Some Practical Issues on Credit Risk

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Agenda

1. New Trends on the Market of Credit Risk
2. Credit Risk Modelling in the New Basle Capital Accord
3. Extending the Basle II model
4. The Measurement of Credit Risk
5. Credit Portfolio Management
6. The Time-inconsistency of the Copula Model
1 New Trends on the Market of Credit Risk

This report is Moody's fifteenth annual study of corporate debt defaults. It comes a critical juncture for the capital markets worldwide. Record defaults — unreached in a number and dollar volume since the Great Depression — have culminated in the bankruptcies of well-known firms whose rapid collapse caught investors by surprise. In the wake of these failures, concern for credit quality has grown to a level not seen in seventy years.

- The default rate for all Moody's-rated corporate bond issuers ended 2001 at 3.7%. For speculative-grade rated issuers, the default rate reached 10.2%.
- Rating downgrades exceed rating upgrades 1.9 to 1 in 2001.
- The average recovery rate of defaulted bonds fell to a record low of 21% of par.
### 1.2 Credit Derivatives

![Graph showing Notional Outstanding ($ bn) for CDOs in Europe from 1996 to 2002.

<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>24</td>
<td>50</td>
<td>133</td>
<td>144</td>
</tr>
<tr>
<td>Volume ($ bn)</td>
<td>5</td>
<td>5.7</td>
<td>4.5</td>
<td>29.2</td>
<td>63.2</td>
<td>106</td>
<td>143.4</td>
</tr>
</tbody>
</table>

*Source: Moody's Investor Service*
2 Credit Risk Modelling in the New Basle Capital Accord
2.1 The New Basel Capital Accord

The 1988 Capital Accord concerns only credit risk (and market risk — Amendment of January 1996) ⇒ the Cooke Ratio requires capital to be at least 8 percent of the “risk” of the bank.

- January 2001: proposal for a New Basel Capital Accord (credit risk measurement will be more risk sensitive + explicit capital calculations for operational risk)
- November 2002: QIS 3 (Quantitative Impact Study)

⇒ The objectives of the New Accord are the following:

1. Capital calculations will be more risk sensitive.
2. Convergence between economic capital (internal measure) and regulatory capital.
The McDonough ratio

It is defined as follows:

\[
\frac{\text{Capital (Tier I + Tier II)}}{\text{credit risk + market risk + operational risk}} \geq 8\%
\]

The aim of allocation for the industry is

<table>
<thead>
<tr>
<th>Risk</th>
<th>January 2001</th>
<th>Now</th>
</tr>
</thead>
<tbody>
<tr>
<td>Credit</td>
<td>75%</td>
<td>83%</td>
</tr>
<tr>
<td>Market</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>Operational</td>
<td>20%</td>
<td>12%</td>
</tr>
</tbody>
</table>
The measurement methods

Risk weighted assets are calculated as follows:

\[ \text{RWA} = \text{EaD} \times \text{RW} \]

1. **Standardized Approach (SA)**
   
   The risk weights are based on external ratings:

<table>
<thead>
<tr>
<th>Rating</th>
<th>AAA/AA-</th>
<th>A+/A-</th>
<th>BBB+/BBB-</th>
<th>BB+/B-</th>
<th>B-/C</th>
<th>non rated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sovereign</td>
<td>0%</td>
<td>20%</td>
<td>50%</td>
<td>100%</td>
<td>150%</td>
<td>100%</td>
</tr>
<tr>
<td>Bank</td>
<td>1</td>
<td>20%</td>
<td>50%</td>
<td>100%</td>
<td>150%</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>20%</td>
<td>50%</td>
<td>100%</td>
<td>150%</td>
<td>50%</td>
</tr>
<tr>
<td></td>
<td>2 (-3M)</td>
<td>20%</td>
<td>20%</td>
<td>50%</td>
<td>150%</td>
<td>20%</td>
</tr>
<tr>
<td>Corporate</td>
<td>20%</td>
<td>50%</td>
<td>BBB+/BB-</td>
<td>B+/C</td>
<td>150%</td>
<td>100%</td>
</tr>
</tbody>
</table>

2. **Internal Rating Based Approach (IRB)**

   \[ \text{RW} = c \cdot \text{LGD} \cdot \text{RC} (PD) \]

   1. *foundation approach*
   
   2. *advanced approach*
Percentage Changes in Capital Requirements for G10 Banks

(a) Standardised Approach

(b) IRB Foundation Approach

Source: QIS2, BCBS, BIS (November 2001)
2.2 The IRB Approach

The (original) IRB risk weights are

$$RW = \min \left( \frac{LGD}{50} \times BRW(PD), 12.5 \times LGD \right)$$

where BRW is a benchmark function calibrated on a 50% LGD

$$BRW(PD) = 976.5 \times \Phi \left( 1.118 \times \Phi^{-1}(PD) + 1.288 \right) \times \left( 1 + 0.470 \times \frac{1 - PD}{PD^{0.44}} \right)$$
Let us consider a portfolio $\Pi$ with $I$ loans. The loss is

$$L = \sum_{i=1}^{I} \text{EaD}_i \cdot \text{LGD}_i \cdot 1\{\tau_i \leq t_i\}$$

We assume that the default probability $P_i = \Pr\{\tau_i \leq t_i\}$ is $P_i(X)$ where $X$ is the systematic factor with distribution $H$. For the Infinitely fine-grained portfolio $\Pi_\infty$ ‘equivalent’ to the original portfolio $\Pi$, we have

$$\Pr\{L_\infty = \mathbb{E}[L] \mid X\} = 1$$

If $P_i$ are increasing functions with respect to $X$, the percentile $\alpha$ of the loss distribution is

$$F^{-1}_\infty(\alpha) \ := \ \sum_{i=1}^{I} \text{EaD}_i \cdot \mathbb{E}[\text{LGD}_i] \cdot P_i(H^{-1}(\alpha))$$

**risk contribution of the loan $i$**
IRB approach explained (from Wilde [2001])

Merton/Vasicek model

\[ Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i \]

\[ D_i = 1 \{ \tau_i \leq t_i \} \Leftrightarrow Z_i < B_i \]

\( P_i \) is the unconditional default probability

\[ P_i(X) = \Pr \{ D_i = 1 \mid X \} = \Phi \left( \frac{\Phi^{-1}(P_i) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right) \]

\[ \text{RC}_i = \text{EaD}_i \cdot \mathbb{E} [\text{LGD}_i] \cdot \Phi \left( \frac{\Phi^{-1}(P_i) - \sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}} \right) \]

With \( \alpha = 99.5\% \) and \( \rho = 20\% \), we have

\[ \text{RC}_i = \text{EaD}_i \cdot \mathbb{E} [\text{LGD}_i] \cdot \Phi \left( 1.118\Phi^{-1}(P_i) + 1.288 \right) \]
If $RW(PD = 0.7\%, LGD = 50\%) = 100\%$, we have

$$BRW(PD) = \frac{619.59}{3Y \text{ scaling factor}} \times \Phi \left( 1.118 \times \Phi^{-1} \left( 1 - (1 - PD)^{3} \right) + 1.288 \right)$$

3Y conditional default probability

**Basel’s approximation formula:**

$$BRW(PD) = \frac{976.5}{1Y \text{ scaling factor}} \times \Phi \left( 1.118 \times \Phi^{-1} (PD) + 1.288 \right)$$

1Y conditional default probability

$$\times \left( 1 + 0.470 \times \frac{1 \text{ } - \text{ } PD}{PD^{0.44}} \right)$$

3Y maturity adjustment
Infinitely fine-grained portfolio by the example

Vasicek model — Rating CCC

\[ \text{EAD} = 1 - \text{LGD} \sim B(3,3) \quad \text{PD} = 17.5\% - \rho = 20\% \]

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\mathbb{E}[L_n \mid X]$</th>
<th>$\text{VaR}[L_n]$</th>
<th>rel. diff.</th>
<th>$\mathbb{E}[L_n \mid X]$</th>
<th>$\text{VaR}[L_n]$</th>
<th>rel. diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n = 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.370</td>
<td>0.286</td>
<td>29.502</td>
<td>7.402</td>
<td>7.361</td>
<td>0.553</td>
</tr>
<tr>
<td>75%</td>
<td>0.599</td>
<td>0.720</td>
<td>-16.853</td>
<td>11.979</td>
<td>12.146</td>
<td>-1.374</td>
</tr>
<tr>
<td>90%</td>
<td>0.858</td>
<td>1.213</td>
<td>-29.304</td>
<td>17.153</td>
<td>17.566</td>
<td>-2.349</td>
</tr>
<tr>
<td>99%</td>
<td>1.368</td>
<td>2.114</td>
<td>-35.314</td>
<td>27.354</td>
<td>28.280</td>
<td>-3.275</td>
</tr>
<tr>
<td>99.5%</td>
<td>1.490</td>
<td>2.333</td>
<td>-36.121</td>
<td>29.800</td>
<td>30.820</td>
<td>-3.310</td>
</tr>
<tr>
<td>99.9%</td>
<td>1.729</td>
<td>2.763</td>
<td>-37.431</td>
<td>34.577</td>
<td>35.642</td>
<td>-2.989</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n = 500$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>37.008</td>
<td>37.101</td>
<td>-0.249</td>
<td>370.085</td>
<td>369.291</td>
<td>0.215</td>
</tr>
<tr>
<td>75%</td>
<td>59.895</td>
<td>60.112</td>
<td>-0.362</td>
<td>598.947</td>
<td>597.975</td>
<td>0.163</td>
</tr>
<tr>
<td>90%</td>
<td>85.765</td>
<td>86.164</td>
<td>-0.463</td>
<td>857.649</td>
<td>857.280</td>
<td>0.043</td>
</tr>
<tr>
<td>95%</td>
<td>102.993</td>
<td>103.579</td>
<td>-0.566</td>
<td>1029.929</td>
<td>1030.332</td>
<td>-0.039</td>
</tr>
<tr>
<td>99%</td>
<td>136.768</td>
<td>137.854</td>
<td>-0.788</td>
<td>1367.684</td>
<td>1367.906</td>
<td>-0.016</td>
</tr>
<tr>
<td>99.5%</td>
<td>149.000</td>
<td>150.168</td>
<td>-0.777</td>
<td>1490.005</td>
<td>1489.127</td>
<td>0.059</td>
</tr>
<tr>
<td>99.9%</td>
<td>172.884</td>
<td>174.333</td>
<td>-0.831</td>
<td>1728.844</td>
<td>1716.432</td>
<td>0.723</td>
</tr>
</tbody>
</table>
Vasicek model — Rating BBB
EAD = 1 – LGD ∼ B(3, 3) – PD = 0.20% – ρ = 20%

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 5</td>
<td></td>
<td></td>
<td>n = 100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.002</td>
<td>0.000</td>
<td></td>
<td>0.032</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>75%</td>
<td>0.005</td>
<td>0.000</td>
<td></td>
<td>0.099</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>0.012</td>
<td>0.000</td>
<td></td>
<td>0.249</td>
<td>0.434</td>
<td>-42.604</td>
<td></td>
</tr>
<tr>
<td>95%</td>
<td>0.021</td>
<td>0.000</td>
<td></td>
<td>0.415</td>
<td>0.702</td>
<td>-40.856</td>
<td></td>
</tr>
<tr>
<td>99%</td>
<td>0.050</td>
<td>0.000</td>
<td></td>
<td>0.998</td>
<td>1.489</td>
<td>-32.992</td>
<td></td>
</tr>
<tr>
<td>99.5%</td>
<td>0.067</td>
<td>0.513</td>
<td>-86.924</td>
<td>1.340</td>
<td>1.912</td>
<td>-29.912</td>
<td></td>
</tr>
<tr>
<td>99.9%</td>
<td>0.118</td>
<td>0.782</td>
<td>-84.918</td>
<td>2.359</td>
<td>3.135</td>
<td>-24.733</td>
<td></td>
</tr>
<tr>
<td></td>
<td>n = 500</td>
<td></td>
<td></td>
<td>n = 5000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.161</td>
<td>0.000</td>
<td></td>
<td>1.614</td>
<td>1.574</td>
<td>2.583</td>
<td></td>
</tr>
<tr>
<td>75%</td>
<td>0.496</td>
<td>0.585</td>
<td>-15.185</td>
<td>4.961</td>
<td>5.036</td>
<td>-1.485</td>
<td></td>
</tr>
<tr>
<td>90%</td>
<td>1.245</td>
<td>1.442</td>
<td>-13.653</td>
<td>12.454</td>
<td>12.658</td>
<td>-1.616</td>
<td></td>
</tr>
<tr>
<td>95%</td>
<td>2.075</td>
<td>2.353</td>
<td>-11.827</td>
<td>20.750</td>
<td>20.913</td>
<td>-0.783</td>
<td></td>
</tr>
<tr>
<td>99%</td>
<td>4.988</td>
<td>5.447</td>
<td>-8.421</td>
<td>49.884</td>
<td>50.372</td>
<td>-0.969</td>
<td></td>
</tr>
<tr>
<td>99.5%</td>
<td>6.701</td>
<td>7.170</td>
<td>-6.530</td>
<td>67.014</td>
<td>67.108</td>
<td>-0.141</td>
<td></td>
</tr>
<tr>
<td>99.9%</td>
<td>11.797</td>
<td>12.491</td>
<td>-5.559</td>
<td>117.967</td>
<td>116.891</td>
<td>0.920</td>
<td></td>
</tr>
</tbody>
</table>

Some Practical Issues on Credit Risk
Credit Risk Modelling in the New Basle Capital Accord
Vasicek model — Rating AA

\[ \text{EAD} = 1 - \text{LGD} \sim \mathcal{B}(3, 3) - \text{PD} = 0.03\% - \rho = 20\% \]

| \( \alpha \) | \( \mathbb{E}[L_n | X] \) | \( \text{VaR}[L_n] \) | rel. diff. | \( \mathbb{E}[L_n | X] \) | \( \text{VaR}[L_n] \) | rel. diff. |
|---|---|---|---|---|---|---|
| \( n = 5 \) | | | | \( n = 100 \) | | |
| 50% | 0.000 | 0.000 | | 0.003 | 0.000 | |
| 75% | 0.001 | 0.000 | | 0.012 | 0.000 | |
| 90% | 0.002 | 0.000 | | 0.035 | 0.000 | |
| 95% | 0.003 | 0.000 | | 0.064 | 0.000 | |
| 99% | 0.009 | 0.000 | | 0.188 | 0.591 | -68.253 |
| 99.5% | 0.014 | 0.000 | | 0.270 | 0.732 | -63.097 |
| 99.9% | 0.027 | 0.459 | -94.030 | | 0.548 | 1.180 | -53.538 |
| \( n = 500 \) | | | | \( n = 5000 \) | | |
| 50% | 0.016 | 0.000 | | 0.156 | 0.000 | |
| 75% | 0.058 | 0.000 | | 0.583 | 0.658 | -11.471 |
| 90% | 0.174 | 0.132 | 31.750 | | 1.743 | 1.922 | -9.330 |
| 95% | 0.322 | 0.591 | -45.531 | | 3.220 | 3.504 | -8.102 |
| 99% | 0.938 | 1.349 | -30.465 | | 9.383 | 9.776 | -4.018 |
| 99.5% | 1.351 | 1.814 | -25.487 | | 13.514 | 13.781 | -1.937 |
Vasicek model — Rating BBB
EAD = 1 – LGD ~ \( B(3,3) \) – PD = 0.20% – \( \rho = 80\% \)

| \( \alpha \) | \( \mathbb{E}[L_n | X] \) | \( \text{VaR}[L_n] \) | rel. diff. | \( \mathbb{E}[L_n | X] \) | \( \text{VaR}[L_n] \) | rel. diff. |
|--------------|-----------------|-----------------|------------|-----------------|-----------------|------------|
| \( n = 5 \)  |                 |                 |            |                 |                 |            |
| 50%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 75%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 90%          | 0.000           | 0.000           |            | 0.003           | 0.000           |            |
| 95%          | 0.002           | 0.000           |            | 0.041           | 0.000           |            |
| 99%          | 0.093           | 0.000           |            | 1.864           | 2.015           | -7.457     |
| 99.5%        | 0.249           | 0.372           | -33.079    | 4.978           | 4.979           | -0.021     |
| 99.9%        | 0.998           | 1.241           | -19.548    | 19.962          | 19.540          | 2.159      |
| \( n = 100 \)|                 |                 |            |                 |                 |            |
| 50%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 75%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 90%          | 0.000           | 0.000           |            | 0.003           | 0.000           |            |
| 95%          | 0.002           | 0.000           |            | 0.041           | 0.000           |            |
| 99%          | 0.093           | 0.000           |            | 1.864           | 2.015           | -7.457     |
| 99.5%        | 0.249           | 0.372           | -33.079    | 4.978           | 4.979           | -0.021     |
| 99.9%        | 0.998           | 1.241           | -19.548    | 19.962          | 19.540          | 2.159      |
| \( n = 500 \)|                 |                 |            |                 |                 |            |
| 50%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 75%          | 0.000           | 0.000           |            | 0.000           | 0.000           |            |
| 90%          | 0.013           | 0.000           |            | 0.135           | 0.000           |            |
| 95%          | 0.207           | 0.000           |            | 2.069           | 2.047           | 1.060      |
| 99%          | 9.322           | 9.356           | -0.362     | 93.219          | 92.604          | 0.664      |
| 99.5%        | 24.888          | 25.179          | -1.155     | 248.881         | 242.187         | 2.764      |
| 99.9%        | 99.811          | 92.704          | 7.667      | 998.114         | 1028.333        | -2.939     |

Some Practical Issues on Credit Risk
Credit Risk Modelling in the New Basle Capital Accord 2-12
3 Extending Basel II model
3.1 A new formulation of the Basle II model

\[ Z_i = \sqrt{\rho} X + \sqrt{1 - \rho} \varepsilon_i \]

\[ P_i(t, X) = \Phi \left( \frac{\Phi^{-1}(1 - S_i(t)) - \sqrt{\rho} X}{\sqrt{1 - \rho}} \right) \]

\( Z = (Z_1, \ldots, Z_I) \) is a Gaussian vector with a covariance matrix \( \Sigma = C_I(\rho) \) which is equal to

\[
\Sigma = \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \rho \\
\rho & \cdots & \rho & 1
\end{pmatrix}
\]
The joint default probability is

\[ P_{1,\ldots,I} = \Pr \{ D_1 = 1, \ldots, D_I = 1 \} \]
\[ = \Pr \{ Z_1 \leq B_1, \ldots, Z_I \leq B_I \} \]
\[ = \Phi (B_1, \ldots, B_I; \Sigma) \]
\[ = \Phi (\Phi^{-1}(P_1), \ldots, \Phi^{-1}(P_I); \Sigma) \]
\[ = C (P_1, \ldots, P_I; \Sigma) \]

with \( C \) the Normal copula with the matrix of canonical correlations \( C_I(\rho) \).

If we now consider the joint survival function of default times, we have

\[ S(t_1, \ldots, t_I) = \Pr \{ \tau_1 > t_1, \ldots, \tau_I > t_I \} \]
\[ = \Pr \{ Z_1 > \Phi^{-1}(P_1(t_1)), \ldots, Z_I > \Phi^{-1}(P_I(t_I)) \} \]
\[ = C (1 - P_1(t_1), \ldots, 1 - P_I(t_I); \Sigma) \]
\[ = C (S_1(t_1), \ldots, S_I(t_I); \Sigma) \]
3.2 The loss distribution

If we consider ‘zero coupon’ loans, we have

\[ L = \sum_{i=1}^{I} x_i \cdot (1 - R_i) \cdot 1 \{\tau_i \leq t_i\} \]

where \( x_i \) is the notional of the loan and \( R_i \) and \( \tau_i \) are the recovery rate and the default time of the firm. The random variables are \( R_1, \ldots, R_I \) and \( \tau_1, \ldots, \tau_I \).

Assumptions:

1. The distributions of these random variables are given (because of internal credit rating system).
2. \( R_i \perp \tau_i \)
3. We have informations about default correlations between ‘sectors’.
Introducing stochastic recovery rate

The standard of the industry is the Beta distribution:

\[ f(x) = \frac{x^{a-1} (1-x)^{b-1}}{\int_0^1 x^{a-1} (1-x)^{b-1} \, dx} \]

Given the first two moments \( \mu(R) \) and \( \sigma(R) \) of the recovery rate, we may estimate the parameters by the method of moments:

\[ a = \frac{\mu^2(R) (1 - \mu(R))}{\sigma^2(R)} - \mu(R) \]
\[ b = \frac{\mu^2(R) (1 - \mu(R))^2}{\mu(R) \sigma^2(R)} - (1 - \mu(R)) \]
Density of the Beta Distribution
Proposition 1 Given a random variable \( U \) in \([0, 1]\), there exists (almost) always a random variable \( B \) with a Beta distribution such that \( \mathbb{E}[B] = \mathbb{E}[U] \) and \( \sigma[B] = \sigma[U] \).

The idea of the proof is the following. Because \( \mathbb{E}[U^2] \leq \mathbb{E}[U] \), we have \( \sigma[U] \leq \sigma^+(\mathbb{E}[U]) = \sqrt{\mathbb{E}[U](1 - \mathbb{E}[U])} \). For the Beta distribution, because \( a > 0 \) and \( b > 0 \), we have

\[
\sigma(B) < \sqrt{\mathbb{E}[B](1 - \mathbb{E}[B])}
\]
**Influence of the LGD Distribution on a Portfolio of 10 Loans with 5Y Maturity**

1. \( \mu = 50\% \quad \sigma = 40\% \)
2. \( \mu = 60\% \quad \sigma = 40\% \)
3. \( \mu = 70\% \quad \sigma = 40\% \)
4. \( \mu = 70\% \quad \sigma = 10\% \)
Influence of the LGD distribution
The Non Granularity Case
Modelling dependence of default times

We assume that $Z_i$ depends on one factor:

$$Z_i = \sum_{j=1}^{J} \beta_{i,j} X_j + \varepsilon_i$$

with $\sum_{j=1}^{J} 1\{\beta_{i,j} = 0\} = J - 1$

with $X_j \perp \varepsilon_i$, but $X_{j_1}$ and $X_{j_2}$ are not necessarily independent.

Let $j = m(i)$ be the mapping function between the loan $i$ and its sector $j$.

The survival time copula $(\tau_1, \ldots, \tau_I)$ is the Normal copula with the following matrix of canonical correlations:

$$\Sigma = \begin{pmatrix} 1 & \rho(m(1), m(2)) & \cdots & \rho(m(1), m(I)) \\ \rho(m(1), m(2)) & 1 & \cdots & \rho(m(2), m(I)) \\ \vdots & \cdots & \cdots & \cdots \\ \rho(m(I-1), m(I)) & \rho(m(I), m(1)) & \cdots & 1 \end{pmatrix}$$
Consider the example with 4 sectors

<table>
<thead>
<tr>
<th>Sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30%</td>
<td>20%</td>
<td>10%</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>40%</td>
<td>30%</td>
<td>20%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>50%</td>
<td>10%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>60%</td>
</tr>
</tbody>
</table>

and 7 loans

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = m(i) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

The matrix of canonical correlations is then

\[
\Sigma = \begin{pmatrix}
1.00 & 0.30 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\
1.00 & 0.20 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 \\
1.00 & 0.30 & 0.30 & 0.30 & 0.30 & 0.20 & 0.10 \\
1.00 & 0.50 & 0.50 & 0.10 & 0.10 & 0.10 & 0.10 \\
1.00 & 0.50 & 0.10 & 0.10 & 1.00 & 0.10 & 1.00 \\
1.00 & 0.10 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00
\end{pmatrix}
\]
The fast [Sloane] algorithm (from Riboulet and Roncalli [2002])

We want to simulate r.v. \((u_1, \ldots, u_I)\) from the Normal copula.

The [CHOL] algorithm is

\[
\begin{align*}
P &= \text{chol}(\Sigma) \\
z &= P\varepsilon \quad \text{with} \quad \varepsilon_{i_1} \perp \varepsilon_{i_2} \\
u_i &= \Phi(z_i)
\end{align*}
\]

This algorithm is time-consuming and memory-consuming:

\[
\begin{array}{c|ccc}
I & 100 & 1000 & 10000 \\
Memory size & 78.125 \text{ Kb} & 7.629 \text{ Mb} & 762.94 \text{ Mb} \\
\end{array}
\]

If \(\Sigma\) is \(C_1(\rho)\), the \([\sqrt{\rho}]\) algorithm is more efficient:

\[
\begin{align*}
z_i &= \sqrt{\rho}x + \sqrt{1-\rho}\varepsilon_i \quad \text{with} \quad x \perp \varepsilon_{i_1} \perp \varepsilon_{i_2} \\
u_i &= \Phi(z_i)
\end{align*}
\]
Let $\rho^*$ be the symmetric matrix with $\rho^*_{j,j}$ the intra-sector canonical correlations and $\rho^*_{j_1,j_2}$ the inter-sector canonical correlations. $\rho^*$ is not a correlation matrix.

The [Sloane] algorithm is the following:

$$\rho^* = V^* \Lambda^* V^* \top \quad \text{(eigensystem)}$$

$$A^* = V^* (\Lambda^*)^{\frac{1}{2}} \quad \text{($V^*$ is the } L_2\text{-normalized matrix of } V^*)$$

$$z_i = \sum_{j=1}^{J} A^*_{m(i),j} x_j + \sqrt{1 - \rho^*(m(i),m(i))} \varepsilon_i \text{ with } x_{j_1} \perp x_{j_2} \perp \varepsilon_{i_1} \perp \varepsilon_{i_2}$$

$$u_i = \Phi(z_i)$$

If $J = 1$, [Sloane] $= [\sqrt{\rho}]$.

**Proposition 2** If the eigenvalues $\lambda^*_j$ are positive, then $\Sigma$ is a correlation matrix.
The algorithm order of [CHOL] is $I^2$.

The algorithm order of [Sloane] is $I$ (because $J$ is fixed).

<table>
<thead>
<tr>
<th></th>
<th>Dimension of the matrix</th>
<th>Number of random variates</th>
<th>Number of + operations</th>
<th>Number of × operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>[CHOL]</td>
<td>$I \times I$</td>
<td>$I$</td>
<td>$I \times (I - 1)$</td>
<td>$I \times I$</td>
</tr>
<tr>
<td>[Sloane]</td>
<td>$J \times J$</td>
<td>$I + J$</td>
<td>$I \times J$</td>
<td>$I \times J$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10000 loans + 20 sectors</td>
<td></td>
</tr>
<tr>
<td>[CHOL]</td>
<td>$10^8$</td>
<td>10000</td>
<td>$\approx 10^8$</td>
<td>$10^8$</td>
</tr>
<tr>
<td>[Sloane]</td>
<td>400</td>
<td>10020</td>
<td>$2 \times 10^5$</td>
<td>$2 \times 10^5$</td>
</tr>
</tbody>
</table>
3.3 An example
500 loans, 5Y maturity, EaD = 1000, $\mu(R) = 50\%$, $\sigma(R) = 20\%$.

The $\rho^*$ matrix is

<table>
<thead>
<tr>
<th>Sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30%</td>
<td>20%</td>
<td>10%</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>40%</td>
<td>30%</td>
<td>20%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>50%</td>
<td>10%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>60%</td>
</tr>
</tbody>
</table>

The repartition by ratings is

<table>
<thead>
<tr>
<th>Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of loans</td>
<td>5%</td>
<td>15%</td>
<td>20%</td>
<td>30%</td>
<td>15%</td>
<td>10%</td>
<td>5%</td>
</tr>
</tbody>
</table>

The repartition by sectors is

<table>
<thead>
<tr>
<th>Sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of loans</td>
<td>20%</td>
<td>30%</td>
<td>10%</td>
<td>40%</td>
</tr>
</tbody>
</table>
Frequency of the Loss Distribution (Normal Copula/Sector Correlations)
4 The Measurement of Credit Risk
4.1 The Credit Risk Measure

1. **Value-at-Risk**
   \[
   \text{CreditVaR}(\alpha) = \inf \{L : \Pr \{L(t) \leq L\} \geq \alpha\}
   \]

2. **Expected Regret**
   \[
   ER(\bar{L}) = \mathbb{E}[L(t) \mid L(t) \geq \bar{L}]
   \]

3. **Expected Shortfall**
   \[
   ES(\alpha) = \mathbb{E}[L(t) \mid L(t) \geq \text{CreditVaR}(\alpha)]
   \]

4. **Unexpected Loss**
   \[
   UL(\alpha) = \text{CreditVaR}(\alpha) - \mathbb{E}[L(t)]
   \]
Portfolio Credit VaR
Influence of the LGD distribution
The Granularity Case
4.2 The Risk Contribution

The discrete marginal contribution is defined as follows:

$$\text{RC}(i) = \text{Risk} \left( x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_I \right) - \text{Risk} \left( x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_I \right)$$

We have

$$\text{Risk} \neq \sum_{i=1}^{I} \text{RC}(i)$$

In the following table, we report the values of $\frac{\sum_{i=1}^{I} \text{RC}(i)}{\text{Risk}}$:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>CreditVaR</th>
<th>ES</th>
<th>CreditVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normal copula</td>
<td>Normal copula</td>
<td>$t_6$ copula</td>
</tr>
<tr>
<td>95%</td>
<td>88.7%</td>
<td>99.0%</td>
<td>99.0%</td>
</tr>
<tr>
<td>99%</td>
<td>81.1%</td>
<td>99.1%</td>
<td>92.4%</td>
</tr>
<tr>
<td>99.9%</td>
<td>79.4%</td>
<td>99.4%</td>
<td>135%</td>
</tr>
</tbody>
</table>
For a set $\mathcal{A}$ of loans, we have

$$RC(\mathcal{A}) = \text{Risk} - \text{Risk}(x_i \notin \mathcal{A})$$

For example, with the CreditVaR measure, we have

<table>
<thead>
<tr>
<th>Sector</th>
<th>$\sum_{i \in \mathcal{A}} RC(i)$</th>
<th>$\text{RC}(\mathcal{A})$</th>
<th>$\sum_{i \in \mathcal{A}} RC(i)$</th>
<th>$\text{RC}(\mathcal{A})$</th>
<th>$\sum_{i \in \mathcal{A}} RC(i)$</th>
<th>$\text{RC}(\mathcal{A})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5075</td>
<td>5208</td>
<td>5295</td>
<td>5721</td>
<td>5829</td>
<td>4836</td>
</tr>
<tr>
<td>2</td>
<td>12317</td>
<td>12908</td>
<td>16158</td>
<td>16168</td>
<td>28797</td>
<td>20356</td>
</tr>
<tr>
<td>3</td>
<td>4440</td>
<td>4495</td>
<td>4484</td>
<td>5046</td>
<td>8086</td>
<td>6153</td>
</tr>
<tr>
<td>4</td>
<td>18118</td>
<td>17195</td>
<td>26448</td>
<td>24683</td>
<td>31600</td>
<td>36756</td>
</tr>
<tr>
<td>$\sum_{\mathcal{A}}$</td>
<td>39950</td>
<td>39806</td>
<td>52384</td>
<td>51619</td>
<td>74314</td>
<td>68100</td>
</tr>
<tr>
<td>Risk</td>
<td>45063</td>
<td></td>
<td>64592</td>
<td></td>
<td>93581</td>
<td></td>
</tr>
</tbody>
</table>
Risk Contribution (Credit VaR 99%)
4.3 The Risk Sensitivity

We have

\[ DR(i) = \frac{\partial \text{Risk}(x_1, \ldots, x_I)}{\partial x_i} \]

For example, with the ER measure, we have

\[ DR(i) = \frac{E[(1 - R_i) \cdot 1\{\tau_i \leq t_i\} \cdot 1\{L(t) \geq \bar{L}\}]}{\Pr\{L(t) \geq \bar{L}\}} \]

and

\[ \sum_{i=1}^{I} cRC(i) = \sum_{i=1}^{I} x_i \cdot DR(i) = ER(\bar{L}) \]

cRC(i) is the continuous marginal contribution.
The CreditVaR sensitivity

Theoretical result of Gouriéroux, Laurent and Scaillet [2000]

**Theorem 1** Let \((\varepsilon_1, \ldots, \varepsilon_I)\) be a random vector and \((x_1, \ldots, x_I)\) a vector in \(\mathbb{R}^I\). We consider the loss \(L\) defined by

\[
L = \sum_{i=1}^{I} x_i \cdot \varepsilon_i
\]

Let \(Q(L; \alpha)\) the percentile \(\alpha\) of \(L\). We have

\[
\frac{\partial Q(L; \alpha)}{\partial x_i} = \mathbb{E}[\varepsilon_i \mid L = Q(L; \alpha)]
\]
The Gaussian case \( L = \sum_{i=1}^{I} x_i \cdot \varepsilon_i \) with \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_I) \sim \mathcal{N}(\mu, \Sigma) \). We have \( L \sim \mathcal{N}(x^\top \mu, x^\top \Sigma x) \) and \( Q(L; \alpha) = x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x} \). The derivatives are

\[
\frac{\partial Q(L; \alpha)}{\partial x} = \mu + \Phi^{-1}(\alpha) \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}
\]

We remark that

\[
\begin{pmatrix} \varepsilon \\ L \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu \\ x^\top \mu \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma x \\ x^\top \Sigma & x^\top \Sigma x \end{pmatrix} \right)
\]

It comes that \( \varepsilon \mid L = \ell \sim \mathcal{N}(\mu_{\varepsilon \mid L}, \Sigma_{\varepsilon \mid L}) \) with \( \mu_{\varepsilon \mid L} = \mu + \Sigma x (x^\top \Sigma x)^{-1} (\ell - x^\top \mu) \) and \( \Sigma_{\varepsilon \mid L} = \Sigma - \Sigma x (x^\top \Sigma x)^{-1} x^\top \Sigma \). We deduce that

\[
\mathbb{E}[\varepsilon \mid L = Q(L; \alpha)] = \mathbb{E}[\varepsilon \mid L = x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x}]
\]

\[
= \mu + \Sigma x (x^\top \Sigma x)^{-1} \left( x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x} - x^\top \mu \right)
\]

\[
= \mu + \Phi^{-1}(\alpha) \Sigma x \frac{\sqrt{x^\top \Sigma x}}{(x^\top \Sigma x)^{-1}}
\]

\[
= \frac{\partial Q(L; \alpha)}{\partial x}
\]
Application to the credit loss

We have

\[
\frac{\partial \text{CreditVaR}(\alpha)}{\partial x_k} = \mathbb{E} \left[ (1 - R_k) \cdot 1 \{ \tau_k \leq t_k \} \mid \sum_{i=1}^{I} x_i \cdot (1 - R_i) \cdot 1 \{ \tau_i \leq t_i \} = \text{CreditVaR}(\alpha) \right]
\]
Numerical computation

\[ \text{CreditVaR}(\alpha) = L_{n\alpha:n} \]

If \( n\alpha = \lfloor n\alpha \rfloor \), we have \( \text{CreditVaR}(\alpha) = L_{\kappa_{n\alpha}} \) and

\[
\frac{\partial \text{CreditVaR}(\alpha)}{\partial x_i} = (1 - R_{i,\kappa_{n\alpha}}) \cdot 1 \{ \tau_{i,\kappa_{n\alpha}} \leq t_i \}
\]

If \( n\alpha > \lfloor n\alpha \rfloor \), we use the linear interpolation

\[ \text{CreditVaR}(\alpha) = (1 - n\alpha + \lfloor n\alpha \rfloor) L_{\kappa_{\lfloor n\alpha \rfloor}} + (n\alpha - \lfloor n\alpha \rfloor) L_{\kappa_{\lfloor n\alpha \rfloor}+1} \]

\[ = L_{\kappa_{\lfloor n\alpha \rfloor}} + (n\alpha - \lfloor n\alpha \rfloor) \left( L_{\kappa_{\lfloor n\alpha \rfloor}+1} - L_{\kappa_{\lfloor n\alpha \rfloor}} \right) \]

We have

\[
\frac{\partial \text{CreditVaR}(\alpha)}{\partial x_i} = (1 - n\alpha + \lfloor n\alpha \rfloor) \left( (1 - R_{i,\kappa_{\lfloor n\alpha \rfloor}}) \cdot 1 \{ \tau_{i,\kappa_{\lfloor n\alpha \rfloor}} \leq t_i \} \right) + \\
(n\alpha - \lfloor n\alpha \rfloor) \left( (1 - R_{i,\kappa_{\lfloor n\alpha \rfloor}+1}) \cdot 1 \{ \tau_{i,\kappa_{\lfloor n\alpha \rfloor}+1} \leq t_i \} \right)
\]

\( \Rightarrow \) large variance of estimates.
The localization method

We suppose that

$$\text{CreditVaR}(\alpha) = \sum_{m \in M} p_m L_m$$

where $\sum_{m \in M} p_m = 1$. Under the measure of probability $\{p_m, m \in M\}$, we have

$$\frac{\partial \text{CreditVaR}(\alpha)}{\partial x_i} = \sum_{m \in M} p_m \frac{L_{i,m}(t)}{x_i}$$

We pose $M = \{\kappa_{\lfloor n\alpha \rfloor - h}, \ldots, \kappa_{\lfloor n\alpha \rfloor}, \kappa_{\lfloor n\alpha \rfloor + 1}, \ldots, \kappa_{\lfloor n\alpha \rfloor + h}\}$ with a triangular kernel:

$$p_{\kappa_{\lfloor n\alpha \rfloor} + k} = \begin{cases} \frac{h+1-k}{h+1-(n\alpha-\lfloor n\alpha \rfloor)} & \text{if } k > 0 \\ \frac{h+k}{h+(n\alpha-\lfloor n\alpha \rfloor)} & \text{if } k \leq 0 \end{cases}$$

or a uniform kernel:

$$p_{\kappa_{\lfloor n\alpha \rfloor} + k} = \frac{1}{2h}$$
Triangular and Uniform Kernels
Numerical experiments

\[ L = \sum_{i=1}^{2} x_i \cdot \varepsilon_i \]

with

\[
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{pmatrix}
\sim \mathcal{N}
\left(
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & 0.5 \\
0.5 & 1
\end{pmatrix}
\right)
\]

and \( x_1 = 100 \) and \( x_2 = 50 \).

Analytical calculus gives CreditVaR (99%) = 307.7469, DR (1) = 2.1981921 and DR (2) = 1.7585537.

We remark that

\[ 307.7469 = x_1 \times 2.1981921 + x_2 \times 1.7585537 \]
Risk Sensitivity (Gaussian Case — CreditVaR 99%)
Main result

Proposition 3  Because the CreditVaR is expressed in terms of order statistics, we have

$$\text{CreditVaR}(\alpha) = \sum_{i=1}^{I} x_i \frac{\partial \text{CreditVaR}(\alpha)}{\partial x_i}$$

<table>
<thead>
<tr>
<th>rating/sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total by rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0</td>
<td>81</td>
<td>13</td>
<td>170</td>
<td>264</td>
</tr>
<tr>
<td>AA</td>
<td>40</td>
<td>725</td>
<td>137</td>
<td>2752</td>
<td>3654</td>
</tr>
<tr>
<td>A</td>
<td>328</td>
<td>1849</td>
<td>199</td>
<td>7061</td>
<td>9437</td>
</tr>
<tr>
<td>BBB</td>
<td>1308</td>
<td>6718</td>
<td>1430</td>
<td>16661</td>
<td>26117</td>
</tr>
<tr>
<td>BB</td>
<td>1362</td>
<td>6988</td>
<td>1592</td>
<td>13488</td>
<td>23430</td>
</tr>
<tr>
<td>B</td>
<td>2275</td>
<td>4211</td>
<td>3019</td>
<td>10323</td>
<td>19827</td>
</tr>
<tr>
<td>CCC</td>
<td>1502</td>
<td>4983</td>
<td>902</td>
<td>4561</td>
<td>11948</td>
</tr>
<tr>
<td>Total by sector</td>
<td>6816</td>
<td>25554</td>
<td>7291</td>
<td>55015</td>
<td>94676 = CreditVaR</td>
</tr>
</tbody>
</table>

Some Practical Issues on Credit Risk
The Measurement of Credit Risk 4-12
5  Credit Portfolio Management
5.1 The pair Risk/return

We define the Risk Adjusted Performance measure by

$$\text{RAPM} = \frac{(\text{Euribor} + \text{Sp})}{\text{Risk}}$$

For a loan, we have

$$\text{RAPM}(i) = \frac{x_i \cdot (\text{Euribor} + \text{Sp}(i))}{\text{Risk}(i)}$$

For a portfolio, we have

$$\text{RAPM}(x_1, \ldots, x_I) = \frac{\sum_{i=1}^{I} x_i \cdot (\text{Euribor} + \text{Sp}(i))}{\text{Risk}(x_1, \ldots, x_I)}$$

For a loan in a portfolio, we have

$$\text{RAPM}(i; x_1, \ldots, x_I) = \frac{x_i \cdot (\text{Euribor} + \text{Sp}(i))}{\text{RC}(i)}$$
RAPM of Individual Loans
5.2 The Efficient Frontier
The problem is ($C \in \mathbb{R}_+$ and $x \in \Omega$)

$$\max \text{ ExReturn} (x_1, \ldots, x_I)$$
$$\text{u.c.} \quad \text{Risk} (x_1, \ldots, x_I) \leq C$$

The simulation method Naive algorithm / Frontier-based algorithm

The optimisation method The ES problem

$$\min \text{ ES} (x_1, \ldots, x_I)$$
$$\text{u.c.} \quad \text{ExReturn} (x_1, \ldots, x_I) \geq C$$

may be solved by LP technique:

$$\min \quad \psi + (1 - \alpha)^{-1} \frac{1}{S} \sum_{s=1}^{S} z_s$$
$$\text{u.c.} \quad \text{ExReturn} (x_1, \ldots, x_I) \geq C$$
$$x \in \Omega$$
$$z_s \geq \sum_{i=1}^{I} x_i R_i^s D_i^s - \psi$$
$$z_s \geq 0$$

Building the CreditVaR frontier with the ES/ER optimisation problem

Some Practical Issues on Credit Risk
Credit Portfolio Management 5-3
The Frontier-Based Simulation Algorithm
5.3 Other Techniques

The method of contributions

The method of Lagrange multipliers
6 The Time-inconsistency of the Copula Model

The Stationarity of the Default Probability

Let \( \tau_1 \) and \( \tau_2 \) be two default times with the joint survival function:

\[
S(t_1, t_2) = \tilde{C}(S_1(t_1), S_2(t_2))
\]

We have

\[
S_1(t \mid \tau_2 = t^*) = \partial_2 \tilde{C}(S_1(t), S_2(t^*))
\]

If \( C \neq C^\perp \), the probability of default of one firm changes when another firm defaults (Schmidt and Ward [2002]).

Remark 1

Next computations are performed with the generator \( \Lambda \) of the Markov chain associated with the annual S&P TM. Let \( K \) be the state of default and \( i \) the initial rating of the firm. We have

\[
S_i(t) = 1 - e_i^\top \exp(t\Lambda)e_K
\]

The hasard rate is defined by

\[
\lambda(t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \Pr \{ t \leq \tau \leq t + \Delta \mid \tau \geq t \} = \frac{f(t)}{S(t)}
\]

Using a Normal copula, we have

\[
S_{i_1}(t \mid \tau_{i_2} = t^*) = \Phi \left( \frac{\Phi^{-1}(S_{i_1}(t)) - \rho \Phi^{-1}(S_{i_2}(t^*))}{\sqrt{1 - \rho^2}} \right)
\]
Hazard rate of the ratings
A firm rated AAA defaults - $\rho = 5\%$
A firm rated AAA defaults – $\rho = 50\%$
A firm rated BB defaults - $\rho = 50\%$
A firm rated CCC defaults - ρ = 50%
The Stationarity of the Survival Copula (from Jouanin [2002])

If the survival copula at time $t_0$ is $\tilde{C}$, and if no defaults occur between $t_0$ and $t$, the conditional survival copula at time $t$ is not necessarily $\tilde{C}$ (Giesecke [2000]).

We have

$$S(t_1, t_2 \mid \tau_1, \tau_2 > t) = \frac{\tilde{C}(S_1(t_1), S_2(t_2))}{\tilde{C}(S_1(t), S_2(t))}$$

and we would like to have

$$S(t_1, t_2 \mid \tau_1, \tau_2 > t) = \tilde{C}(S_1(t_1 \mid \tau_1, \tau_2 > t), S_2(t_2 \mid \tau_1, \tau_2 > t))$$

If $\tilde{C} = C^\perp$, this property is verified.

To overcome the lack of Markov property, we may look for a copula family such that the conditional survival copula belongs to the same family. With exponential survival times, one solution is the Gumbel-Barnett copula. Let $\theta$ be the copula parameter at time $t_0$. The copula parameter at time $t$ is

$$\theta(t) = \frac{\theta}{(1 + \theta \lambda_1 t)(1 + \theta \lambda_2 t)}$$
Kendall’s tau of the conditional ‘Markov’ copula
7 References


