4 Operational Risk

4.1 Estimation of the loss severity distribution

We consider a sample of $n$ individual losses $\{x_1, \ldots, x_n\}$. We assume that they can be described by different probability distributions:

(i) $X$ follows a log-normal distribution $\mathcal{LN}(\mu, \sigma^2)$.

(ii) $X$ follows a Pareto distribution $\mathcal{P}(\alpha, x^{-})$ defined by:

$$\Pr\{X \leq x\} = 1 - \left(\frac{x}{x^{-}}\right)^{-\alpha}$$

with $x \geq x^{-}$ and $\alpha > 0$.

(iii) $X$ follows a gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \, dt$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss $X$ follows a gamma distribution: $\ln X \sim \Gamma(\alpha; \beta)$.

1. We consider the case (i).

   (a) Show that the probability density function is:

   $$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

   (b) Calculate the two first moments of $X$. Deduce the orthogonal conditions of the generalized method of moments.

   (c) Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$.

2. We consider the case (ii).

   (a) Calculate the two first moments of $X$. Deduce the GMM conditions for estimating the parameter $\alpha$.

   (b) Find the maximum likelihood estimator $\hat{\alpha}$. 

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Financial Risk Management

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3. We consider the case (iii). Write the log-likelihood function associated to the sample of individual losses \( \{x_1, \ldots, x_n\} \). Deduce the first-order conditions of the maximum likelihood estimators \( \hat{\alpha} \) and \( \hat{\beta} \).

4. We consider the case (iv). Show that the probability density function of \( X \) is:
   \[
   f(x) = \frac{\beta^\alpha (\ln x)^{\alpha - 1}}{\Gamma(\alpha)} x^{\beta + 1}
   \]
   What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses \( \{x_1, \ldots, x_n\} \).

5. We now assume that the losses \( \{x_1, \ldots, x_n\} \) have been collected beyond a threshold \( H \) meaning that \( X \geq H \).
   
   (a) What becomes the generalized method of moments in the case (i).
   
   (b) Calculate the maximum likelihood estimator \( \hat{\alpha} \) in the case (ii).
   
   (c) Write the log-likelihood function in the case (iii).

4.2 Estimation of the loss frequency distribution

We consider a dataset of individual losses \( \{x_1, \ldots, x_n\} \) corresponding to a sample of \( T \) annual loss numbers \( \{N_{Y_1}, \ldots, N_{Y_T}\} \). This implies that:

\[
\sum_{t=1}^T N_{Y_t} = n
\]

If we measure the number of losses per quarter \( \{N_{Q_1}, \ldots, N_{Q_4T}\} \), we use the notation:

\[
\sum_{t=1}^{4T} N_{Q_t} = n
\]

1. We assume that the annual number of losses follows a Poisson distribution \( P(\lambda_Y) \). Calculate the maximum likelihood estimator \( \hat{\lambda}_Y \) associated to the sample \( \{N_{Y_1}, \ldots, N_{Y_T}\} \).

2. We assume that the quarterly number of losses follows a Poisson distribution \( P(\lambda_Q) \). Calculate the maximum likelihood estimator \( \hat{\lambda}_Q \) associated to the sample \( \{N_{Q_1}, \ldots, N_{Q_{4T}}\} \).

3. What is the impact of considering a quarterly or annual basis on the computation of the capital charge?

4. What does this result become if we consider a method of moments based on the first moment?

5. Same question if we consider a method of moments based on the second moment.

5 Asset Liability Management Risk

5.1 Computation of the amortization functions \( S(t, u) \) and \( S^*(t, u) \)

In what follows, we consider a debt instrument, whose remaining maturity is equal to \( m \). We note \( t \) the current date and \( T = t + m \) the maturity date.

1. We consider a bullet repayment debt. Define its amortization function \( S(t, u) \). Calculate the survival function \( S^*(t, u) \) of the stock. Show that:

\[
S^*(t, u) = \mathbb{1} \{ t \leq u < t + m \} \cdot \left( 1 - \frac{u - t}{m} \right)
\]

in the case where the new production is constant. Comment on this result.
2. Same question if we consider a debt instrument, whose amortization rate is constant.

3. Same question if we assume\(^1\) that the amortization function is exponential with parameter \(\lambda\).

4. Find the expression of \(D^*(t)\) when the new production is constant.

5. Calculate the durations \(D(t)\) and \(D^*(t)\) for the three previous cases.

6. Calculate the corresponding dynamics \(dN(t)\).

5.2 Impact of prepayment on the amortization scheme of the CAM

We recall that the outstanding balance at time \(t\) is given by:

\[
N(t) = 1 \{t < m\} \cdot N_0 \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}
\]

1. Find the dynamics \(dN(t)\).

2. We note \(\tilde{N}(t)\) the modified outstanding balance that takes into account the prepayment risk. Let \(\lambda_p(t)\) be the prepayment rate at time \(t\). Write the dynamics of \(\tilde{N}(t)\).

3. Show that \(\tilde{N}(t) = N(t) S_p(t)\) where \(S_p(t)\) is the prepayment-based survival function.

4. Calculate the liquidity duration \(\tilde{D}(t)\) associated to the outstanding balance \(\tilde{N}(t)\) when the hazard rate of prepayments is constant and equal to \(\lambda_p\).

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\(^1\)By definition of the exponential amortization, we have \(m = +\infty\).