A note about the conjecture about Spearman's rho and Kendall's tau

V. Durrleman
Operations Research and Financial Engineering, Princeton University, USA

A. Nikeghbali
University Paris VI, France

T. Roncalli*
Groupe de Recherche Opérationnelle, Crédit Lyonnais, France

October 23, 2000

Abstract

In this paper, we consider the open question on Spearman's rho and Kendall's tau of Nelsen [1991]. Using a technical hypothesis, we can answer in the positive. One question remain open: how can we understand the technical hypothesis? Because this hypothesis is not right in general, we could find some pathological cases which contradicts the Nelsen's conjecture.

1 Nelsen's conjecture

We consider the open question of Nelsen [1991]:

> *It is well-known that when sampling from a bivariate population in which X and Y are independent (or nearly so), the sample statistic corresponding to ϱ is about 50% larger than that corresponding to τ. [...] suppose $\{\mathbf{C}_\theta(x,y)\}$ is a family of copulas induced by the (possibly multidimensional) parameter θ such that $C_{\theta_0} = C^\perp$ — the product copula — and C_θ is a continuous function of θ at θ_0. For all such families considered in this section, we have*
>
> $$\lim_{\theta \longrightarrow \theta_0} \frac{\mathrm{d}\varrho}{\mathrm{d}\tau} = \frac{3}{2} \tag{1}$$
>
> *Does this always hold for such families of copulas?*

1.1 The general framework

We are given a family of copulas indexed by a parameter $\theta \in A \subset \mathbb{R}^k$ where A is a rectangle containing θ_0. which correspond to the case $\mathbf{C}_{\theta_0} = \mathbf{C}^\perp$. Let us denote $\mathcal{C}$ the set of all copulas. We assume that the function $\theta \in A \longmapsto \mathbf{C}_\theta \in \mathcal{C}$ is continuous. We shall also assume that $\varrho_\theta \neq 0$ for every $\theta \in A \setminus \{\theta_0\}$.

We assume eventually that for every θ in A, $\mathbf{C}_\theta$ is absolutely continuous with respect to Lebesgue's measure. We make also the assumption that $\partial_x \mathbf{C}_\theta \, \partial_y \mathbf{C}_\theta$ is integrable for every θ in A.

* *Corresponding author:* Groupe de Recherche Opérationnelle, Bercy-Expo — Immeuble Bercy Sud — 4$^{\text{è}}$ étage, 90 quai de Bercy — 75613 Paris Cedex 12 — France; *E-mail adress:* thierry.roncalli@creditlyonnais.fr

1.2 The technical hypothesis

Let $\xi_\theta(x,y) = \mathbf{C}_\theta(x,y) - \mathbf{C}^\perp(x,y)$. The result we will state in sequel holds under the following technical hypothesis:

$$\int_0^1 \int_0^1 \frac{\partial_x \xi_\theta(x,y)\ \partial_y \xi_\theta(x,y)}{\int_0^1 \int_0^1 \xi_\theta(x,y)\ \mathrm{d}x\,\mathrm{d}y} \mathrm{d}x\,\mathrm{d}y \longrightarrow 0 \tag{2}$$

when $\theta \longrightarrow \theta_0$. Note that this expression makes sense since we assume that $\varrho_\theta \neq 0$ for every $\theta \in A \setminus \{\theta_0\}$. Its understanding is not clear for the moment, but we give some basic examples where it is satisfied:

- We assume that $k = 1$ and $\theta_0 = 0$. If $\xi_\theta(x,y) = \theta f(x,y)$ for some real function f, the expression in (2) equals to
$$\theta \int_0^1 \int_0^1 \frac{\partial_x f(x,y)\ \partial_y f(x,y)}{\int_0^1 \int_0^1 f(x,y)\ \mathrm{d}x\,\mathrm{d}y} \mathrm{d}x\,\mathrm{d}y \tag{3}$$
and tends to 0 as $\theta \longrightarrow \theta_0$.

- We assume that $k = 1$ and $\theta_0 = 0$. If $\partial_x \mathbf{C}_\theta, \partial_y \mathbf{C}_\theta \in C([0,1]\times[0,1],[0,1])$ and $\mathbf{C}_\theta$ are differentiable at point θ_0, then
$$\begin{aligned} \xi_\theta(x,y) &= \theta f(x,y) + o(\theta) \\ \frac{\partial \xi_\theta}{\partial x}(x,y) &= \theta g_x(x,y) + o(\theta) \\ \frac{\partial \xi_\theta}{\partial y}(x,y) &= \theta g_y(x,y) + o(\theta) \end{aligned} \tag{4}$$
and the expression in (2) equals to
$$\theta \int_0^1 \int_0^1 \frac{[g_x(x,y) + o(1)]\,[g_y(x,y) + o(1)]}{\int_0^1 \int_0^1 [f(x,y) + o(1)]\ \mathrm{d}x\,\mathrm{d}y} \mathrm{d}x\,\mathrm{d}y \tag{5}$$
where the $o(1)$'s are uniform in (x,y). The hypothesis is also satisfied.

- We do not assume that $k = 1$. In order to satisfy the hypothesis, we want to apply a limit theorem. Let us define $\varpi(x,y)$ as follows
$$\varpi(x,y) = \frac{\partial_x \xi_\theta(x,y)\ \partial_y \xi_\theta(x,y)}{\int_0^1 \int_0^1 \xi_\theta(x,y)\ \mathrm{d}x\,\mathrm{d}y} \tag{6}$$
The hypothesis that must be checked are then $\varpi(x,y) \longrightarrow 0$ when $\theta \longrightarrow \theta_0$ for almost all (x,y) and $\varpi(x,y)$ is uniformly integrable.

1.3 The main result

Theorem 1 *With the above framework and hypothesis, we state that the assumption (1) holds.*

Proof. We denote c the density of the copula $\mathbf{C}_\theta$ which exists by virtue of the hypothesis. Because $\partial_x \mathbf{C}_\theta\, \partial_y \mathbf{C}_\theta$ is integrable, we can write

$$\begin{aligned} \tau &= 1 - 4 \int_0^1 \int_0^1 \int_0^x \int_0^y c(x,v)\, c(u,y)\ \mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\ &= -4 \int_0^1 \int_0^1 \int_0^x \int_0^y (c(x,v)\, c(u,y) - 1)\ \mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\ &= -4 \int_0^1 \int_0^1 \int_0^x \int_0^y ((c(x,v) - 1)(c(u,y) - 1) + c(u,y) - 1 + c(x,v) - 1)\ \mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{7}$$

and we also have

$$\varrho = 12\int_0^1\int_0^1\int_0^x\int_0^y (c(u,v)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \tag{8}$$

We need then a lemma.

Lemma 2 *We verify that*

$$\begin{aligned}\int_0^1\int_0^1\int_0^x\int_0^y (c(u,y)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_0^1\int_0^x\int_0^y (c(x,v)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_0^1\int_0^1\int_0^x\int_0^y (c(u,v)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \end{aligned}\tag{9}$$

Proof. We first prove that the first term equals to the third one. The first term reduces to

$$\int_0^1\int_0^1\int_0^x yc(u,y)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{4} \tag{10}$$

Using Fubini's theorem[1] for the third term, we get

$$\begin{aligned}\int_0^1\int_0^x\int_0^1\int_v^1 (c(u,v)-1)\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x &= \int_0^1\int_0^x\int_0^1 (1-v)(c(u,v)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x \\ &= \frac{1}{2} - \int_0^1\int_0^1\int_0^x vc(u,v)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{4}\end{aligned}\tag{11}$$

We then prove that the second term equals to the third one. The second term reduces after two uses of Fubini's theorem to

$$\int_0^1\int_0^1\int_0^y\int_0^x c(x,v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x - \frac{1}{4} = \int_0^1\int_0^1\int_0^y xc(x,v)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x - \frac{1}{4} \tag{12}$$

Using Fubini's theorem in the third expression, we get

$$\begin{aligned}\int_0^1\int_0^y\int_0^1\int_u^1 (c(u,v)-1)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y &= \int_0^1\int_0^y\int_0^1 (1-u)(c(u,v)-1)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y \\ &= \frac{1}{2} - \int_0^1\int_0^1\int_0^y uc(u,v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y - \frac{1}{4}\end{aligned}\tag{13}$$

and this completes the proof of the lemma. ■

Using the result of the lemma and the calculations made for τ and ϱ, we have

$$\begin{aligned}\tau &= \frac{2}{3}\varrho - 4\int_0^1\int_0^1\int_0^x\int_0^y (c(x,v)-1)(c(u,y)-1)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{3}\varrho - 4\int_0^1\int_0^1 \partial_x\xi_\theta(x,y)\,\partial_y\xi_\theta(x,y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}\tag{14}$$

and we conclude that the assumption (1) holds as soon as the technical hypothesis (2) is satisfied. ■

2 More results

2.1 The independent case

$\frac{\tau}{\varrho} = \frac{2}{3}$ is not the only way to go continuously to the independent copula. In fact as we shall see, there is another way.

[1] Every quantity is positive here.

For an obvious reason, we were in the case where $\varrho \neq 0$ for every $\theta \in A \setminus \{\theta_0\}$. We can wonder whether we may continuously go to the independent copula with $\varrho = 0$ and $\tau \neq 0$. The answer is given by the following family

$$\begin{aligned} \mathbf{C}_\theta (u,v) &= uv + \frac{\theta}{32} \left(4 \left(\left(u - \frac{1}{2} \right)^2 + \left(v - \frac{1}{2} \right)^2 \right) - 1 \right)^3 \times \\ & \left(20 \left(\left(u - \frac{1}{2} \right)^2 + \left(v - \frac{1}{2} \right)^2 \right) - 1 \right) \mathbf{1}_{\left[\left(u - \frac{1}{2} \right)^2 + \left(v - \frac{1}{2} \right)^2 < \frac{1}{4} \right]} \end{aligned} \tag{15}$$

for $\theta \in (-1, 1)$.

2.2 The case where $\varrho = 0$ and $\tau = 0$

We have shown that t**here are only two ways to go regularly to the independent copula**. We may be interested in knowing whether this remarkable property of the independent copula holds for the more general class of copulas for which $\varrho = 0$ and $\tau = 0$. The answer is **no**.

We can try to do the same work as we did for the independent copula. Let $\mathbf{C}^\star$ be the copula for which $\varrho = 0$ and $\tau = 0$ and C_θ be a family of copulas defined as follows

$$\mathbf{C}_\theta (u,v) = \mathbf{C}^\star (u,v) + \theta f (u,v) \tag{16}$$

where f is a suitable "perturbation". The same calculations lead to the following equation:

$$\frac{\tau}{\varrho} = \frac{2}{3} \frac{\iint_{\mathbf{I}^2} \partial^2_{x,y} \mathbf{C}^\star (u,v) \ f(u,v) \ \mathrm{d}u \, \mathrm{d}v}{\iint_{\mathbf{I}^2} f(u,v) \ \mathrm{d}u \, \mathrm{d}v} - \frac{\theta}{3} \frac{\iint_{\mathbf{I}^2} \partial^2_{x,y} f(u,v) \ f(u,v) \ \mathrm{d}u \, \mathrm{d}v}{\iint_{\mathbf{I}^2} f(u,v) \ \mathrm{d}u \, \mathrm{d}v} \tag{17}$$

The fact that $\partial^2_{x,y} \mathbf{C}^\star (u,v) = 1$ for the independent copula cannot be used here. Nevertheless, we give an example where we go to the following copula in a very unusual way.

Let $\mathbf{C}$ be the two place function defined by

$$\mathbf{C}(u,v) = uv + \alpha \left[u(u-1)(2u-1) \right] \left[v(v-1)(2v-1) \right] \tag{18}$$

with $\alpha \in [-1, 2]$.

Theorem 3 $\mathbf{C}$ *is a copula function.*

Proof. The family given by equation (18) is a sub-family of one defined by Nelsen [1998]:

$$\mathbf{C}(u,v) = uv + uv(u-1)v(v-1) \left[a + b(1-2u)(1-2v) \right] \tag{19}$$

where $b \in [-1, 2]$ and $a \in \mathbb{R}$ such that $|a| \leq b + 1$ for $b \in \left[-1, \frac{1}{2}\right]$ and $|a| \leq \sqrt{6b - 3b^2}$ for $b \in \left[\frac{1}{2}, 2\right]$. If we set a to zero, it comes that $b \geq -1$ for $b \in \left[-1, \frac{1}{2}\right]$ and $b \leq 2$ for $b \in \left[\frac{1}{2}, 2\right]$. The two place function (18) is also a **copula with cubic sections** (see section 3.2.5 of Nelsen [1998]). $\blacksquare$

In figure 1, we have represented the density of the cubic copula[2] with α equal respectively to -1, -0.5, 1 and 2. The contours of density correspond to figure 2. We remark that the mass distribution is symmetric about the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, which explain that $\varrho = 0$ and $\tau = 0$. In order to understand more easily this dependence structure, we have plotted the density of the bivariate random variables (X, Y) when the margins are normal in figure 3.

[2] which is equal to

$$\frac{\partial^2}{\partial u \, \partial v} \mathbf{C}(u,v) = 1 + \alpha \quad 6u^2 - 6u + 1 \quad 6v^2 - 6v + 1 \tag{20}$$

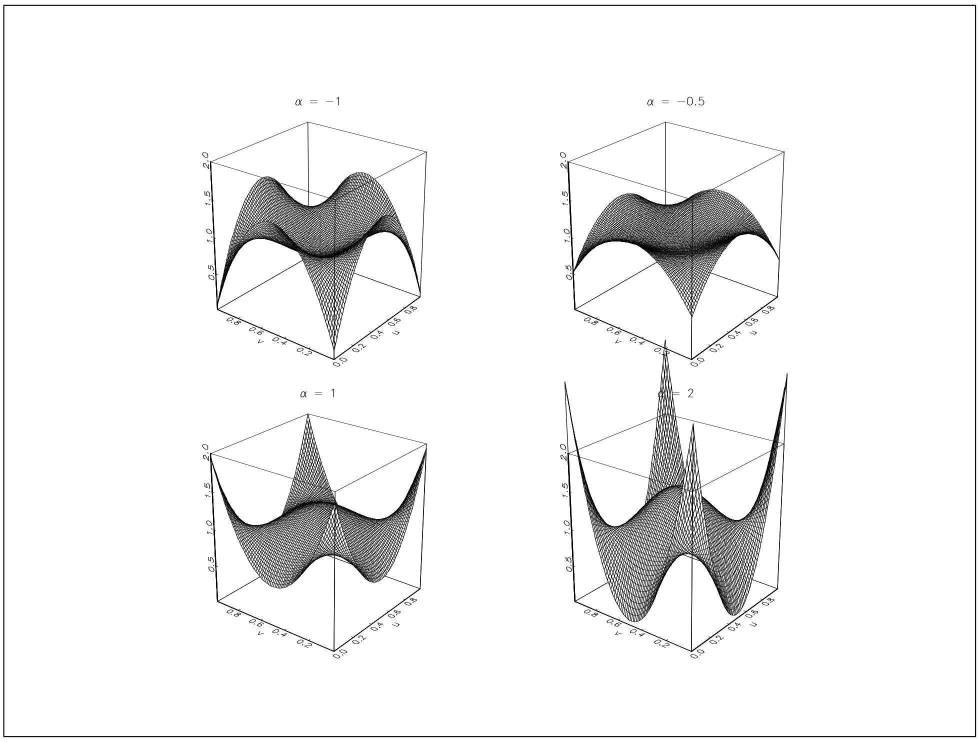

Figure 1: Density function for the cubic copula

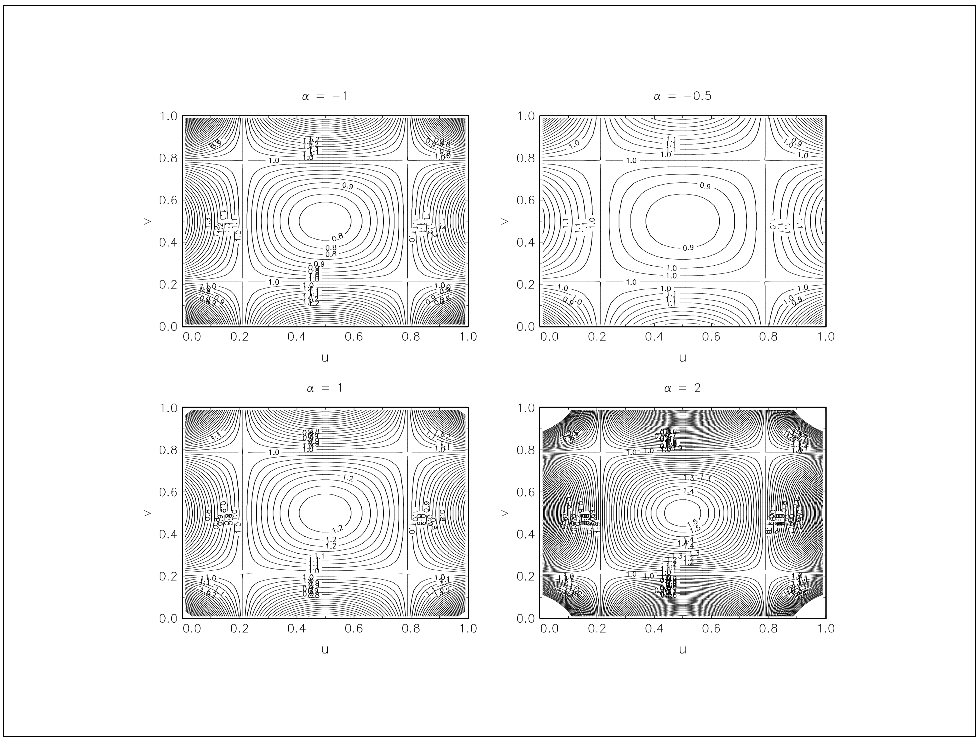

Figure 2: Contours of density for the cubic copula

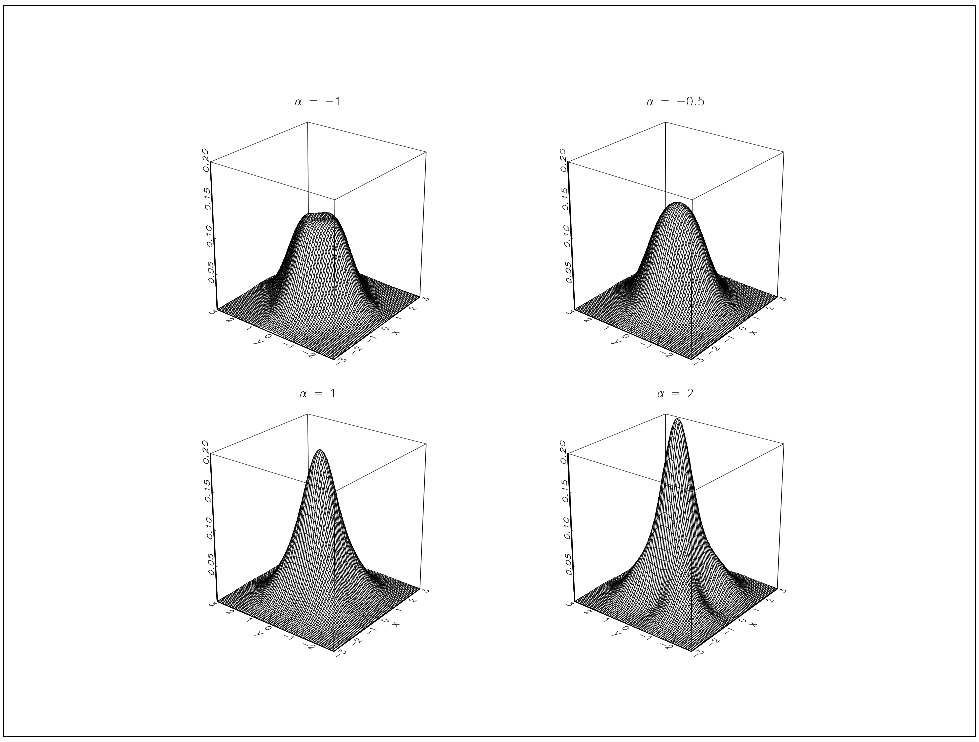

Figure 3: Probability density function with Gaussian marginals

Theorem 4 *The Kendall's tau and the Spearman's rho of the cubic copula (18) is 0 for any value of α.*

Proof. If a copula $\mathbf{C}$ satisfies $\mathbf{C}(u,v) = u - \mathbf{C}(u,1-v)$ or $\mathbf{C}(u,v) = v - \mathbf{C}(1-u,v)$, we know that $\tau = \varrho = 0$ (see exercise 5.19 of Nelsen [1998]). We can check that the cubic copula satisfy the two conditions:

$$\begin{aligned} u - \mathbf{C}(u,1-v) &= u - [u(1-v) + \alpha[u(u-1)(2u-1)][-v(1-v)(1-2v)]] \\ &= uv + \alpha[u(u-1)(2u-1)][v(v-1)(2v-1)] \\ &= \mathbf{C}(u,v) \end{aligned} \tag{21}$$

■

Let $\mathbf{C}^{\star} = \mathbf{C}_1$ and

$$f(u,v) = uv(1-u)(1-v)(u-2v)\left(u - \frac{716}{1235}v\right) \tag{22}$$

After routine calculations, it turns out that

$$\varrho = \frac{6}{6175}\theta \tag{23}$$

and

$$\tau = -\frac{6}{6175}\theta + \frac{329606}{343175625}\theta^2 \tag{24}$$

Here $\theta \in (-1.2, 1.7)$. We have represented the contours of density of this copula family in figure 4. The $\tau - \varrho$ region corresponds to figure 5.

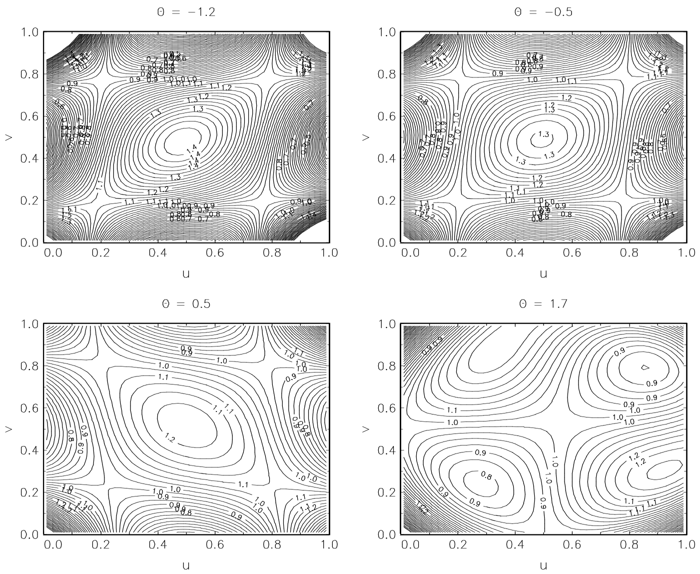

Figure 4: Contours of density for the ‘perturbed’ cubic copula

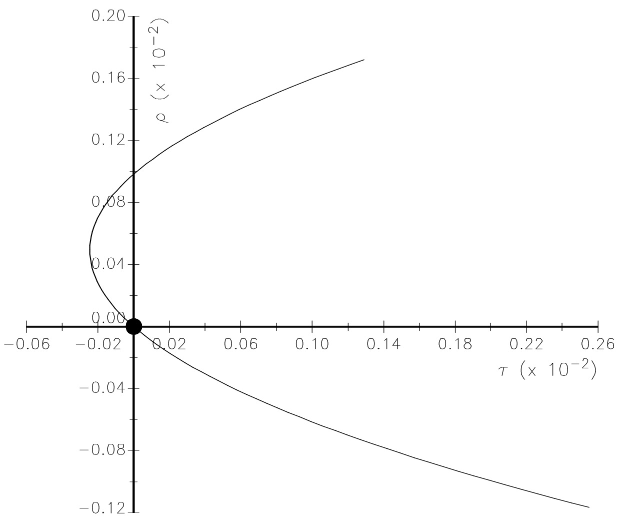

Figure 5: $\tau - \varrho$ region of the ‘perturbed’ cubic copula

2.3 A remark on the power of tests for the independence of two random variables

The example of the cubic copula (18) shows that Kendall's tau and Speraman's rho can not be used to test the independence. We may consider now the power of tests for independence if the copula of the corresponding randoms variables verifies $\varrho = \tau = 0$ but is not the product copula. For example, if we use the χ^2 test for contingency table data, we obtain figure 6 for the density of the associated test[3]. In particular, we remark the very poor power[4] of the χ^2 test when α is closed to 0. If we use standard dependence measures[5] as the Schweizer and Wolff's σ or the Hoeffding's index Φ^2, we obtain a similar result. We have represented them[6] in figure 7. Figure 8 shows the distribution of the Hoeffding's index estimator based on the empirical copula[7] with 250 observations. We remark that the power of the test[8] is very closed to the Type I error when α is small!

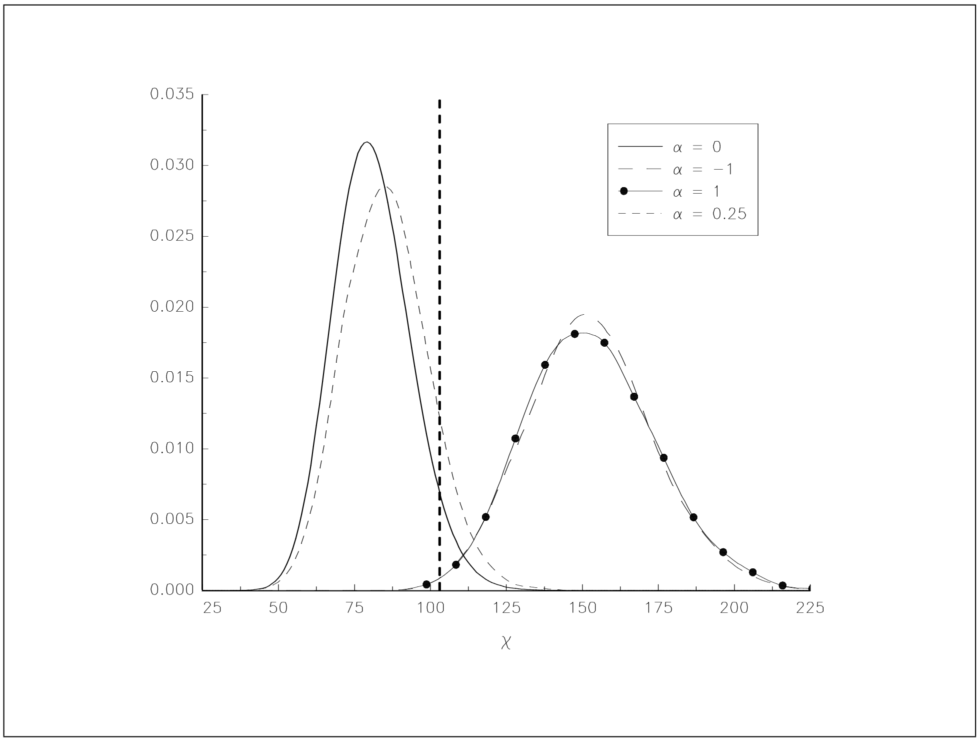

Figure 6: χ^2 test for independence and the cubic copula

[3]The number of observations is fixed to 2000. To build the two-way table, we consider 10 classes of same length.

[4]For $\alpha = 0.25$ and 2000 observations, the power of the test is equal to 0.11 when the Type I error is 5%, which corresponds to the vertical solid line in figure 6.

[5]see [2], [3] and [6].

[6]We have

$$\Phi = \frac{1}{7\sqrt{10}} |\alpha| \tag{25}$$

and

$$\sigma = \frac{3}{64} |\alpha| \tag{26}$$

[7]Let $\hat{\mathbf{C}}$ be the empirical (or Deheuvels) copula. An estimator of the Hoeffding's index is given by the following formula

$$\hat{\Phi} = \sqrt{\frac{90}{T^2 - 1} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \left(\hat{\mathbf{C}} \left(\frac{t_1}{T}, \frac{t_2}{T} \right) - \frac{t_1 t_2}{T^2} \right)^2} \tag{27}$$

where T is the number of observations.

[8]For $\alpha = 0.25$ and 250 observations, the power of the test is equal to 0.06 when the Type I error is 5%, which corresponds to the vertical solid line in figure 8.

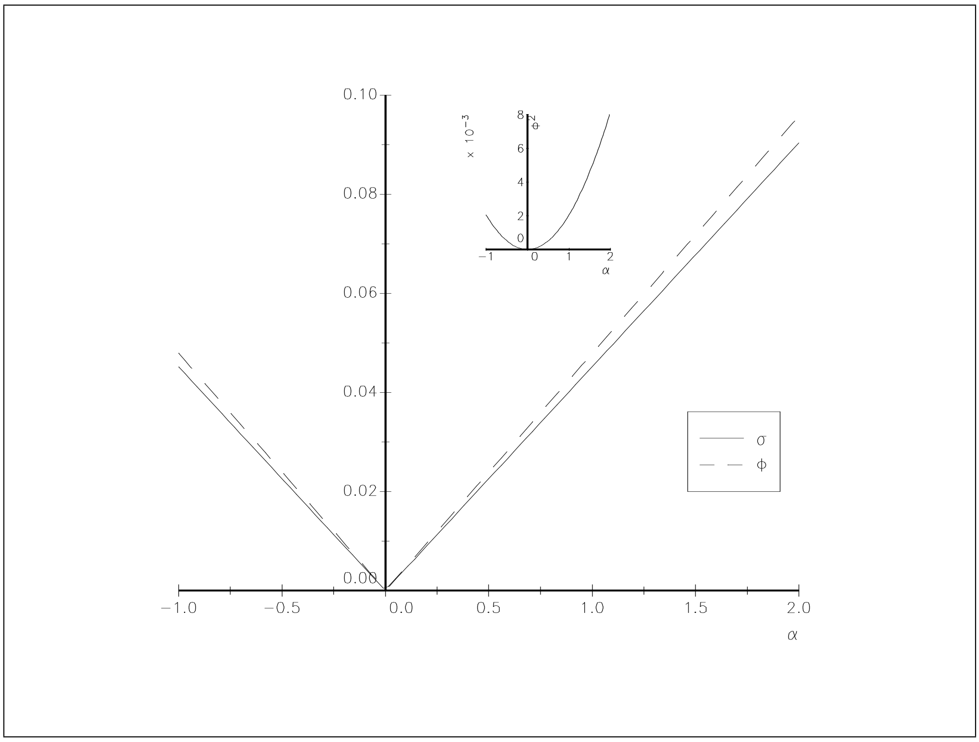

Figure 7: Dependence measures of the cubic copula

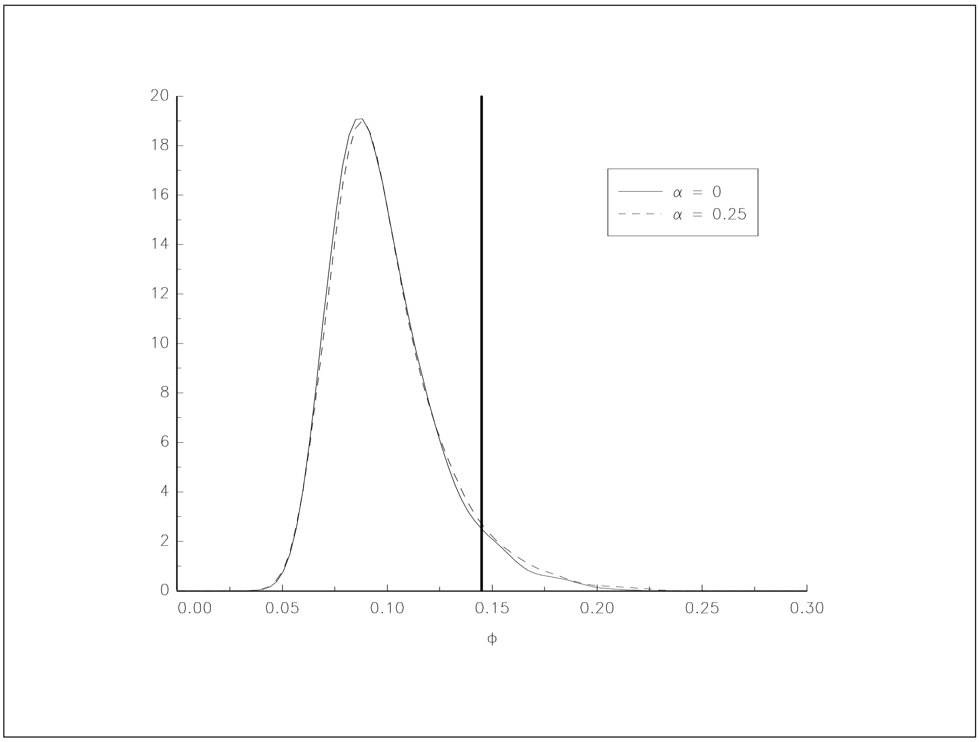

Figure 8: Φ test for independence and the cubic copula

References

[1] DALL'AGLIO, G., S. KOTZ and G. SALINETTI [1991], Advances in Probability Distributions with Given Marginals (Beyond the Copulas), Kluwer Academic Publishers, Dordrecht

[2] FISHER, N.I. and P.K. SEN (Eds.) [1994], The Collected Works of Wassily Hoeffding, *Springer Series in Statistics*, Springer-Verlag, New York

[3] HOEFFDING, W. [1940], Masstabinvariante Korrelationtheorie, *Schriften des Mathematischen Instituts und des Instituts für Angewandte Mathematik der Universität Berlin*, **5**, 181-233

[4] NELSEN, R.B. [1991], Copulas and association, in G. Dall'Aglio, S. Kotz and G. Salinetti (Eds.), *Advances in Probability Distributions with Given Marginals (Beyond the Copulas)*, Kluwer Academic Publishers, Dordrecht

[5] NELSEN, R.B. [1998], An Introduction to Copulas, *Lectures Notes in Statistics*, **139**, Springer Verlag, New York

[6] SCHWEIZER, B. and E. WOLFF [1981], On nonparametric measures of dependence for random variables, *Annals of Statistics*, **9**, 879-885