Financial Risk Management

Tutorial Class — Session 5

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6 Copulas and Stochastic Dependence Modeling

6.1 The bivariate Pareto copula

We consider the bivariate Pareto distribution:

$$\begin{aligned}\mathbf{F}(x_1, x_2) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + x_2}{\theta_2} - 1\right)^{-\alpha}\end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\alpha > 0$.

1. Show that the marginal functions of $\mathbf{F}(x_1, x_2)$ correspond to univariate Pareto distributions.
2. Find the copula function associated to the bivariate Pareto distribution.
3. Deduce the copula density function.
4. Show that the bivariate Pareto copula function has no lower tail dependence, but an upper tail dependence.
5. Do you think that the bivariate Pareto copula family can reach the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$? Justify your answer.
6. Let X_1 and X_2 be two Pareto-distributed random variables, whose parameters are (α_1, θ_1) and (α_2, θ_2).
   (a) Show that the linear correlation between X_1 and X_2 is equal to 1 if and only if the parameters α_1 and α_2 are equal.
   (b) Show that the linear correlation between X_1 and X_2 can never reached the lower bound -1.
   (c) Build a new bivariate Pareto distribution by assuming that the marginal distributions are $\mathcal{P}(\alpha_1, \theta_1)$ and $\mathcal{P}(\alpha_2, \theta_2)$ and the dependence is a bivariate Pareto copula function with parameter α. What is the relevance of this approach for building bivariate Pareto distributions?

6.2 Calculation of correlation bounds

1. Give the mathematical definition of the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$. What is the probabilistic interpretation of these copulas?

2. We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and LGD $\sim \mathcal{U}_{[0,1]}$.

   (a) Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:

   $$\text{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

   (b) Show that the linear correlation $\rho(\boldsymbol{\tau}, \text{LGD})$ verifies the following inequality:

   $$|\rho \langle \boldsymbol{\tau}, \text{LGD} \rangle| \leq \frac{\sqrt{3}}{2}$$

   (c) Comment on these results.

3. We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

   (a) We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relation is true:

   $$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

   (b) Show that there exists a function f such that $\boldsymbol{\tau}_2 = f(\boldsymbol{\tau}_2)$ when the dependence function is $\mathbf{C}^-$.

   (c) Show that the lower and upper bounds of the linear correlation satisfy the following relationship:

   $$-1 < \rho \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle \leq 1$$

   (d) In the more general case, show that the linear correlation of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

4. We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

   (a) Find the probability distribution of $X_1 + X_2$.

   (b) Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

   $$\text{cov}(Y_1, Y_2) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \left(e^{\rho \sigma_1 \sigma_2} - 1 \right)$$

   (c) Deduce the correlation between Y_1 and Y_2.

   (d) For which values of θ does the equality $\rho \langle Y_1, Y_2 \rangle = +1$ hold? Same question when $\rho \langle Y_1, Y_2 \rangle = -1$.

   (e) We consider the bivariate Black-Scholes model:

   $$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

   with $\mathbb{E}[W_1(t) W_2(t)] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t \to \infty} \rho \langle S_1(t), S_2(t) \rangle$.

   (f) Comment on these results.

7 Extreme Value Theory

7.1 Extreme value theory in the bivariate case

1. What is an extreme value (EV) copula $\mathbf{C}$?
2. Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are EV copulas. Why $\mathbf{C}^{-}$ can not be an EV copula?
3. We define the Gumbel-Hougaard copula as follows:

$$\mathbf{C}(u_1,u_2)=\exp\left(-\left[(-\ln u_1)^{\theta}+(-\ln u_2)^{\theta}\right]^{1/\theta}\right)$$

   with $\theta \geq 1$. Verify that it is an EV copula.
4. What is the definition of the upper tail dependence λ? What is its usefulness in multivariate extreme value theory?
5. Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$. If $g'(x_0)\neq 0$, L'Hospital's rule states that:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

   Deduce that the upper tail dependence λ of the Gumbel-Hougaard copula is $2-2^{1/\theta}$. What is the correlation of two extremes when $\theta=1$?
6. We define the Marshall-Olkin copula as follows:

$$\mathbf{C}(u_1,u_2)=u_1^{1-\theta_1}u_2^{1-\theta_2}\min\left(u_1^{\theta_1},u_2^{\theta_2}\right)$$

   with $\{\theta_1,\theta_2\}\in[0,1]^2$.

   (a) Verify that it is an EV copula.

   (b) Find the upper tail dependence λ of the Marshall-Olkin copula.

   (c) What is the correlation of two extremes when $\min(\theta_1,\theta_2)=0$?

   (d) In which case are two extremes perfectly correlated?

7.2 Maximum domain of attraction in the bivariate case

1. We consider the following distributions of probability:

Distribution		$\mathbf{F}(x)$
Exponential	$\mathcal{E}(\lambda)$	$1-e^{-\lambda x}$
Uniform	$\mathcal{U}_{[0,1]}$	x
Pareto	$\mathcal{P}(\alpha,\theta)$	$1-\left(\frac{\theta+x}{\theta}\right)^{-\alpha}$

   For each distribution, we give the normalization parameters a_n and b_n of the Fisher-Tippet theorem and the corresponding limit distribution distribution $\mathbf{G}(x)$:

Distribution	a_n	b_n	$\mathbf{G}(x)$
Exponential	λ^{-1}	$\lambda^{-1}\ln n$	$\mathbf{\Lambda}(x)=e^{-e^{-x}}$
Uniform	n^{-1}	$1-n^{-1}$	$\mathbf{\Psi}_1(x-1)=e^{x-1}$
Pareto	$\theta\alpha^{-1}n^{1/\alpha}$	$\theta n^{1/\alpha}-\theta$	$\mathbf{\Phi}_\alpha\left(1+\frac{x}{\alpha}\right)=e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}$

   We note $\mathbf{G}(x_1,x_2)$ the asymptotic distribution of the bivariate random vector $(X_{1,n:n},X_{2,n:n})$ where $X_{1,i}$ (resp. $X_{2,i}$) are *iid* random variables.

(a) What is the expression of $\mathbf{G}(x_1, x_2)$ when $X_{1,i}$ and $X_{2,i}$ are independent, $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{U}_{[0,1]}$?

(b) Same question when $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

(c) Same question when $X_{1,i} \sim \mathcal{U}_{[0,1]}$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

2. What becomes the previous results when the dependence function between $X_{1,i}$ and $X_{2,i}$ is the Normal copula with parameter $\rho < 1$?

3. Same question when the parameter of the Normal copula is equal to one.

4. Find the expression of $\mathbf{G}(x_1, x_2)$ when the dependence function is the Gumbel-Hougaard copula.

8 Monte Carlo Simulation Methods

8.1 Simulation of the bivariate Normal copula

Let $X = (X_1, X_2)$ be a standard Gaussian vector with correlation ρ. We note $U_1 = \Phi(X_1)$ and $U_2 = \Phi(X_2)$.

1. We note Σ the matrix defined as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Calculate the Cholesky decomposition of Σ. Deduce an algorithm to simulate X.

2. Show that the copula of (X_1, X_2) is the same that the copula of the random vector (U_1, U_2).

3. Deduce an algorithm to simulate the Normal copula with parameter ρ.

4. Calculate the conditional distribution of X_2 knowing that $X_1 = x$. Then show that:

$$\Phi_2(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1 - \rho^2}}\right) \phi(x) \, \mathrm{d}x$$

5. Deduce an expression of the Normal copula.

6. Calculate the conditional copula function $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the Normal copula with parameter ρ.

7. Show that this algorithm is equivalent to the Cholesky algorithm found in Question 3.

9 Stress Testing and Scenario Analysis

9.1 Construction of a stress scenario with the GEV distribution

1. We note a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

(a) Find the limit distribution $\mathbf{G}$ when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$.

(b) Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

(c) Same question when X is a Pareto distribution:

$$\mathbf{F}(x) = 1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha},$$

$a_n = \theta \alpha^{-1} n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

2. We denote by $\mathbf{G}$ the GEV probability distribution:

$$\mathbf{G}(x) = \exp\left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}$$

What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1, \ldots, x_T\}$.

3. Show that for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the Gumbel distribution:

$$\mathbf{\Lambda}(x) = \exp\left(- \exp\left(- \left(\frac{x - \mu}{\sigma} \right) \right) \right)$$

4. We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

   (a) Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

   $$r(\mathcal{T}) \simeq - \left(\hat{\mu} + \left(\frac{\mathcal{T}}{n} - 1 \right) \hat{\sigma} \right)$$

   where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of GEV parameters.

   (b) We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

   Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.