*.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `www.thierry-roncalli.com/AssetManagementCourse.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- **Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences**
- Lecture 7: Machine Learning in Asset Management

We consider an investment universe of 8 issuers. In the table below, we report the carbon emissions $\mathcal{CE}_{i,j}$ (in $\mathrm{ktCO_2e}$) of these companies and their revenues Y_i (in \$ bn), and we indicate in the last row whether the company belongs to sector $\mathcal{S}ector_1$ or $\mathcal{S}ector_2$:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25
$\mathcal{S}ector$	1	2	1	1	2	1	2	2

The benchmark b of this investment universe is defined as:

$$b = (22\%, 19\%, 17\%, 13\%, 11\%, 8\%, 6\%, 4\%)$$

In what follows, we consider long-only portfolios.

Question 1

We want to compute the carbon intensity of the benchmark.

Question (a)

Compute the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes 1, 2 and 3.

We have:

$$\mathcal{CI}_{i,j} = \frac{\mathcal{CE}_{i,j}}{Y_i}$$

For instance, if we consider the 8^{th} issuer, we have[2]:

$$\begin{aligned}
\mathcal{CI}_{8,1} &= \frac{\mathcal{CE}_{8,1}}{Y_8} = \frac{5}{25} = 0.20 \text{ tCO}_2\text{e/\$ mn} \\
\mathcal{CI}_{8,2} &= \frac{\mathcal{CE}_{8,2}}{Y_8} = \frac{64}{25} = 2.56 \text{ tCO}_2\text{e/\$ mn} \\
\mathcal{CI}_{8,3} &= \frac{\mathcal{CE}_{8,3}}{Y_8} = \frac{199}{25} = 7.96 \text{ tCO}_2\text{e/\$ mn}
\end{aligned}$$

[2] Because 1 $\text{ktCO}_2\text{e/\$}$ bn $= 1$ $\text{tCO}_2\text{e/\$ mn}$.

Since we have:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25

we obtain:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,2}$	0.25	15.24	8.24	3.50	22.50	0.05	18.69	2.56
$\mathcal{CI}_{i,3}$	80.00	45.73	9.68	5.50	2.50	1.83	280.37	7.96

Question (b)

Deduce the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes $1+2$ and $1+2+3$.

We have:

$$\mathcal{CI}_{i,1-2} = \frac{\mathcal{CE}_{i,1} + \mathcal{CE}_{i,2}}{Y_i} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2}$$

and:

$$\mathcal{CI}_{i,1-3} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2} + \mathcal{CI}_{i,3}$$

We deduce that:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,1-2}$	0.50	30.49	14.00	4.00	35.00	0.29	299.07	2.76
$\mathcal{CI}_{i,1-3}$	80.50	76.22	23.68	9.50	37.50	2.12	579.44	10.72

Question (c)

Deduce the weighted average carbon intensity (WACI) of the benchmark if we consider the scope $1 + 2 + 3$.

We have:

$$\begin{aligned}
\mathcal{CI}(b) &= \sum_{i=1}^{8} b_i \mathcal{CI}_i \\
&= 0.22 \times 80.50 + 0.19 \times 76.2195 + 0.17 \times 23.68 + 0.13 \times 9.50 + \\
&\quad 0.11 \times 37.50 + 0.08 \times 2.1275 + 0.06 \times 579.4393 + 0.04 \times 10.72 \\
&= 76.9427 \text{ tCO}_2\text{e}/\$ \text{ mn}
\end{aligned}$$

Question (d)

We assume that the market capitalization of the benchmark portfolio is equal to \$10 tn and we invest \$1 bn.

Question (d).i

Deduce the market capitalization of each company (expressed in \$ bn).

We have:

$$b_i = \frac{MC_i}{\sum_{k=1}^{8} \mathrm{MC}_k}$$

and $\sum_{k=1}^{8} \mathrm{MC}_k = \$10$ tn. We deduce that:

$$\mathrm{MC}_i = 10 \times b_i$$

We obtain the following values of market capitalization expressed in \$ bn:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
MC_i	2 200	1 900	1 700	1 300	1 100	800	600	400

Question (d).ii

Compute the ownership ratio for each asset (expressed in bps).

Let W be the wealth invested in the benchmark portfolio b. The wealth invested in asset i is equal to $b_i W$. We deduce that the ownership ratio is equal to:

$$\varpi_i = \frac{b_i W}{\mathrm{MC}_i} = \frac{b_i W}{b_i \sum_{k=1}^{n} \mathrm{MC}_k} = \frac{W}{\sum_{k=1}^{n} \mathrm{MC}_k}$$

When we invest in a capitalization-weighted portfolio, the ownership ratio is the same for all the assets. In our case, we have:

$$\varpi_i = \frac{1}{10 \times 1000} = 0.01\%$$

The ownership ratio is equal to 1 basis point.

Question (d).iii

Compute the carbon emissions of the benchmark portfolio[a] if we invest \$1 bn and we consider the scope $1 + 2 + 3$.

[a] We assume that the float percentage is equal to 100% for all the 8 companies.

Using the financed emissions approach, the carbon emissions of our investment is equal to:

$$\begin{aligned}\mathcal{CE}\,(\$1\text{ bn}) &= 0.01\% \times (75 + 75 + 24\,000) + \\ &\quad 0.01\% \times (5\,000 + 5\,000 + 15\,000) + \\ &\quad \ldots + \\ &\quad 0.01\% \times (5 + 64 + 199) \\ &= 12.3045 \text{ ktCO}_2\text{e}\end{aligned}$$

Question (d).iv

Compare the (exact) carbon intensity of the benchmark portfolio with the WACI value obtained in Question 1.(c).

We compute the revenues of our investment:

$$Y\left(\$1 \text{ bn}\right) = 0.01\% \sum_{i=1}^{8} Y_i = \$0.1287 \text{ bn}$$

We deduce that the exact carbon intensity is equal to:

$$\mathcal{CI}\left(\$1 \text{ bn}\right) = \frac{\mathcal{CE}\left(\$1 \text{ bn}\right)}{Y\left(\$1 \text{ bn}\right)} = \frac{12.3045}{0.1287} = 95.6061 \ \mathrm{tCO_2e/\$ \ mn}$$

We notice that the WACI of the benchmark underestimates the exact carbon intensity of our investment by 19.5%:

$$76.9427 < 95.6061$$

Question 2

We want to manage an equity portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We assume that the volatility of the stocks is respectively equal to 22%, 20%, 25%, 18%, 40%, 23%, 13% and 29%. The correlation matrix between these stocks is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 60\% & 100\% \end{pmatrix}$$

Question (a)

Compute the covariance matrix Σ.

The covariance matrix $\Sigma = (\Sigma_{i,j})$ is defined by:

$$\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$$

We obtain the following numerical values (expressed in bps):

$$\Sigma = \begin{pmatrix} 484.0 & 352.0 & 385.0 & 237.6 & 616.0 & 253.0 & 200.2 & 382.8 \\ 352.0 & 400.0 & 375.0 & 234.0 & 400.0 & 276.0 & 130.0 & 377.0 \\ 385.0 & 375.0 & 625.0 & 360.0 & 700.0 & 402.5 & 227.5 & 507.5 \\ 237.6 & 234.0 & 360.0 & 324.0 & 612.0 & 331.2 & 175.5 & 391.5 \\ 616.0 & 400.0 & 700.0 & 612.0 & 1600.0 & 552.0 & 416.0 & 754.0 \\ 253.0 & 276.0 & 402.5 & 331.2 & 552.0 & 529.0 & 149.5 & 466.9 \\ 200.2 & 130.0 & 227.5 & 175.5 & 416.0 & 149.5 & 169.0 & 226.2 \\ 382.8 & 377.0 & 507.5 & 391.5 & 754.0 & 466.9 & 226.2 & 841.0 \end{pmatrix}$$

Question (b)

Write the optimization problem if the objective function is to minimize the tracking error risk under the constraint of carbon intensity reduction.

The tracking error variance of portfolio w with respect to benchmark b is equal to:

$$\sigma^2 (w \mid b) = (w - b)^\top \Sigma (w - b)$$

The carbon intensity constraint has the following expression:

$$\sum_{i=1}^{8} w_i \mathcal{CI}_i \leq (1 - \mathcal{R}) \mathcal{CI}(b)$$

where $\mathcal{R}$ is the reduction rate and $\mathcal{CI}(b)$ is the carbon intensity of the benchmark. Let $\mathcal{CI}^\star = (1 - \mathcal{R}) \mathcal{CI}(b)$ be the target value of the carbon footprint. The optimization problem is then:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} \sigma^2 (w \mid b) \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{8} w_i \mathcal{CI}_i \leq \mathcal{CI}^\star \\ \sum_{i=1}^{8} w_i = 1 \\ 0 \leq w_i \leq 1 \end{array} \right. \end{aligned}$$

We add the second and third constraints in order to obtain a long-only portfolio.

Question (c)

Give the QP formulation of the optimization problem.

The objective function is equal to:

$$f(w) = \frac{1}{2}\sigma^2(w \mid b) = \frac{1}{2}(w-b)^\top \Sigma (w-b) = \frac{1}{2}w^\top \Sigma w - w^\top \Sigma b + \frac{1}{2}b^\top \Sigma b$$

while the matrix form of the carbon intensity constraint is:

$$\mathcal{CI}^\top w \leq \mathcal{CI}^\star$$

where $\mathcal{CI} = (\mathcal{CI}_1, \ldots, \mathcal{CI}_8)$ is the column vector of carbon intensities. Since $b^\top \Sigma b$ is a constant and does not depend on w, we can cast the previous optimization problem into a QP problem:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} w^\top Q w - w^\top R \\ &\text{s.t.} \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^- \leq w \leq w^+ \end{array} \right. \end{aligned}$$

We have $Q = \Sigma$, $R = \Sigma b$, $A = \mathbf{1}_8^\top$, $B = 1$, $C = \mathcal{CI}^\top$, $D = \mathcal{CI}^\star$, $w^- = \mathbf{0}_8$ and $w^+ = \mathbf{1}_8$.

Question (d)

$\mathcal{R}$ is equal to 20%. Find the optimal portfolio if we target scope $1+2$. What is the value of the tracking error volatility?

We have:

$$\begin{aligned}\mathcal{CI}(b) &= 0.22 \times 0.50 + 0.19 \times 30.4878 + \ldots + 0.04 \times 2.76 \\ &= 30.7305 \text{ tCO}_2\text{e}/\$ \text{ mn}\end{aligned}$$

We deduce that:

$$\mathcal{CI}^{\star} = (1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 30.7305 = 24.5844 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

Therefore, the inequality constraint of the QP problem is:

$$\begin{pmatrix} 0.50 & 30.49 & 14.00 & 4.00 & 35.00 & 0.29 & 299.07 & 2.76 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_7 \\ w_8 \end{pmatrix} \leq 24.5844$$

We obtain the following optimal solution:

$$w^{\star} = \begin{pmatrix} 23.4961\% \\ 17.8129\% \\ 17.1278\% \\ 15.4643\% \\ 10.4037\% \\ 7.5903\% \\ 4.0946\% \\ 4.0104\% \end{pmatrix}$$

The minimum tracking error volatility $\sigma\left(w^{\star} \mid b\right)$ is equal to 15.37 bps.

Question (e)

Same question if $\mathcal{R}$ is equal to 30%, 50%, and 70%.

Table 1: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-2}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.4961	24.2441	25.7402	30.4117
w_2	19.0000	17.8129	17.2194	16.0323	9.8310
w_3	17.0000	17.1278	17.1917	17.3194	17.8348
w_4	13.0000	15.4643	16.6964	19.1606	23.3934
w_5	11.0000	10.4037	10.1055	9.5091	7.1088
w_6	8.0000	7.5903	7.3854	6.9757	6.7329
w_7	6.0000	4.0946	3.1418	1.2364	0.0000
w_8	4.0000	4.0104	4.0157	4.0261	4.6874
$\mathcal{CI}(w)$	30.7305	24.5844	21.5114	15.3653	9.2192
$\sigma(w \mid b)$	0.00	15.37	23.05	38.42	72.45

In Table 1, we report the optimal solution $w^{\star}$ (expressed in %) of the optimization problem for different values of $\mathcal{R}$. We also indicate the carbon intensity of the portfolio (in $\mathrm{tCO_2e/\$\ mn}$) and the tracking error volatility (in bps). For instance, if $\mathcal{R}$ is set to 50%, the weights of assets $\#1$, $\#3$, $\#4$ and $\#8$ increase whereas the weights of assets $\#2$, $\#5$, $\#6$ and $\#7$ decrease. The carbon intensity of this portfolio is equal to 15.3653 $\mathrm{tCO_2e/\$\ mn}$. The tracking error volatility is below 40 bps, which is relatively low.

Question (f)

We target scope $1 + 2 + 3$. Find the optimal portfolio if $\mathcal{R}$ is equal to $20\%, 30\%$, 50% and 70%. Give the value of the tracking error volatility for each optimized portfolio.

In this case, the inequality constraint $Cw \leq D$ is defined by:

$$C = \mathcal{CI}_{1-3}^{\top} = \begin{pmatrix} 80.5000 \\ 76.2195 \\ 23.6800 \\ 9.5000 \\ 37.5000 \\ 2.1275 \\ 579.4393 \\ 10.7200 \end{pmatrix}^{\top}$$

and:

$$D = (1 - \mathcal{R}) \times 76.9427$$

We obtain the results given in Table 2.

Table 2: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-3}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.9666	24.9499	26.4870	13.6749
w_2	19.0000	17.4410	16.6615	8.8001	0.0000
w_3	17.0000	17.1988	17.2981	19.4253	24.1464
w_4	13.0000	16.5034	18.2552	25.8926	41.0535
w_5	11.0000	10.2049	9.8073	7.1330	3.5676
w_6	8.0000	7.4169	7.1254	7.0659	8.8851
w_7	6.0000	3.2641	1.8961	0.0000	0.0000
w_8	4.0000	4.0043	4.0065	5.1961	8.6725
$\mathcal{CI}(w)$	76.9427	61.5541	53.8599	38.4713	23.0828
$\sigma(w \mid b)$	0.00	21.99	32.99	104.81	414.48

Question (g)

Compare the optimal solutions obtained in Questions 2.(e) and 2.(f).

Figure 1: Impact of the scope on the tracking error volatility

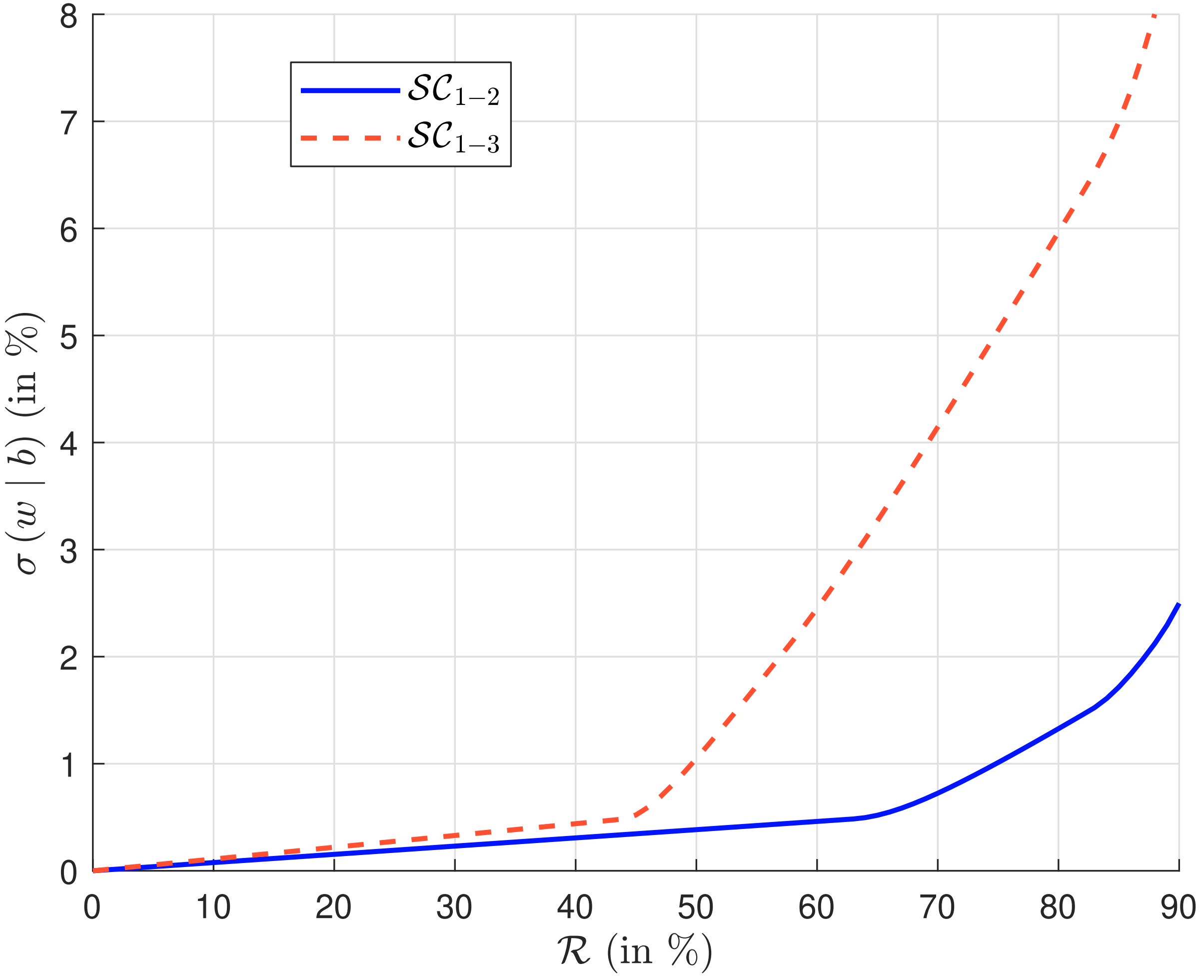
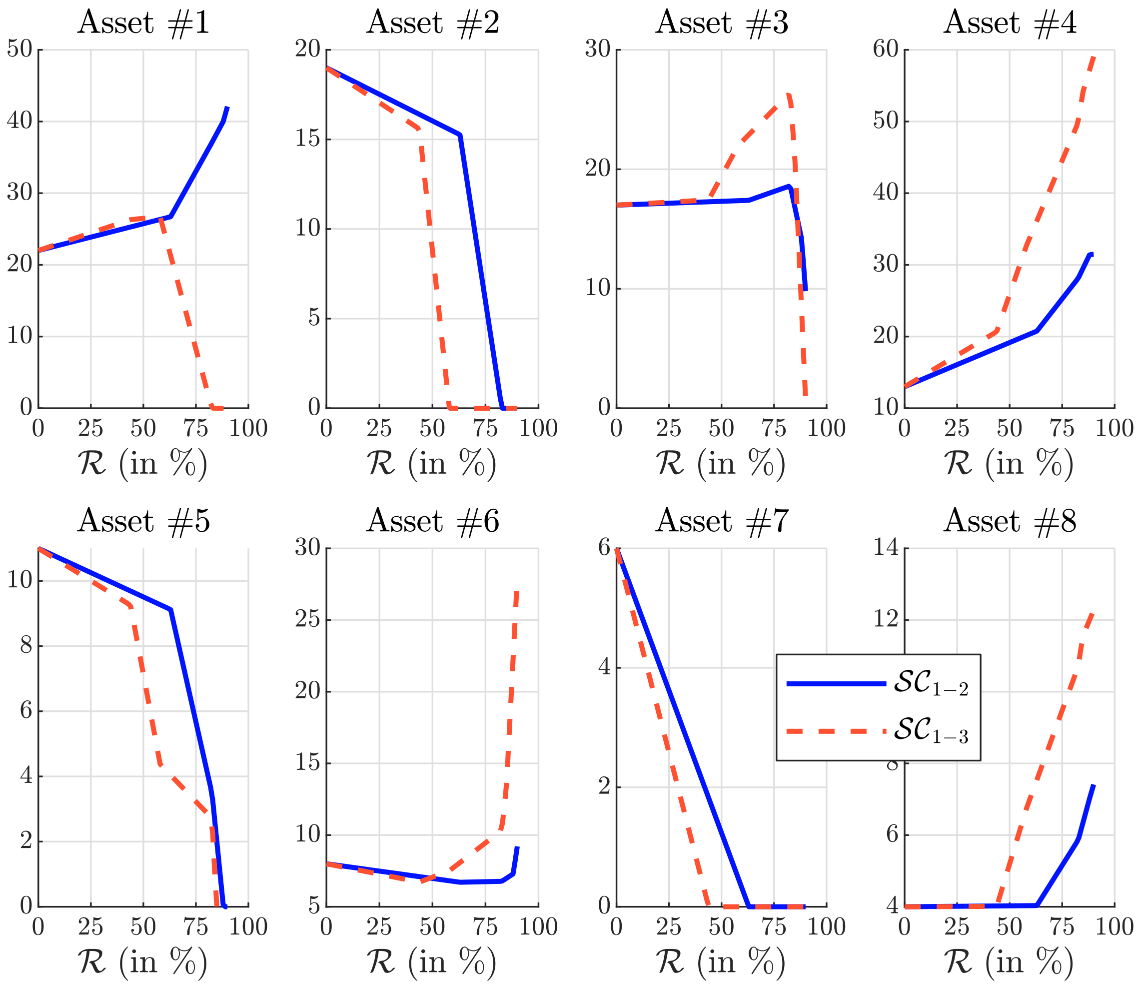


Figure 2: Impact of the scope on the portfolio allocation (in %)



In Figure 1, we report the relationship between the reduction rate $\mathcal{R}$ and the tracking error volatility $\sigma(w \mid b)$. The choice of the scope has little impact when $\mathcal{R} \leq 45\%$. Then, we notice a high increase when we consider the scope $1+2+3$. The portfolio's weights are given in Figure 2. For assets $\#1$ and $\#3$, the behavior is divergent when we compare scopes $1+2$ and $1+2+3$.

Question 3

We want to manage a bond portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We use the scope $1+2+3$. In the table below, we report the modified duration MD_i and the duration-times-spread factor DTS_i of each corporate bond i:

Asset	#1	#2	#3	#4	#5	#6	#7	#8
MD_i (in years)	3.56	7.48	6.54	10.23	2.40	2.30	9.12	7.96
DTS_i (in bps)	103	155	75	796	89	45	320	245
$\mathcal{S}ector$	1	2	1	1	2	1	2	2

Question 3 (Cont'd)

We remind that the active risk can be calculated using three functions. For the active share, we have:

$$\mathcal{R}_{\mathrm{AS}}(w \mid b) = \sigma^2_{\mathrm{AS}}(w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2$$

We also consider the MD-based tracking error risk:

$$\mathcal{R}_{\mathrm{MD}}(w \mid b) = \sigma^2_{\mathrm{MD}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \,\mathrm{MD}_i \right)^2$$

and the DTS-based tracking error risk:

$$\mathcal{R}_{\mathrm{DTS}}(w \mid b) = \sigma^2_{\mathrm{DTS}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \,\mathrm{DTS}_i \right)^2$$

Question 3 (Cont'd)

Finally, we define the synthetic risk measure as a combination of AS, MD and DTS active risks:

$$\mathcal{R}(w \mid b) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

where $\varphi_{\mathrm{AS}} \geq 0$, $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the weight of each risk. In what follows, we use the following numerical values: $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 1$. The reduction rate $\boldsymbol{\mathcal{R}}$ of the weighted average carbon intensity is set to 50% for the scope $1 + 2 + 3$.

Question (a)

Compute the modified duration $\mathrm{MD}(b)$ and the duration-times-spread factor $\mathrm{DTS}(b)$ of the benchmark.

We have:

$$\begin{aligned}
\mathrm{MD}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{MD}_i \\
&= 0.22 \times 3.56 + 0.19 \times 7.48 + \ldots + 0.04 \times 7.96 \\
&= 5.96 \text{ years}
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{DTS}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{DTS}_i \\
&= 0.22 \times 103 + 0.19 \times 155 + \ldots + 0.04 \times 155 \\
&= 210.73 \text{ bps}
\end{aligned}$$

Question (b)

Let w_{ew} be the equally-weighted portfolio. Compute[a] $\mathrm{MD}\left(w_{\mathrm{ew}}\right)$, $\mathrm{DTS}\left(w_{\mathrm{ew}}\right)$, $\sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right)$.

[a] Precise the corresponding unit (years, bps or %) for each metric.

We have:

$$\left\{\begin{array}{l}\mathrm{MD}\left(w_{\mathrm{ew}}\right)=6.20 \text{ years} \\ \mathrm{DTS}\left(w_{\mathrm{ew}}\right)=228.50 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right)=17.03\% \\ \sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right)=1.00 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right)=36.19 \text{ bps}\end{array}\right.$$

Question (c)

We consider the following optimization problem:

$$
\begin{aligned}
w^\star &= \arg\min \frac{1}{2}\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) \\
\text{s.t.} & \quad \left\{
\begin{array}{l}
\sum_{i=1}^n w_i = 1 \\
\mathrm{MD}\left(w\right) = \mathrm{MD}\left(b\right) \\
\mathrm{DTS}\left(w\right) = \mathrm{DTS}\left(b\right) \\
\boldsymbol{\mathcal{CI}}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \boldsymbol{\mathcal{CI}}\left(b\right) \\
0 \leq w_i \leq 1
\end{array}
\right.
\end{aligned}
$$

Give the analytical value of the objective function. Find the optimal portfolio $w^\star$. Compute $\mathrm{MD}\left(w^\star\right)$, $\mathrm{DTS}\left(w^\star\right)$, $\sigma_{\mathrm{AS}}\left(w^\star \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^\star \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^\star \mid b\right)$.

We have:

$$\begin{aligned}\mathcal{R}_{\mathrm{AS}}(w \mid b) &= (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\ &\quad (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2\end{aligned}$$

The objective function is then:

$$f(w) = \frac{1}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b)$$

The optimal solution is equal to:

$$w^\star = (17.30\%, 17.41\%, 20.95\%, 14.41\%, 10.02\%, 11.09\%, 0\%, 8.81\%)$$

The risk metrics are:

$$\left\{\begin{array}{l}\mathrm{MD}(w^\star) = 5.96 \text{ years} \\ \mathrm{DTS}(w^\star) = 210.73 \text{ bps} \\ \sigma_{\mathrm{AS}}(w^\star \mid b) = 10.57\% \\ \sigma_{\mathrm{MD}}(w^\star \mid b) = 0.43 \text{ years} \\ \sigma_{\mathrm{DTS}}(w^\star \mid b) = 15.21 \text{ bps}\end{array}\right.$$

Question (d)

We consider the following optimization problem:

$$\begin{array}{rcl} w^{\star} & = & \arg\min \dfrac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) + \dfrac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) \\ & & \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} w_i = 1 \\ \mathrm{DTS}\left(w\right) = \mathrm{DTS}\left(b\right) \\ \mathcal{CI}\left(w\right) \leq \left(1 - \mathcal{R}\right)\mathcal{CI}\left(b\right) \\ 0 \leq w_i \leq 1 \end{array} \right. \end{array}$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have[3]:

$$
\begin{aligned}
\mathcal{R}_{\mathrm{MD}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{MD}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{MD}_i\right)^2 \\
&= \left(\sum_{i=1,3,4,6} w_i\,\mathrm{MD}_i - \sum_{i=1,3,4,6} b_i\,\mathrm{MD}_i\right)^2 + \\
&\quad \left(\sum_{i=2,5,7,8} w_i\,\mathrm{MD}_i - \sum_{i=2,5,7,8} b_i\,\mathrm{MD}_i\right)^2 \\
&= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + \\
&\quad (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2
\end{aligned}
$$

The objective function is then:

$$
f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b)
$$

[3] We verify that $3.4089 + 2.5508 = 5.9597$ years.

The optimal solution is equal to:

$$w^{\star} = (16.31\%, 18.44\%, 17.70\%, 13.82\%, 11.67\%, 11.18\%, 0\%, 10.88\%)$$

The risk metrics are:

$$\left\{\begin{array}{l}
\mathrm{MD}\left(w^{\star}\right) = 5.93 \text{ years} \\
\mathrm{DTS}\left(w^{\star}\right) = 210.73 \text{ bps} \\
\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right) = 11.30\% \\
\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right) = 0.03 \text{ years} \\
\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right) = 3.70 \text{ bps}
\end{array}\right.$$

Question (e)

We consider the following optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\sum_{i=1}^{n} w_i = 1 \\
\boldsymbol{\mathcal{CI}}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{CI}}\left(b\right) \\
0 \leq w_i \leq 1
\end{array}
\right.
\end{aligned}
$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have[4]:

$$\begin{aligned}
\mathcal{R}_{\mathrm{DTS}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 \\
&= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\
&\quad (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2
\end{aligned}$$

The objective function is then:

$$f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

[4] We verify that $142.49 + 68.24 = 210.73$ bps.

The optimal solution is equal to:

$$w^{\star} = (16.98\%, 17.21\%, 18.26\%, 13.45\%, 12.10\%, 9.46\%, 0\%, 12.55\%)$$

The risk metrics are:

$$\left\{ \begin{array}{l} \mathrm{MD}\left(w^{\star}\right) = 5.97 \text{ years} \\ \mathrm{DTS}\left(w^{\star}\right) = 210.68 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right) = 11.94\% \\ \sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right) = 0.03 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right) = 0.06 \text{ bps} \end{array} \right.$$

Question (f)

Comment on the results obtained in Questions 3.(c), 3.(d) and 3.(e).

Table 3: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(c)	3.(d)	3.(e)
w_1	22.0000	17.3049	16.3102	16.9797
w_2	19.0000	17.4119	18.4420	17.2101
w_3	17.0000	20.9523	17.6993	18.2582
w_4	13.0000	14.4113	13.8195	13.4494
w_5	11.0000	10.0239	11.6729	12.1008
w_6	8.0000	11.0881	11.1792	9.4553
w_7	6.0000	0.0000	0.0000	0.0000
w_8	4.0000	8.8075	10.8769	12.5464
$\mathrm{MD}(w)$	5.9597	5.9597	5.9344	5.9683
$\mathrm{DTS}(w)$	210.7300	210.7300	210.7300	210.6791
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	10.5726	11.3004	11.9400
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.4338	0.0254	0.0308
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	15.2056	3.7018	0.0561
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713	38.4713

Question (g)

How to find the previous solution of Question 3.(e) using a QP solver?

The goal is to write the objective function into a quadratic function:

$$
\begin{aligned}
f(w) &= \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b) \\
&= \frac{1}{2} w^\top Q(b) w - w^\top R(b) + c(b)
\end{aligned}
$$

where:

$$
\begin{aligned}
\mathcal{R}_{\mathrm{AS}}(w \mid b) &= (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\
&\quad (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2 \\
\mathcal{R}_{\mathrm{MD}}(w \mid b) &= (3.56 w_1 + 6.54 w_3 + 10.23 w_4 + 2.30 w_6 - 3.4089)^2 + \\
&\quad (7.48 w_2 + 2.40 w_5 + 9.12 w_7 + 7.96 w_8 - 2.5508)^2 \\
\mathcal{R}_{\mathrm{DTS}}(w \mid b) &= (103 w_1 + 75 w_3 + 796 w_4 + 45 w_6 - 142.49)^2 + \\
&\quad (155 w_2 + 89 w_5 + 320 w_7 + 245 w_8 - 68.24)^2
\end{aligned}
$$

We use the analytical approach which is described in Section 11.1.2 on pages 332-339. Moreover, we rearrange the universe such that the first fourth assets belong to the first sector and the last fourth assets belong to the second sector. In this case, we have:

$$w = \left(\underbrace{w_1, w_3, w_4, w_6}_{\mathcal{S}ector_1}, \underbrace{w_2, w_5, w_7, w_8}_{\mathcal{S}ector_2} \right)$$

The matrix $Q(b)$ is block-diagonal:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{4,4} \\ \mathbf{0}_{4,4} & Q_2 \end{pmatrix}$$

where the matrices Q_1 and Q_2 are equal to:

$$Q_1 = \begin{pmatrix} 11\,025.8400 & 8\,307.0600 & 82\,898.4700 & 4\,839.7000 \\ 8\,307.0600 & 6\,794.2900 & 61\,372.6050 & 3\,751.0500 \\ 82\,898.4700 & 61\,372.6050 & 636\,332.3225 & 36\,408.2250 \\ 4\,839.7000 & 3\,751.0500 & 36\,408.2250 & 2\,257.2500 \end{pmatrix}$$

and:

$$Q_2 = \begin{pmatrix} 25\,523.7600 & 14\,243.8000 & 51\,305.4400 & 39\,463.5200 \\ 14\,243.8000 & 8\,165.0000 & 29\,027.2000 & 22\,282.6000 \\ 51\,305.4400 & 29\,027.2000 & 104\,579.3600 & 80\,214.8800 \\ 39\,463.5200 & 22\,282.6000 & 80\,214.8800 & 61\,709.0400 \end{pmatrix}$$

The vector $R(b)$ is defined as follows:

$$R(b) = \begin{pmatrix} 15\,001.8621 \\ 11\,261.1051 \\ 114\,306.8662 \\ 6\,616.0617 \\ 11\,073.1996 \\ 6\,237.4080 \\ 22\,424.3824 \\ 17\,230.4092 \end{pmatrix}$$

Finally, the value of $c(b)$ is equal to:

$$c(b) = 12\,714.3386$$

Using a QP solver, we obtain the following numerical solution:

$$\begin{pmatrix} w_1 \\ w_3 \\ w_4 \\ w_6 \\ w_2 \\ w_5 \\ w_7 \\ w_8 \end{pmatrix} = \begin{pmatrix} 16.9796 \\ 18.2582 \\ 13.4494 \\ 9.4553 \\ 17.2102 \\ 12.1009 \\ 0.0000 \\ 12.5464 \end{pmatrix} \times 10^{-2}$$

We observe some small differences (after the fifth digit) because the QP solver is more efficient than a traditional nonlinear solver.

Question 4

We consider a variant of Question 3 and assume that the synthetic risk measure is:

$$\mathcal{D}(w \mid b) = \varphi_{\mathrm{AS}} \mathcal{D}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}} \mathcal{D}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}} \mathcal{D}_{\mathrm{DTS}}(w \mid b)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}(w \mid b) &= \frac{1}{2} \sum_{i=1}^{n} \left| w_i - b_i \right| \\
\mathcal{D}_{\mathrm{MD}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right| \\
\mathcal{D}_{\mathrm{DTS}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{DTS}_i \right|
\end{aligned}$$

Question (a)

Define the corresponding optimization problem when the objective is to minimize the active risk and reduce the carbon intensity of the benchmark by $\mathcal{R}$.

The optimization problem is:

$$\begin{aligned} w^\star &= \arg\min \mathcal{D}\left(w \mid b\right) \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_8^\top w = 1 \\ \boldsymbol{\mathcal{CI}}^\top w \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \boldsymbol{\mathcal{CI}}\left(b\right) \\ \mathbf{0}_8 \leq w \leq \mathbf{1}_8 \end{array} \right. \end{aligned}$$

Question (b)

Give the LP formulation of the optimization problem.

We use the absolute value trick and obtain the following optimization problem:

$$
\begin{array}{rcl}
w^{\star} & = & \arg\min \dfrac{1}{2}\varphi_{\mathrm{AS}} \displaystyle\sum_{i=1}^{8} \tau_{i,w} + \varphi_{\mathrm{MD}} \sum_{j=1}^{2} \tau_{j,\mathrm{MD}} + \varphi_{\mathrm{DTS}} \sum_{j=1}^{2} \tau_{j,\mathrm{DTS}} \\
& \text{s.t.} & \left\{
\begin{array}{l}
\mathbf{1}_8^\top w = 1 \\
\mathbf{0}_8 \leq w \leq \mathbf{1}_8 \\
\boldsymbol{\mathcal{CI}}^\top w \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\
\left| w_i - b_i \right| \leq \tau_{i,w} \\
\left| \sum_{i \in \boldsymbol{\mathcal{S}}ector_j} (w_i - b_i) \mathrm{MD}_i \right| \leq \tau_{j,\mathrm{MD}} \\
\left| \sum_{i \in \boldsymbol{\mathcal{S}}ector_j} (w_i - b_i) \mathrm{DTS}_i \right| \leq \tau_{j,\mathrm{DTS}} \\
\tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0
\end{array}
\right.
\end{array}
$$

We can now formulate this problem as a standard LP problem:

$$\begin{aligned} x^{\star} &= \arg\min c^{\top} x \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where x is the 20×1 vector defined as follows:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

The 20×1 vector c is equal to:

$$c = \begin{pmatrix} \mathbf{0}_8 \\ \dfrac{1}{2}\varphi_{\mathrm{AS}}\mathbf{1}_8 \\ \varphi_{\mathrm{MD}}\mathbf{1}_2 \\ \varphi_{\mathrm{DTS}}\mathbf{1}_2 \end{pmatrix}$$

The equality constraint is defined by $A = \begin{pmatrix} \mathbf{1}_8^\top & \mathbf{0}_8^\top & \mathbf{0}_2^\top & \mathbf{0}_2^\top \end{pmatrix}$ and $B = 1$. The bounds are $x^- = \mathbf{0}_{20}$ and $x^+ = \infty \cdot \mathbf{1}_{20}$.

For the inequality constraint, we have[5]:

$$Cx \leq D \Leftrightarrow \begin{pmatrix} I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ -I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ -C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ -C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ \mathcal{CI}^\top & \mathbf{0}_{1,8} & 0 & 0 \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ \mathrm{MD}^\star \\ -\mathrm{MD}^\star \\ \mathrm{DTS}^\star \\ -\mathrm{DTS}^\star \\ (1-\mathcal{R})\,\mathcal{CI}(b) \end{pmatrix}$$

where:

$$C_{\mathrm{MD}} = \begin{pmatrix} 3.56 & 0.00 & 6.54 & 10.23 & 0.00 & 2.30 & 0.00 & 0.00 \\ 0.00 & 7.48 & 0.00 & 0.00 & 2.40 & 0.00 & 9.12 & 7.96 \end{pmatrix}$$

and:

$$C_{\mathrm{DTS}} = \begin{pmatrix} 103 & 0 & 75 & 796 & 0 & 45 & 0 & 0 \\ 0 & 155 & 0 & 0 & 89 & 0 & 320 & 245 \end{pmatrix}$$

The 2×1 vectors $\mathrm{MD}^\star$ and $\mathrm{DTS}^\star$ are respectively equal to $(3.4089, 2.5508)$ and $(142.49, 68.24)$.

[5] C is a 25×8 matrix and D is a 25×1 vector.

Question (c)

Find the optimal portfolio when $\mathcal{R}$ is set to 50%. Compare the solution with this obtained in Question 3.(e).

We obtain the following solution:

$$
\begin{aligned}
w^\star &= (18.7360, 15.8657, 17.8575, 13.2589, 11, 9.4622, 0, 13.8196) \times 10^{-2} \\
\tau_w^\star &= (3.2640, 3.1343, 0.8575, 0.2589, 0, 1.4622, 6, 9.8196) \times 10^{-2} \\
\tau_{\mathrm{MD}} &= (0, 0) \\
\tau_{\mathrm{DTS}} &= (0, 0)
\end{aligned}
$$

Table 4: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(e)	4.(c)
w_1	22.0000	16.9796	18.7360
w_2	19.0000	17.2102	15.8657
w_3	17.0000	18.2582	17.8575
w_4	13.0000	13.4494	13.2589
w_5	11.0000	12.1009	11.0000
w_6	8.0000	9.4553	9.4622
w_7	6.0000	0.0000	0.0000
w_8	4.0000	12.5464	13.8196
$\mathrm{MD}(w)$	5.9597	5.9683	5.9597
$\mathrm{DTS}(w)$	210.7300	210.6791	210.7300
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	11.9400	12.4837
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.0308	0.0000
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0561	0.0000
$\mathcal{D}_{\mathrm{AS}}(w \mid b)$	0.0000	25.6203	24.7964
$\mathcal{D}_{\mathrm{MD}}(w \mid b)$	0.0000	0.0426	0.0000
$\mathcal{D}_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0608	0.0000
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713

In Table 4, we compare the two solutions[6]. They are very close. In fact, we notice that the LP solution matches perfectly the MD and DTS constraints, but has a higher AS risk $\sigma_{\mathrm{AS}}(w \mid b)$. If we note the two solutions $w^\star(\mathcal{L}_1)$ and $w^\star(\mathcal{L}_2)$, we have:

$$\left\{\begin{array}{c} \mathcal{R}(w^\star(\mathcal{L}_2) \mid b) = 1.4524 < \mathcal{R}(w^\star(\mathcal{L}_1) \mid b) = 1.5584 \\ \mathcal{D}(w^\star(\mathcal{L}_2) \mid b) = 13.9366 > \mathcal{D}(w^\star(\mathcal{L}_1) \mid b) = 12.3982 \end{array}\right.$$

There is a trade-off between the $\mathcal{L}_1$- and $\mathcal{L}_2$-norm risk measures. This is why we cannot say that one solution dominates the other.

[6] The units are the following: % for the weights w_i, and the active share metrics $\sigma_{\mathrm{AS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{AS}}(w \mid b)$; years for the modified duration metrics $\mathrm{MD}(w)$, $\sigma_{\mathrm{MD}}(w \mid b)$ and $\mathcal{D}_{\mathrm{MD}}(w \mid b)$; bps for the duration-times-spread metrics $\mathrm{DTS}(w)$, $\sigma_{\mathrm{DTS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{DTS}}(w \mid b)$; $\mathrm{tCO_2e/\$\ mn}$ for the carbon intensity $\mathrm{DTS}(w)$.