Beyond Conditionally Independent Defaults

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1 Introduction

When dealing with basket credit derivatives, one is quickly compelled to cope with the joint default probability for some counterparties. Until a quite recent time, the *structural approach* could be considered as the predominant methodology. Within this framework, the default occurs at the first crossing time of the value process through a barrier. Thus, it looks quite legitimate to model the joint default of two firms as KMV and CreditMetrics assume, that is when both value process cross their own default barrier. As a consequence of such an approach, A Gaussian (or log-normal) assumption on the value processes induces a Gaussian dependence between the defaults.

The main alternative to this *structural approach* tries to take into account the unpredictability of the default random time. Then, when dealing with a single reference credit, one has to pay attention to the way one is going to model the instantaneous default probability. In that approach, which is commonly called the *intensity based approach* (or *reduced-form*), a default event is said to occur when an intensity process exceeds an unobserved threshold random variable. For one-firm derivatives, it provides a quite flexible framework and can be easily fitted to actual term structure of credit spreads.

This paper tackles the problem of modelling correlated default events within the intensity framework. The problem is trivial when default times are assumed to be (conditionally) independent. But it is well known among market practitioners that such an hypothesis does not allow to fit the observed default correlations. We will recall on a toy example this idea that correlating even perfectly the firms' spreads will not guarantee an implied high dependence between default times. But when one wants to weaken this assumption the problem becomes much more involved. Our method is first to estimate the marginal probability distribution of each individual default and then to use copulae to model the joint distribution, following the recent works of Giesecke [2001] and Schönbucher & Schubert [2001].

The copula approach has the advantage of splitting the distribution of each single intensity process and the joint law of default times, in such a way that the calibration of individual intensities to term structures remains easy. More intricate will be the calibration of the parameter of the copula which models the dependence between all triggers. A natural way to estimate this parameter would be to use correlation products, such as first-to-defaults. Yet these products are still not liquid enough in the market to perform such a calibration. We thus turn to a technique commonly used among securitization market practitioners: *Moody's diversity score*. It enables us to discuss about the choice of the copula function.

2 Conditionally independent defaults

We begin to recall the basic ideas on (one then two firms) intensity models. A huge literature is available on this subject. Not to give a too long list of references, we send the reader to Bielecki & Rutkowski [2001], where a very thorough account of the intensity framework can be found (and also some elements about the case of dependent defaults). In the case of conditionally independent defaults, once the intensity processes are fixed (those are the latent variables of the model), the residual sources of randomness which may affect the firms' default times are assumed independent. Thus, all possible dependence between defaults will result from the correlation between intensities. But we show that high default correlations cannot be achieved this way. In particular, it is worthless to compute historical spread correlations to derive an accurate value of default correlations.

2.1 the intensity framework

We first recall the well-known result for the pricing of a derivative security, when there is only one defaultable firm. We denote by $(\mathcal{F}_t)$ the filtration, i.e the information generated by all state variables (economic variables, interest rates, currencies, etc.). Along the whole paper, we assume the existence of a risk-neutral probability measure $\mathbb{P}$. Then, the basic elements of the intensity model are an $(\mathcal{F}_t)$-adapted, non-negative and continuous process (λ_t^1) (firm 1's *intensity*), and θ_1 (the *threshold*), an exponential random variable of parameter 1 independent from $\mathcal{F}_\infty$. For example, when using a (multi-) factor interest rate model, we can use the factor(s) for also driving the intensity process λ in order to provide correlations between interest rates and the default process. Then firm 1's default time is defined by (provided the firm has not defaulted yet)

$$\tau_1 := \inf \left\{ t : \int_0^t \lambda_s^1 \, \mathrm{d}s \geq \theta_1 \right\} \tag{1}$$

Then the defaultable zero-coupon price of firm 1 is given by

$$B_1(t,T) = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E}\left[e^{-\int_t^T (r_s + \lambda_s^1)\, \mathrm{d}s} | \mathcal{F}_t \right], \tag{2}$$

which allows to identify the intensity process as firm 1's spread. Lando LANDO [1998] derived similar formulae for firm 1 derivatives.

2.2 Drawbacks: low default correlations

The preceding framework is readily generalized to the case of I defaultable firms. For the sake of simplicity, we consider only the case of two firms. Therefore, the default times of two firms 1 and 2 are defined through (1) where we have this time two intensity processes λ^i, and two random thresholds θ_i which are still assumed to be independent from $\mathcal{F}_\infty$ and **mutually independent**. This last assumption is usually made in practice for its tractability.

We choose quadratic intensities $\lambda_t^i = \sigma_i (W_t^i)^2$ where $\mathbf{W} = \left(W^1, W^2\right)$ is a vector of 2 correlated $(\mathcal{F}_t)$ Brownian motions — we shall note ρ for the correlation. The parameters σ_i are fixed such as we match the cumulated default probabilities using

$$\mathbb{P}(\tau_i > t) = \frac{1}{\sqrt{\cosh\left(\sigma_i t \sqrt{2}\right)}}$$

In the following numerical application, we take $\sigma_i = 0.04$ which induces cumulated default probabilities quite close to historical data relative to BBB rated firms.

Since an explicit form of the joint distribution of the default times $\tau = (\tau_1, \tau_2)$ can be explicitly derived[1], one may be able to compute efficiently any correlation measures. The first one is the *discrete default correlation* which corresponds to $\operatorname{cor}\left(\mathbf{1}_{\{\tau_1 < t\}}, \mathbf{1}_{\{\tau_2 < t\}}\right)$ whereas the second one is the correlation between the two random survival times $\operatorname{cor}(\tau_1, \tau_2)$, called the *survival time correlation* by LI [2000]. We remark in the Figure 1 that this simple model does not suffice to produce significant correlations between defaults (moreover, the correlations are only positive). We shall try other ways to incorporate more dependency in these models, in the next section.

3 How to get more dependence ?

We no longer assume that the random thresholds are independent variables. By coupling the dependence of the thresholds together with the correlation between the intensity processes, we achieve to produce more realistic correlations between defaults. As the triggers are not market data, we shed light on the relationship between the input distribution on the thresholds and the output distribution of the defaults. But, we have first to give some useful definitions about copulae, which are an efficient tool to model dependence for non-Gaussian variables. Many ideas developed in this section were first explored by LI [2000], GIESECKE [2001], and SCHÖNBUCHER & SCHUBERT [2001].

[1] For an explicit form of $\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2)$ we send the reader to [4] Jouanin et al. [2001].

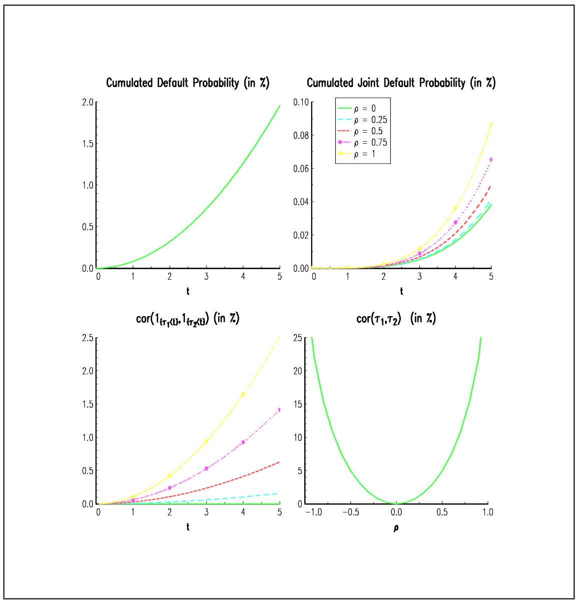

Figure 1: Influence of the correlation parameter on the first default time

3.1 basic ideas about copulae

Copulae are user-friendly tools for modelling dependence and turn to be widespread in finance. We just give here a quick definition, refering the interested reader to Nelsen [1999], which provides a more rigorous account.

First, we introduce some notations that will remain along the paper: for any two-dimensional random variable $X = (X_1, X_2)$, we denote for the marginal and joint survival probabilities:

$$\begin{cases} S^X(x_1, x_2) := \mathbb{P}(X_1 > x_1, X_2 > x_2), \\ \\ S_1^X(x_1) := \mathbb{P}(X_1 > x_1), \quad S_2^X(x_2) := \mathbb{P}(X_2 > x_2). \end{cases}$$

Now, a *survival copula* is almost the joint survival probability of any two-dimensional *uniform* random variables (U_1, U_2),

$$\breve{\mathbf{C}}^U(u_1, u_2) := S^U(1 - u_1, 1 - u_2).$$

For any two-dimensional random variables $X = (X_1, X_2)$, it is obvious for any user of Monte Carlo simulation methods that $S_1^X(X_1)$ and $S_2^X(X_2)$ are uniform random variates. They have therefore a survival copula, which we call the survival copula of X and write $\breve{\mathbf{C}}^X$ and we get Sklar's lemma

$$S^X(x_1, x_2) = \breve{\mathbf{C}}^{\mathbf{X}}(S_1^X(x_1), S_2^X(x_2)). \tag{3}$$

From a practical viewpoint, Sklar's lemma shows how copulae can be used to split the margins and the dependence of the joint distribution. From a theoretical angle, it also provides an easy way to derive copulae from well-known joint distributions. We give here two examples, with the Normal and Cook-Johnson families:

$$\breve{\mathbf{C}}_\rho^G(u_1, u_2) = \Phi_\rho(\phi^{-1}(u_1), \phi^{-1}(u_2))$$
$$\breve{\mathbf{C}}_\alpha^{CJ}(u_1, u_2) = \left(u_1^{-\alpha} + u_2^{-\alpha} - 1\right)^{-\alpha^{-1}}$$

with Φ_ρ the bi-variate normal distribution of parameter ρ, and ϕ the standard Gaussian cdf. We finally remark that the Cook-Johnson family enables to correlate univariate rare events.

3.2 the threshold approach

In this section, we still consider two defaultable firms, the default times of which are modelled as in (1). Here, we propose to model the dependence between default process in a two-step algorithm. As is the case with conditionally independent defaults, the first step consists in correlating the intensity processes. Then the second step deals with the choice of the copula $\breve{\mathbf{C}}^\theta$ of the thresholds (θ_1, θ_2) which are assumed to be independent from $\mathcal{F}_\infty$.

Within the threshold framework, we can derive a new pricing formula for firm 1's zero-coupon alternative to (2) on $\{\tau_1 > t, \tau_2 > t\}$:

$$B_1(t, T) = \mathbb{E}\left[e^{-\int_t^T r_s\,\mathrm{d}s}\,\frac{\breve{\mathbf{C}}^\theta\left(e^{-\int_0^T \lambda_s^1\,\mathrm{d}s}, e^{-\int_0^t \lambda_s^2\,\mathrm{d}s}\right)}{\breve{\mathbf{C}}^\theta\left(e^{-\int_0^t \lambda_s^1\,\mathrm{d}s}, e^{-\int_0^t \lambda_s^2\,\mathrm{d}s}\right)} \mid \mathcal{F}_t\right].$$

What is striking in this formula is the role of firm 2's intensity in the valuation of a claim depending *a priori* on firm 1's default only. In the case of the independent copula $\breve{\mathbf{C}}(u_1, u_2) = u_1 u_2$, we retrieve the usual formula (2) (corresponding to the case of conditionally independent defaults). Of course, a similar valuation formula still holds for more general contingent claims depending on both firms defaults.

3.3 linking threshold and default copulae

As the random triggers are not observable variables, it may be useful to derive a relationship between their copula and the implied dependence of default times. This will enable us to measure the spectrum of dependence between defaults allowed by our method and thus what we gained compared to the assumption of conditionally independent defaults.

Denoting $\breve{\mathbf{C}}^{\tau}$ for defaults' survival copula, we get from GIESECKE [2001]

$$\breve{\mathbf{C}}^{\tau}\left(S_1^{\tau}(t_1), S_2^{\tau}(t_2)\right) = \mathbb{E}\left[\breve{\mathbf{C}}^{\theta}\left(e^{-\int_0^{t_1}\lambda_s^1\,\mathrm{d}s}, e^{-\int_0^{t_2}\lambda_s^2\,\mathrm{d}s}\right)\right].$$

A special case of this formula is worth noticing: if intensities are deterministic, both survival copulae are equal. In particular, this shows that we can achieve high dependence between defaults, as we announced. This also suggests an alternative computational method, which consists in directly imposing the copula $\breve{\mathbf{C}}^{\tau}$ of default times. This technique is Li's original survival method, which we do not develop here because it requires heavy Monte-Carlo simulations for computing the price of contingent claims.

Having a look at the figure 2, we see that the threshold approach enables to reach a wide range of correlation for the multivariate default times distribution. So, one could be optimistic when calibrating the parameters. That means we can hope that the real correlation must be attainable within the above described framework.

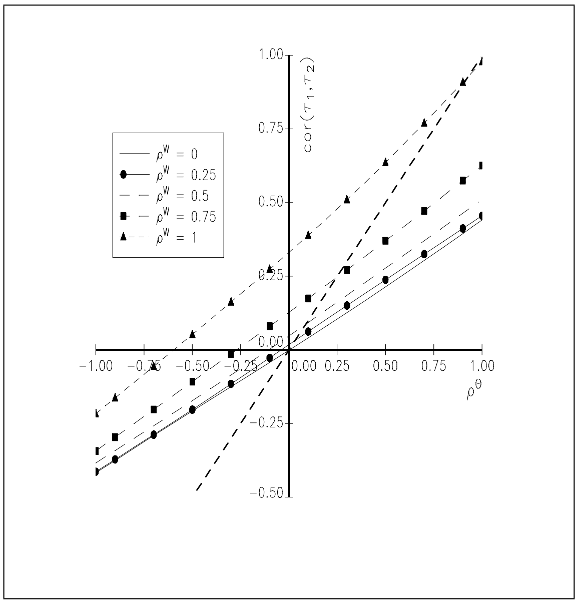

Figure 2: Influence of the correlation parameter on the first default time

4 Calibration issues

We end this paper with a discussion of the calibration procedure. We will only discuss the calibration of the thresholds' copula. Indeed the calibration of the intensity process can be carried out with available market data, such as CDS or risky bonds prices (we remark that at time 0 the zero-coupon price of firm 1 is still given by formula (2)). For the copula, a natural idea is to rely on correlation products, such as first-to-defaults. But as the market lacks liquidity, the information on correlations is still too much scarce to be useful.

As in DAVIS and LO [2001], we will assume that more reliable information can be drawn from Moody's diversity score. Practitioners assume a pre-specified contagion mechanism which enables them to cope with joint default probability issue. Those *a priori* models may rely either on economic intuition or statistical arguments. Of course, the quality of this calibration procedure will remain highly questionable.

4.1 using Moody's binomial technique

Recent waves of securitization on credit market may look like an efficient source of information. Indeed, returns of the different *tranches* of CDO are relevant to the default contagion in a pool of credits. Each *tranche* is noted following the Moody's *Binomial Expansion Technique.* That approach is based on the assumption that the distribution of the number of defaults among I risky issuers of the same sector before the maturity T could be summarized using only D independent issuers, i.e.

$$\sum_{i=1}^{I}\mathbf{1}_{\{\tau_i\le T\}} \overset{\text{law}}{=} \frac{I}{D}\sum_{d=1}^{D}\mathbf{1}_{\{\bar{\tau}_i\le T\}} \tag{4}$$

where $\left(\mathbf{1}_{\{\tau_1\le T\}},\ldots,\mathbf{1}_{\{\tau_I\le T\}}\right)$ are dependent Bernoulli random variables with parameters p_i ($i\in\{1,\ldots,I\}$) and $\left(\mathbf{1}_{\{\bar{\tau}_1\le T\}},\ldots,\mathbf{1}_{\{\bar{\tau}_D\le T\}}\right)$ are *i.i.d.* Bernoulli random variables with parameter $\bar{p}$. This technique is quite analogous to a classical Principal Component Analysis for Gaussian variables and D can be seen as a measure of the concentration of the defaults. According to Moody's, D is computed in order to match the two first moments on both sides of (4) on empirical observations.

Thus, in a first step, it occurred to us to try to calibrate the thresholds' copula to that market consensus. First we have to compute each firm's default conditional probability before T. Then we have to choose some *a priori* copula family $(\breve{\mathbf{C}}^{\theta}_{\alpha})_{\alpha\in A}$, the parameter α of which we want to estimate. Since Moody's technique is based on the knowledge of the default probability one has to condition the expectations by $\mathcal{F}_\infty$ during calibration procedure so that

$$p_i = 1 - \exp\left(-\int_0^T \lambda^i_s \,\mathrm{d}s\right).$$

The two first moments procedure induce the following equalities[2]

$$\sum_{i=1}^{I} p_i = \bar{p}I$$

$$\sum_{i<j} \breve{\mathbf{C}}^{\theta}_{\alpha}\left(1, p_i, \ldots, p_j, 1\right) = \frac{\bar{p}I(\bar{p}I-1)}{2} + \frac{\bar{p}(1-\bar{p})I^2}{2D} \quad (5)$$

First equation gives the common default probability $\bar{p}$ then an implied value for the parameter α stems from the second equation.

4.2 Discussion of the Calibration

Three factors are likely to vary in (5): the choice of the copula family $\breve{\mathbf{C}}^{\theta}$, the default probabilities p_i and the number of firms I.

Since we aim at matching the diversity score of Moody's given by

$$D_{\text{Moody's}} = \frac{-1+\sqrt{1+8I}}{2}$$

we are going to optimize α so that one meets (5) as best as possible for all $I \leq 10$. Moody's diversity score holds for all $\bar{p}$ so that a quite legitimate property the copula should satisfy is to be robust with respect to the average default probability $\bar{p}$ (especially for small $\bar{p}$). In particular, the copula should be able to correlate rare events.

Let us define the the lower tail dependence function $\lambda^{\theta}_L(u) = \frac{\mathbf{C}^{\theta}(u,u)}{u}$. When u tends to zero, $\lambda^{\theta}_L(u)$ represents the probability that one variable is very small given that the other is very small, so that in our context it may be linked to the probability of default of an issuer given that another issuer has already defaulted. If $\lambda^{\theta}_L(u)$ has limit zero, then the chosen copula does not enable to correlate the default processes in a better way than the conditionally independent approach does.

Another view of the same problem is to look at the relation between the diversity score and lower tail dependence function, assuming that each issuer gets the same default probability p, we get from (5):

$$D = \frac{(1-p)I}{(1-pI)+(I-1)\lambda^{\theta}_L(p)}.$$

Once again, if $\lambda^{\theta}_L(p)$ tends to zero (e.g. for the Normal copula) then D tends to I, which is not relevant with Moody's diversity score.

Finally, as far as the choice of the copula is concerned, we first acknowledge a possible non negligible model risk in the choice of the copula family. In KMV or CreditMetrics methodologies, it is usually assumed that latent variables are correlated through a Normal copula. As we mentioned before, the calibration will not succeed for this family, since this copula behaves like the independent copula in the range of very low probabilities. A more realistic family may be the Cook-Johnson family. In the figure 3, we can notice that the Cook-Johnson copula is more robust with respect to default probability than the Normal copula. Indeed, the calibration procedure leads to a quite steady optimal parameter when p ranges from 1 to 500 bps, whereas the implied correlation of the Normal copula tends to 1 when p tends to 0.

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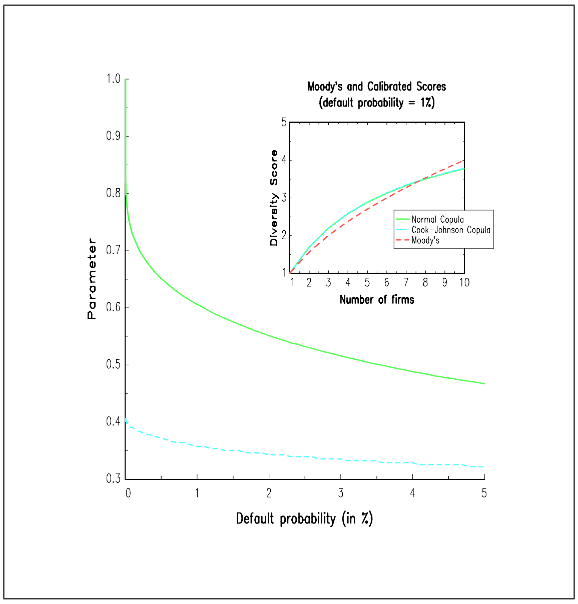

Figure 3: Influence of the default probability and the choice of the copula

www.gro.creditlyonnais.fr/content/wp/copula-intensity.pdf

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