Chapter 2

Market Risk

This chapter begins with the presentation of the regulatory framework. It will help us to understand how the supervision on market risk is organized and how the capital charge is computed. Then we will study the different statistical approaches to measure the value-at-risk and the expected shortfall. Specifically, a section is dedicated to the risk management of derivatives and exotic products. We will see the main concepts, but we will present the more technical details later in Chapter 9 dedicated to model risk. Advanced topics like Monte Carlo methods and stress testing models will also be addressed in Part II. Finally, the last part of the chapter is dedicated to risk allocation.

2.1 Regulatory framework

We recall that the original Basel Accord only concerned credit risk in 1988. However, the occurrences of market shocks were more important and the rapid development of derivatives created some stress events at the end of the eighties and the beginning of the nineties. On 19 October 1987, stock markets crashed and the Dow Jones Industrial Average index dropped by more than 20% in the day. In 1990, the collapse of the Japanese asset price bubble (both in stock and real estate markets) caused a lot of damage in the Japanese banking system and economy. The unexpected rise of US interest rates in 1994 resulted in a bond market massacre and difficulties for banks, hedge funds and money managers. In 1994-1995, several financial disasters occurred, in particular the bankruptcy of Barings and the Orange County affair (Jorion, 2007).

In April 1993, the Basel Committee published a first consultative paper to incorporate market risk in the Cooke ratio. Two years later, in April 1995, it accepted the idea to compute the capital charge for market risks with an internal model. This decision is mainly due to the publication of *RiskMetrics* by J.P. Morgan in October 1994. Finally, the Basel Committee published the amendment to the capital accord to incorporate market risks in January 1996. This proposal has remained the supervisory framework for market risk during many years. However, the 2008 Global Financial Crisis had a big impact in terms of market risk. Just after the crisis, a new approach called Basel 2.5 has been accepted. In 2012, the Basel Committee launched a major project: the fundamental review of the trading book (FRTB). These works resulted in the publication of a new comprehensive framework in January 2019 (BCBS, 2019). This is the Basel III framework for computing the minimum capital requirements for market risk as of January 2022.

According to BCBS (2019), market risk is defined as "*the risk of losses (in on- and off-balance sheet positions) arising from movements in market prices. The risks subject to market risk capital requirements include but are not limited to:*

- *default risk, interest rate risk, credit spread risk, equity risk, foreign exchange (FX) risk and commodities risk for trading book instruments*;
- *FX risk and commodities risk for banking book instruments*."

The following table summarizes the perimeter of markets risks that require regulatory capital:

Portfolio	Fixed Income	Equity	Currency	Commodity	Credit
Trading	✓	✓	✓	✓	✓
Banking			✓	✓	

The Basel Committee makes the distinction between the trading book and the banking book. Instruments to be included in the trading book are subject to market risk capital requirements, while instruments to be included in the banking book are subject to credit risk capital requirements (with the exception of foreign exchange and commodity instruments). The trading book refers to positions in assets held with trading intent or for hedging other elements of the trading book. These assets are systematically valuated on a fair value (mark-to-market or mark-to-model) basis, are actively managed and their holding is intentionally for short-term resale. Examples are proprietary trading, market-making activities, hedging portfolios of derivatives products, listed equities, repo transactions, etc. The banking book refers to positions in assets that are expected to be held until the maturity. These assets may be valuated at their historic cost or with a fair value approach. Examples are unlisted equities, real estate holdings, hedge funds, etc.

The first task of the bank is therefore to define trading book assets and banking book assets. For instance, if the bank sells an option on the Libor rate to a client, a capital charge for the market risk is required. If the bank provides a personal loan to a client with a fixed interest rate, there is a market risk if the interest rate risk is not hedged. However, a capital charge is not required in this case, because the exposure concerns the banking book. Exposures on stocks may be included in the banking book if the objective is a long-term investment.

2.1.1 The Basel I/II framework

To compute the capital charge, banks have the choice between two approaches:

1. the standardized measurement method (SMM);
2. the internal model-based approach (IMA).

The standardized measurement method has been implemented by banks at the end of the nineties. However, banks quickly realized that they can sharply reduce their capital requirements by adopting internal models. This explained that SMM was only used by a few number of small banks in the 2000s.

2.1.1.1 Standardized measurement method

Five main risk categories are identified: interest rate risk, equity risk, currency risk, commodity risk and price risk on options and derivatives. For each category, a capital charge is computed to cover the general market risk, but also the specific risk. According to the Basel Committee, specific risk includes the risk "*that an individual debt or equity security moves by more or less than the general market in day-to-day trading and event risk (e.g. takeover risk or default risk)*". The use of internal models is subject to the approval of the supervisor and the bank can mix the two approaches under some conditions. For instance, the bank may use SMM for the specific risk and IMA for the general market risk.

In this approach, the capital charge $\mathcal{K}$ is equal to the risk exposure E times the capital charge weight K:

$$\mathcal{K} = E \cdot K$$

For the specific risk, the risk exposure corresponds to the notional of the instrument, whether it is a long or a short position. For the general market risk, long and short positions on different instruments can be offset. In what follows, we give the main guidelines and we invite the reader to consult BCBS (1996a, 2006) to obtain the computational details.

Interest rate risk Let us first consider the specific risk. The Basel Committee makes the distinction between sovereign and other fixed income instruments. In the case of government instruments, the capital charge weights are:

Rating	AAA to AA−	A+ to BBB−			BB+ to B−	Below B−	NR
Maturity		0−6M	6M−2Y	2Y+			
K	0%	0.25%	1.00%	1.60%	8%	12%	8%

This capital charge depends on the rating and also the residual maturity for A+ to BBB− issuers[1]. The category NR stands for non-rated issuers. In the case of other instruments issued by public sector entities, banks and corporate companies, the capital charge weights are:

Rating	AAA to BBB−			BB+ to BB−	Below BB−	NR
Maturity	0−6M	6M−2Y	2Y+			
K	0.25%	1.00%	1.60%	8%	12%	8%

Example 4 *We consider a trading portfolio with the following exposures: a long position of* \$50 *mn on Euro-Bund futures, a short position of* \$100 *mn on three-month T-Bills and a long position of* \$10 *mn on an investment grade (IG) corporate bond with a three-year residual maturity.*

The underlying asset of Euro-Bund futures is a German bond with a long maturity (higher than 6 years). We deduce that the capital charge for specific risk for the two sovereign exposures is equal to zero, because both Germany and US are rated above A+. Concerning the corporate bond, we obtain:

$$\mathcal{K} = 10 \times 1.60\% = \$160\,000$$

For the general market risk, the bank has the choice between two methods: the maturity approach and the duration approach. In the maturity approach, long and short positions are slotted into a maturity-based ladder comprising fifteen time-bands (less than one month, between one and three months, ... between 12 and 20 years, greater than 20 years). The risk weights depend on the time band and the value of the coupon[2], and apply to the net exposure on each time band. For example, a capital charge of 8% is used for the net

[1] Three maturity periods are defined: 6 months or less, greater than 6 months and up to 24 months, more than 24 months.

[2] We distinguish coupons less than 3% (small coupons or SC) and coupons 3% or more (big coupons or BC).

exposure of instruments (with small coupons), whose maturity is between 12 and 20 years. For reflecting basis and gap risks, the bank must also include a 10% capital charge to the smallest exposure of the matched positions. This adjustment is called the '*vertical disallowance*'. The Basel Committee considers a second adjustment for horizontal offsetting (the '*horizontal disallowance*'). For that, it defines 3 zones (less than 1 year, one year to four years and more than four years). The offsetting can be done within and between the zones. The adjustment coefficients are 30% within the zones 2 and 3, 40% within the zone 1, between the zones 1 and 2, and between the zones 2 and 3, and 100% between the zones 1 and 3. Therefore, the regulatory capital for the general market risk is the sum of the three components:

$$\mathcal{K} = \mathcal{K}^{\mathrm{OP}} + \mathcal{K}^{\mathrm{VD}} + \mathcal{K}^{\mathrm{HD}}$$

where $\mathcal{K}^{\mathrm{OP}}$, $\mathcal{K}^{\mathrm{VD}}$ and $\mathcal{K}^{\mathrm{HD}}$ are the required capital for the overall net open position, the vertical disallowance and the horizontal disallowance.

With the duration approach, the bank computes the price sensitivity of each position with respect to a change in yield Δy, slots the sensitivities into a duration-based ladder and applies adjustments for vertical and horizontal disallowances. The computation of the required capital is exactly the same as previously, but with a different definition of time bands and zones.

Equity risk For equity exposures, the capital charge for specific risk is 4% if the portfolio is liquid and well-diversified and 8% otherwise. For the general market risk, the risk weight is equal to 8% and applies to the net exposure.

Example 5 *We consider a* \$100 *mn short exposure on the S&P 500 index futures contract and a* \$60 *mn long exposure on the Apple stock.*

The capital charge for specific risk is[3]:

$$\begin{aligned} \mathcal{K}^{\mathrm{Specific}} &= 100 \times 4\% + 60 \times 8\% \\ &= 4 + 4.8 \\ &= 8.8 \end{aligned}$$

The net exposure is $-\$40$ mn. We deduce that the capital charge for the general market risk is:

$$\begin{aligned} \mathcal{K}^{\mathrm{General}} &= |-40| \times 8\% \\ &= 3.2 \end{aligned}$$

It follows that the total capital charge for this equity portfolio is \$12 mn.

Remark 1 *Under Basel 2.5, the capital charge for specific risk is set to 8% whatever the liquidity of the portfolio.*

Foreign exchange risk The Basel Committee includes gold in this category and not in the commodity category because of its specificity in terms of volatility and its status of safe-heaven currency. The bank has first to calculate the net position (long or short) of each currency. The capital charge is then 8% of the global net position defined as the sum of:

[3]We assume that the S&P 500 index is liquid and well-diversified, whereas the exposure on the Apple stock is not diversified.

- the maximum between the aggregated value $\mathcal{L}_{\mathrm{FX}}$ of long positions and the aggregated value $\mathcal{S}_{\mathrm{FX}}$ of short positions and,
- the absolute value of the net position $\mathcal{N}_{\mathrm{Gold}}$ in gold.

We have:

$$\mathcal{K} = 8\% \times (\max(\mathcal{L}_{\mathrm{FX}}, \mathcal{S}_{\mathrm{FX}}) + |\mathcal{N}_{\mathrm{Gold}}|)$$

Example 6 *We consider a bank which has the following long and short positions expressed in \$ mn*[4]*:*

Currency	EUR	JPY	GBP	CHF	CAD	AUD	ZAR	Gold
$\mathcal{L}_i$	170	0	25	37	11	3	8	33
$\mathcal{S}_i$	80	50	12	9	28	0	8	6

We first compute the net exposure $\mathcal{N}_i$ for each currency:

$$\mathcal{N}_i = \mathcal{L}_i - \mathcal{S}_i$$

We obtain the following figures:

Currency	EUR	JPY	GBP	CHF	CAD	AUD	ZAR	Gold
$\mathcal{N}_i$	90	-50	13	28	-17	3	0	27

We then calculate the aggregated long and short positions:

$$\begin{aligned} \mathcal{L}_{\mathrm{FX}} &= 90 + 13 + 28 + 3 + 0 = 134 \\ \mathcal{S}_{\mathrm{FX}} &= 50 + 17 = 67 \\ \mathcal{N}_{\mathrm{Gold}} &= 27 \end{aligned}$$

We finally deduce that the capital charge is equal to \$12.88 mn:

$$\begin{aligned} \mathcal{K} &= 8\% \times (\max(134, 67) + |27|) \\ &= 8\% \times 161 \\ &= 12.88 \end{aligned}$$

Commodity risk Commodity risk concerns both physical and derivative positions (forward, futures[5] and options). This includes energy products (oil, gas, ethanol, etc.), agricultural products (grains, oilseeds, fiber, livestock, etc.) and metals (industrial and precious), but excludes gold which is covered under foreign exchange risk. The Basel Committee makes the distinction between the risk of spot or physical trading, which is mainly affected by the directional risk and the risk of derivative trading, which includes the directional risk, the basis risk, the cost-of-carry and the forward gap (or time spread) risk. The SMM for commodity risk includes two options: the simplified approach and the maturity ladder approach.

Under the simplified approach, the capital charge for directional risk is 15% of the absolute value of the net position in each commodity. For the other three risks, the capital charge is equal to 3% of the global gross position. We have:

$$\mathcal{K} = 15\% \times \sum_{i=1}^{m} |\mathcal{L}_i - \mathcal{S}_i| + 3\% \times \sum_{i=1}^{m} (\mathcal{L}_i + \mathcal{S}_i)$$

[4] We implicity assume that the reporting currency of the bank is the US dollar.

[5] The most traded futures contracts are crude oil, brent, heating oil, gas oil, natural oil, rbob gasoline silver, platinum, palladium, zinc, lead, aluminium, cocoa, soybeans, corn, cotton, wheat, sugar, live cattle, coffee and soybean oil.

where m is the number of commodities, $\mathcal{L}_i$ is the long position on commodity i and $\mathcal{S}_i$ is the short position on commodity i.

Example 7 *We consider a portfolio of five commodities. The mark-to-market exposures expressed in* \$ *mn are the following:*

Commodity	Crude Oil	Coffee	Natural Gas	Cotton	Sugar
$\mathcal{L}_i$	23	5	3	8	11
$\mathcal{S}_i$	0	0	19	2	6

The aggregated net exposure $\sum_{i=1}^{5} \lvert \mathcal{L}_i - \mathcal{S}_i \rvert$ is equal to \$55 mn whereas the gross exposure $\sum_{i=1}^{5} (\mathcal{L}_i + \mathcal{S}_i)$ is equal to \$77 mn. We deduce that the required capital is $15\% \times 55 + 3\% \times 77$ or \$10.56 mn.

Under the maturity ladder approach, the bank should spread long and short exposures of each commodity to seven time bands: 0-1M, 1M-3M, 3M-6M, 6M-1Y, 1Y-2Y, 2Y-3Y, 3Y+. For each time band, the capital charge for the basis risk is equal to 1.5% of the matched positions (long and short). Nevertheless, the residual net position of previous time bands may be carried forward to offset exposures in next time bands. In this case, a surcharge of 0.6% of the residual net position is added at each time band to cover the time spread risk. Finally, a capital charge of 15% is applied to the global net exposure (or the residual unmatched position) for directional risk.

Option's market risk There are three approaches for the treatment of options and derivatives. The first method, called the simplified approach, consists of calculating separately the capital charge of the position for the option and the associated underlying. In the case of an hedged exposure (long cash and long put, short cash and long call), the required capital is the standard capital charge of the cash exposure less the amount of the in-the-money option. In the case of a non-hedged exposure, the required capital is the minimum value between the mark-to-market of the option and the standard capital charge for the underlying.

Example 8 *We consider a variant of Example 5. We have a* \$100 *mn short exposure on the S&P 500 index futures contract and a* \$60 *mn long exposure on the Apple stock. We assume that the current stock price of Apple is* \$120. *Six months ago, we have bought* 400 000 *put options on Apple with a strike of* \$130 *and a one-year maturity. We also decide to buy* 10 000 *ATM call options on Google. The current stock price of Google is* \$540 *and the market value of the option is* \$45.5.

We deduce that we have 500 000 shares of the Apple stock. This implies that \$48 mn of the long exposure on Apple is hedged by the put options. Concerning the derivative exposure on Google, the market value is equal to \$0.455 mn. We can therefore decompose this portfolio into three main exposures:

- a directional exposure composed by the \$100 mn short exposure on the S&P 500 index and the \$12 mn remaining long exposure on the Apple stock;
- a \$48 mn hedged exposure on the Apple stock;
- a \$0.455 mn derivative exposure on the Google stock.

For the directional exposure, we compute the capital charge for specific and general market risks[6]:

$$\begin{aligned}\mathcal{K} &= (100 \times 4\% + 12 \times 8\%) + 88 \times 8\% \\ &= 4.96 + 7.04 \\ &= 12\end{aligned}$$

For the hedged exposure, we proceed as previously but we deduce the in-the-money value[7]:

$$\begin{aligned}\mathcal{K} &= 48 \times (8\% + 8\%) - 4 \\ &= 3.68\end{aligned}$$

The market value of the Google options is \$0.455 mn. We compare this value to the standard capital charge[8] to determine the capital charge:

$$\begin{aligned}\mathcal{K} &= \min(5.4 \times 16\%, 0.455) \\ &= 0.455\end{aligned}$$

We finally deduce that the required capital is \$16.135 mn.

The second approach is the delta-plus method. In this case, the directional exposure of the option is calculated by its delta. Banks will also required to compute an additional capital charge for gamma and vega risks. We consider different options and we note $j \in \mathcal{A}_i$ when the option j is written on the underlying asset i. We first compute the (signed) capital charge for the 4 risks at the asset level:

$$\begin{aligned}\mathcal{K}_i^{\text{Specific}} &= \left(\sum_{j \in \mathcal{A}_i} N_j \cdot \boldsymbol{\Delta}_j\right) \cdot S_i \cdot K_i^{\text{Specific}} \\ \mathcal{K}_i^{\text{General}} &= \left(\sum_{j \in \mathcal{A}_i} N_j \cdot \boldsymbol{\Delta}_j\right) \cdot S_i \cdot K_i^{\text{General}} \\ \mathcal{K}_i^{\text{Gamma}} &= \frac{1}{2}\left(\sum_{j \in \mathcal{A}_i} N_j \cdot \boldsymbol{\Gamma}_j\right) \cdot \left(S_i \cdot K_i^{\text{Gamma}}\right)^2 \\ \mathcal{K}_i^{\text{Vega}} &= \sum_{j \in \mathcal{A}_i} N_j \cdot \boldsymbol{\upsilon}_j \cdot (25\% \cdot \Sigma_j)\end{aligned}$$

where S_i is the current market value of the asset i, K_i^{Specific} and K_i^{General} are the corresponding standard capital charge for specific and general market risk and K_i^{Gamma} is the capital charge for gamma impact[9]. Here, N_j, $\boldsymbol{\Delta}_j$, $\boldsymbol{\Gamma}_j$ and $\boldsymbol{\upsilon}_j$ are the exposure, delta, gamma and vega of the option j. For the vega risk, the shift corresponds to $\pm 25\%$ of the implied volatility Σ_j. For a portfolio of assets, the traditional netting rules apply to specific and general market risks. The total capital charge for gamma risk corresponds to the opposite of the sum of the negative individual capital charges for gamma risk whereas the total capital charge for vega risk corresponds to the sum of the absolute value of individual capital charges for vega risk.

[6]The net short exposure is equal to \$88 mn.

[7]It is equal to $400\,000 \times \max(130 - 120, 0)$.

[8]It is equal to $10\,000 \times 540 \times (8\% + 8\%)$.

[9]It is equal to 8% for equities, 8% for currencies and 15% for commodities. In the case of interest rate risk, it corresponds to the standard value $K(t)$ for the time band t (see the table on page 8 in BCBS (1996a)).

Example 9 *We consider a portfolio of 4 options written on stocks with the following characteristics:*

Option	Stock	Exposure	Type	Price	Strike	Maturity	Volatility
1	A	−5	call	100	110	1.00	20%
2	A	−10	call	100	100	2.00	20%
3	B	10	call	200	210	1.00	30%
4	B	8	put	200	190	1.25	35%

This means that we have 2 assets. For stock A, we have a short exposure on 5 call options with a one-year maturity and a short exposure on 10 call options with a two-year maturity. For stock B, we have a long exposure on 10 call options with a one-year maturity and a long exposure on 8 put options with a maturity of one year and three months.

Using the Black-Scholes model, we first compute the Greek coefficients for each option j. Because the options are written on single stocks, the capital charges K_i^{Specific}, K_i^{General} and K_i^{Gamma} are all equal to 8%. Using the previous formulas, we then deduce the individual capital charges for each option[10]:

j	1	2	3	4
$\boldsymbol{\Delta}_j$	0.45	0.69	0.56	−0.31
$\boldsymbol{\Gamma}_j$	0.02	0.01	0.01	0.00
$\boldsymbol{\upsilon}_j$	39.58	49.91	78.85	79.25
$\mathcal{K}_j^{\text{Specific}}$	−17.99	−55.18	89.79	−40.11
$\mathcal{K}_j^{\text{General}}$	−17.99	−55.18	89.79	−40.11
$\mathcal{K}_j^{\text{Gamma}}$	−3.17	−3.99	8.41	4.64
$\mathcal{K}_j^{\text{Vega}}$	−9.89	−24.96	59.14	55.48

We can now aggregate the previous individual capital charges for each stock. We obtain:

Stock	$\mathcal{K}_i^{\text{Specific}}$	$\mathcal{K}_i^{\text{General}}$	$\mathcal{K}_i^{\text{Gamma}}$	$\mathcal{K}_i^{\text{Vega}}$
A	−73.16	−73.16	−7.16	−34.85
B	49.69	49.69	13.05	114.61
Total	122.85	23.47	7.16	149.46

To compute the total capital charge, we apply the netting rule for the general market risk, but not for the specific risk. This means that $\mathcal{K}^{\text{Specific}} = |-73.16| + |49.69| = 122.85$ and $\mathcal{K}^{\text{General}} = |-73.16 + 49.69| = 23.47$. For gamma risk, we only consider negative impacts and we have $\mathcal{K}^{\text{General}} = |-7.16| = 7.16$. For vega risk, there is no netting rule: $\mathcal{K}^{\text{Vega}} = |-34.85| + |114.61| = 149.46$. We finally deduce that the overall capital is 302.94.

The third method is the scenario approach. In this case, we evaluate the profit and loss (P&L) for simultaneous changes in the underlying price and in the implied volatility of the option. For defining these scenarios, the ranges are the standard shifts used previously. For instance, we use the following ranges for equities:

		S_i −8%	S_i +8%
Σ_j	−25%		
	+25%		

[10] For instance, the individual capital charge of the second option for the gamma risk is

$$\mathcal{K}_j^{\text{Gamma}} = \frac{1}{2} \times (-10) \times 0.0125 \times (100 \times 8\%)^2 = -3.99$$

The scenario matrix corresponds to intermediate points on the 2×2 grid. For each cell of the scenario matrix, we calculate the P&L of the option exposure[11]. The capital charge is then the largest loss.

Securitization instruments The treatment of specific risk of securitization positions is revised in Basel 2.5 and is based on external ratings. For instance, the capital charge for securitization exposures is 1.6% if the instrument is rated from AAA to AA−. For resecuritization exposures, it is equal to 3.2%. If the rating of the instrument is from BB+ to BB−, the risk capital charges becomes respectively[12] 28% and 52%.

2.1.1.2 Internal model-based approach

The use of an internal model is conditional upon the approval of the supervisory authority. In particular, the bank must meet certain criteria concerning different topics. These criteria concerns the risk management system, the specification of market risk factors, the properties of the internal model, the stress testing framework, the treatment of the specific risk and the backtesting procedure. In particular, the Basel Committee considers that the bank must have "*sufficient numbers of staff skilled in the use of sophisticated models not only in the trading area but also in the risk control, audit, and if necessary, back office areas*". We notice that the Basel Committee first insists on the quality of the trading department, meaning that the trader is the first level of risk management. The validation of an internal model does not therefore only concern the risk management department, but the bank as a whole.

Qualitative criteria BCBS (1996a) defines the following qualitative criteria:

- "*The bank should have an independent risk control unit that is responsible for the design and implementation of the bank's risk management system. [...] This unit must be independent from business trading units and should report directly to senior management of the bank*".
- The risk management department produces and analyzes daily reports, is responsible for the backtesting procedure and conducts stress testing analysis.
- The internal model must be used to manage the risk of the bank in the daily basis. It must be completed by trading limits expressed in risk exposure.
- The bank must document internal policies, controls and procedures concerning the risk measurement system (including the internal model).

It is today obvious that the risk management department should not report to the trading and sales department. Twenty-five years ago, it was not the case. Most of risk management units were incorporated to business units. It has completely changed because of the regulation and risk management is now independent from the front office. The risk management function has really emerged with the amendment to incorporate market risks and even more with the Basel II reform, whereas the finance function has long been developed in banks. For instance, it's very recent that the head of risk management[13] is also a member of the executive committee of the bank whereas the head of the finance department[14] has always been part of the top management.

[11] It may include the cash exposure if the option is used for hedging purposes.
[12] See pages 4-7 of BCBS (2009b) for the other risk capital charges.
[13] He is called the chief risk officer or CRO.
[14] He is called the chief financial officer or CFO.

From the supervisory point of view, an internal model does not reduce to measure the risk. It must be integrated in the management of the risk. This is why the Basel Committee points out the importance between the outputs of the model (or the risk measure), the organization of the risk management and the impact on the business.

Quantitative criteria The choice of the internal model is left to the bank, but it must respect the following quantitative criteria:

- The value-at-risk (VaR) is computed on a daily basis with a 99% confidence level. The minimum holding period of the VaR is 10 trading days. If the bank computes a VaR with a shorter holding period, it can use the square-root-of-time rule.
- The risk measure can take into account diversification, that is the correlations between the risk categories.
- The model must capture the relevant risk factors and the bank must pay attention to the specification of the appropriate set of market risk factors.
- The sample period for calculating the value-at-risk is at least one year and the bank must update the data set frequently (every month at least).
- In the case of options, the model must capture the non-linear effects with respect to the risk factors and the vega risk.
- "*Each bank must meet, on a daily basis, a capital requirement expressed as the higher of (i) its previous day's value-at-risk number [...] and (ii) an average of the daily value-at-risk measures on each of the preceding sixty business days, multiplied by a multiplication factor*".
- The value of the multiplication factor depends on the quality of the internal model with a range between 3 and 4. The quality of the internal model is related to its ex-post performance measured by the backtesting procedure.

The holding period to define the capital is 10 trading days. However, it is difficult to compute the value-at-risk for such holding period. In practice, the bank computes the one-day value-at-risk and converts this number into a ten-day value-at-risk using the square-root-of-time rule:

$$\mathrm{VaR}_\alpha\left(w;\text{ten days}\right)=\sqrt{10}\times\mathrm{VaR}_\alpha\left(w;\text{one day}\right)$$

This rule comes from the scaling property of the volatility associated to a geometric Brownian motion. It has the advantage to be simple and objective, but it generally underestimates the risk when the loss distribution exhibits fat tails[15].

The required capital at time t is equal to:

$$\mathcal{K}_t=\max\left(\mathrm{VaR}_{t-1},(3+\xi)\cdot\frac{1}{60}\sum_{i=1}^{60}\mathrm{VaR}_{t-i}\right) \tag{2.1}$$

where VaR_t is the value-at-risk calculated at time t and ξ is the penalty coefficient ($0\leq\xi\leq1$). In normal periods where $\mathrm{VaR}_{t-1}\simeq\mathrm{VaR}_{t-i}$, the required capital is the average of the last 60 value-at-risk values times the multiplication factor[16] $m_c=3+\xi$. In this case, we have:

$$\mathcal{K}_t=\mathcal{K}_{t-1}+\frac{m_c}{60}\cdot\left(\mathrm{VaR}_{t-1}-\mathrm{VaR}_{t-61}\right)$$

[15] See for instance Diebold *et al.* (1998), Daníelsson and Zigrand (2006) or Wang *et al.* (2011).
[16] The complementary factor is explained on page 88.

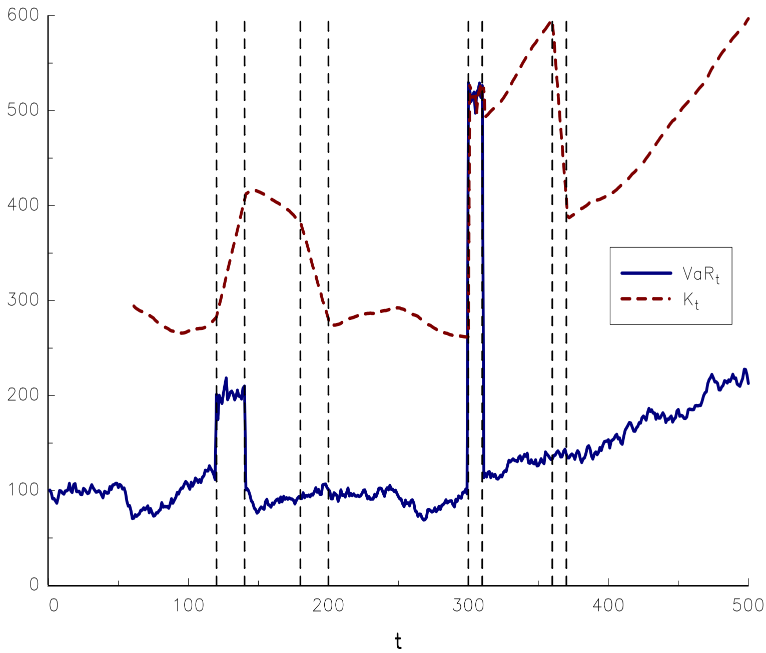

FIGURE 2.1: Calculation of the required capital with the VaR

The impact of VaR_{t-1} is limited because the factor $(3+\xi)/60$ is smaller than 6.7%. The required capital can only be equal to the previous day's value-at-risk if the bank faces a stress $\text{VaR}_{t-1} \gg \text{VaR}_{t-i}$. We also notice that a shock on the VaR vanishes after 60 trading days. To understand the calculation of the capital, we report an illustration in Figure 2.1. The solid line corresponds to the value-at-risk VaR_t whereas the dashed line corresponds to the capital $\boldsymbol{\mathcal{K}}_t$. We assume that $\xi = 0$ meaning that the multiplication factor is equal to 3. When $t < 120$, the value-at-risk varies around a constant. The capital is then relatively smooth and is three times the average VaR. At time $t = 120$, we observe a shock on the value-at-risk, which lasts 20 days. Immediately, the capital increases until $t \leq 140$. Indeed, at this time, the capital takes into account the full period of the shocked VaR (between $t = 120$ and $t = 139$). The full effect of this stressed period continues until $t \leq 180$, but this effect becomes partial when $t > 180$. The impact of the shock vanishes when $t = 200$. We then observe a period of 100 days where the capital is smooth because the daily value-at-risk does not change a lot. A second shock on the value-at-risk occurs at time $t = 300$, but the magnitude of the shock is larger than previously. During 10 days, the required capital is exactly equal to the previous day's value-at-risk. After 10 days, the bank succeeds to reduce the risk of its portfolio. However, the daily value-at-risk increases from $t = 310$ to $t = 500$. As previously, the impact of the second shock vanishes 60 days after the end of shock. However, the capital increases strongly at the end of the period. This is due to the effect of the multiplication factor m_c on the value-at-risk.

Stress testing Stress testing is a simulation method to identify events that could have a great impact on the soundness of the bank. The framework consists of applying stress scenarios and low-probability events on the trading portfolio of the bank and to evaluate the maximum loss. Contrary to the value-at-risk[17], stress testing is not used to compute the

[17]The 99% VaR is considered as a risk measure in normal markets and therefore ignores stress events.

required capital. The underlying idea is more to identify the adverse scenarios for the bank, evaluate the corresponding losses, reduce eventually the too risky exposures and anticipate the management of such stress periods.

Stress tests should incorporate both market and liquidity risks. The Basel Committee considers two types of stress tests:

1. supervisory stress scenarios;
2. stress scenarios developed by the bank itself.

The supervisory stress scenarios are standardized and apply to the different banks. This allows the supervisors to compare the vulnerability between the different banks. The bank must complement them by its own scenarios in order to evaluate the vulnerability of its portfolio according to the characteristics of the portfolio. In particular, the bank may be exposed to some political risks, regional risks or market risks that are not taken into account by standardized scenarios. The banks must report their test results to the supervisors in a quarterly basis.

Stress scenarios may be historical or hypothetical. In the case of historical scenarios, the bank computes the worst-case loss associated to different crisis: the Black Monday (1987), the European monetary system crisis (1992), the bond market sell-off (1994), the internet bubble (2000), the subprime mortgage crisis (2007), the liquidity crisis due to Lehman Brothers collapse (2008), the Euro zone crisis (2011-2012), etc. Hypothetical scenarios are more difficult to calibrate, because they must correspond to extreme but also plausible events. Moreover, the multidimensional aspect of stress scenarios is an issue. Indeed, the stress scenario is defined by the extreme event, but the corresponding loss is evaluated with respect to the shocks on market risk factors. For instance, if we consider a severe Middle East crisis, this event will have a direct impact on the oil price, but also indirect impacts on other market risk factors (equity prices, US dollar, interest rates). Whereas historical scenarios are objective, hypothetical scenarios are by construction subjective and their calibration will differ from one financial institution to another. In the case of the Middle East crisis, one bank may consider that the oil price could fall by 30% whereas another bank may use a price reduction of 50%.

In 2009, the Basel Committee revised the market risk framework. In particular, it introduces the stressed value-at-risk measure. The stressed VaR has the same characteristics than the traditional VaR (99% confidence level and 10-day holing period), but the model inputs are "*calibrated to historical data from a continuous 12-month period of significant financial stress relevant to the bank's portfolio*". For instance, a typical period is the 2008 year which both combines the subprime mortgage crisis and the Lehman Brothers bankruptcy. This implies that the historical period to compute the SVaR is completely different than the historical period to compute the VaR (see Figure 2.2). In Basel 2.5, the capital requirement for stressed VaR is:

$$\mathcal{K}_t^{\mathrm{SVaR}} = \max\left(\mathrm{SVaR}_{t-1}, m_s \cdot \frac{1}{60}\sum_{i=1}^{60}\mathrm{SVaR}_{t-i}\right)$$

where SVaR_t is the stressed VaR measure computed at time t. Like the coefficient m_c, the multiplication factor m_s for the stressed VaR is also calibrated with respect to the backtesting outcomes, meaning that we have $m_s = m_c$ in many cases.

Specific risk and other risk charges In the case where the internal model does not take into account the specific risk, the bank must compute a specific risk charge (SRC) using

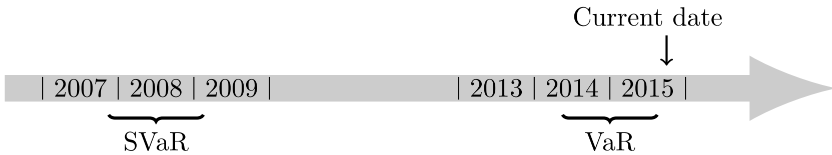

FIGURE 2.2: Two different periods to compute the VaR and the SVaR

the standardized measurement method. To be validated as a value-at-risk measure with specific risks, the model must satisfy at least the following criteria: it captures concentrations (magnitude and changes in composition), it captures name-related basis and event risks and it considers the assessment of the liquidity risk. For instance, an internal model built with a general market risk factor[18] does not capture specific risk. Indeed, the risk exposure of the portfolio is entirely determined by the beta of the portfolio with respect to the market risk factor. This implies that two portfolios with the same beta but with a different composition, concentration or liquidity have the same value-at-risk.

Basel 2.5 established a new capital requirement "*in response to the increasing amount of exposure in banks' trading books to credit-risk related and often illiquid products whose risk is not reflected in value-at-risk*" (BCBS, 2009b). The incremental risk charge (IRC) measures the impact of rating migrations and defaults, corresponds to a 99.9% value-at-risk for a one-year time horizon and concerns portfolios of credit vanilla trading (bonds and CDS). The IRC may be incorporated into the internal model or it may be treated as a surcharge from a separate calculation. Also under Basel 2.5, the Basel Committee introduced the comprehensive risk measure (CRM), which corresponds to a supplementary capital charge for credit exotic trading portfolios[19]. The CRM is also a 99.9% value-at-risk for a one-year time horizon. For IRC and CRM, the capital charge is the maximum between the most recent risk measure and the average of the risk measure over 12 weeks[20]. We finally obtain the following formula to compute the capital charge for the market risk under Basel 2.5:

$$\mathcal{K}_t = \mathcal{K}_t^{\mathrm{VaR}} + \mathcal{K}_t^{\mathrm{SVaR}} + \mathcal{K}_t^{\mathrm{SRC}} + \mathcal{K}_t^{\mathrm{IRC}} + \mathcal{K}_t^{\mathrm{CRM}}$$

where $\mathcal{K}_t^{\mathrm{VaR}}$ is given by Equation (2.1) and $\mathcal{K}_t^{\mathrm{SRC}}$ is the specific risk charge. In this formula, $\mathcal{K}_t^{\mathrm{SRC}}$ and/or $\mathcal{K}_t^{\mathrm{IRC}}$ may be equal to zero if the modeling of these two risks is included in the value-at-risk internal model.

Backtesting and the ex-post evaluation of the internal model The backtesting procedure is described in the document *Supervisory Framework for the Use of Backtesting in Conjunction with the Internal Models Approach to Market Risk Capital Requirements* published by the Basel Committee in January 1996. It consists of verifying that the internal model is consistent with a 99% confidence level. The idea is then to compare the outcomes of the risk model with realized loss values. For instance, we expect that the realized loss exceeds the VaR figure once every 100 observations on average.

The backtesting is based on the one-day holding period and compares the previous day's value-at-risk with the daily realized profit and loss. An exception occurs if the loss exceeds the value-at-risk. For a given period, we compute the number of exceptions. Depending of the frequency of exceptions, the supervisor determines the value of the penalty function between

[18] This is the case of the capital asset pricing model (CAPM) developed by Sharpe (1964).

[19] This concerns correlation trading activities on credit derivatives.

[20] Contrary to the VaR and SVaR measures, the risk measure is not scaled by a multiplication factor for IRC and CRM.

0 and 1. In the case of a sample based on 250 trading days, the Basel Committee defines three zones and proposes the values given in Table 2.1. The green zone corresponds to a number of exceptions less or equal to 4. In this case, the Basel Committee considers that there is no problem and the penalty coefficient ξ is set to 0. If the number of exceptions belongs to the yellow zone (between 5 and 9 exceptions), it may indicate that the confidence level of the internal model could be lower than 99% and implies that ξ is greater than zero. For instance, if the number of exceptions for the last 250 trading days is 6, the Basel Committee proposes that the penalty coefficient ξ is set to 0.50, meaning that the multiplication coefficient m_c is equal to 3.50. The red zone is a concern. In this case, the supervisor must investigate the reasons of such large number of exceptions. If the problem comes from the relevancy of the model, the supervisor can invalidate the internal model-based approach.

TABLE 2.1: Value of the penalty coefficient ξ for a sample of 250 observations

Zone	Number of exceptions	ξ
Green	0 – 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	10+	1.00

The definition of the color zones comes from the statistical analysis of the exception frequency. We note w the portfolio, $L_t(w)$ the daily loss at time t and $\mathrm{VaR}_\alpha(w;h)$ the value-at-risk calculated at time $t-1$. By definition, $L_t(w)$ is the opposite of the P&L $\Pi_t(w)$:

$$\begin{aligned} L_t(w) &= -\Pi_t(w) \\ &= \mathrm{MtM}_{t-1} - \mathrm{MtM}_t \end{aligned}$$

where MtM_t is the mark-to-market of the trading portfolio at time t. By definition, we have:

$$\Pr\{L_t(w) \geq \mathrm{VaR}_\alpha(w;h)\} = 1-\alpha$$

where α is the confidence level of the value-at-risk. Let e_t be the random variable which is equal to 1 if there is an exception and 0 otherwise. e_t is a Bernoulli random variable with parameter p:

$$\begin{aligned} p &= \Pr\{e_t = 1\} \\ &= \Pr\{L_t(w) \geq \mathrm{VaR}_\alpha(w;h)\} \\ &= 1-\alpha \end{aligned}$$

In the case of the Basel framework, α is set to 99% meaning that we have a probability of 1% to observe an exception every trading day. For a given period $[t_1, t_2]$ of n trading days, the probability to observe exactly m exceptions is given by the binomial formula:

$$\Pr\{N_e(t_1;t_2) = m\} = \binom{n}{m}(1-\alpha)^m \alpha^{n-m}$$

where $N_e(t_1;t_2) = \sum_{t=t_1}^{t_2} e_t$ is the number of exceptions for the period $[t_1, t_2]$. We obtain this result under the assumption that the exceptions are independent across time. $N_e(t_1;t_2)$

is then the binomial random variable $\mathcal{B}(n; 1-\alpha)$. We deduce that the probability to have up to m exceptions is:

$$\Pr\{N_e(t_1;t_2) \le m\} = \sum_{j=0}^{m} \binom{n}{j} (1-\alpha)^i \alpha^{n-j}$$

The three previous zones are then defined with respect to the statistical confidence level of the assumption $\mathcal{H}: \alpha = 99\%$. The green zone corresponds to the 95% confidence level: $\Pr\{N_e(t_1;t_2) \le m\} < 95\%$. In this case, the hypothesis $\mathcal{H}: \alpha = 99\%$ is not rejected at the 95% confidence level. The yellow and red zones are respectively defined by $95\% \le \Pr\{N_e(t_1;t_2) \le m\} < 99.99\%$ and $\Pr\{N_e(t_1;t_2) \le m\} \ge 99.99\%$. This implies that the hypothesis $\mathcal{H}: \alpha = 99\%$ is rejected at the 99.99% confidence level if the number of exceptions belongs to the red zone.

TABLE 2.2: Probability distribution (in %) of the number of exceptions ($n = 250$ trading days)

	$\alpha = 99\%$		$\alpha = 98\%$	
m	$\Pr\{N_e = m\}$	$\Pr\{N_e \le m\}$	$\Pr\{N_e = m\}$	$\Pr\{N_e \le m\}$
0	8.106	8.106	0.640	0.640
1	20.469	28.575	3.268	3.908
2	25.742	54.317	8.303	12.211
3	21.495	75.812	14.008	26.219
4	13.407	89.219	17.653	43.872
5	6.663	95.882	17.725	61.597
6	2.748	98.630	14.771	76.367
7	0.968	99.597	10.507	86.875
8	0.297	99.894	6.514	93.388
9	0.081	99.975	3.574	96.963
10	0.020	99.995	1.758	98.720

If we apply the previous statistical analysis when n is equal to 250 trading days, we obtain the results given in Table 2.2. For instance, the probability to have zero exception is 8.106%, the probability to have one exception is 20.469%, etc. We retrieve the three color zones determined by the Basel Committee. The green zone corresponds to the interval $[0, 4]$, the yellow zone is defined by the interval $[5, 9]$ and the red zone involves the interval $[10, 250]$. We notice that the color zones can vary significantly if the confidence level of the value-at-risk is not equal to 99%. For instance, if it is equal to 98%, the green zone corresponds to less than 9 exceptions. In Figure 2.3, we have reported the color zones with respect to the size n of the sample.

Example 10 *Calculate the color zones when n is equal to 1 000 trading days and $\alpha = 99\%$.*

We have $\Pr\{N_e \le 14\} = 91.759\%$ and $\Pr\{N_e \le 15\} = 95.213\%$. This implies that the green zones ends at 14 exceptions whereas the yellow zone begins at 15 exceptions. Because $\Pr\{N_e \le 23\} = 99.989\%$ and $\Pr\{N_e \le 24\} = 99.996\%$, we also deduce that the red zone begins at 24 exceptions.

Remark 2 *The statistical approach of backtesting ignores the effects of intra-day trading. Indeed, we make the assumption that the portfolio remains unchanged from $t-1$ to t, which is not the case in practice. This is why the Basel Committee proposes to compute the loss*

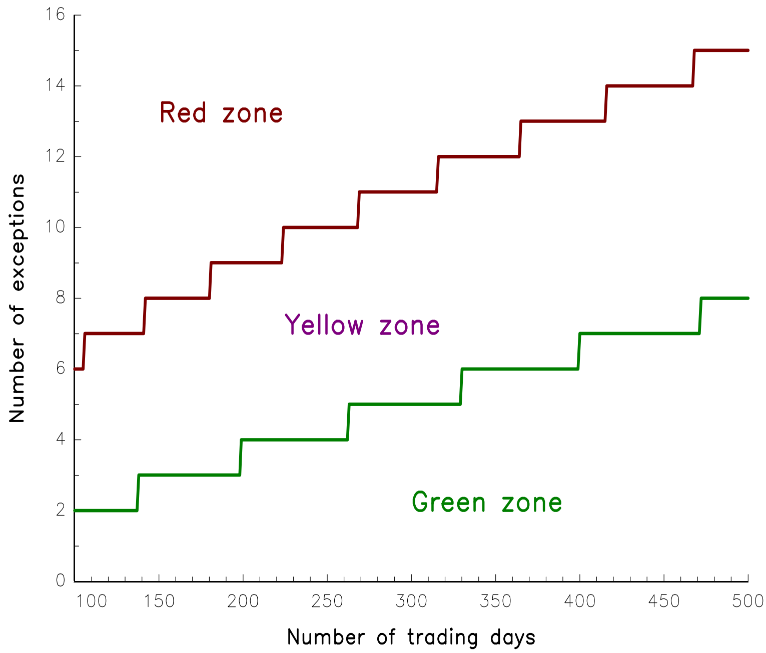

FIGURE 2.3: Color zones of the backtesting procedure ($\alpha = 99\%$)

in two different ways. The first approach uses the official realized P&L, whereas the second approach consists in separating the P&L of the previous's day portfolio and the P&L due to the intra-day trading activities.

2.1.2 The Basel III framework

The finalization of the reform for computing the market risk capital charge has taken considerable time. After the 2008 crisis, the market risk is revised by the Basel Committee, which adds new capital charges (Basel 2.5) in addition to those defined in the Basel I framework. In the same time, the Basel Committee published a new framework called Basel III, which focused on liquidity and leverage risks. In 2013, the Basel Committee launched a vast project called the fundamental review of the trading book (FRTB). During long time, the banking industry believed that these discussions were the basis of new reforms in order to prepare a Basel IV Accord. However, the Basel Committee argued that these changes are simply completing the Basel III reforms. As for the Basel I Accord, banks have the choice between two approaches for computing the capital charge:

1. a standardized method (SA-TB[21]);
2. an internal model-based approach (IMA).

Contrary to the previous framework, the SA-TB method is very important even if banks calculate the capital charge with the IMA method. Indeed, the bank must implement SA-TB in order to meet the output floor requirement[22], which is set at 72.5% in January 2027.

[21] TB means trading book.

[22] The mechanism of capital floor is explained on page 22.

2.1.2.1 Standardized approach

The standardized capital charge is the sum of three components: sensitivity-based method capital, the default risk capital (DRC) and the residual risk add-on (RRAO). The first component must be viewed as the pure market risk and is the equivalent of the capital charge for the general market risk in the Basel I Accord. The second component captures the jump-to-default risk (JTD) and replaces the specific risk that we find in the Basel I framework. The last component captures specific risks that are difficult to measure in practice.

Sensitivity-based capital requirement This method consists in calculating a capital charge for delta, vega and curvature risks, and then aggregating the three capital requirements:

$$\mathcal{K} = \mathcal{K}^{\text{Delta}} + \mathcal{K}^{\text{Vega}} + \mathcal{K}^{\text{Curvature}}$$

Seven risk classes are defined by the Basel Committee: (1) general interest rate risk (GIRR), (2) credit spread risk (CSR) on non-securitization products, (3) CSR on non-correlation trading portfolio (non-CTP), (4) CSR on correlation trading portfolio (CTP), (5) equity risk, (6) commodity risk and (7) foreign exchange risk. The sensitivities of the different instruments of one risk class are risk-weighted and then aggregated. The first level of aggregation concerns the risk buckets, defined as risk factors with common characteristics. For example, the bucket #1 for credit spread risk corresponds to all instruments that are exposed to the IG sovereign credit spread. The second level of aggregation is done by considering the different buckets that compose the risk class. For example, the credit spread risk is composed of 18 risk buckets (8 investment grade buckets, 7 high yield buckets, 2 index buckets and one other sector bucket).

For delta and vega components, we first begin to calculate the weighted sensitivity of each risk factor $\mathcal{F}_j$:

$$\text{WS}_j = S_j \cdot \text{RW}_j$$

where S_j and RW_j are the net sensitivity of the portfolio with respect to the risk factor and the risk weight of $\mathcal{F}_j$. More precisely, we have $S_j = \sum_i S_{i,j}$ where $S_{i,j}$ is the sensitivity of the instrument i with respect to $\mathcal{F}_j$. Second, we calculate the capital requirement for the risk bucket $\mathcal{B}_k$:

$$\mathcal{K}_{\mathcal{B}_k} = \sqrt{\max\left(\sum_j \text{WS}_j^2 + \sum_{j' \neq j} \rho_{j,j'} \,\text{WS}_j \,\text{WS}_{j'}, 0\right)}$$

where $\mathcal{F}_j \in \mathcal{B}_k$. We recognize the formula of a standard deviation[23]. Finally, we aggregate the different buckets for a given risk class[24]:

$$\mathcal{K}^{\text{Delta/Vega}} = \sqrt{\sum_k \mathcal{K}_{\mathcal{B}_k}^2 + \sum_{k' \neq k} \gamma_{k,k'} \,\text{WS}_{\mathcal{B}_k} \,\text{WS}_{\mathcal{B}_{k'}}}$$

where $\text{WS}_{\mathcal{B}_k} = \sum_{j \in \mathcal{B}_k} \text{WS}_j$ is the weighted sensitivity of the bucket $\mathcal{B}_k$. Again, we recognize the formula of a standard deviation. Therefore, the capital requirement for delta and vega risks can be viewed as a Gaussian risk measure with the following parameters:

1. the sensitivities S_j of the risk factors that are calculated by the bank;
2. the risk weights RW_j of the risk factors;

[23] The variance is floored at zero, because the correlation matrix formed by the cross-correlations $\rho_{j,j'}$ is not necessarily positive definite.

[24] If the term under the square root is negative, the Basel Committee proposes an alternative formula.

3. the correlation $\rho_{j,j'}$ between risk factors within a bucket;

4. the correlation $\gamma_{k,k'}$ between the risk buckets.

For the curvature risk, the methodology is different because it is based on two adverse scenarios. We note $P_i(\mathcal{F}_j)$ the price of the instrument i when the current level of the risk factor is $\mathcal{F}_j$. We calculate $P_i^+(\mathcal{F}_j) = P_i\left(\mathcal{F}_j + \Delta\mathcal{F}_j^+\right)$ and $P_i^-(\mathcal{F}_j) = P_i\left(\mathcal{F}_j - \Delta\mathcal{F}_j^-\right)$ the price of instrument i when the risk factor is shocked upward by $\Delta\mathcal{F}_j^+$ and downward by $\Delta\mathcal{F}_j^-$. The curvature risk capital requirement for the risk factor $\mathcal{F}_j$ is equal to:

$$\mathrm{CVR}_j^{\pm} = -\sum_i \left(P_i^{\pm}(\mathcal{F}_j) - P_i(\mathcal{F}_j) - S_{i,j} \cdot \mathrm{RW}_j^{\mathrm{CRV}}\right)$$

where $S_{i,j}$ is the delta sensitivity[25] of instrument i with respect to the risk factor $\mathcal{F}_j$ and $\mathrm{RW}_j^{\mathrm{CRV}}$ is the curvature risk weight of $\mathcal{F}_j$. CVR_j^+ and CVR_j^- play the role of WS_j in the delta/vega capital computation. The capital requirement for the bucket (or risk class) $\mathcal{B}_k$ is:

$$\mathcal{K}_{\mathcal{B}_k}^{\pm} = \sqrt{\max\left(\sum_j \left(\max\left(\mathrm{CVR}_j^{\pm}, 0\right)\right)^2 + \sum_{j' \neq j} \rho_{j,j'} \psi\left(\mathrm{CVR}_j^{\pm}, \mathrm{CVR}_{j'}^{\pm}\right), 0\right)}$$

where $\psi(\mathrm{CVR}_j, \mathrm{CVR}_{j'})$ is equal to 0 if the two arguments are both negative or is equal to $\mathrm{CVR}_j \times \mathrm{CVR}_{j'}$ otherwise. Then, the capital requirement for the risk bucket $\mathcal{B}_k$ is the maximum of the two adverse scenarios:

$$\mathcal{K}_{\mathcal{B}_k} = \max\left(\mathcal{K}_{\mathcal{B}_k}^+, \mathcal{K}_{\mathcal{B}_k}^-\right)$$

At this stage, one scenario is selected: the upward scenario if $\mathcal{K}_{\mathcal{B}_k}^+ > \mathcal{K}_{\mathcal{B}_k}^-$ or the downward scenario if $\mathcal{K}_{\mathcal{B}_k}^+ < \mathcal{K}_{\mathcal{B}_k}^-$. And we define the curvature risk $\mathrm{CVR}_{\mathcal{B}_k}$ for each bucket as follows:

$$\begin{aligned} \mathrm{CVR}_{\mathcal{B}_k} &= \mathbb{1}\left\{\mathcal{K}_{\mathcal{B}_k}^+ > \mathcal{K}_{\mathcal{B}_k}^-\right\} \cdot \sum_{j \in \mathcal{B}_k} \mathrm{CVR}_j^+ + \\ &\quad \mathbb{1}\left\{\mathcal{K}_{\mathcal{B}_k}^+ < \mathcal{K}_{\mathcal{B}_k}^-\right\} \cdot \sum_{j \in \mathcal{B}_k} \mathrm{CVR}_j^- \end{aligned}$$

Finally, the capital requirement for the curvature risk is equal to:

$$\mathcal{K}^{\mathrm{Curvature}} = \sqrt{\max\left(\sum_k \mathcal{K}_{\mathcal{B}_k}^2 + \sum_{k' \neq k} \gamma_{k,k'} \psi\left(\mathrm{CVR}_{\mathcal{B}_k}, \mathrm{CVR}_{\mathcal{B}_{k'}}\right), 0\right)}$$

We conclude that we use the same methodology for delta, vega and curvature risks with three main differences: the computation of the sensitivities, the scale of risk weights, and the use of two scenarios for the curvature risk.

The first step consists in defining the risk factors. The Basel Committee gives a very precise list of risk factors by asset classes (BCBS, 2019). For instance, the equity delta risk factors are the equity spot prices and the equity repo rates, the equity vega risk factors

[25] For FX and equity risk classes, $S_{i,j}$ is the delta sensitivity of instrument i. For the other risk classes, $S_{i,j}$ is the sum of delta sensitivities of instrument i with respect to the risk factor $\mathcal{F}_j$.

are the implied volatilities of options, and the equity curvature risk factors are the equity spot prices. We retrieve the notions of delta, vega and gamma that we encounter in the theory of options. In the case of the interest rate risk class (GIRR), the risk factors include the yield curve[26], a flat curve of market-implied inflation rates for each currency and some cross-currency basis risks. For the other categories, the delta risk factors are credit spread curves, commodity spot prices and exchange rates. As for equities, vega and curvature risk factors correspond to implied volatilities of options and aggregated delta risk factors.

The second step consists in calculating the sensitivities. The equity delta sensitivity of the instrument i with respect to the equity risk factor $\mathcal{F}_j$ is given by:

$$S_{i,j} = \boldsymbol{\Delta}_i \left(\mathcal{F}_j\right) \cdot \mathcal{F}_j$$

where $\boldsymbol{\Delta}_i \left(\mathcal{F}_j\right)$ measures the (discrete) delta[27] of the instrument i by shocking the equity risk factor $\mathcal{F}_j$ by 1%. If the instrument i corresponds to a stock, the sensitivity is exactly the price of this stock when the risk factor is the stock price, and zero otherwise. If the instrument i corresponds to an European option on this stock, the sensitivity is the traditional delta of the option times the stock price. The previous formula is also valid for FX and commodity risks. For interest rate and credit risks, the delta corresponds to the PV01, that is a change of the interest rate and credit spread by 1 bp. For the vega sensitivity, we have:

$$S_{i,j} = \boldsymbol{\upsilon}_i \left(\mathcal{F}_j\right) \cdot \mathcal{F}_j$$

where $\mathcal{F}_j$ is the implied volatility.

The third step consists in calculating the risk-weighted sensitivities WS_j. For that, we use the tables given in BCBS (2019). For example, the risk weight for the 3M interest rate is equal to 1.7% while the risk weight for the 30Y interest rate is equal to 1.1% (BCBS, 2019, Table 1, page 38). For equity spot prices, the risk weight goes from 15% for large cap DM indices to 70% for small cap EM stocks (BCBS, 2019, Table 10, page 47). The fourth step computes the capital charge for each bucket. In this case, we need the '*factor*' correlations $\rho_{j,j'}$ between the risk factors within the same bucket. For example, the yield curve correlations between the 10 tenors of the same currency are given in Table 2 on page 38 in BCBS (2019). For the equity risk, $\rho_{j,j'}$ goes from 7.5% to 80%. Finally, we can compute the capital by considering the '*bucket*' correlations. For example, $\gamma_{k,k'}$ is set to 50% between the different currencies in the case of the interest rate risk. We must note that the values given by the Basel Committee correspond to a medium correlation scenario. The Basel Committee observes that correlations may increase or decrease in period of a stressed market, and impose that the bank must use the maximum of capital requirement under three correlation scenarios: medium, high and low. Under the high correlation scenario, the correlations are increased: $\rho_{j,j'}^{\mathrm{High}} = \min\left(1.25 \times \rho_{j,j'}, 1\right)$ and $\gamma_{k,k'}^{\mathrm{High}} = \min\left(1.25 \times \gamma_{k,k'}, 1\right)$. Under the low correlation scenario, the correlations are decreased: $\rho_{j,j'}^{\mathrm{Low}} = \max\left(2 \times \rho_{j,j'} - 1, 0.75 \times \rho_{j,j'}\right)$ and $\gamma_{k,k'}^{\mathrm{Low}} = \max\left(2 \times \gamma_{k,k'} - 1, 0.75 \times \gamma_{k,k'}\right)$. Figure 2.4 shows how the medium correlation is scaled to high and low correlation scenarios.

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[26] The risk factors correspond to the following tenors of the yield curve: 3M, 6M, 1Y, 2Y, 3Y, 5Y, 10Y, 15Y, 20Y and 30Y.

[27] It follows that:

$$\begin{aligned} S_{i,j} &= \frac{P_i\left(1.01 \cdot \mathcal{F}_j\right) - P_i\left(\mathcal{F}_j\right)}{1.01 \cdot \mathcal{F}_j - \mathcal{F}_j} \cdot \mathcal{F}_j \\ &= \frac{P_i\left(1.01 \cdot \mathcal{F}_j\right) - P_i\left(\mathcal{F}_j\right)}{0.01} \end{aligned}$$

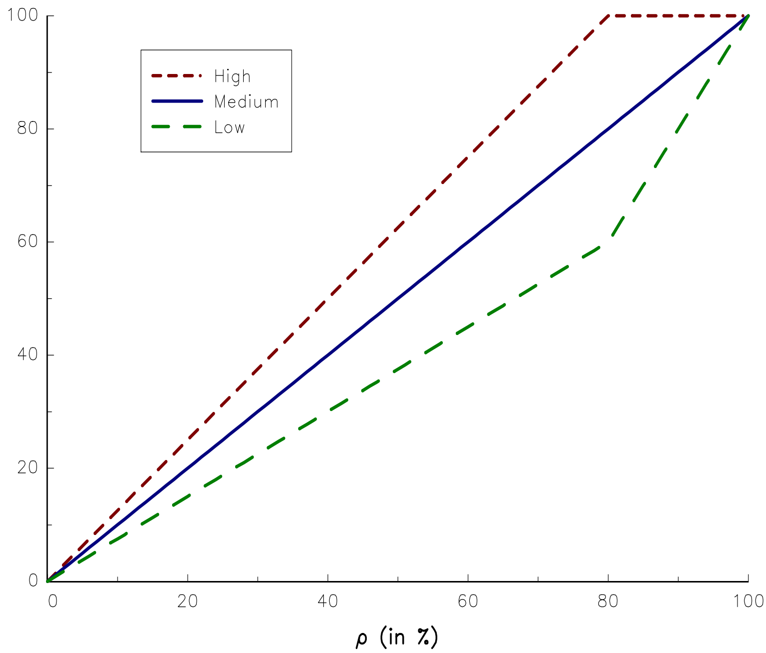


FIGURE 2.4: High, medium and low correlation scenarios

Default risk capital The gross jump-to-default (JTD) risk is computed by differentiating long and short exposures[28]:

$$\mathrm{JTD}^{\mathcal{L}\mathrm{ong}} = \max\left(N \cdot \mathrm{LGD} + \Pi, 0\right)$$

and:

$$\mathrm{JTD}^{\mathcal{S}\mathrm{hort}} = \min\left(N \cdot \mathrm{LGD} + \Pi, 0\right)$$

where N is the notional, LGD is the loss given default[29] and Π is the current P&L. Then, we offset long and short exposures to the same obligor under some conditions of seniority and maturity. At this stage, we obtain net JTD exposures, that can be positive (long) or negative (short). Three buckets are defined: (1) corporates, (2) sovereigns and (3) local governments and municipalities. For each bucket $\mathcal{B}_k$, the capital charge is calculated as follows:

$$\mathcal{K}_{\mathcal{B}_k}^{\mathrm{DRC}} = \max\left(\sum_{i \in \mathcal{L}\mathrm{ong}} \mathrm{RW}_i \cdot \mathrm{JTD}_i^{\mathrm{Net}} - \mathrm{HBR} \sum_{i \in \mathcal{S}\mathrm{hort}} \mathrm{RW}_i \cdot \left|\mathrm{JTD}_i^{\mathrm{Net}}\right|, 0\right) \tag{2.2}$$

where the risk weight depends on the rating of the obligor:

Rating	AAA	AA	A	BBB	BB	B	CCC	NR
RW	0.5%	2%	3%	6%	15%	30%	50%	15%

[28] A long exposure implies that the default results in a loss, whereas a short exposure implies that the default results in a gain.

[29] The default values are 100% for equity and non-senior debt instruments, 75% for senior debt instruments, 25% for covered bonds and 0% for FX instruments.

and HBR is the hedge benefit ratio:

$$\text{HBR} = \frac{\sum_{i \in \mathcal{L}\text{ong}} \text{JTD}_i^{\text{Net}}}{\sum_{i \in \mathcal{L}\text{ong}} \text{JTD}_i^{\text{Net}} + \sum_{i \in \mathcal{S}\text{hort}} \left| \text{JTD}_i^{\text{Net}} \right|}$$

At first sight, Equation (2.2) seems to be complicated. In order to better understand this formula, we assume that there is no short credit exposure and the P&L of each instrument is equal to zero. Therefore, the capital charge for the bucket $\mathcal{B}_k$ is equal to:

$$\mathcal{K}_{\mathcal{B}_k}^{\text{DRC}} = \sum_{i \in \mathcal{B}_k} \underbrace{N_i \cdot \text{LGD}_i}_{\text{EAD}_i} \cdot \text{RW}_i$$

We recognize the formula for computing the credit risk capital when we replace the exposure at default by the product of the notional and the loss given default. In the case of a portfolio of loans, the exposures are always positive. In the case of a trading portfolio, we face more complex situations because we can have both long and short credit exposures. The introduction of the hedge benefit ratio allows to mitigate the risk of long credit exposures.

Remark 3 *The previous framework is valid for non-securitization instruments. For securitization, a similar approach is followed, but the LGD factor disappears in order to avoid double counting. Moreover, the treatment of offsetting differs for non-CTP and CTP products.*

Residual risk add-on The idea of this capital charge is to capture market risks which are not taken into account by the two previous methods. Residual risks concerns instruments with an exotic underlying (weather, natural disasters, longevity, etc.), payoffs that are not a linear combination of vanilla options (spread options, basket options, best-of, worst-of, etc.), or products that present significant gap, correlation or behavioral risks (digital options, barrier options, embedded options, etc.). We have:

$$\mathcal{K}_i^{\text{RRAO}} = N_i \cdot \text{RW}_i$$

where RW_i is equal to 1% for instruments with an exotic underlying and 10 bps for the other residual risks.

2.1.2.2 Internal model-based approach

As in the first Basel Accord, the Basel III framework includes general criteria, qualitative standards, quantitative criteria, backtesting procedures and stress testing approaches. The main difference concerning general criteria is the introduction of trading desks. According to BCBS (2019), a trading desk is "*an unambiguously defined group of traders or trading accounts that implements a well-defined business strategy operating within a clear risk management structure*". Internal models are implemented at the trading desk level. Within a bank, some trading desks are then approved for the use of internal models, while other trading desks must use the SA-TB approach. The Basel Committee reinforces the role of the model validation unit, the process of the market risk measurement system (documentation, annual independent review, etc.) and the use of stress scenarios.

Capital requirement for modellable risk factors Concerning capital requirements, the value-at-risk at the 99% confidence level is replaced by the expected shortfall at the

TABLE 2.3: Liquidity horizon (Basel III)

Liquidity class k	Liquidity horizon h_k
1	10
2	20
3	40
4	60
5	120

97.5% confidence level. Moreover, the 10-day holding period is not valid for all instruments. Indeed, the expected shortfall must take into account the liquidity risk and we have:

$$\mathrm{ES}_\alpha(w) = \sqrt{\sum_{k=1}^{5}\left(\mathrm{ES}_\alpha(w;h_k)\sqrt{\frac{h_k - h_{k-1}}{h_1}}\right)^2}$$

where:

- $\mathrm{ES}_\alpha(w;h_1)$ is the expected shortfall of the portfolio w at horizon 10 days by considering all risk factors;
- $\mathrm{ES}_\alpha(w;h_k)$ is the expected shortfall of the portfolio w at horizon h_k days by considering the risk factors $\mathcal{F}_j$ that belongs to the liquidity class k;
- h_k is the horizon of the liquidity class k, which is given in Table 2.3 (h_0 is set to zero).

This expected shortfall framework is valid for modellable risk factors. Within this framework, all instruments are classified into 5 buckets (10, 20, 40, 60 and 120 days), which are defined by BCBS (2019) as follows:

1. Interest rates (specified currencies[30] and domestic currency of the bank), equity prices (large caps), FX rates (specified currency pairs[31]).
2. Interest rates (unspecified currencies), equity prices (small caps) and volatilities (large caps), FX rates (currency pairs), credit spreads (IG sovereigns), commodity prices (energy, carbon emissions, precious metals, non-ferrous metals).
3. FX rates (other types), FX volatilities, credit spreads (IG corporates and HY sovereigns).
4. Interest rates (other types), IR volatility, equity prices (other types) and volatilities (small caps), credit spreads (HY corporates), commodity prices (other types) and volatilities (energy, carbon emissions, precious metals, non-ferrous metals).
5. Credit spreads (other types) and credit spread volatilities, commodity volatilities and prices (other types).

The expected shortfall must reflect the risk measure for a period of stress. For that, the Basel Committee proposes an indirect approach:

$$\mathrm{ES}_\alpha(w;h) = \mathrm{ES}_\alpha^{(\text{reduced,stress})}(w;h)\cdot\min\left(\frac{\mathrm{ES}_\alpha^{(\text{full,current})}(w;h)}{\mathrm{ES}_\alpha^{(\text{reduced,current})}(w;h)},1\right)$$

[30] The specified currencies are composed of EUR, USD, GBP, AUD, JPY, SEK and CAD.

[31] They correspond to the 20 most liquid currencies: USD, EUR, JPY, GBP, AUD, CAD, CHF, MXN, CNY, NZD, RUB, HKD, SGD, TRY, KRW, SEK, ZAR, INR, NOK and BRL.

where $\text{ES}_\alpha^{(\text{full,current})}$ is the expected shortfall based on the current period with the full set of risk factors, $\text{ES}_\alpha^{(\text{reduced,current})}$ is the expected shortfall based on the current period with a restricted set of risk factors and $\text{ES}_\alpha^{(\text{reduced,stress})}$ is the expected shortfall based on the stress period[32] with the restricted set of risk factors. The Basel Committee recognizes that it is difficult to calculate directly $\text{ES}_\alpha^{(\text{full,stress})}(w;h)$ on the stress period with the full set of risk factors. Therefore, the previous formula assumes that there is a proportionality factor between the full set and the restricted set of risk factors[33]:

$$\frac{\text{ES}_\alpha^{(\text{full,stress})}(w;h)}{\text{ES}_\alpha^{(\text{full,current})}(w;h)} \approx \frac{\text{ES}_\alpha^{(\text{reduced,stress})}(w;h)}{\text{ES}_\alpha^{(\text{reduced,current})}(w;h)}$$

Example 11 *In the table below, we have calculated the 10-day expected shortfall for a given portfolio:*

Set of risk factors	*Period*	*Liquidity class* 1	2	3	4	5
Full	*Current*	100	75	34	12	6
Reduced	*Current*	88	63	30	7	5
Reduced	*Stress*	112	83	47	9	7

As expected, the expected shortfall decreases with the liquidity horizon, because there are less and less risk factors that belong to the liquidity class. We also verify that the ES for the reduced set of risk factors is lower than the ES for the full set of risk factors.

TABLE 2.4: Scaled expected shortfall

k	Sc_k	Full Current	Reduced Current	Reduced Stress	Full/Stress (not scaled)	Full Stress
1	1	100.00	88.00	112.00	127.27	127.27
2	1	75.00	63.00	83.00	98.81	98.81
3	$\sqrt{2}$	48.08	42.43	66.47	53.27	75.33
4	$\sqrt{2}$	16.97	9.90	12.73	15.43	21.82
5	$\sqrt{6}$	14.70	12.25	17.15	8.40	20.58
Total		135.80	117.31	155.91		180.38

Results are given in Table 2.4. For each liquidity class k, we have reported the scaling factor $Sc_k = \sqrt{(h_k - h_{k-1})/h_1}$, the scaled expected shortfall $\text{ES}_\alpha^\star(w;h_k) = Sc_k \cdot \text{ES}_\alpha(w;h_k)$ (columns 3, 4 and 5) and the total expected shortfall $\text{ES}_\alpha(w) = \sqrt{\sum_{k=1}^5 \left(\text{ES}_\alpha^\star(w;h_k)\right)^2}$. It is respectively equal to 135.80, 117.31 and 155.91 for the full/current, reduced/current and reduced/stress case. Since the proportionality factor is equal to $135.80/117.31 = 1.1576$, we deduce that the ES for the full set of risk factors and the stress period is equal to $1.1576 \times 155.91 = 180.48$. Another way to calculate the ES is first to compute the ES for the full set of risk factors and the stress period for each liquidity class k and deduce the scaled expected shortfall (columns 6 and 7). In this case, the ES for the full set of risk factors and the stress period is equal to 180.38.

[32] The bank must consider the most severe 12-month period of stress available.

[33] However, the Basel Committee indicates that the reduced set of risk factors must explain al leat 75% of the risk in periods of stress.

The final step for computing the capital requirement (also known as the 'internally modelled capital charge') is to apply this formula:

$$\mathrm{IMCC} = \varrho \cdot \mathrm{IMCC}_{global} + (1 - \varrho) \cdot \sum_{k=1}^{5} \mathrm{IMCC}_k$$

where ϱ is equal to 50%, IMCC_{global} is the stressed ES calculated with the internal model and cross-correlations between risk classes, IMCC_k is the stressed ES calculated at the risk class level (interest rate, equity, foreign exchange, commodity and credit spread). IMCC is then an average of two capital charges: one that takes into account cross-correlations and another one that ignores diversification effects.

Capital requirement for non-modellable risk factors Concerning non-modellable risk factors, the capital requirement is based on stress scenarios, that are equivalent to a stressed expected shortfall. The Basel Committee distinguish three types of non-modellable risk factors:

1. Non-modellable idiosyncratic credit spread risk factors ($i = 1, \ldots, m_{\mathrm{c}}$);
2. Non-modellable idiosyncratic equity risk factors ($j = 1, \ldots, m_{\mathrm{e}}$);
3. Remaining non-modellable risk factors ($k = 1, \ldots, m_{\mathrm{o}}$).

The capital requirement for non-modellable risk factors is then equal to:

$$\mathrm{SES} = \mathrm{SES}^{\mathrm{Credit}} + \mathrm{SES}^{\mathrm{Equity}} + \mathrm{SES}^{\mathrm{Other}}$$

where $\mathrm{SES}^{\mathrm{Credit}} = \sqrt{\sum_{i=1}^{m_{\mathrm{c}}} \mathrm{SES}_i^2}$, $\mathrm{SES}^{\mathrm{Equity}} = \sqrt{\sum_{j=1}^{m_e} \mathrm{SES}_j^2}$ and:

$$\mathrm{SES}^{\mathrm{Other}} = \sqrt{\varrho^2 \cdot \left(\sum_{k=1}^{m_{\mathrm{o}}} \mathrm{SES}_k \right)^2 + (1 - \varrho^2) \cdot \sum_{k=1}^{m_{\mathrm{o}}} \mathrm{SES}_k^2}$$

For non-modellable credit or equity risks, we assume a zero correlation. For the remaining non-modellable risks, the correlation ϱ is set to 60%. An important issue for computing SES is the liquidity horizon. The Basel Committee imposes to consider the same values used for modellable risk factors, with a floor of 20 days. For idiosyncratic credit spreads, the liquidity horizon is set to 120 days.

Capital requirement for default risk The default risk capital (DRC) is calculated using a value-at-risk model with a 99.9% confidence level. The computation must be done using the same default probabilities that are used for the IRB approach. This implies that default risk is calculated under the historical probability measure, and not under the risk-neutral probability measure. This is why market-implied default probabilities are prohibited.

Capital requirement for the market risk For eligible trading desks that are approved to use the IMA approach, the capital requirement for market risk is equal to:

$$\begin{aligned} \mathcal{K}_t^{\mathrm{IMA}} &= \max \left(\mathrm{IMCC}_{t-1} + \mathrm{SES}_{t-1}, \frac{m_c \sum_{i=1}^{60} \mathrm{IMCC}_{t-i} + \sum_{i=1}^{60} \mathrm{SES}_{t-i}}{60} \right) + \\ & \quad \mathrm{DRC} \end{aligned} \tag{2.3}$$

where $m_c = 1.5 + \xi$ and $0 \leq \xi \leq 0.5$. This formula is similar to the one defined in the Basel I Accord. We notice that the magnitude of the multiplication factor m_c has changed since we have $1.5 \leq m_c \leq 2$.

TABLE 2.5: Value of the penalty coefficient ξ in Basel III

Zone	Number of exceptions	ξ
Green	0 – 4	0.00
Amber	5	0.20
	6	0.26
	7	0.33
	8	0.38
	9	0.42
Red	10+	0.50

Backtesting The backtesting procedure continues to be based on the daily VaR with a 99% confidence level and a sample of the last 250 observations. Table 2.5 presents the definition of the color zones. We notice that the amber zone replaces the yellow zone, and the values of the penalty coefficient ξ have changed. The value of the multiplier $m_c = 1.5+\xi$ depends then on the one-year backtesting procedure at the bank-wide level. However, the bank must also conduct backtesting exercises for each eligible trading desk because of two reasons. First, the P&L attribution (PLA) is one of the pillars for the approval of trading desks by supervisory authorities. It is highly reinforced with several PLA tests, that distinguish actual P&L (including intra-day trading activities) and hypothetical P&L (static portfolio). Second, if one eligible trading desk is located in the amber zone, the formula (2.3) is modified in order to take into account a capital surcharge. Moreover, if one eligible trading desk has more than 12 exceptions[34], the bank must use the SA-TB approach for calculating the capital charge of this trading desk.

2.2 Statistical estimation methods of risk measures

We have seen that Basel I is based on the value-at-risk while Basel III uses the expected shortfall for computing the capital requirement for market risk. In this section, we define precisely what a risk measure is and we analyze the value-at-risk and the expected shortfall, which are the two regulatory risk measures. In particular, we present the three statistical approaches (historical, analytical and Monte Carlo) that are available. The last part of this section is dedicated to options and exotic products.

2.2.1 Definition

2.2.1.1 Coherent risk measures

Let $\mathcal{R}(w)$ be the risk measure of portfolio w. In this section, we define the different properties that should satisfy the risk measure $\mathcal{R}(w)$ in order to be acceptable in terms of capital allocation. Following Artzner *et al.* (1999), $\mathcal{R}$ is said to be '*coherent*' if it satisfies the following properties:

[34]The Basel Committee adds a second inclusive condition: the trading desk must have less than 30 exceptions at the 97.5% confidence level. This remark shows that the bank must in fact conduct two backtesting procedures at the trading desk level: one based at the 99% confidence level and another one based at the 97.5% confidence level.

1. Subadditivity
$$\mathcal{R}(w_1 + w_2) \leq \mathcal{R}(w_1) + \mathcal{R}(w_2)$$
The risk of two portfolios should be less than adding the risk of the two separate portfolios.

2. Homogeneity
$$\mathcal{R}(\lambda w) = \lambda \mathcal{R}(w) \quad \text{if } \lambda \geq 0$$
Leveraging or deleveraging of the portfolio increases or decreases the risk measure in the same magnitude.

3. Monotonicity
$$\text{if } w_1 \prec w_2, \text{ then } \mathcal{R}(w_1) \geq \mathcal{R}(w_2)$$
If portfolio w_2 has a better return than portfolio w_1 under all scenarios, risk measure $\mathcal{R}(w_1)$ should be higher than risk measure $\mathcal{R}(w_2)$.

4. Translation invariance
$$\text{if } m \in \mathbb{R}, \text{ then } \mathcal{R}(w + m) = \mathcal{R}(w) - m$$
Adding a cash position of amount m to the portfolio reduces the risk by m. This implies that we can hedge the risk of the portfolio by considering a capital that is equal to the risk measure:
$$\mathcal{R}(w + \mathcal{R}(w)) = \mathcal{R}(w) - \mathcal{R}(w) = 0$$

The definition of coherent risk measures led to a considerable interest in the quantitative risk management. Thus, Föllmer and Schied (2002) propose to replace the homogeneity and subadditivity conditions by a weaker condition called the convexity property:
$$\mathcal{R}(\lambda w_1 + (1 - \lambda) w_2) \leq \lambda \mathcal{R}(w_1) + (1 - \lambda) \mathcal{R}(w_2)$$
This condition means that diversification should not increase the risk.

We can write the loss of a portfolio as $L(w) = -P_t(w) R_{t+h}(w)$ where $P_t(w)$ and $R_{t+h}(w)$ are the current value and the future return of the portfolio. Without loss of generality[35], we assume that $P_t(w)$ is equal to 1. In this case, the expected loss $\mathbb{E}[L(w)]$ is the opposite of the expected return $\mu(w)$ of the portfolio and the standard deviation $\sigma(L(w))$ is equal to the portfolio volatility $\sigma(w)$. We consider then different risk measures:

- Volatility of the loss
$$\mathcal{R}(w) = \sigma(L(w)) = \sigma(w)$$
The volatility of the loss is the standard deviation of the portfolio loss.

- Standard deviation-based risk measure
$$\mathcal{R}(w) = \mathrm{SD}_c(w) = \mathbb{E}[L(w)] + c \cdot \sigma(L(w)) = -\mu(w) + c \cdot \sigma(w)$$
To obtain this measure, we scale the volatility by factor $c > 0$ and subtract the expected return of the portfolio.

[35] The homogeneity property implies that:
$$\mathcal{R}\left(\frac{w}{P_t(w)}\right) = \frac{\mathcal{R}(w)}{P_t(w)}$$
We can therefore calculate the risk measure using the absolute loss (expressed in \$) or the relative loss (expressed in %). The two approaches are perfectly equivalent.

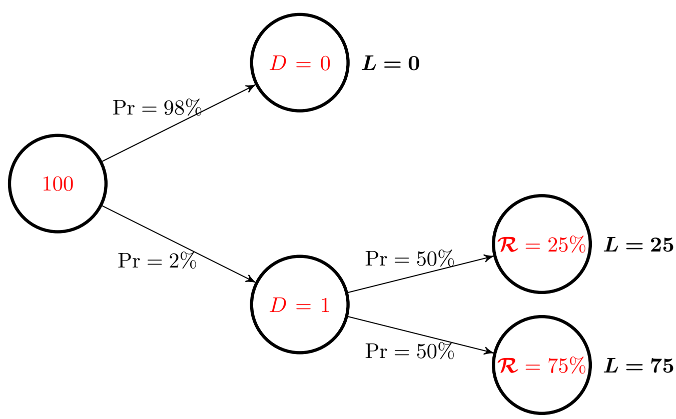
- Value-at-risk
$$\mathcal{R}(w) = \mathrm{VaR}_\alpha(w) = \inf\{\ell : \Pr\{L(w) \le \ell\} \ge \alpha\}$$
The value-at-risk is the α-quantile of the loss distribution $\mathbf{F}$ and we note it $\mathbf{F}^{-1}(\alpha)$.

- Expected shortfall
$$\mathcal{R}(w) = \mathrm{ES}_\alpha(w) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(w)\,\mathrm{d}u$$
The expected shortfall is the average of the VaRs at level α and higher (Acerbi and Tasche, 2002). We note that it is also equal to the expected loss given that the loss is beyond the value-at-risk:
$$\mathrm{ES}_\alpha(w) = \mathbb{E}\left[L(w) \mid L(w) \ge \mathrm{VaR}_\alpha(w)\right]$$
By definition, the expected shortfall is greater or equal than the value-at-risk for a given confidence level.

We can show that the standard deviation-based risk measure and the expected shortfall satisfy the previous coherency and convexity conditions. For the value-at-risk, the subadditivity property does not hold in general. This is a problem because the portfolio risk may have be meaningful in this case. More curiously, the volatility is not a coherent risk measure because it does not verify the translation invariance axiom.

Example 12 *We consider a* \$100 *defaultable zero-coupon bond, whose default probability is equal to* 200 *bps. We assume that the recovery rate* $\boldsymbol{\mathcal{R}}$ *is a binary random variable with* $\Pr\{\boldsymbol{\mathcal{R}} = 0.25\} = \Pr\{\boldsymbol{\mathcal{R}} = 0.75\} = 50\%$.

Below, we have represented the probability tree diagram of the loss L of the zero-coupon bond. We deduce that $\mathbf{F}(0) = \Pr\{L \le 0\} = 98\%$, $\mathbf{F}(25) = \Pr\{L_i \le 25\} = 99\%$ and $\mathbf{F}(75) = \Pr\{L_i \le 75\} = 100\%$.



It follows that the 99% value-at-risk is equal to \$25, and we have:
$$\begin{aligned}
\mathrm{ES}_{99\%}(L) &= \mathbb{E}[L \mid L \ge 25] \\
&= \frac{25+75}{2} \\
&= \$50
\end{aligned}$$

We assume now that the portfolio contains two zero-coupon bonds, whose default times are independent. The probability density function of (L_1, L_2) is given below:

	$L_1 = 0$	$L_1 = 25$	$L_1 = 75$	
$L_2 = 0$	96.04%	0.98%	0.98%	98.00%
$L_2 = 25$	0.98%	0.01%	0.01%	1.00%
$L_2 = 75$	0.98%	0.01%	0.01%	1.00%
	98.00%	1.00%	1.00%	

We deduce that the probability distribution function of $L = L_1 + L_2$ is:

ℓ	0	25	50	75	100	150
$\Pr\{L = \ell\}$	96.04%	1.96%	0.01%	1.96%	0.02%	0.01%
$\Pr\{L \leq \ell\}$	96.04%	98%	98.01%	99.97%	99.99%	100%

It follows that $\mathrm{VaR}_{99\%}(L) = 75$ and:

$$\begin{aligned}\mathrm{ES}_{99\%}(L) &= \frac{75 \times 1.96\% + 100 \times 0.02\% + 150 * 0.01\%}{1.96\% + 0.02\% + 0.01\%} \\ &= \$75.63\end{aligned}$$

For this example, the value-at-risk does not satisfy the subadditivity property, which is not the case of the expected shortfall[36].

For this reason, the value-at-risk has been frequently criticized by academics. They also pointed out that it does not capture the tail risk of the portfolio. This led the Basel Committee to replace the 99% value-at-risk by the 97.5% expected shortfall for the internal model-based approach in Basel III (BCBS, 2019).

2.2.1.2 Value-at-risk

The value-at-risk $\mathrm{VaR}_\alpha(w; h)$ is defined as the potential loss which the portfolio w can suffer for a given confidence level α and a fixed holding period h. Three parameters are necessary to compute this risk measure:

- the holding period h, which indicates the time period to calculate the loss;
- the confidence level α, which gives the probability that the loss is lower than the value-at-risk;
- the portfolio w, which gives the allocation in terms of risky assets and is related to the risk factors.

Without the first two parameters, it is not possible to interpret the amount of the value-at-risk, which is expressed in monetary units. For instance, a portfolio with a VaR of \$100 mn may be regarded as highly risky if the VaR corresponds to a 90% confidence level and a one-day holding period, but it may be a low risk investment if the confidence level is 99.9% and the holding period is one year.

We note $P_t(w)$ the mark-to-market value of the portfolio w at time t. The profit and loss between t and $t+h$ is equal to:

$$\Pi(w) = P_{t+h}(w) - P_t(w)$$

[36] We have $\mathrm{VaR}_{99\%}(L_1) + \mathrm{VaR}_{99\%}(L_2) = 50$, $\mathrm{VaR}_{99\%}(L_1 + L_2) > \mathrm{VaR}_{99\%}(L_1) + \mathrm{VaR}_{99\%}(L_2)$, $\mathrm{ES}_{99\%}(L_1) + \mathrm{ES}_{99\%}(L_2) = 100$ and $\mathrm{ES}_{99\%}(L_1 + L_2) < \mathrm{ES}_{99\%}(L_1) + \mathrm{ES}_{99\%}(L_2)$.

We define the loss of the portfolio as the opposite of the P&L: $L(w) = -\Pi(w)$. At time t, the loss is not known and is therefore random. From a statistical point of view, the value-at-risk $\mathrm{VaR}_\alpha(w;h)$ is the quantile[37] of the loss for the probability α:

$$\Pr\{L(w) \leq \mathrm{VaR}_\alpha(w;h)\} = \alpha$$

This means that the probability that the random loss is lower than the VaR is exactly equal to the confidence level. We finally obtain:

$$\mathrm{VaR}_\alpha(w;h) = \mathbf{F}_L^{-1}(\alpha)$$

where $\mathbf{F}_L$ is the distribution function of the loss[38].

We notice that the previous analysis assumes that the portfolio remains unchanged between t and $t+h$. In practice, it is not the case because of trading and rebalancing activities. The holding period h depends then on the nature of the portfolio. The Basel Committee has set h to one trading day for performing the backtesting procedure in order to minimize rebalancing impacts. However, h is equal to 10 trading days for capital requirements in Basel I. It is the period which is considered necessary to ensure the rebalancing of the portfolio if it is too risky or if it costs too much regulatory capital. The confidence level α is equal to 99% meaning that there is an exception every 100 trading days. It is obvious that it does not correspond to an extreme risk measure. From the point of view of regulators, the 99% value-at-risk gives then a measure of the market risk in the case of normal conditions.

2.2.1.3 Expected shortfall

The expected shortfall $\mathrm{ES}_\alpha(w;h)$ is defined as the expected loss beyond the value-at-risk of the portfolio:

$$\mathrm{ES}_\alpha(w;h) = \mathbb{E}\left[L(w) \mid L(w) \geq \mathrm{VaR}_\alpha(w;h)\right]$$

Therefore, it depends on the three parameters (h, α and w) of the VaR. Since we have $\mathrm{ES}_\alpha(w;h) \geq \mathrm{VaR}_\alpha(w;h)$, the expected shortfall is considered as a risk measure under more extreme conditions than the value-at-risk. By construction, we also have:

$$\alpha_1 > \alpha_2 \Rightarrow \mathrm{ES}_{\alpha_1}(w;h) \geq \mathrm{VaR}_{\alpha_2}(w;h)$$

However, it is impossible de compare the expected shortfall and the value-at-risk when the ES confidence level is lower than the VaR confidence level ($\alpha_1 < \alpha_2$). This is why it is difficult to compare the ES in Basel III ($\alpha = 97.5\%$) and the VaR in Basel I ($\alpha = 99\%$).

2.2.1.4 Estimator or estimate?

To calculate the value-at-risk or the expected shortfall, we first have to identify the risk factors that affect the future value of the portfolio. Their number can be large or small depending on the market, but also on the portfolio. For instance, in the case of an equity portfolio, we can use the one-factor model (CAPM), a multi-factor model (industry risk factors, Fama-French risk factors, etc.) or we can have a risk factor for each individual stock. For interest rate products, the Basel Committee imposes that the bank uses at least

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[37] If the distribution of the loss is not continuous, the statistical definition of the quantile function is:

$$\mathrm{VaR}_\alpha(w;h) = \inf\{x : \Pr\{L(w) \leq x\} \geq \alpha\}$$

[38] In a similar way, we have $\Pr\{\Pi(w) \geq -\mathrm{VaR}_\alpha(w;h)\} = \alpha$ and $\mathrm{VaR}_\alpha(w;h) = -\mathbf{F}_\Pi^{-1}(1-\alpha)$ where $\mathbf{F}_\Pi$ is the distribution function of the P&L.

six factors to model the yield curve risk in Basel I and ten factors in Basel III. This contrasts with currency and commodity portfolios where we must take into account one risk factor by exchange rate and by currency. Let $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ be the vector of risk factors. We assume that there is a pricing function g such that:

$$P_t(w) = g(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t}; w)$$

We deduce that the expression of the random loss is the difference between the current value and the future value of the portfolio:

$$\begin{aligned} L(w) &= P_t(w) - g(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w) \\ &= \ell(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w) \end{aligned}$$

where ℓ is the loss function. The big issue is then to model the future values of risk factors. In practice, the distribution $\mathbf{F}_L$ is not known because the multidimensional distribution of the risk factors is not known. This is why we have to estimate $\mathbf{F}_L$ meaning that the calculated VaR and ES are also two estimated values:

$$\widehat{\mathrm{VaR}}_\alpha(w;h) = \hat{\mathbf{F}}_L^{-1}(\alpha) = -\hat{\mathbf{F}}_\Pi^{-1}(1-\alpha)$$

and:

$$\widehat{\mathrm{ES}}_\alpha(w;h) = \frac{1}{1-\alpha}\int_\alpha^1 \hat{\mathbf{F}}_L^{-1}(u)\,\mathrm{d}u$$

Therefore, we have to make the difference between the estimator and the estimate. Indeed, the calculated value-at-risk or expected shortfall is an estimate, meaning that it is a realization of the corresponding estimator. In practice, there are three approaches to calculate the risk measure depending on the method used to estimate $\hat{\mathbf{F}}_L$:

1. the historical value-at-risk/expected shortfall, which is also called the empirical or non-parametric VaR/ES;

2. the analytical (or parametric) value-at-risk/expected shortfall;

3. the Monte Carlo (or simulated) value-at-risk/expected shortfall.

The historical approach is the most widely used method by banks for computing the capital charge. This is an unbiased estimator, but with a large variance. On the contrary, the analytical estimator is biased, because it assumes a parametric function for the risk factors, but it has a lower variance than the historical estimator. Finally, the Monte Carlo estimator can produce an unbiased estimator with a small variance. However, it could be difficult to put in place because it requires large computational times.

Remark 4 *In this book, we use the statistical expressions* $\mathrm{VaR}_\alpha(w;h)$ *and* $\mathrm{ES}_\alpha(w;h)$ *in place of* $\widehat{\mathrm{VaR}}_\alpha(w;h)$ *and* $\widehat{\mathrm{ES}}_\alpha(w;h)$ *in order to reduce the amount of notation.*

2.2.2 Historical methods

The historical VaR corresponds to a non-parametric estimate of the value-at-risk. For that, we consider the empirical distribution of the risk factors observed in the past. Let $(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s})$ be the vector of risk factors observed at time $s < t$. If we calculate the future P&L with this historical scenario, we obtain:

$$\Pi_s(w) = g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w) - P_t(w)$$

If we consider n_S historical scenarios ($s = 1, \ldots, n_S$), the empirical distribution $\hat{\mathbf{F}}_\Pi$ is described by the following probability distribution:

$\Pi(w)$	$\Pi_1(w)$	$\Pi_2(w)$	$\cdots$	$\Pi_{n_S}(w)$
p_s	$1/n_S$	$1/n_S$		$1/n_S$

because each probability of occurrence is the same for all the historical scenarios. To calculate the empirical quantile $\hat{\mathbf{F}}_L^{-1}(\alpha)$, we can use two approaches: the order statistic approach and the kernel density approach.

2.2.2.1 The order statistic approach

Let $X_1, \ldots, X_n$ be a sample from a continuous distribution $\mathbf{F}$. Suppose that for a given scalar $\alpha \in \left]0,1\right[$, there exists a sequence $\{a_n\}$ such that $\sqrt{n}(a_n - n\alpha) \to 0$. Lehmann (1999) shows that:

$$\sqrt{n}\left(X_{(a_n:n)} - \mathbf{F}^{-1}(\alpha)\right) \to \mathcal{N}\left(0, \frac{\alpha(1-\alpha)}{f^2\left(\mathbf{F}^{-1}(\alpha)\right)}\right) \tag{2.4}$$

This result implies that we can estimate the quantile $\mathbf{F}^{-1}(\alpha)$ by the mean of the $n\alpha^{\text{th}}$ order statistic. Let us apply the previous result to our problem. We calculate the order statistics associated to the P&L sample $\{\Pi_1(w), \ldots, \Pi_{n_S}(w)\}$:

$$\min_s \Pi_s(w) = \Pi_{(1:n_S)} \leq \Pi_{(2:n_S)} \leq \cdots \leq \Pi_{(n_S-1:n_S)} \leq \Pi_{(n_S:n_S)} = \max_s \Pi_s(w)$$

The value-at-risk with a confidence level α is then equal to the opposite of the $n_S(1-\alpha)^{\text{th}}$ order statistic of the P&L:

$$\mathrm{VaR}_\alpha(w;h) = -\Pi_{(n_S(1-\alpha):n_S)} \tag{2.5}$$

If $n_S(1-\alpha)$ is not an integer, we consider the interpolation scheme:

$$\mathrm{VaR}_\alpha(w;h) = -\left(\Pi_{(q:n_S)} + (n_S(1-\alpha) - q)\left(\Pi_{(q+1:n_S)} - \Pi_{(q:n_S)}\right)\right)$$

where $q = q_\alpha(n_S) = \lfloor n_S(1-\alpha) \rfloor$ is the integer part of $n_S(1-\alpha)$. For instance, if $n_S = 100$, the 99% value-at-risk corresponds to the largest loss. In the case where we use 250 historical scenarios, the 99% value-at-risk is the mean between the second and third largest losses:

$$\begin{aligned}
\mathrm{VaR}_\alpha(w;h) &= -\left(\Pi_{(2:250)} + (2.5-2)\left(\Pi_{(3:250)} - \Pi_{(2:250)}\right)\right) \\
&= -\frac{1}{2}\left(\Pi_{(2:250)} + \Pi_{(3:250)}\right) \\
&= \frac{1}{2}\left(L_{(249:250)} + L_{(248:250)}\right)
\end{aligned}$$

Remark 5 *We reiterate that* $\mathrm{VaR}_\alpha(w;h)$ *defined by Equation (2.5) is an estimator with an asymptotic variance given by Equation (2.4). Suppose that the loss of the portfolio is Gaussian and* $L(w) \sim \mathcal{N}(0,1)$*. The exact value-at-risk is* $\Phi^{-1}(\alpha)$ *and takes the values* 1.28 *or* 2.33 *if* α *is equal to 90% or 99%. The standard deviation of the estimator depends on the number* n_S *of historical scenarios:*

$$\sigma(\mathrm{VaR}_\alpha(w;h)) \approx \frac{\sqrt{\alpha(1-\alpha)}}{\sqrt{n_S}\phi\left(\Phi^{-1}(\alpha)\right)}$$

In Figure 2.5, we have reported the density function of the VaR estimator. We notice that the estimation error decreases with n_S*. Moreover, it is lower for* $\alpha = 90\%$ *than for* $\alpha = 99\%$*, because the density of the Gaussian distribution at the point* $x = 1.28$ *is larger than at the point* $x = 2.33$*.*

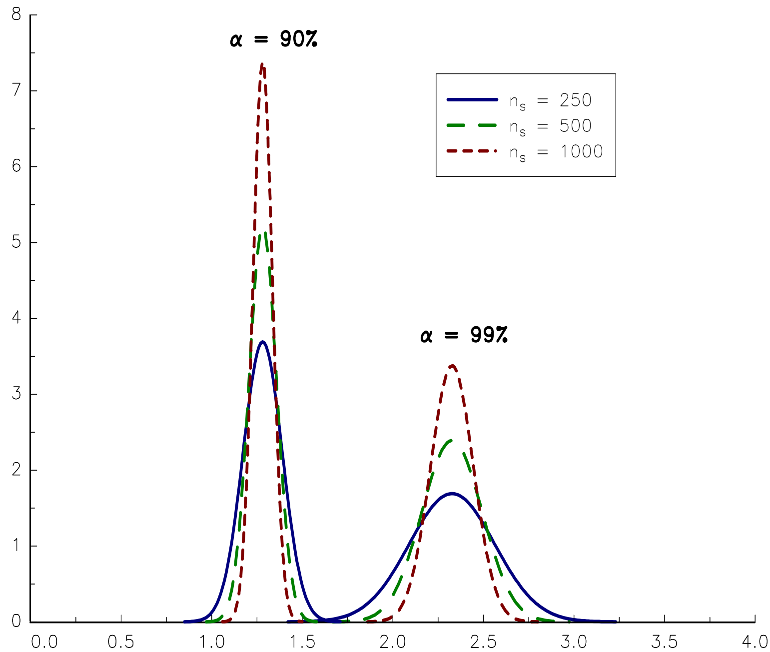

FIGURE 2.5: Density of the VaR estimator (Gaussian case)

Example 13 *We consider a portfolio composed of* 10 *stocks Apple and* 20 *stocks Coca-Cola. The current date is 2 January 2015.*

The mark-to-market of the portfolio is:

$$P_t(w) = 10 \times P_{1,t} + 20 \times P_{2,t}$$

where $P_{1,t}$ and $P_{2,t}$ are the stock prices of Apple and Coca-Cola. We assume that the market risk factors corresponds to the daily stock returns $R_{1,t}$ and $R_{2,t}$. We deduce that the P&L for the scenario s is equal to:

$$\Pi_s(w) = \underbrace{10 \times P_{1,s} + 20 \times P_{2,s}}_{g(R_{1,s},R_{2,s};w)} - P_t(w)$$

where $P_{i,s} = P_{i,t} \times (1 + R_{i,s})$ is the simulated price of stock i for the scenario s. In Table 2.6, we have reported the values of the first ten historical scenarios[39]. Using these scenarios, we can calculate the simulated price $P_{i,s}$ using the current price of the stocks (\$109.33 for Apple and \$42.14 for Coca-Cola). For instance, in the case of the 9$^{\text{th}}$ scenario, we obtain:

$$\begin{aligned} P_{1,s} &= 109.33 \times (1 - 0.77\%) = \$108.49 \\ P_{2,s} &= 42.14 \times (1 - 1.04\%) = \$41.70 \end{aligned}$$

[39] For instance, the market risk factor for the first historical scenario and for Apple is calculated as follows:

$$R_{1,1} = \frac{109.33}{110.38} - 1 = -0.95\%$$

We then deduce the simulated mark-to-market $\mathrm{MtM}_s(w) = g(R_{1,s}, R_{2,s}; w)$, the current value of the portfolio[40] and the P&L $\Pi_s(w)$. These data are given in Table 2.7. In addition to the first ten historical scenarios, we also report the results for the six worst cases and the last scenario[41]. We notice that the largest loss is reached for the 236th historical scenario at the date of 28 January 2014. If we rank the scenarios, the worst P&Ls are -84.34, -51.46, -43.31, -40.75, -35.91 and -35.42. We deduce that the daily historical VaR is equal to:

$$\mathrm{VaR}_{99\%}(w; \text{one day}) = \frac{1}{2}(51.46 + 43.31) = \$47.39$$

If we assume that $m_c = 3$, the corresponding capital charge represents 23.22% of the portfolio value:

$$\mathcal{K}_t^{\mathrm{VaR}} = 3 \times \sqrt{10} \times 47.39 = \$449.54$$

TABLE 2.6: Computation of the market risk factors $R_{1,s}$ and $R_{2,s}$

s	Date	Apple		Coca-Cola	
		Price	$R_{1,s}$	Price	$R_{2,s}$
1	2015-01-02	109.33	−0.95%	42.14	−0.19%
2	2014-12-31	110.38	−1.90%	42.22	−1.26%
3	2014-12-30	112.52	−1.22%	42.76	−0.23%
4	2014-12-29	113.91	−0.07%	42.86	−0.23%
5	2014-12-26	113.99	1.77%	42.96	0.05%
6	2014-12-24	112.01	−0.47%	42.94	−0.07%
7	2014-12-23	112.54	−0.35%	42.97	1.46%
8	2014-12-22	112.94	1.04%	42.35	0.95%
9	2014-12-19	111.78	−0.77%	41.95	−1.04%
10	2014-12-18	112.65	2.96%	42.39	2.02%

Under Basel 2.5, we have to compute a second capital charge for the stressed VaR. If we assume that the stressed period is from 9 October 2007 to 9 March 2009, we obtain 356 stressed scenarios. By applying the previous method, the six largest simulated losses are[42] 219.20 (29/09/2008), 127.84 (17/09/2008), 126.86 (07/10/2008), 124.23 (14/10/2008), 115.24 (23/01/2008) and 99.55 (29/09/2008). The 99% SVaR corresponds to the 3.56th order statistic. We deduce that:

$$\begin{aligned}\mathrm{SVaR}_{99\%}(w; \text{one day}) &= 126.86 + (3.56 - 3) \times (124.23 - 126.86) \\ &= \$125.38\end{aligned}$$

It follows that:

$$\mathcal{K}_t^{\mathrm{SVaR}} = 3 \times \sqrt{10} \times 125.38 = \$1\,189.49$$

The total capital requirement under Basel 2.5 is then:

$$\mathcal{K}_t = \mathcal{K}_t^{\mathrm{VaR}} + \mathcal{K}_t^{\mathrm{SVaR}} = \$1\,639.03$$

It represents 84.6% of the current mark-to-market!

[40] We have:

$$P_t(w) = 10 \times 109.33 + 20 \times 42.14 = \$1\,936.10$$

[41] We assume that the value-at-risk is calculated using 250 historical scenarios (from 2015-01-02 to 2014-01-07).

[42] We indicate in brackets the scenario day of the loss.

TABLE 2.7: Computation of the simulated P&L $\Pi_s(w)$

s	Date	Apple		Coca-Cola		$\mathrm{MtM}_s(w)$	$\Pi_s(w)$
		$R_{1,s}$	$P_{1,s}$	$R_{2,s}$	$P_{2,s}$		
1	2015-01-02	−0.95%	108.29	−0.19%	42.06	1 924.10	−12.00
2	2014-12-31	−1.90%	107.25	−1.26%	41.61	1 904.66	−31.44
3	2014-12-30	−1.22%	108.00	−0.23%	42.04	1 920.79	−15.31
4	2014-12-29	−0.07%	109.25	−0.23%	42.04	1 933.37	−2.73
5	2014-12-26	1.77%	111.26	0.05%	42.16	1 955.82	19.72
6	2014-12-24	−0.47%	108.82	−0.07%	42.11	1 930.36	−5.74
7	2014-12-23	−0.35%	108.94	1.46%	42.76	1 944.57	8.47
8	2014-12-22	1.04%	110.46	0.95%	42.54	1 955.48	19.38
9	2014-12-19	−0.77%	108.49	−1.04%	41.70	1 918.91	−17.19
10	2014-12-18	2.96%	112.57	2.02%	42.99	1 985.51	49.41
23	2014-12-01	−3.25%	105.78	−0.62%	41.88	1 895.35	−40.75
69	2014-09-25	−3.81%	105.16	−1.16%	41.65	1 884.64	−51.46
85	2014-09-03	−4.22%	104.72	0.34%	42.28	1 892.79	−43.31
108	2014-07-31	−2.60%	106.49	−0.83%	41.79	1 900.68	−35.42
236	2014-01-28	−7.99%	100.59	0.36%	42.29	1 851.76	−84.34
242	2014-01-17	−2.45%	106.65	−1.08%	41.68	1 900.19	−35.91
250	2014-01-07	−0.72%	108.55	0.30%	42.27	1 930.79	−5.31

Remark 6 *As the previous example has shown, directional exposures are highly penalized under Basel 2.5. More generally, it is not always evident that capital requirements are lower with IMA than with SMM (Crouhy* et al.*, 2013).*

Since the expected shortfall is the expected loss beyond the value-at-risk, it follows that the historical expected shortfall is given by:

$$\mathrm{ES}_\alpha(w;h) = \frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{L_s \geq \mathrm{VaR}_\alpha(w;h)\} \cdot L_s$$

or:

$$\mathrm{ES}_\alpha(w;h) = -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{\Pi_s \leq -\mathrm{VaR}_\alpha(w;h)\} \cdot \Pi_s$$

where $q_\alpha(n_S) = \lfloor n_s(1-\alpha) \rfloor$ is the integer part of $n_s(1-\alpha)$. We deduce that:

$$\mathrm{ES}_\alpha(w;h) = -\frac{1}{q_\alpha(n_S)} \sum_{i=1}^{q_\alpha(n_S)} \Pi_{(i:n_S)}$$

Computing the historical expected shortfall consists then in averaging the first $q_\alpha(n_S)$ order statistics of the P&L. For example, if n_S is equal to 250 scenarios and $\alpha = 97.5\%$, we obtain $n_s(1-\alpha) = 6.25$ and $q_\alpha(n_S) = 6$. In Basel III, computing the historical ES is then equivalent to average the 6 largest losses of the 250 historical scenarios. In the table below, we indicate the value of $q_\alpha(n_S)$ for different values of n_S and α:

α / n_S	100	150	200	250	300	350	400	450	500	1000
90.0%	9	14	19	24	29	34	39	44	49	99
95.0%	5	7	10	12	15	17	20	22	25	50
97.5%	2	3	5	6	7	8	10	11	12	25
99.0%	1	1	2	2	3	3	4	4	5	10

Let us consider Example 13 on page 68. We have found that the historical value-at-risk $\mathrm{VaR}_{99\%}(w; \text{one day})$ of the Apple/Coca-Cola portfolio was equal to \$47.39. The 99% expected shortfall is the average of the two largest losses:

$$\mathrm{ES}_{99\%}(w; \text{one day}) = \frac{84.34 + 51.46}{2} = \$67.90$$

However, the confidence level is set to 97.5% in Basel III, meaning that the expected shortfall is the average of the six largest losses:

$$\begin{aligned}\mathrm{ES}_{97.5\%}(w; \text{one day}) &= \frac{84.34 + 51.46 + 43.31 + 40.75 + 35.91 + 35.42}{6} \\ &= \$48.53\end{aligned}$$

2.2.2.2 The kernel approach

Let $\{x_1, \ldots, x_n\}$ be a sample of the random variable X. In Section 10.1.4.1 on page 637, we show that we can estimate the empirical distribution $\hat{\mathbf{F}}(x) = n^{-1}\sum_{i=1}^{n} \mathbb{1}\{x_i \leq x\}$ by the kernel estimator:

$$\hat{\mathbf{F}}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}\left(\frac{x - x_i}{\boldsymbol{h}}\right)$$

where $\mathcal{I}$ is the integrated kernel function and $\boldsymbol{h}$ is the bandwidth.

To estimate the value-at-risk with a confidence level α, Gouriéroux *et al.* (2000) solves the equation $\hat{\mathbf{F}}_L(\mathrm{VaR}_\alpha(w; h)) = \alpha$ or:

$$\frac{1}{n_S}\sum_{s=1}^{n_S} \mathcal{I}\left(\frac{-\mathrm{VaR}_\alpha(w; h) - \Pi_s(w)}{\boldsymbol{h}}\right) = 1 - \alpha$$

If we consider Example 13 on page 68 with the last 250 historical scenarios, we obtain the results given in Figure 2.6. We have reported the estimated distribution $\hat{\mathbf{F}}_\Pi$ of $\Pi(w)$ based on order statistic and Gaussian kernel methods[43]. We verify that the kernel approach produces a smoother distribution. If we zoom on the 1% quantile, we notice that the two methods give similar results. The daily VaR with the kernel approach is equal to \$47.44 whereas it was equal to \$47.39 with the order statistic approach.

For computing the non-parametric expected shortfall, we use the following result[44]:

$$\mathbb{E}[X \cdot \mathbb{1}\{X \leq x\}] \approx \frac{1}{n}\sum_{i=1}^{n} x_i \mathcal{I}\left(\frac{x - x_i}{\boldsymbol{h}}\right)$$

Therefore, Scaillet (2004) shows that the kernel estimator of the expected shortfall is equal to:

$$\mathrm{ES}_\alpha(w; h) = -\frac{1}{(1-\alpha)\, n_S}\sum_{s=1}^{n_S} \Pi_s \mathcal{I}\left(\frac{-\mathrm{VaR}_\alpha(w; h) - \Pi_s}{\boldsymbol{h}}\right)$$

In the case of the Apple/Coca-Cola example, we obtain $\mathrm{ES}_{99\%}(w; h) = \$60.53$ and $\mathrm{ES}_{97.5\%}(w; h) = \$45.28$. With the kernel approach, we can estimate the value-at-risk and the expected shortfall with a high confidence level α. For instance, if $\alpha = 99.25\%$, we have $(1 - \alpha)\, n_s = 0.625 < 1$. Therefore, it is impossible to estimate the VaR or the ES with 250 observations, which is not the case with the kernel estimator. In our example, we obtain $\mathrm{VaR}_{99.75\%}(w; h) = \$58.27$ and $\mathrm{ES}_{99.75\%}(w; h) = \$77.32$.

[43] We consider the Gaussian kernel defined by $\mathcal{K}(u) = \phi(u)$ and $\mathcal{I}(u) = \Phi(u)$. The estimated standard deviation $\hat{\sigma}(\Pi)$ is equal to 17.7147, while the bandwidth is $\boldsymbol{h} = 1.364 \times n^{-1/5} \times \hat{\sigma}(\Pi) = 8.0027$.

[44] See Exercise 2.4.12 on page 124.

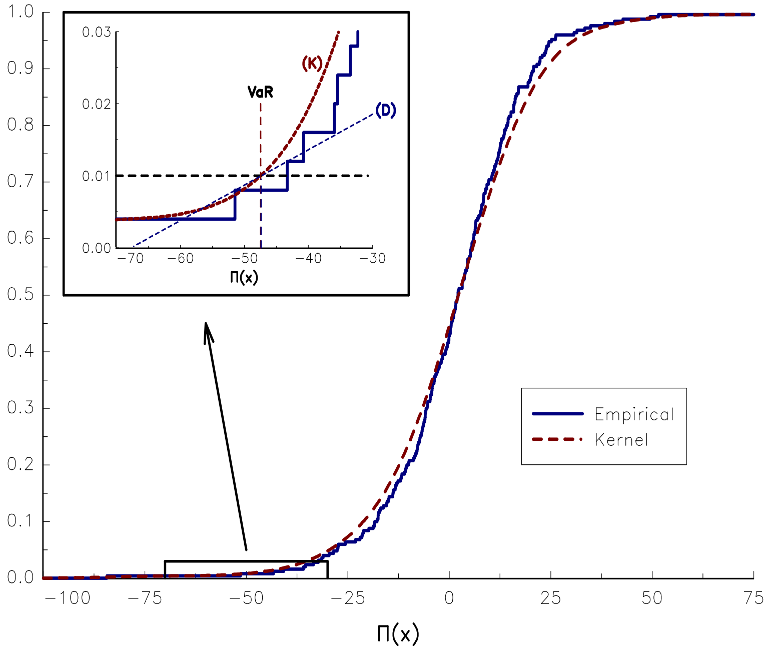

FIGURE 2.6: Kernel estimation of the historical VaR

Remark 7 *Monte Carlo simulations reveal that the kernel method reduces the variance of the VaR estimation, but not the variance of the ES estimation (Chen, 2007). In practice, the kernel approach gives similar figures than the order statistic approach, especially when the number of scenarios is large. However, the two estimators may differ in the presence of fat tails. For large confidence levels, the method based on order statistics seems to be more conservative.*

2.2.3 Analytical methods

2.2.3.1 Derivation of the closed-form formula

Gaussian value-at-risk We speak about analytical value-at-risk when we are able to find a closed-form formula of $\mathbf{F}_L^{-1}(\alpha)$. Suppose that $L(w) \sim \mathcal{N}\left(\mu(L), \sigma^2(L)\right)$. In this case, we have $\Pr\left\{L(w) \leq \mathbf{F}_L^{-1}(\alpha)\right\} = \alpha$ or:

$$\Pr\left\{\frac{L(w) - \mu(L)}{\sigma(L)} \leq \frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right\} = \alpha \Leftrightarrow \Phi\left(\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right) = \alpha$$

We deduce that:

$$\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)} = \Phi^{-1}(\alpha) \Leftrightarrow \mathbf{F}_L^{-1}(\alpha) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$$

The expression of the value-at-risk is then[45]:

$$\mathrm{VaR}_\alpha(w; h) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L) \tag{2.6}$$

[45]We also have $\mathrm{VaR}_\alpha(w;h) = -\mu(\Pi) + \Phi^{-1}(\alpha)\,\sigma(\Pi)$ because the P&L $\Pi(x)$ is the opposite of the portfolio loss $L(x)$ meaning that $\mu(\Pi) = -\mu(L)$ and $\sigma(\Pi) = \sigma(L)$.

This formula is known as the Gaussian value-at-risk. For instance, if $\alpha = 99\%$ (resp. 95%), $\Phi^{-1}(\alpha)$ is equal to 2.33 (resp. 1.65) and we have:

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + 2.33 \times \sigma(L)$$

Remark 8 *We notice that the value-at-risk depends on the parameters $\mu(L)$ and $\sigma(L)$. This is why the analytical value-at-risk is also called the parametric value-at-risk. In practice, we don't know these parameters and we have to estimate them. This implies that the analytical value-at-risk is also an estimator. For the Gaussian distribution, we obtain:*

$$\widehat{\mathrm{VaR}}_\alpha(w;h) = \hat{\mu}(L) + \Phi^{-1}(\alpha)\,\hat{\sigma}(L)$$

In practice, it is extremely difficult to estimate the mean and we set $\hat{\mu}(L) = 0$.

Example 14 *We consider a short position of \$1 mn on the S&P 500 futures contract. We estimate that the annualized volatility $\hat{\sigma}_{\mathrm{SPX}}$ is equal to 35%. Calculate the daily value-at-risk with a 99% confidence level.*

The portfolio loss is equal to $L(w) = N \times R_{\mathrm{SPX}}$ where N is the exposure amount ($-\$1$ mn) and R_{SPX} is the (Gaussian) return of the S&P 500 index. We deduce that the annualized loss volatility is $\hat{\sigma}(L) = |N| \times \hat{\sigma}_{\mathrm{SPX}}$. The value-at-risk for a one-year holding period is:

$$\mathrm{VaR}_{99\%}(w;\text{one year}) = 2.33 \times 10^6 \times 0.35 = \$815\,500$$

By using the square-root-of-time rule, we deduce that:

$$\mathrm{VaR}_{99\%}(w;\text{one day}) = \frac{815\,500}{\sqrt{260}} = \$50\,575$$

This means that we have a 1% probability to lose more than \$50 575 per day.

In finance, the standard model is the Black-Scholes model where the price S_t of the asset is a geometric Brownian motion:

$$\mathrm{d}S_t = \mu_S S_t\,\mathrm{d}t + \sigma_S S_t\,\mathrm{d}W_t$$

and W_t is a Wiener process. We can show that:

$$\ln S_{t_2} - \ln S_{t_1} = \left(\mu_S - \frac{1}{2}\sigma_S^2\right)(t_2 - t_1) + \sigma_S\left(W_{t_2} - W_{t_1}\right)$$

for $t_2 \geq t_1$. We have $W_{t_2} - W_{t_1} = \sqrt{t_2 - t_1}\varepsilon$ where $\varepsilon \sim \mathcal{N}(0,1)$. We finally deduce that $\operatorname{var}(\ln S_{t_2} - \ln S_{t_1}) = \sigma_S^2(t_2 - t_1)$. Let $R_S(\Delta t)$ be a sample of log-returns measured at a regular time interval Δt. It follows that:

$$\hat{\sigma}_S = \frac{1}{\sqrt{\Delta t}} \cdot \sigma(R_S(\Delta t))$$

If we consider two sample periods Δt and $\Delta t'$, we obtain the following relationship:

$$\sigma(R_S(\Delta t')) = \sqrt{\frac{\Delta t'}{\Delta t}} \cdot \sigma(R_S(\Delta t))$$

For the mean, we have $\hat{\mu}_S = \Delta t^{-1} \cdot \mathbb{E}[R_S(\Delta t)]$ and $\mathbb{E}(R_S(\Delta t')) = (\Delta t'/\Delta t) \cdot \mathbb{E}(R_S(\Delta t))$. We notice that the square-root-of-time rule is only valid for the volatility and therefore for risk measures that are linear with respect to the volatility. In practice, there is no other solution and this explains why this rule continues to be used even if we know that the approximation is poor when the portfolio loss is not Gaussian.

Gaussian expected shortfall By definition, we have:

$$\begin{aligned}\mathrm{ES}_{\alpha}(w) &= \mathbb{E}\left[L(w) \mid L(w) \geq \mathrm{VaR}_{\alpha}(w)\right] \\ &= \frac{1}{1-\alpha}\int_{\mathbf{F}_L^{-1}(\alpha)}^{\infty} x f_L(x)\,\mathrm{d}x\end{aligned}$$

where f_L and $\mathbf{F}_L$ are the density and distribution functions of the loss $L(w)$. In the Gaussian case $L(w) \sim \mathcal{N}\left(\mu(L), \sigma^2(L)\right)$, we have $\mathrm{VaR}_{\alpha}(w) = \mathbf{F}_L^{-1}(\alpha) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$ and:

$$\mathrm{ES}_{\alpha}(w) = \frac{1}{1-\alpha}\int_{\mu(L)+\Phi^{-1}(\alpha)\sigma(L)}^{\infty} \frac{x}{\sigma(L)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu(L)}{\sigma(L)}\right)^2\right)\mathrm{d}x$$

With the variable change $t = \sigma(L)^{-1}(x - \mu(L))$, we obtain:

$$\begin{aligned}\mathrm{ES}_{\alpha}(w) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty}(\mu(L)+\sigma(L)\,t)\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\ &= \frac{\mu(L)}{1-\alpha}\left[\Phi(t)\right]_{\Phi^{-1}(\alpha)}^{\infty} + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} t\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\ &= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^2\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\ &= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^2\right)\end{aligned}$$

The expected shortfall of the portfolio w is then:

$$\mathrm{ES}_{\alpha}(w) = \mu(L) + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\sigma(L)$$

When the portfolio loss is Gaussian, the value-at-risk and the expected shortfall are both a standard deviation-based risk measure. They coincide when the scaling parameters $c_{\mathrm{VaR}} = \Phi^{-1}(\alpha_{\mathrm{VaR}})$ and $c_{\mathrm{ES}} = \phi\left(\Phi^{-1}(\alpha_{\mathrm{ES}})\right)/(1-\alpha_{\mathrm{ES}})$ are equal[46]. In Table 2.8, we report the values taken by c_{VaR} and c_{ES}. We notice that the 97.5% Gaussian expected shortfall is very close to the 99% Gaussian value-at-risk.

TABLE 2.8: Scaling factors c_{VaR} and c_{ES}

α (in %)	95.0	96.0	97.0	97.5	98.0	98.5	99.0	99.5
c_{VaR}	1.64	1.75	1.88	1.96	2.05	2.17	**2.33**	2.58
c_{ES}	2.06	2.15	2.27	**2.34**	2.42	2.52	2.67	2.89

Remark 9 *In the Gaussian case, the Basel III framework consists in replacing the scaling factor* 2.33 *by* 2.34. *In what follows, we focus on the VaR, because the ES figures can be directly deduced.*

[46]The equality is achieved when $(\alpha_{\mathrm{VaR}}, \alpha_{\mathrm{ES}})$ is equal to $(90\%, 75.44\%)$, $(95\%, 87.45\%)$, $(99\%, 97.42\%)$, $(99.9\%, 99.74\%)$, etc.

2.2.3.2 Linear factor models

We consider a portfolio of n assets and a pricing function g which is linear with respect to the asset prices. We have:

$$g(\mathcal{F}_t; w) = \sum_{i=1}^{n} w_i P_{i,t}$$

We deduce that the random P&L is:

$$\begin{aligned} \Pi(w) &= P_{t+h}(w) - P_t(w) \\ &= \sum_{i=1}^{n} w_i P_{i,t+h} - \sum_{i=1}^{n} w_i P_{i,t} \\ &= \sum_{i=1}^{n} w_i (P_{i,t+h} - P_{i,t}) \end{aligned}$$

Here, $P_{i,t}$ is known whereas $P_{i,t+h}$ is stochastic. The first idea is to choose the factors as the future prices. The problem is that prices are far to be stationary meaning that we will face some issues to model the distribution $\mathbf{F}_{\Pi}$. Another idea is to write the future price as follows:

$$P_{i,t+h} = P_{i,t}(1 + R_{i,t+h})$$

where $R_{i,t+h}$ is the asset return between t and $t+h$. In this case, we obtain:

$$\Pi(w) = \sum_{i=1}^{n} w_i P_{i,t} R_{i,t+h}$$

In this approach, the asset returns are the market risk factors and each asset has its own risk factor.

The covariance model Let R_t be the vector of asset returns. We note $W_{i,t} = w_i P_{i,t}$ the wealth invested (or the nominal exposure) in asset i and $W_t = (W_{1,t}, \ldots, W_{n,t})$. It follows that:

$$\Pi(w) = \sum_{i=1}^{n} W_{i,t} R_{i,t+h} = W_t^\top R_{t+h}$$

If we assume that $R_{t+h} \sim \mathcal{N}(\mu, \Sigma)$, we deduce that $\mu(\Pi) = W_t^\top \mu$ and $\sigma^2(\Pi) = W_t^\top \Sigma W_t$. Using Equation (2.6), the expression of the value-at-risk is[47]:

$$\mathrm{VaR}_\alpha(w; h) = -W_t^\top \mu + \Phi^{-1}(\alpha)\sqrt{W_t^\top \Sigma W_t}$$

In this approach, we only need to estimate the covariance matrix of asset returns to compute the value-at-risk. This explains the popularity of this model, especially when the P&L of the portfolio is a linear function of the asset returns[48].

Let us consider our previous Apple/Coca-Cola example. The nominal exposures[49] are \$1 093.3 (Apple) and \$842.8 (Coca-Cola). If we consider the historical prices from 2014-01-07 to 2015-01-02, the estimated standard deviation of daily returns is equal to 1.3611% for

[47] For the expected shortfall formula, we replace $\Phi^{-1}(\alpha)$ by $\phi\left(\Phi^{-1}(\alpha)\right)/(1-\alpha)$.

[48] For instance, this approach is frequently used by asset managers to measure the risk of equity portfolios.

[49] These figures are equal to 10×109.33 and 20×42.14.

Apple and 0.9468% for Coca-Cola, whereas the cross-correlation is equal to 12.0787%. It follows that:

$$\begin{aligned}
\sigma^2(\Pi) &= W_t^\top \Sigma W_t \\
&= 1\,093.3^2 \times \left(\frac{1.3611}{100}\right)^2 + 842.8^2 \times \left(\frac{0.9468}{100}\right)^2 + \\
&\quad 2 \times \frac{12.0787}{100} \times 1\,093.3 \times 842.8 \times \frac{1.3611}{100} \times \frac{0.9468}{100} \\
&= 313.80
\end{aligned}$$

If we omit the term of expected return $-W_t^\top \mu$, we deduce that the 99% daily value-at-risk[50] is equal to \$41.21. We obtain a lower figure than with the historical value-at-risk, which was equal to \$47.39. We explain this result, because the Gaussian distribution underestimates the probability of extreme events and is not adapted to take into account tail risk.

The factor model We consider the standard linear factor model where asset returns R_t are related to a set of risk factors $\mathcal{F}_t = (\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ in the following way:

$$R_t = B\mathcal{F}_t + \varepsilon_t$$

where $\mathbb{E}(\mathcal{F}_t) = \mu(\mathcal{F})$, $\operatorname{cov}(\mathcal{F}_t) = \Omega$, $\mathbb{E}(\varepsilon_t) = \mathbf{0}$ and $\operatorname{cov}(\varepsilon_t) = D$. $\mathcal{F}_t$ represents the common risks whereas ε_t is the vector of specific or idiosyncratic risks. This implies that $\mathcal{F}_t$ and ε_t are independent and D is a diagonal matrix[51]. B is a $(n \times m)$ matrix that measures the sensitivity of asset returns with respect to the risk factors. The first two moments of R_t are given by:

$$\mu = \mathbb{E}[R_t] = B\mu(\mathcal{F})$$

and[52]:

$$\Sigma = \operatorname{cov}(R_t) = B\Omega B^\top + D$$

If we assume that asset returns are Gaussian, we deduce that[53]:

$$\mathrm{VaR}_\alpha(w;h) = -W_t^\top B\mu(\mathcal{F}) + \Phi^{-1}(\alpha)\sqrt{W_t^\top \left(B\Omega B^\top + D\right) W_t}$$

The linear factor model plays a major role in financial modeling. The capital asset pricing model (CAPM) developed by Sharpe (1964) is a particular case of this model when there is a single factor, which corresponds to the market portfolio. In the arbitrage pricing theory (APT) of Ross (1976), $\mathcal{F}_t$ corresponds to a set of (unknown) arbitrage factors. They may be macro-economic, statistical or characteristic-based factors. The three-factor model of

[50] We have:

$$\mathrm{VaR}_{99\%}(w;\text{one day}) = \Phi^{-1}(0.99)\sqrt{313.80} = \$41.21$$

[51] In the following, we note $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ where $\tilde{\sigma}_i$ is the idiosyncratic volatility of asset i.

[52] We have:

$$\begin{aligned}
\Sigma &= \mathbb{E}\left[(R_t - \mu)(R_t - \mu)^\top\right] \\
&= \mathbb{E}\left[(B(\mathcal{F}_t - \mu(\mathcal{F}) + \varepsilon_t))(B(\mathcal{F}_t - \mu(\mathcal{F}) + \varepsilon_t))^\top\right] \\
&= B\mathbb{E}\left[(\mathcal{F}_t - \mu(\mathcal{F}))(\mathcal{F}_t - \mu(\mathcal{F}))^\top\right] B_t^\top + \mathbb{E}\left[\varepsilon_t \varepsilon_t^\top\right] \\
&= B\Omega B^\top + D
\end{aligned}$$

[53] For the expected shortfall formula, we replace $\Phi^{-1}(\alpha)$ by $\phi\left(\Phi^{-1}(\alpha)\right)/(1-\alpha)$.

Fama and French (1993) is certainly the most famous application of APT. In this case, the factors are the market factor, the size factor corresponding to a long/short portfolio between small stocks and large stocks and the value factor, which is the return of stocks with high book-to-market values minus the return of stocks with low book-to-market values. Since its publication, the original Fama-French factor has been extended to many other factors including momentum, quality or liquidity factors[54].

BCBS (1996a) makes direct reference to CAPM. In this case, we obtain a single-factor model:

$$R_t = \alpha + \beta R_{m,t} + \varepsilon_t$$

where $R_{m,t}$ is the return of the market and $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of beta coefficients. Let σ_m be the volatility of the market risk factor. We have $\operatorname{var}(R_{i,t}) = \beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2$ and $\operatorname{cov}(R_{i,t}, R_{j,t}) = \beta_i \beta_j \sigma_m^2$. By omitting the mean, we obtain:

$$\mathrm{VaR}_\alpha(w; h) = \Phi^{-1}(\alpha) \sqrt{\sigma_m^2 \left(\sum_{i=1}^n \tilde{\beta}_i^2 + 2 \sum_{j>i} \tilde{\beta}_i \tilde{\beta}_j \right) + \sum_{i=1}^n W_{i,t}^2 \tilde{\sigma}_i^2}$$

where $\tilde{\beta}_i = W_{i,t} \beta_i$ is the beta exposure of asset i expressed in \$. With the previous formula, we can calculate the VaR due to the market risk factor by omitting the specific risk[55].

If we consider our previous example, we can choose the S&P 500 index as the market risk factor. For the period 2014-01-07 to 2015-01-02, the beta coefficient is equal to 0.8307 for Apple and 0.4556 for Coca-Cola, whereas the corresponding idiosyncratic volatilities are 1.2241% (Apple) and 0.8887% (Coca-Cola). As the market volatility is estimated at 0.7165%, the daily value-at-risk is equal to \$41.68 if we include specific risks. Otherwise, it is equal to \$21.54 if we only consider the effect of the market risk factor.

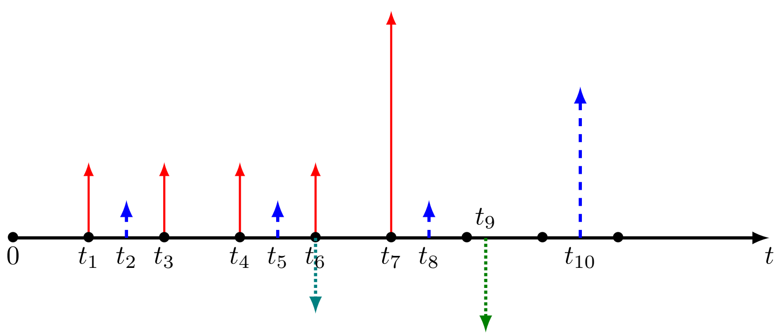

FIGURE 2.7: Cash flows of two bonds and two short exposures

Application to a bond portfolio We consider a portfolio of bonds from the same issuer. In this instance, we can model the bond portfolio by a stream of n_C coupons $C(t_m)$ with fixed dates $t_m \geq t$. Figure 2.7 presents an example of aggregating cash flows with two bonds with a fixed coupon rate and two short exposures. We note $B_t(T)$ the price of a zero-coupon bond at time t for the maturity T. We have $B_t(T) = e^{-(T-t)R_t(T)}$ where $R_t(T)$ is the zero-coupon rate. The sensitivity of the zero-coupon bond is:

$$\frac{\partial B_t(T)}{\partial R_t(T)} = -(T-t) B_t(T)$$

[54]See Cazalet and Roncalli (2014) for a survey.
[55]We set $\tilde{\sigma}_i$ to 0.

For a small change in yield, we obtain:

$$\Delta_h B_{t+h}(T) \approx -(T-t) B_t(T) \Delta_h R_{t+h}(T)$$

The value of the portfolio is:

$$P_t(w) = \sum_{m=1}^{n_C} C(t_m) B_t(t_m)$$

We deduce that:

$$\begin{aligned} \Pi(w) &= P_{t+h}(w) - P_t(w) \\ &= \sum_{m=1}^{n_C} C(t_m)(B_{t+h}(t_m) - B_t(t_m)) \end{aligned}$$

Let us consider the following approximation:

$$\begin{aligned} \Pi(w) &\approx -\sum_{m=1}^{n_C} C(t_m)(t_m - t) B_t(t_m) \Delta_h R_{t+h}(t_m) \\ &= \sum_{m=1}^{n_C} W_{i,t_m} \Delta_h R_{t+h}(t_m) \end{aligned}$$

where $W_{i,t_m} = -C(t_m)(t_m - t) B_t(t_m)$. This expression of the P&L is similar to this obtained with a portfolio of stocks. If we assume that the yield variations are Gaussian, the value-at-risk is equal to:

$$\mathrm{VaR}_\alpha(w;h) = -W_t^\top \mu + \Phi^{-1}(\alpha)\sqrt{W_t^\top \Sigma W_t}$$

where μ and Σ are the mean and the covariance matrix of the vector of yield changes $(\Delta_h R_{t+h}(t_1), \ldots, \Delta_h R_{t+h}(t_{n_c}))$.

Example 15 *We consider an exposure on a US bond at* 31 *December* 2014*. The notional of the bond is* 100 *whereas the annual coupons are equal to* 5*. The remaining maturity is five years and the fixing dates are at the end of December. The number of bonds held in the portfolio is* 10 000*.*

Using the US zero-coupon rates[56], we obtain the following figures for one bond at 31 December 2014:

$t_m - t$	$C(t_m)$	$R_t(t_m)$	$B_t(t_m)$	W_{t_m}
1	5	0.431%	0.996	−4.978
2	5	0.879%	0.983	−9.826
3	5	1.276%	0.962	−14.437
4	5	1.569%	0.939	−18.783
5	105	1.777%	0.915	−480.356

At the end of December 2014, the one-year zero-coupon rate is 0.431%, the two-year zero-coupon rate is 0.879%, etc. We deduce that the bond price is \$115.47 and the total exposure is \$1 154 706. Using the historical period of year 2014, we estimate the covariance matrix

[56] The data comes from the Datastream database. The zero-coupon interest rate of maturity yy years and mm months corresponds to the code USyyYmm.

between daily changes of the five zero-coupon rates[57]. We deduce that the Gaussian VaR of the bond portfolio is equal to \$4 971. If the multiplicative factor m_c is set to 3, the required capital $\mathcal{K}_t^{\mathrm{VaR}}$ is equal to \$47 158 or 4.08% of the mark-to-market. We can compare these figures with those obtained with the historical value-at-risk. In this instance, the daily value-at-risk is higher and equal to \$5 302.

Remark 10 *The previous analysis assumes that the risk factors correspond to the yield changes, meaning that the calculated value-at-risk only concerns interest rate risk. Therefore, it cannot capture all the risks if the bond portfolio is subject to credit risk.*

Defining risk factors with the principal component analysis In the previous paragraph, the bond portfolio was very simple with only one bond and one yield curve. In practice, the bond portfolio contains streams of coupons for many maturities and yield curves. It is therefore necessary to reduce the dimension of the VaR calculation. The underlying idea is that we don't need to use the comprehensive set of zero-coupon rates to represent the set of risk factors that affects the yield curve. For instance, Nelson and Siegel (1987) propose a three-factor parametric model to define the yield curve. Another representation of the yield curve has been formulated by Litterman and Scheinkman (1991), who have proposed to characterize the factors using the principal component analysis (PCA).

Let Σ be the covariance matrix associated to the random vector X_t of dimension n. We consider the eigendecomposition $\Sigma = V \Lambda V^\top$ where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and V is an orthornormal matrix. In the principal component analysis, the (endogenous) risk factors are $\mathcal{F}_t = V^\top X_t$. The reduction method by PCA consists in selecting the first m risk factors with $m \leq n$. When applied to the value-at-risk calculation, it can be achieved in two different ways:

1. In the parametric approach, the covariance matrix Σ is replaced by $\Sigma^\star = V \Lambda^\star V^\top$ where $\Lambda^\star = \operatorname{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$.
2. In the historical method, we only consider the first m PCA factors $\mathcal{F}_t^\star = (\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ or equivalently the modified random vector[58] $X_t^\star = V \mathcal{F}_t^\bullet$ where $\mathcal{F}_t^\bullet = (\mathcal{F}_t^\star, \mathbf{0}_{n-m})$.

If we apply this extracting method of risk factors to Example 15, the eigenvalues are equal to 47.299×10^8, 0.875×10^8, 0.166×10^8, 0.046×10^8, 0.012×10^8 whereas the matrix V of eigenvectors is:

$$V = \begin{pmatrix} 0.084 & -0.375 & -0.711 & 0.589 & 0.002 \\ 0.303 & -0.610 & -0.215 & -0.690 & -0.114 \\ 0.470 & -0.389 & 0.515 & 0.305 & 0.519 \\ 0.567 & 0.103 & 0.195 & 0.223 & -0.762 \\ 0.599 & 0.570 & -0.381 & -0.183 & 0.371 \end{pmatrix}$$

[57] The standard deviation is respectively equal to 0.746 bps for $\Delta_h R_t(t+1)$, 2.170 bps for $\Delta_h R_t(t+2)$, 3.264 bps for $\Delta_h R_t(t+3)$, 3.901 bps for $\Delta_h R_t(t+4)$ and 4.155 bps for $\Delta_h R_t(t+5)$ where h corresponds to one trading day. For the correlation matrix, we get:

$$\rho = \begin{pmatrix} 100.000 & & & & \\ 87.205 & 100.000 & & & \\ 79.809 & 97.845 & 100.000 & & \\ 75.584 & 95.270 & 98.895 & 100.000 & \\ 71.944 & 92.110 & 96.556 & 99.219 & 100.000 \end{pmatrix}$$

[58] Because we have $V^{-1} = V^\top$.

We deduce that:

$$\begin{aligned}\mathcal{F}_{1,t} &= 0.084 \times R_t(t+1) + 0.303 \times R_t(t+2) + \cdots + 0.599 \times R_t(t+5) \\ &\vdots \\ \mathcal{F}_{5,t} &= 0.002 \times R_t(t+1) - 0.114 \times R_t(t+2) + \cdots + 0.371 \times R_t(t+5)\end{aligned}$$

We retrieve the three factors of Litterman and Scheinkman, which are a level factor $\mathcal{F}_{1,t}$, a slope factor $\mathcal{F}_{2,t}$ and a convexity or curvature factor $\mathcal{F}_{3,t}$. In the following table, we report the incremental VaR of each risk factor, which is defined as difference between the value-at-risk including the risk factor and the value-at-risk excluding the risk factor:

VaR	$\mathcal{F}_{1,t}$	$\mathcal{F}_{2,t}$	$\mathcal{F}_{3,t}$	$\mathcal{F}_{4,t}$	$\mathcal{F}_{5,t}$	Sum
Gaussian	4934.71	32.94	2.86	0.17	0.19	4970.87
Historical	5857.39	−765.44	216.58	−7.98	1.41	5301.95

We notice that the value-at-risk is principally explained by the first risk factor, that is the general level of interest rates, whereas the contribution of the slope and convexity factors is small and the contribution of the remaining risk factors is marginal. This result can be explained by the long-only characteristics of the portfolio. Nevertheless, even if we consider a more complex bond portfolio, we generally observe that a number of factors is sufficient to model all the risk dimensions of the yield curve. An example is provided in Figure 2.8 with a stream of long and short exposures[59]. Using the period January 2014 – December 2014, the convergence of the value-at-risk is achieved with six factors. This result is connected to the requirement of the Basel Committee that "*banks must model the yield curve using a minimum of six risk factors*".

2.2.3.3 Volatility forecasting

The challenge of the Gaussian value-at-risk is the estimation of the loss volatility or the covariance matrix of asset returns/risk factors. The issue is not to consider the best estimate for describing the past, but to use the best estimate for forecasting the loss distribution. In the previous illustrations, we use the empirical covariance matrix or the empirical standard deviation. However, other estimators have been proposed by academics and professionals.

The original approach implemented in RiskMetrics used an exponentially weighted moving average (EWMA) for modeling the covariance between asset returns[60]:

$$\hat{\Sigma}_t = \lambda \hat{\Sigma}_{t-1} + (1-\lambda) R_{t-1} R_{t-1}^{\top}$$

where the parameter $\lambda \in [0,1]$ is the decay factor, which represents the degree of weighting decrease. Using a finite sample, the previous estimate is equivalent to a weighted estimator:

$$\hat{\Sigma}_t = \sum_{s=1}^{n_S} \omega_s R_{t-s} R_{t-s}^{\top}$$

where:

$$\omega_s = \frac{(1-\lambda)}{(1-\lambda^{n_S})} \lambda^{s-1}$$

In Figure 2.9, we represent the weights ω_s for different values of λ when the number n_S of historical scenarios is equal to 250. We verify that this estimator gives more importance to

[59] We have $C_t(t+1/2) = 400$, $C_t(t+1) = 300$, $C_t(t+3/2) = 200$, $C_t(t+2) = -200$, $C_t(t+3) = -300$, $C_t(t+4) = -500$, $C_t(t+5) = 500$, $C_t(t+6) = 400$, $C_t(t+7) = -300$, $C_t(t+10) = -700$, $C_t(t+10) = 300$ and $C_t(t+30) = 700$.

[60] We assume that the mean of expected returns is equal to $\mathbf{0}$.

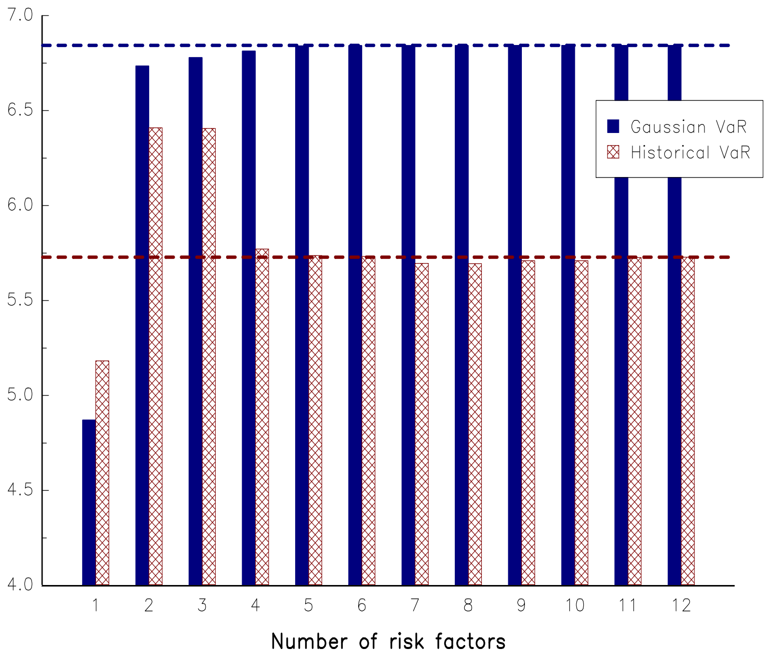

FIGURE 2.8: Convergence of the VaR with PCA risk factors

the current values than to the past values. For instance, if λ is equal to 0.94[61], 50% of the weights corresponds to the twelve first observations and the half-life is 16.7 days. We also observe that the case $\lambda = 1$ corresponds to the standard covariance estimator with uniform weights.

Another approach to model volatility in risk management is to consider that the volatility is time-varying. In 1982, Engle introduced a class of stochastic processes in order to take into account the heteroscedasticity of asset returns[62]:

$$R_{i,t} = \mu_i + \varepsilon_t \quad \text{where} \quad \varepsilon_t = \sigma_t e_t \quad \text{and} \quad e_t \sim \mathcal{N}(0,1)$$

The time-varying variance $h_t = \sigma_t^2$ satisfies the following equation:

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

where $\alpha_j \geq 0$ for all $j \geq 0$. We note that the conditional variance of ε_t is not constant and depends on the past values of ε_t. A substantial impact on the asset return $R_{i,t}$ implies an increase of the conditional variance of ε_{t+1} at time $t+1$ and therefore an increase of the probability to observe another substantial impact on $R_{i,t+1}$. Therefore, this means that the volatility is persistent, which is a well-known stylized fact in finance (Chou, 1988). This type of stochastic processes, known as ARCH models (Autoregressive Conditional Heteroscedasticity), has been extended by Bollerslev (1986) in the following way:

$$h_t = \alpha_0 + \gamma_1 h_{t-1} + \gamma_2 h_{t-2} + \cdots + \gamma_p h_{t-p} + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

In this case, the conditional variance depends also on its past values and we obtain a GARCH(p,q) model. If $\sum_{i=1}^{p} \gamma_i + \sum_{i=1}^{q} \alpha_i = 1$, we may show that the process ε_t^2 has a unit

[61] It was the original value of the RiskMetrics system (J.P. Morgan, 1996).
[62] See Section 10.2.4.1 on page 664 for a comprehensive presentation of ARCH and GARCH models.

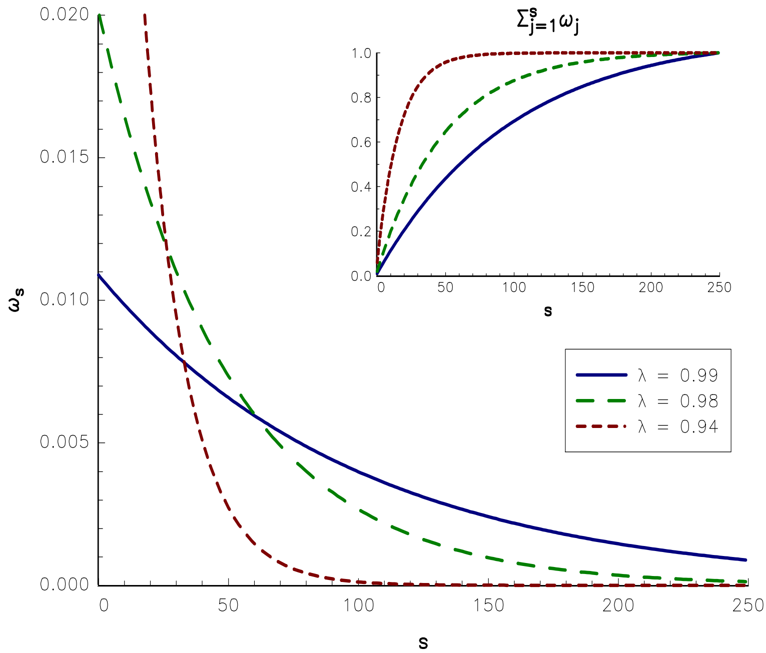

FIGURE 2.9: Weights of the EWMA estimator

root and the model is called an integrated GARCH (or IGARCH) process. If we neglect the constant term, the expression of the IGARCH(1,1) process is $h_t = (1-\alpha)\,h_{t-1} + \alpha R_{i,t-1}^2$ or equivalently:

$$\sigma_t^2 = (1-\alpha)\,\sigma_{t-1}^2 + \alpha R_{i,t-1}^2$$

This estimator is then an exponentially weighted moving average with a factor λ equal to $1-\alpha$.

In Figure 2.10, we have reported the annualized volatility of the S&P 500 index estimated using the GARCH model (first panel). The ML estimates of the parameters are $\hat{\gamma}_1 = 0.8954$ and $\hat{\alpha}_1 = 0.0929$. We verify that this estimated model is close to an IGARCH process. In the other panels, we compare the GARCH volatility with the empirical one-year historical volatility, the EWMA volatility (with $\lambda = 0.94$) and a short volatility based on 20 trading days. We observe large differences between the GARCH volatility and the one-year historical volatility, but the two others estimators (EWMA and short volatility) give similar results to the GARCH estimator. To compare the out-of-sample forecasting accuracy of these different models, we consider respectively a long and a short exposure on the S&P 500 index. At time t, we compute the value-at-risk for the next day and we compare this figure with the realized mark-to-market. Table 2.9 show the number of exceptions per year for the different models: (1) GARCH(1,1) model, (2) Gaussian value-at-risk with a one-year historical volatility, (3) EWMA model with $\lambda = 0.94$, (4) Gaussian value-at-risk with a twenty-day short volatility and (5) historical value-at-risk based on the last 260 trading days. We observe that the GARCH model produces the smallest number of exceptions, whereas the largest number of exceptions occurs in the case of the Gaussian value-at-risk with the one-year historical volatility. We also notice that the number of exceptions is smaller for the short exposure than for the long exposure. This is due to the asymmetry of returns, because extreme negative returns are larger than extreme positive returns on average.

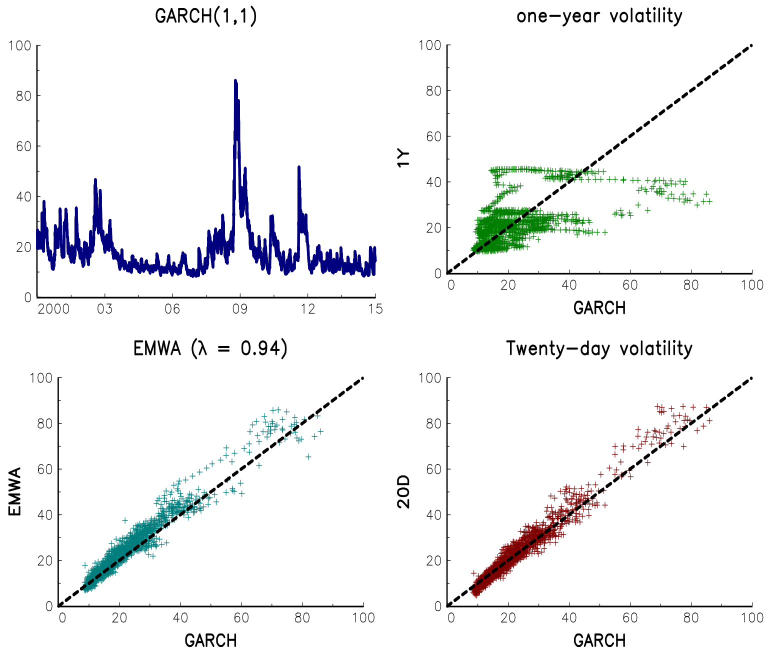

FIGURE 2.10: Comparison of GARCH and EWMA volatilities

TABLE 2.9: Number of exceptions per year for long and short exposures on the S&P 500 index

Year	Long exposure					Short exposure				
	(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
2000	5	5	2	4	4	5	8	4	6	4
2001	4	3	2	3	2	2	4	2	5	2
2002	2	5	2	4	3	5	9	4	6	5
2003	1	0	0	2	0	1	0	1	4	0
2004	2	0	2	6	0	0	0	0	2	1
2005	1	1	2	4	3	1	4	1	6	3
2006	2	4	3	4	4	2	5	3	5	3
2007	6	15	6	10	7	1	9	0	3	7
2008	7	23	5	7	10	4	12	4	3	8
2009	5	0	1	6	0	2	2	2	3	0
2010	7	6	5	8	3	3	5	2	7	3
2011	6	8	6	7	4	2	8	1	6	3
2012	5	1	4	5	0	3	1	2	7	1
2013	4	2	3	9	2	2	2	2	4	1
2014	6	9	7	11	2	2	4	2	2	4

2.2.3.4 Extension to other probability distributions

The Gaussian value-at-risk has been strongly criticized because it depends only on the first two moments of the loss distribution. Indeed, there is a lot of evidence that asset returns and risk factors are not Gaussian (Cont, 2001). They generally present fat tails and skew effects. It is therefore interesting to consider alternative probability distributions, which are more appropriate to take into account these stylized facts.

Let $\mu_r = \mathbb{E}\left[(X - \mathbb{E}[X])^r\right]$ be the r-order centered moment of the random variable X. The skewness $\gamma_1 = \mu_3/\mu_2^{3/2}$ is the measure of the asymmetry of the loss distribution. If $\gamma_1 < 0$ (resp. $\gamma_1 > 0$), the distribution is left-skewed (resp. right-skewed) because the left (resp. right) tail is longer. For the Gaussian distribution, γ_1 is equal to zero. To characterize whether the distribution is peaked or flat relative to the normal distribution, we consider the excess kurtosis $\gamma_2 = \mu_4/\mu_2^2 - 3$. If $\gamma_2 > 0$, the distribution presents heavy tails. In the case of the Gaussian distribution, γ_2 is exactly equal to zero. We have illustrated the skewness and kurtosis statistics in Figure 2.11. Whereas we generally encounter skewness risk in credit and hedge fund portfolios, kurtosis risk has a stronger impact in equity portfolios. For example, if we consider the daily returns of the S&P 500 index, we obtain an empirical distribution[63] which has a higher kurtosis than the fitted Gaussian distribution (Figure 2.12).

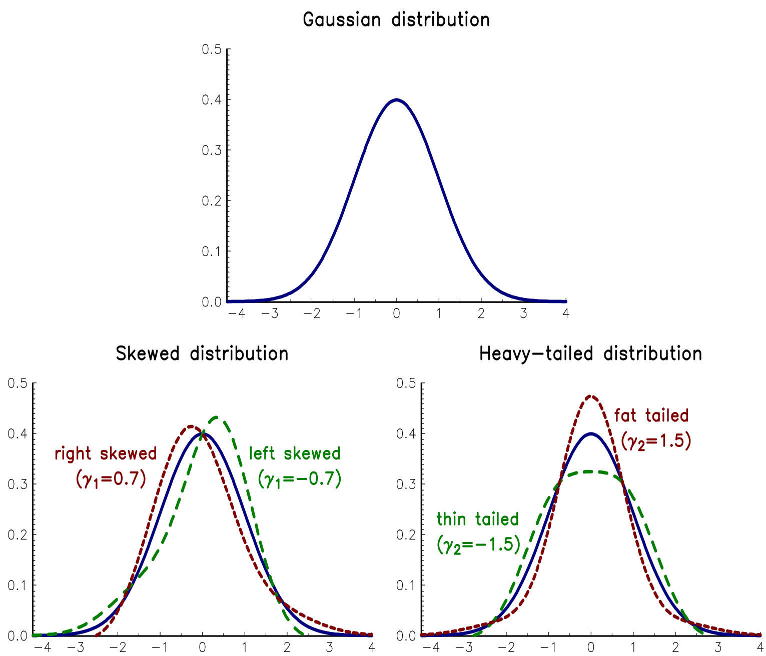

FIGURE 2.11: Examples of skewed and fat tailed distributions

An example of fat-tail distributions is the Student's t probability distribution. If $X \sim t_\nu$, we have $\mathbb{E}[X] = 0$ and $\operatorname{var}(X) = \nu/(\nu - 2)$ for $\nu > 2$. Because X has a fixed mean and variance for a given degrees of freedom, we need to introduce location and scale parameters to model the future loss $L(w) = \xi + \omega X$. To calculate the value-at-risk, we proceed as in the Gaussian case. We have:

$$\Pr\left\{L(w) \leq \mathbf{F}_L^{-1}(\alpha)\right\} = \alpha \Leftrightarrow \Pr\left\{X \leq \frac{\mathbf{F}_L^{-1}(\alpha) - \xi}{\omega}\right\} = \alpha$$

[63] It is estimated using the kernel approach.

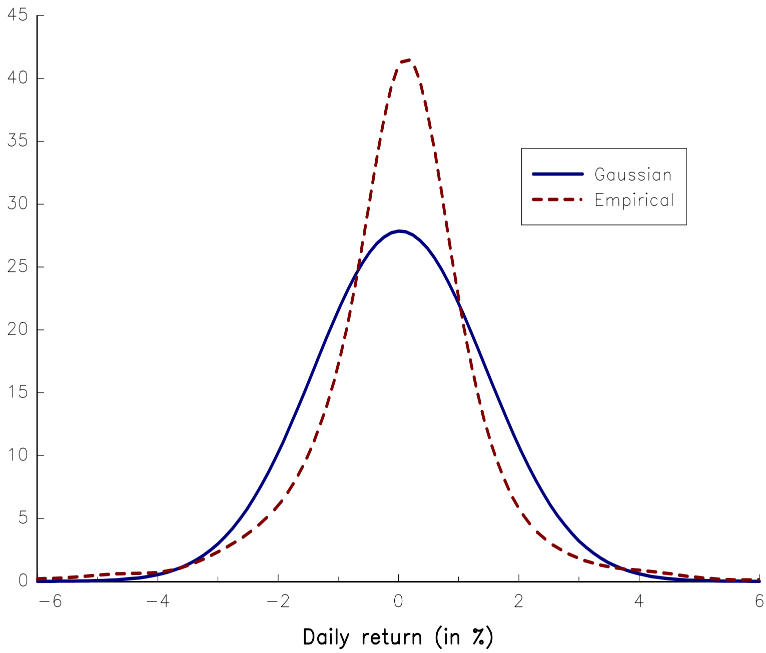

FIGURE 2.12: Estimated distribution of S&P 500 daily returns (2007-2014)

We deduce that:

$$\mathbf{T}\left(\frac{\mathbf{F}_L^{-1}(\alpha)-\xi}{\omega};\nu\right)=\alpha\Leftrightarrow\mathbf{F}_L^{-1}(\alpha)=\xi+\mathbf{T}^{-1}(\alpha;\nu)\,\omega$$

In practice, the parameters ξ and ω are estimated by the method of moments[64]. We finally deduce that:

$$\mathrm{VaR}_\alpha(w;h)=\mu(L)+\mathbf{T}^{-1}(\alpha;\nu)\,\sigma(L)\sqrt{\frac{\nu-2}{\nu}}$$

Let us illustrate the impact of the probability distribution with Example 13. By using different values of ν, we obtain the following daily VaRs:

ν	3.00	3.50	4.00	5.00	6.00	10.00	1000	∞
ω	10.23	11.60	12.53	13.72	14.46	15.84	17.70	17.71
$\mathrm{VaR}_\alpha(w;h)$	46.44	47.09	46.93	46.17	45.46	43.79	41.24	41.21

If $\nu\to\infty$, we verify that the Student's t value-at-risk converges to the Gaussian value-at-risk (\$41.21). If the degrees of freedom is equal to 4, it is closer to the historical value-at-risk (\$47.39).

We can derive closed-form formulas for several probability distributions. However, most of them are not used in practice, because these methods are not appealing from a professional point of view. Nevertheless, one approach is very popular among professionals. Using the Cornish-Fisher expansion of the normal distribution, Zangari (1996) proposes to estimate the value-at-risk in the following way:

$$\mathrm{VaR}_\alpha(w;h)=\mu(L)+\mathfrak{z}(\alpha;\gamma_1(L),\gamma_2(L))\times\sigma(L)\tag{2.7}$$

[64]We have $\mathbb{E}[\xi+\omega X]=\xi$ and $\mathrm{var}(\xi+\omega X)=\left(\omega^2\nu\right)/(\nu-2)$.

where:

$$\mathfrak{z}\left(\alpha;\gamma_1,\gamma_2\right) = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2 - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)\gamma_1^2 \tag{2.8}$$

and $z_\alpha = \Phi^{-1}(\alpha)$. This is the same formula as the one used for the Gaussian value-at-risk but with another scaling parameter[65]. In Equation (2.7), the skewness and excess kurtosis coefficients are those of the loss distribution[66].

TABLE 2.10: Value of the Cornish-Fisher quantile $\mathfrak{z}\left(99\%;\gamma_1,\gamma_2\right)$

γ_1	γ_2							
	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00
−2.00								0.99
−1.00			1.68	1.92	2.15	2.38	2.62	2.85
−0.50		2.10	2.33	2.57	2.80	3.03	3.27	3.50
0.00	2.33	2.56	2.79	3.03	3.26	3.50	3.73	3.96
0.50		2.83	3.07	3.30	3.54	3.77	4.00	4.24
1.00			3.15	3.39	3.62	3.85	4.09	4.32
2.00								3.93

Table 2.10 shows the value of the Cornish-Fisher quantile $\mathfrak{z}\left(99\%;\gamma_1,\gamma_2\right)$ for different values of skewness and excess kurtosis. We cannot always calculate the quantile because Equation (2.8) does not define necessarily a probability distribution if the parameters γ_1 and γ_2 does not satisfy the following condition (Maillard, 2018):

$$\frac{\partial\,\mathfrak{z}\left(\alpha;\gamma_1,\gamma_2\right)}{\partial\, z_\alpha} \geq 0 \Leftrightarrow \frac{\gamma_1^2}{9} - 4\left(\frac{\gamma_2}{8} - \frac{\gamma_1^2}{6}\right)\left(1 - \frac{\gamma_2}{8} + \frac{5\gamma_1^2}{36}\right) \leq 0$$

We have reported the domain of definition in the third panel in Figure 2.13. For instance, Equation (2.8) is not valid if the skewness is equal to 2 and the excess kurtosis is equal to 3. If we analyze results in Table 2.10, we do not observe that there is a monotone relationship between the skewness and the quantile. To understand this curious behavior, we report the partial derivatives of $\mathfrak{z}\left(\alpha;\gamma_1,\gamma_2\right)$ with respect to γ_1 and γ_2 in Figure 2.13. We notice that their signs depend on the confidence level α, but also on the skewness for $\partial_{\gamma_1}\mathfrak{z}\left(\alpha;\gamma_1,\gamma_2\right)$. Another drawback of the Cornish-Fisher approach concerns the statistical moments, which are not necessarily equal to the input parameters if the skewness and the kurtosis are not close to zero[67]. Contrary to what professionals commonly think, the Cornish-Fisher expansion is therefore difficult to implement.

When we consider other probability distribution than the normal distribution, the difficulty concerns the multivariate case. In the previous examples, we directly model the loss

[65]If $\gamma_1 = \gamma_2 = 0$, we retrieve the Gaussian value-at-risk because $\mathfrak{z}(\alpha;0,0) = \Phi^{-1}(\alpha)$.

[66]If we prefer to use the moments of the P&L, we have to consider the relationships $\gamma_1(Ł) = -\gamma_1(\Pi)$ and $\gamma_2(L) = \gamma_2(\Pi)$.

[67]Let Z be a Cornish-Fisher random variable satisfying $\mathbf{F}^{-1}(\alpha) = \mathfrak{z}\left(\alpha;\gamma_1,\gamma_2\right)$. A direct application of the result in Appendix A.2.2.3 gives:

$$\mathbb{E}\left[Z^r\right] = \int_0^1 \mathfrak{z}^r\left(\alpha;\gamma_1,\gamma_2\right)\,\mathrm{d}\alpha$$

Using numerical integration, we can show that $\gamma_1(Z) \neq \gamma_1$ and $\gamma_2(Z) \neq \gamma_2$ if γ_1 and γ_2 are large enough (Maillard, 2018).

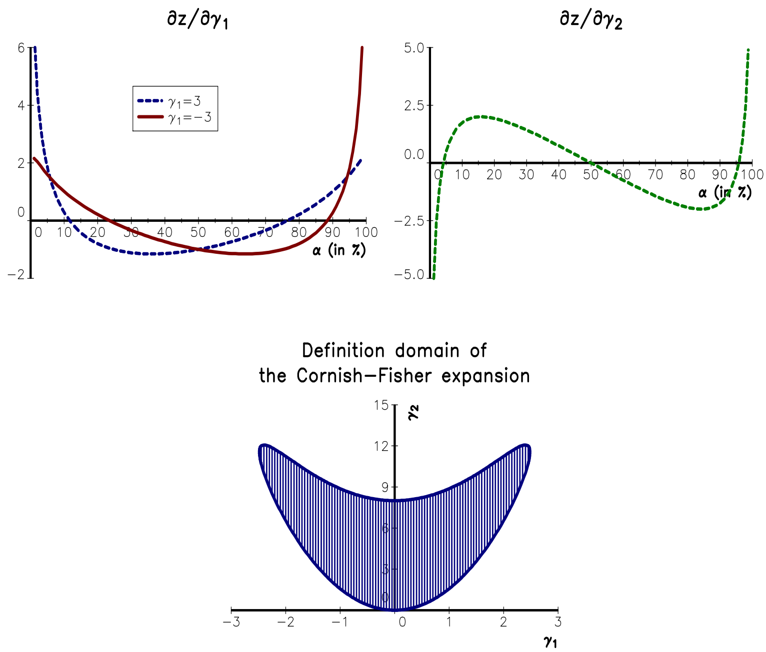

FIGURE 2.13: Derivatives and definition domain of the Cornish-Fisher expansion

distribution, that is the reduced form of the pricing system. To model the joint distribution of risk factors, two main approaches are available. The first approach considers copula functions and the value-at-risk is calculated using the Monte Carlo simulation method (see Chapters 11 and 13). The second approach consists in selecting a multivariate probability distribution, which has some appealing properties. For instance, it should be flexible enough to calibrate the first two moments of the risk factors and should also include asymmetry (positive and negative skewness) and fat tails (positive excess kurtosis) in a natural way. In order to obtain an analytical formula for the value-at-risk, it must be tractable and verify the closure property under affine transformation. This implies that if the random vector X follows a certain class of distribution, then the random vector $Y = A + BX$ belongs also to the same class. These properties reduce dramatically the set of eligible multivariate probability distributions, because the potential candidates are mostly elliptical distributions. Such examples are the skew normal and t distributions presented in Appendix A.2.1 on page 1057.

Example 16 *We consider a portfolio of three assets and assume that their annualized returns follows a multivariate skew normal distribution. The location parameters are equal to* 1%, −2% *and* 15% *whereas the scale parameters are equal to* 5%, 10% *and* 20%. *The correlation parameters to describe the dependence between the skew normal variables are given by the following matrix:*

$$\mathbb{C} = \begin{pmatrix} 1.00 & & \\ 0.35 & 1.00 & \\ 0.20 & -0.50 & 1.00 \end{pmatrix}$$

The three assets have different skewness profiles, and the shape parameters are equal to 0, 10 *and* −15.50.

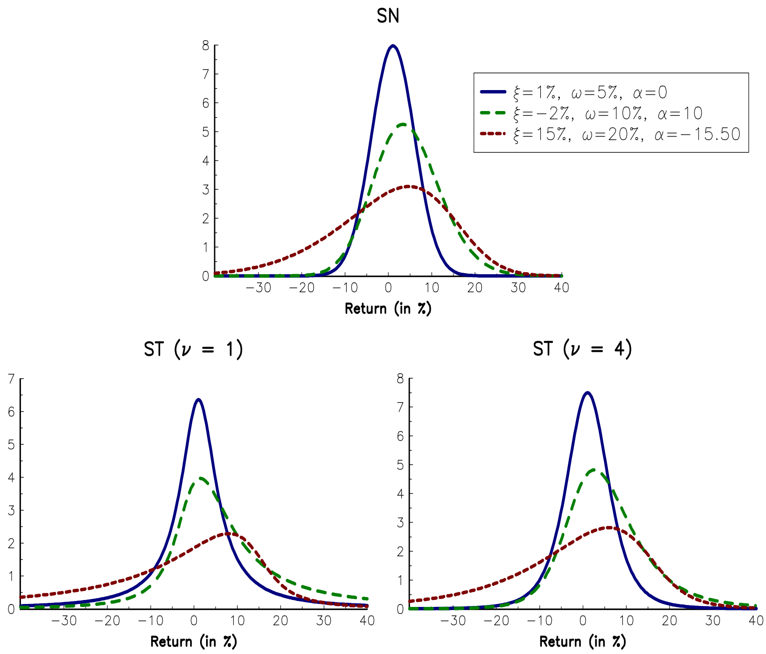

FIGURE 2.14: Skew normal and t distributions of asset returns

In Figure 2.14, we have reported the density function of the three asset returns[68]. The return of the first asset is close to be Gaussian whereas the two other assets exhibit respectively negative and positive skews. Moments are given in the table below:

Asset i	μ_i (in %)	σ_i (in %)	$\gamma_{1,i}$	$\gamma_{2,i}$
1	1.07	5.00	0.00	0.00
2	4.36	7.72	0.24	0.13
3	0.32	13.58	−0.54	0.39

Let us consider the nominal portfolio $w = (\$500, \$200, \$300)$. The annualized P&L $\Pi(w)$ is equal to $w^\top R$ where $R \sim \mathcal{SN}(\xi, \Omega, \eta)$. We deduce that $\Pi(w) \sim \mathcal{SN}(\xi_w, \omega_w, \eta_w)$ with $\xi_w = 46.00$, $\omega_w = 66.14$ and $\eta_w = -0.73$. We finally deduce that the one-year 99% value-at-risk is equal to \$123.91. If we use the multivariate skew t distribution in place of the multivariate skew normal distributions to model asset returns and if we use the same parameter values, the one-year 99% value-at-risk becomes \$558.35 for $\nu = 2$, \$215.21 for $\nu = 5$ and \$130.47 for $\nu = 50$. We verify that the skew t value-at-risk converges to the skew normal value-at-risk when the number of degrees of freedom ν tends to $+\infty$.

The choice of the probability distribution is an important issue and raises the question of model risk. In this instance, the Basel Committee justifies the introduction of the penalty coefficient in order to reduce the risk of a wrong specification (Stahl, 1997). For example, imagine that we calculate the value-at-risk with a probability distribution $\mathbf{F}$ while the true probability distribution of the portfolio loss is $\mathbf{H}$. The multiplication factor m_c defines then a capital buffer such that we are certain that the confidence level of the value-at-risk will be at least equal to α:

$$\Pr\{L(w) \leq \underbrace{m_c \cdot \mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}_{\text{Capital}}\} \geq \alpha \tag{2.9}$$

[68] We also show the density function in the case of the skew t distribution with $\nu = 1$ and $\nu = 4$.

This implies that $\mathbf{H}\left(m_c \cdot \mathrm{VaR}_\alpha^{(\mathbf{F})}(w)\right) \geq \alpha$ and $m_c \cdot \mathrm{VaR}_\alpha^{(\mathbf{F})}(w) \geq \mathbf{H}^{-1}(\alpha)$. We finally deduce that:

$$m_c \geq \frac{\mathrm{VaR}_\alpha^{(\mathbf{H})}(w)}{\mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}$$

In the case where $\mathbf{F}$ and $\mathbf{H}$ are the normal and Student's t distributions, we obtain[69]:

$$m_c \geq \sqrt{\frac{\nu-2}{\nu}} \frac{\mathbf{T}_\nu^{-1}(\alpha)}{\Phi^{-1}(\alpha)}$$

Below is the lower bound of m_c for different values of α and ν.

α/ν	3	4	5	6	10	50	100
90%	0.74	0.85	0.89	0.92	0.96	0.99	1.00
95%	1.13	1.14	1.12	1.10	1.06	1.01	1.01
99%	1.31	1.26	1.21	1.18	1.10	1.02	1.01
99.9%	1.91	1.64	1.48	1.38	1.20	1.03	1.02
99.99%	3.45	2.48	2.02	1.76	1.37	1.06	1.03

For instance, we have $m_c \geq 1.31$ when $\alpha = 99\%$ and $\nu = 3$.

Stahl (1997) considers the general case when $\mathbf{F}$ is the normal distribution and $\mathbf{H}$ is an unknown probability distribution. Let X be a given random variable. The Chebyshev's inequality states that:

$$\Pr\left\{(|X - \mu(X)| > k \cdot \sigma(X))\right\} \leq k^{-2}$$

for any real number $k > 0$. If we apply this theorem to the value-at-risk, we obtain[70]:

$$\Pr\left\{L(w) \leq \sqrt{\frac{1}{1-\alpha}}\sigma(L)\right\} \geq \alpha$$

Using Equation (2.9), we deduce that:

$$m_c = \sqrt{\frac{1}{1-\alpha}} \frac{\sigma(L)}{\mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}$$

In the case of the normal distribution, we finally obtain that the multiplicative factor is:

$$m_c = \frac{1}{\Phi^{-1}(\alpha)} \sqrt{\frac{1}{1-\alpha}}$$

This ratio is the multiplication factor to apply in order to be sure that the confidence level of the value-at-risk is at least equal to α if we use the normal distribution to model the portfolio loss. In the case where the probability distribution is symmetric, this ratio becomes:

$$m_c = \frac{1}{\Phi^{-1}(\alpha)} \sqrt{\frac{1}{2-2\alpha}}$$

In Table 2.11, we report the values of m_c for different confidence levels. If α is equal to 99%, the multiplication factor is equal to 3.04 if the distribution is symmetric and 4.30 otherwise.

[69] We recall that the Gaussian value-at-risk is equal to $\Phi^{-1}(\alpha)\,\sigma(L)$ whereas the Student's t value-at-risk is equal to $\sqrt{(\nu-2)/\nu} \cdot \mathbf{T}_\nu^{-1}(\alpha)\,\sigma(L)$.

[70] We set $\alpha = 1 - k^{-2}$.

TABLE 2.11: Value of the multiplication factor m_c deduced from the Chebyshev's inequality

α (in %)	90.00	95.00	99.00	99.25	99.50	99.75	99.99
Symmetric	1.74	1.92	3.04	3.36	3.88	5.04	19.01
Asymmetric	2.47	2.72	4.30	4.75	5.49	7.12	26.89

Remark 11 *Even if the previous analysis justifies the multiplication factor from a statistical point of view, we face two main issues. First, the multiplication factor assumes that the bank uses a Gaussian value-at-risk. It was the case for many banks in the early 1990s, but they use today historical value-at-risk measures. Some have suggested that the multiplication factor has been introduced in order to reduce the difference in terms of regulatory capital between SMM and IMA and it is certainly the case. The second issue concerns the specificity of the loss distribution. For many positions like long-only unlevered portfolios, the loss is bounded. If we use a Gaussian value-at-risk, the regulatory capital satisfies*[71] $\mathcal{K} = \mathcal{K}^{\mathrm{VaR}} + \mathcal{K}^{\mathrm{SVaR}} > 13.98 \cdot \sigma(L)$ *where* $\sigma(L)$ *is the non-stressed loss volatility. This implies that the value-at-risk is larger than the portfolio value if* $\sigma(L) > 7.2\%$*! There is a direct contradiction here.*

2.2.4 Monte Carlo methods

In this approach, we postulate a given probability distribution $\mathbf{H}$ for the risk factors:

$$(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}) \sim \mathbf{H}$$

Then, we simulate n_S scenarios of risk factors and calculate the simulated P&L $\Pi_s(w)$ for each scenario s. Finally, we estimate the risk measure (VaR/ES) by the method of order statistics. The Monte Carlo method to calculate the VaR/ES is therefore close to the historical method. The only difference is that it uses simulated scenarios instead of historical scenarios. This implies that the Monte Carlo approach is not limited by the number of scenarios. By construction, the Monte Carlo VaR/ES is also similar to the analytical VaR/ES, because they both specify the parametric probability distribution of risk factors. In summary, we can say that:

- the Monte Carlo VaR/ES is a historical VaR/ES with simulated scenarios;
- the Monte Carlo VaR/ES is a parametric VaR/ES for which it is difficult to find an analytical formula.

Let us consider Example 16 on page 87. The expression of the P&L is:

$$\Pi(w) = 500 \times R_1 + 200 \times R_2 + 300 \times R_3$$

Because we know that the combination of the components of a skew normal random vector is a skew normal random variable, we were able to compute the analytical quantile of $\Pi(w)$ at the 1% confidence level. Suppose now that we don't know the analytical distribution of $\Pi(w)$. We can repeat the exercise by using the Monte Carlo method. At each simulation s, we generate the random variates $(R_{1,s}, R_{2,s}, R_{3,s})$ such that:

$$(R_{1,s}, R_{2,s}, R_{3,s}) \sim SN(\xi, \Omega, \eta)$$

[71] Because we have $2 \times m_c \times 2.33 > 13.98$.

and the corresponding P&L $\Pi_s(w) = 500 \times R_{1,s} + 200 \times R_{2,s} + 300 \times R_{3,s}$. The Monte Carlo value-at-risk is the $n_s(1-\alpha)^{\text{th}}$ order statistic:

$$\widehat{\mathrm{VaR}}_\alpha(n_S) = -\Pi_{(n_s(1-\alpha):n_s)}(w)$$

Using the law of large numbers, we can show that the MC estimator converges to the exact VaR:

$$\lim_{n_S \to \infty} \widehat{\mathrm{VaR}}_\alpha(n_S) = \mathrm{VaR}_\alpha$$

In Figure 2.15, we report four Monte Carlo runs with 10 000 simulated scenarios. We notice that the convergence of the Monte Carlo VaR to the analytical VaR is slow[72], because asset returns present high skewness. The convergence will be faster if the probability distribution of risk factors is close to be normal and has no fat tails.

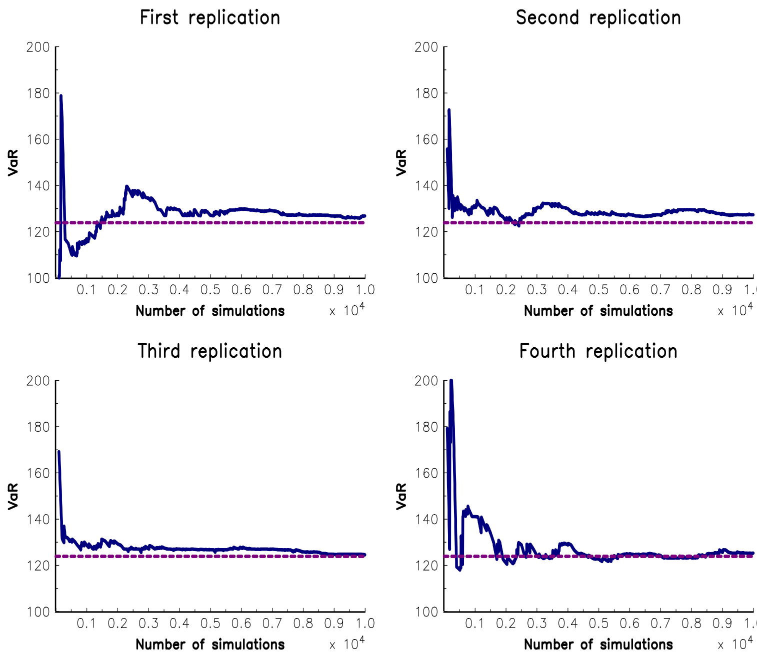

FIGURE 2.15: Convergence of the Monte Carlo VaR when asset returns are skew normal

Remark 12 *The Monte Carlo value-at-risk has been extensively studied with heavy-tailed risk factors (Dupire, 1998; Eberlein* et al.*, 1998; Glasserman* et al.*, 2002). In those cases, one needs to use advanced and specific methods to reduce the variance of the estimator*[73].

Example 17 *We use a variant of Example 15 on page 78. We consider that the bond is exposed to credit risk. In particular, we assume that the current default intensity of the bond issuer is equal to* 200 *bps whereas the recovery rate is equal to* 50%.

In the case of a defaultable bond, the coupons and the notional are paid until the issuer does not default whereas a recovery rate is applied if the issuer defaults before the maturity

[72] We have previously found that the exact VaR is equal to $123.91.
[73] These techniques are presented in Chapter 13.

of the bond. If we assume that the recovery is paid at maturity, we can show that the bond price under default risk is:

$$P_t = \sum_{t_m \geq t} C(t_m) B_t(t_m) \mathbf{S}_t(t_m) + N B_t(T) \left(\mathbf{S}_t(T) + \boldsymbol{\mathcal{R}}_t (1 - \mathbf{S}_t(T)) \right)$$

where $\mathbf{S}_t(t_m)$ is the survival function at time t_m and $\boldsymbol{\mathcal{R}}_t$ is the current recovery rate. We retrieve the formula of the bond price without default risk if $\mathbf{S}_t(t_m) = 1$. Using the numerical values of the parameters, the bond price is equal to \$109.75 and is lower than the non-defaultable bond price[74]. If we assume that the default time is exponential with $\mathbf{S}_t(t_m) = e^{-\lambda_t(t_m - t)}$, we have:

$$\begin{aligned} P_{t+h} &= \sum_{t_m \geq t} C(t_m) e^{(t_m - t - h) R_{t+h}(t_m)} e^{-\lambda_{t+h}(t_m - t - h)} + \\ &\quad N e^{(T - t - h) R_{t+h}(T)} \left(\boldsymbol{\mathcal{R}}_{t+h} + (1 - \boldsymbol{\mathcal{R}}_{t+h}) e^{-\lambda_{t+h}(T - t - h)} \right) \end{aligned}$$

We define the risk factors as the zero-coupon rates, the default intensity and the recovery rate:

$$\begin{aligned} R_{t+h}(t_m) &\simeq R_t(t_m) + \Delta_h R_{t+h}(t_m) \\ \lambda_{t+h} &= \lambda_t + \Delta_h \lambda_{t+h} \\ \boldsymbol{\mathcal{R}}_{t+h} &= \boldsymbol{\mathcal{R}}_t + \Delta_h \boldsymbol{\mathcal{R}}_{t+h} \end{aligned}$$

We assume that the three risk factors are independent and follow the following probability distributions:

$$\begin{aligned} &(\Delta_h R_{t+h}(t_1), \ldots, \Delta_h R_{t+h}(t_n)) \sim \mathcal{N}(0, \Sigma) \\ &\Delta_h \lambda_{t+h} \sim \mathcal{N}\left(0, \sigma_\lambda^2\right) \\ &\Delta_h \boldsymbol{\mathcal{R}}_{t+h} \sim \mathcal{U}_{[a,b]} \end{aligned}$$

We can then simulate the daily P&L $\Pi(w) = w(P_{t+h} - P_t)$ using the above specifications. For the numerical application, we use the covariance matrix given in Footnote 57 whereas the values of σ_λ, a and b are equal to 20 bps, -10% and 10%. In Figure 2.16, we have estimated the density of the daily P&L using 100 000 simulations. IR corresponds to the case when risk factors are only the interest rates[75]. The case IR/S considers that both $R_t(t_m)$ and λ_t are risk factors whereas $\boldsymbol{\mathcal{R}}_t$ is assumed to be constant. Finally, we include the recovery risk in the case IR/S/RR. Using 10 million simulations, we find that the daily value-at-risk is equal to \$4 730 (IR), \$13 460 (IR/S) and \$18 360 (IR/S/RR). We see the impact of taking into account default risk in the calculation of the value-at-risk.

2.2.5 The case of options and derivatives

Special attention should be paid to portfolios of derivatives, because their risk management is much more complicated than a long-only portfolio of traditional assets (Duffie and Pan, 1997). They involve non-linear exposures to risk factors that are difficult to measure, they are sensitive to parameters that are not always observable and they are generally traded on OTC markets. In this section, we provide an overview of the challenges that arise when measuring and managing the risk of these assets. Chapter 9 complements it with a more exhaustive treatment of hedging and pricing issues as well as model risk.

[74] We recall that it was equal to \$115.47.

[75] This implies that we set $\Delta_h \lambda_{t+h}$ and $\Delta_h \boldsymbol{\mathcal{R}}_{t+h}$ to zero in the Monte Carlo procedure.

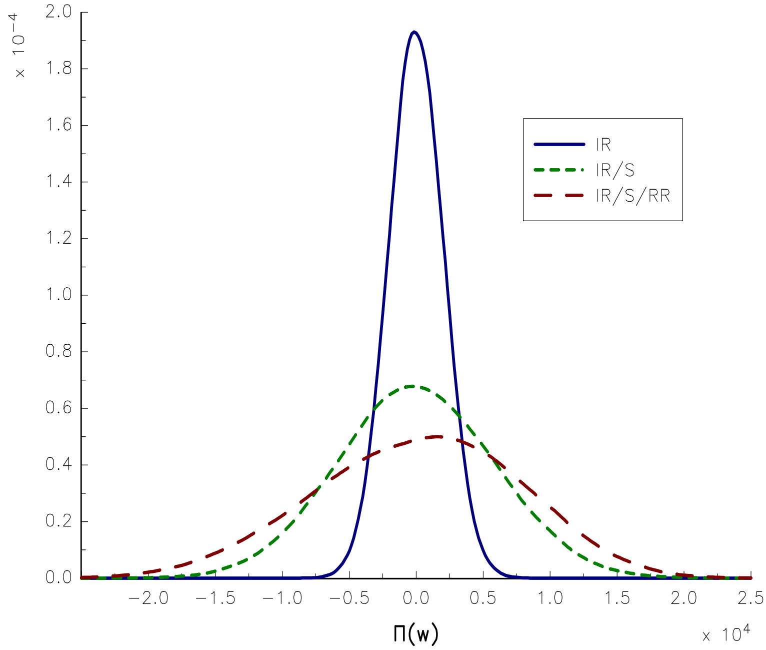

FIGURE 2.16: Probability density function of the daily P&L with credit risk

2.2.5.1 Identification of risk factors

Let us consider an example of a portfolio containing w_S stocks and w_C call options on this stock. We note S_t and $\boldsymbol{\mathcal{C}}_t$ the stock and option prices at time t. The P&L for the holding period h is equal to:

$$\Pi(w) = w_S(S_{t+h} - S_t) + w_C(\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t)$$

If we use asset returns as risk factors, we get:

$$\Pi(w) = w_S S_t R_{S,t+h} + w_C \boldsymbol{\mathcal{C}}_t R_{C,t+h}$$

where $R_{S,t+h}$ and $R_{C,t+h}$ are the returns of the stock and the option for the period $[t, t+h]$. In this approach, we identify two risk factors. The problem is that the option price $\boldsymbol{\mathcal{C}}_t$ is a non-linear function of the underlying price S_t:

$$\boldsymbol{\mathcal{C}}_t = f_C(S_t)$$

This implies that:

$$\begin{aligned} \Pi(w) &= w_S S_t R_{S,t+h} + w_C(f_C(S_{t+h}) - \boldsymbol{\mathcal{C}}_t) \\ &= w_S S_t R_{S,t+h} + w_C(f_C(S_t(1 + R_{S,t+h})) - \boldsymbol{\mathcal{C}}_t) \end{aligned}$$

The P&L depends then on a single risk factor R_S. We notice that we can write the return of the option price as a non-linear function of the stock return:

$$R_{C,t+h} = \frac{f_C(S_t(1 + R_{S,t+h})) - \boldsymbol{\mathcal{C}}_t}{\boldsymbol{\mathcal{C}}_t}$$

The problem is that the probability distribution of R_C is non-stationary and depends on the value of S_t. Therefore, the risk factors cannot be the random vector (R_S, R_C) because they require too complex modeling.

Risk factors are often explicit in primary financial assets (equities, bonds, currencies), which is not the case with derivatives. Previously, we have identified the return of the underlying asset as a risk factor for the call option. In the Black-Scholes model, the price of the call option is given by:

$$C_{\mathrm{BS}}\left(S_t, K, \Sigma_t, T, b_t, r_t\right) = S_t e^{(b_t - r_t)\tau}\Phi\left(d_1\right) - K e^{-r_t\tau}\Phi\left(d_2\right) \tag{2.10}$$

where S_t is the current price of the underlying asset, K is the option strike, Σ_t is the volatility parameter, T is the maturity date, b_t is the cost-of-carry[76] and r_t is the interest rate. The parameter $\tau = T - t$ is the time to maturity whereas the coefficients d_1 and d_2 are defined as follows:

$$\begin{aligned} d_1 &= \frac{1}{\Sigma_t\sqrt{\tau}}\left(\ln\frac{S_t}{K} + b_t\tau\right) + \frac{1}{2}\Sigma_t\sqrt{\tau} \\ d_2 &= d_1 - \Sigma_t\sqrt{\tau} \end{aligned}$$

We can then write the option price as follows:

$$\boldsymbol{\mathcal{C}}_t = f_{\mathrm{BS}}\left(\theta_{\mathrm{contract}};\theta\right)$$

where $\theta_{\mathrm{contract}}$ are the parameters of the contract (strike K and maturity T) and θ are the other parameters than can be objective as the underlying price S_t or subjective as the volatility Σ_t. Any one of these parameters θ may serve as risk factors:

- S_t is obviously a risk factor;
- if Σ_t is not constant, the option price may be sensitive to the volatility risk;
- the option may be impacted by changes in the interest rate or the cost-of-carry.

The risk manager faces here a big issue, because the risk measure will depend on the choice of the risk factors[77]. A typical example is the volatility parameter. We observe a difference between the historical volatility $\hat{\sigma}_t$ and the Black-Scholes volatility Σ_t. Because this implied volatility is not a market price, its value will depend on the option model and the assumptions which are required to calibrate it. For instance, it will be different if we use a stochastic volatility model or a local volatility model. Even if two banks use the same model, they will certainly obtain two different values of the implied volatility, because there is little possibility that they exactly follow the same calibration procedure.

With the underlying asset S_t, the implied volatility Σ_t is the most important risk factor, but other risk factors may be determinant. They concern the dividend risk for equity options, the yield curve risk for interest rate options, the term structure for commodity options or the correlation risk for basket options. In fact, the choice of risk factors is not always obvious because it is driven by the pricing model and the characteristics of the option. We will take a closer look at this point in Chapter 9.

2.2.5.2 Methods to calculate VaR and ES risk measures

The method of full pricing To calculate the value-at-risk or the expected shortfall of option portfolios, we use the same approaches as previously. The difference with primary

[76] The cost-of-carry depends on the underlying asset. We have $b_t = r_t$ for non-dividend stocks and total return indices, $b_t = r_t - d_t$ for stocks paying a continuous dividend yield d_t, $b_t = 0$ for forward and futures contracts and $b_t = r_t - r_t^\star$ for foreign exchange options where $r_t^\star$ is the foreign interest rate.

[77] We encounter the same difficulties for pricing and hedging purposes.

financial assets comes from the pricing function which is non-linear and more complex. In the case of historical and Monte Carlo methods, the P&L of the s^{th} scenario has the following expression:

$$\Pi_s(w) = g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w) - P_t(w)$$

In the case of the introducing example, the P&L becomes then:

$$\Pi_s(w) = \begin{cases} w_S S_t R_s + w_C \left(f_C \left(S_t \left(1 + R_s\right); \Sigma_t\right) - \mathcal{C}_t\right) & \text{with one risk factor} \\ w_S S_t R_s + w_C \left(f_C \left(S_t \left(1 + R_s\right), \Sigma_s\right) - \mathcal{C}_t\right) & \text{with two risk factors} \end{cases}$$

where R_s and Σ_s are the asset return and the implied volatility generated by the s^{th} scenario. If we assume that the interest rate and the cost-of-carry are constant, the pricing function is:

$$f_C(S; \Sigma) = C_{\text{BS}}(S, K, \Sigma, T - h, b_t, r_t)$$

and we notice that the remaining maturity of the option decreases by h days. In the model with two risk factors, we have to simulate the underlying price and the implied volatility. For the single factor model, we use the current implied volatility Σ_t instead of the simulated value Σ_s.

Example 18 *We consider a long position on* 100 *call options with strike* $K = 100$*. The value of the call option is* \$4.14*, the residual maturity*[78] *is* 52 *days and the current price of the underlying asset is* \$100*. We assume that* $\Sigma_t = 20\%$ *and* $b_t = r_t = 5\%$*. The objective is to calculate the daily value-at-risk with a* 99% *confidence level and the daily expected shortfall with a* 97.5% *confidence level. For that, we consider* 250 *historical scenarios, whose first nine values are the following:*

s	1	2	3	4	5	6	7	8	9
R_s	−1.93	−0.69	−0.71	−0.73	1.22	1.01	1.04	1.08	−1.61
$\Delta\Sigma_s$	−4.42	−1.32	−3.04	2.88	−0.13	−0.08	1.29	2.93	0.85

TABLE 2.12: Daily P&L of the long position on the call option when the risk factor is the underlying price

s	R_s (in %)	S_{t+h}	$\mathcal{C}_{t+h}$	Π_s
1	−1.93	98.07	3.09	−104.69
2	−0.69	99.31	3.72	−42.16
3	−0.71	99.29	3.71	−43.22
4	−0.73	99.27	3.70	−44.28
5	1.22	101.22	4.81	67.46
6	1.01	101.01	4.68	54.64
7	1.04	101.04	4.70	56.46
8	1.08	101.08	4.73	58.89
9	−1.61	98.39	3.25	−89.22

Using the price and the characteristics of the call option, we can show that the implied volatility Σ_t is equal to 19.99% (rounded to 20%). We first consider the case of the single risk factor. In Table 2.12, we show the values of the P&L for the first nine scenarios. As an illustration, we provide the detailed calculation for the first scenario. The asset return R_s

[78] We assume that there are 252 trading days per year.

is equal to -1.93%, thus implying that the asset price S_{t+h} is equal to $100 \times (1 - 1.93\%) = 98.07$. The residual maturity τ is equal to $51/252$ years. It follows that:

$$\begin{aligned} d_1 &= \frac{1}{20\% \times \sqrt{51/252}} \left(\ln \frac{98.07}{100} + 5\% \times \frac{51}{252} \right) + \frac{1}{2} \times 20\% \times \sqrt{\frac{51}{252}} \\ &= -0.0592 \end{aligned}$$

and:

$$d_2 = -0.0592 - 20\% \times \sqrt{\frac{51}{252}} = -0.1491$$

We deduce that:

$$\begin{aligned} \boldsymbol{\mathcal{C}}_{t+h} &= 98.07 \times e^{(5\%-5\%)\frac{51}{252}} \times \Phi(-0.0592) - 100 \times e^{5\% \times \frac{51}{252}} \times \Phi(-0.1491) \\ &= 98.07 \times 1.00 \times 0.4764 - 100 \times 1.01 \times 0.4407 \\ &= 3.093 \end{aligned}$$

The simulated P&L for the first historical scenario is then equal to:

$$\Pi_s = 100 \times (3.093 - 4.14) = -104.69$$

Based on the 250 historical scenarios, the 99% value-at-risk is equal to \$154.79, whereas the 97.5% expected shortfall is equal to \$150.04.

Remark 13 *In Figure 2.17, we illustrate that the option return R_C is not a new risk factor. We plot R_S against R_C for the 250 historical scenarios. The points are on the curve of the Black-Scholes formula. The correlation between the two returns is equal to 99.78%, which indicates that R_S and R_C are highly dependent. However, this dependence is non-linear for large positive or negative asset returns. The figure shows also the leverage effect of the call option, because R_C is not of the same order of magnitude as R_S. This illustrates the non-linear characteristic of options. A linear position with a volatility equal to 20% implies a daily VaR around 3%. In our example, the VaR is equal to 37.4% of the portfolio value, which corresponds to a linear exposure in a stock with a volatility of 259%!*

Let us consider the case with two risk factors when the implied volatility changes from t to $t + h$. We assume that the absolute variation of the implied volatility is the right risk factor to model the future implied volatility. It follows that:

$$\Sigma_{t+h} = \Sigma_t + \Delta\Sigma_s$$

In Table 2.13, we indicate the value taken by Σ_{t+h} for the first nine scenarios. This allows us to price the call option and deduce the P&L. For instance, the call option becomes[79] \$2.32 instead of \$3.09 for $s = 1$ because the implied volatility has decreased. Finally, the 99% value-at-risk is equal to \$181.70 and is larger than the previous one due to the second risk factor[80].

The method of sensitivities The previous approach is called *full pricing*, because it consists in re-pricing the option. In the method based on the Greek coefficients, the idea is to approximate the change in the option price by the Taylor expansion. For instance, we define the delta approach as follows[81]:

$$\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t \simeq \boldsymbol{\Delta}_t (S_{t+h} - S_t)$$

[79] We have $d_1 = -0.0986$, $d_2 = -0.1687$, $\Phi(d_1) = 0.4607$, $\Phi(d_2) = 0.4330$ and $\boldsymbol{\mathcal{C}}_{t+h} = 2.318$.

[80] For the expected shortfall, we have $\mathrm{ES}_{97.5\%}(w; \text{one day}) = \$172.09$.

[81] We write the call price as the function $C_{\mathrm{BS}}(S_t, \Sigma_t, T)$.

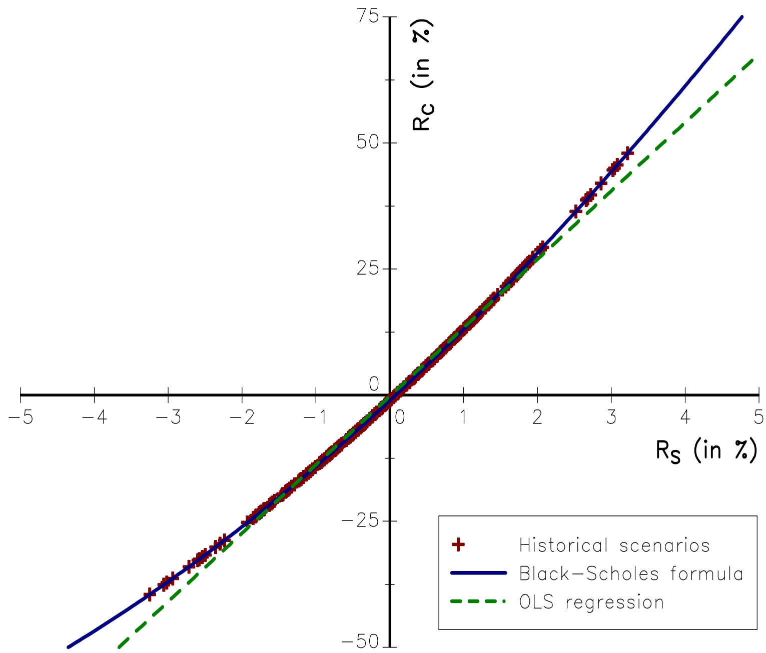

FIGURE 2.17: Relationship between the asset return R_S and the option return R_C

where $\boldsymbol{\Delta}_t$ is the option delta:

$$\boldsymbol{\Delta}_t = \frac{\partial\, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)}{\partial\, S_t}$$

This approximation consists in replacing the non-linear exposure by a linear exposure with respect to the underlying price. As noted by Duffie and Pan (1997), this approach is not satisfactory because it is not accurate for large changes in the underlying price that are the most useful scenarios for calculating the risk measure. The delta approach may be implemented for the three VaR/ES methods. For instance, the Gaussian VaR of the call option is:

$$\mathrm{VaR}_\alpha\left(w; h\right) = \Phi^{-1}\left(\alpha\right) \times \left|\boldsymbol{\Delta}_t\right| \times S_t \times \sigma\left(R_{S,t+h}\right)$$

TABLE 2.13: Daily P&L of the long position on the call option when the risk factors are the underlying price and the implied volatility

s	R_s (in %)	S_{t+h}	$\Delta\Sigma_s$ (in %)	Σ_{t+h}	$\mathcal{C}_{t+h}$	Π_s
1	−1.93	98.07	−4.42	15.58	2.32	−182.25
2	−0.69	99.31	−1.32	18.68	3.48	−65.61
3	−0.71	99.29	−3.04	16.96	3.17	−97.23
4	−0.73	99.27	2.88	22.88	4.21	6.87
5	1.22	101.22	−0.13	19.87	4.79	65.20
6	1.01	101.01	−0.08	19.92	4.67	53.24
7	1.04	101.04	1.29	21.29	4.93	79.03
8	1.08	101.08	2.93	22.93	5.24	110.21
9	−1.61	98.39	0.85	20.85	3.40	−74.21

whereas the Gaussian ES of the call option is:

$$\mathrm{ES}_{\alpha}(w;h) = \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \times |\boldsymbol{\Delta}_t| \times S_t \times \sigma\left(R_{S,t+h}\right)$$

If we consider the introductory example, we have:

$$\begin{aligned}
\Pi(w) &= w_S\left(S_{t+h}-S_t\right)+w_C\left(\boldsymbol{\mathcal{C}}_{t+h}-\boldsymbol{\mathcal{C}}_t\right) \\
&\simeq \left(w_S+w_C\boldsymbol{\Delta}_t\right)\left(S_{t+h}-S_t\right) \\
&= \left(w_S+w_C\boldsymbol{\Delta}_t\right)S_t R_{S,t+h}
\end{aligned}$$

With the delta approach, we aggregate the risk by netting the different delta exposures[82]. In particular, the portfolio is delta neutral if the net exposure is zero:

$$w_S + w_C\boldsymbol{\Delta}_t = 0 \Leftrightarrow w_S = -w_C\boldsymbol{\Delta}_t$$

With the delta approach, the VaR/ES of delta neutral portfolios is then equal to zero.

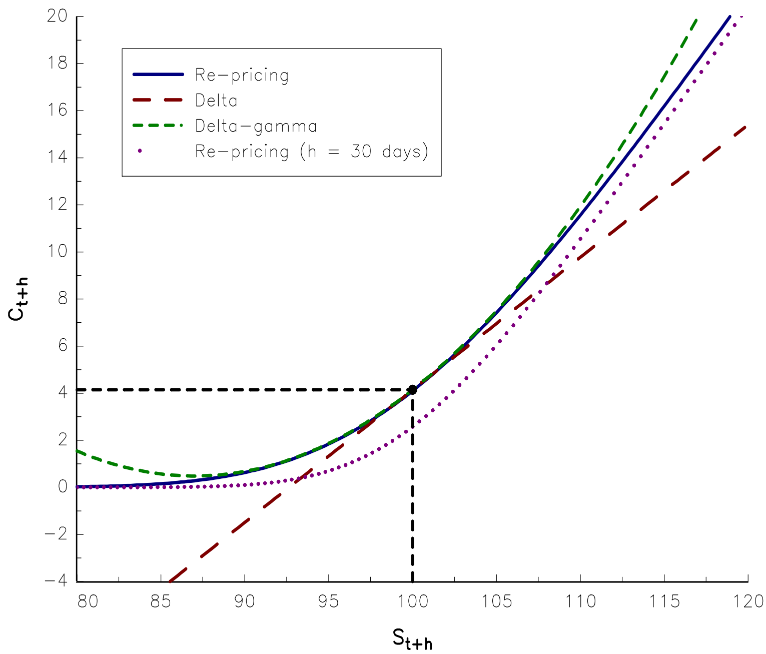

FIGURE 2.18: Approximation of the option price with the Greek coefficients

To overcome this drawback, we can use the second-order approximation or the delta-gamma approach:

$$\boldsymbol{\mathcal{C}}_{t+h}-\boldsymbol{\mathcal{C}}_t \simeq \boldsymbol{\Delta}_t\left(S_{t+h}-S_t\right)+\frac{1}{2}\boldsymbol{\Gamma}_t\left(S_{t+h}-S_t\right)^2$$

where $\boldsymbol{\Gamma}_t$ is the option gamma:

$$\boldsymbol{\Gamma}_t = \frac{\partial^2 C_{\mathrm{BS}}\left(S_t,\Sigma_t,T\right)}{\partial\, S_t^2}$$

[82]A long (or short) position on the underlying asset is equivalent to $\boldsymbol{\Delta}_t = 1$ (or $\boldsymbol{\Delta}_t = -1$).

In Figure 2.18, we compare the two Taylor expansions with the re-pricing method when h is equal to one trading day. We observe that the delta approach provides a bad approximation if the future price S_{t+h} is far from the current price S_t. The inclusion of the gamma helps to correct the pricing error. However, if the time period h is high, the two approximations may be inaccurate even in the neighborhood de S_t (see the case $h = 30$ days in Figure 2.18). It is therefore important to take into account the time or maturity effect:

$$\mathcal{C}_{t+h} - \mathcal{C}_t \simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right) + \frac{1}{2}\boldsymbol{\Gamma}_t \left(S_{t+h} - S_t\right)^2 + \boldsymbol{\Theta}_t h$$

where $\boldsymbol{\Theta}_t = \partial_t \, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)$ is the option theta[83].

The Taylor expansion can be generalized to a set of risk factors $\mathcal{F}_t = \left(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t}\right)$:

$$\begin{aligned} \mathcal{C}_{t+h} - \mathcal{C}_t &\simeq \sum_{j=1}^{m} \frac{\partial\, \mathcal{C}_t}{\partial\, \mathcal{F}_{j,t}} \left(\mathcal{F}_{j,t+h} - \mathcal{F}_{j,t}\right) + \\ & \quad \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial^2\, \mathcal{C}_t}{\partial\, \mathcal{F}_{j,t}\,\partial\, \mathcal{F}_{k,t}} \left(\mathcal{F}_{j,t+h} - \mathcal{F}_{j,t}\right)\left(\mathcal{F}_{k,t+h} - \mathcal{F}_{k,t}\right) \end{aligned}$$

The delta-gamma-theta approach consists in considering the underlying price and the maturity as risk factors. If we add the implied volatility as a new risk factor, we obtain:

$$\begin{aligned} \mathcal{C}_{t+h} - \mathcal{C}_t &\simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right) + \frac{1}{2}\boldsymbol{\Gamma}_t \left(S_{t+h} - S_t\right)^2 + \boldsymbol{\Theta}_t h + \\ & \quad \boldsymbol{\upsilon}_t \left(\Sigma_{t+h} - \Sigma_t\right) \end{aligned}$$

where $\boldsymbol{\upsilon}_t = \partial_{\Sigma_t} C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)$ is the option vega. Here, we have considered that only the second derivative of $\mathcal{C}_t$ with respect to S_t is significant, but we could also include the vanna or volga effect[84].

In the case of the call option, the Black-Scholes sensitivities are equal to:

$$\begin{aligned} \boldsymbol{\Delta}_t &= e^{(b_t - r_t)\tau}\Phi\left(d_1\right) \\ \boldsymbol{\Gamma}_t &= \frac{e^{(b_t - r_t)\tau}\phi\left(d_1\right)}{S_t \Sigma_t \sqrt{\tau}} \\ \boldsymbol{\Theta}_t &= -r_t K e^{-r_t\tau}\Phi\left(d_2\right) - \frac{1}{2\sqrt{\tau}} S_t \Sigma_t e^{(b_t - r_t)\tau}\phi\left(d_1\right) - \\ & \quad \left(b_t - r_t\right) S_t e^{(b_t - r_t)\tau}\Phi\left(d_1\right) \\ \boldsymbol{\upsilon}_t &= e^{(b_t - r_t)\tau} S_t \sqrt{\tau}\phi\left(d_1\right) \end{aligned}$$

If we consider again Example 18 on page 95, we obtain[85] $\boldsymbol{\Delta}_t = 0.5632$, $\boldsymbol{\Gamma}_t = 0.0434$, $\boldsymbol{\Theta}_t = -11.2808$ and $\boldsymbol{\upsilon}_t = 17.8946$. In Table 2.14, we have reported the approximated P&Ls for the first nine scenarios and the one-factor model. The fourth column indicates the P&L obtained by the full pricing method, which were already reported in Table 2.12. $\Pi_s^{\boldsymbol{\Delta}}(w)$, $\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}(w)$ and $\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}(w)$ correspond respectively to delta, delta-gamma, delta-gamma-theta approaches. For example, we have $\Pi_1^{\boldsymbol{\Delta}}(w) = 100 \times 0.5632 \times (98.07 - 100) = -108.69$, $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}(w) = -108.69 + 100 \times \frac{1}{2} \times 0.0434 \times (98.07 - 100)^2 = -100.61$ and $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}(w) =$

[83] An equivalent formula is $\boldsymbol{\Theta}_t = -\partial_T \, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right) = -\partial_\tau \, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)$ because the maturity T (or the time to maturity τ) is moving in the opposite way with respect to the time t.

[84] The vanna coefficient corresponds to the cross-derivative of $\mathcal{C}_t$ with respect to S_t and Σ_t whereas the volga effect is the second derivative of $\mathcal{C}_t$ with respect to Σ_t.

[85] We have $d_1 = 0.1590$, $\Phi\left(d_1\right) = 0.5632$, $\phi\left(d_1\right) = 0.3939$, $d_2 = 0.0681$ and $\Phi\left(d_2\right) = 0.5272$.

$-100.61 - 11.2808 \times 1/252 = -105.09$. We notice that we obtain a good approximation with the delta, but it is more accurate to combine delta, gamma and theta sensibilities. Finally, the 99% VaRs for a one-day holding period are \$171.20 and \$151.16 and \$155.64. This is the delta-gamma-theta approach which gives the closest result[86]. If the set of risk factors includes the implied volatility, we obtain the results in Table 2.15. We notice that the vega effect is very significant (fifth column). As an illustration, we have $\Pi_1^{\upsilon}(w) = 100 \times 17.8946 \times (15.58\% - 20\%) = -79.09$, implying that the volatility risk explains 43.4% of the loss of \$182.25 for the first scenario. Finally, the VaR is equal to \$183.76 with the delta-gamma-theta-vega approach whereas we found previously that it was equal to \$181.70 with the full pricing method.

TABLE 2.14: Calculation of the P&L based on the Greek sensitivities

s	R_s (in %)	S_{t+h}	Π_s	$\Pi_s^{\boldsymbol{\Delta}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}$
1	−1.93	98.07	−104.69	−108.69	−100.61	−105.09
2	−0.69	99.31	−42.16	−38.86	−37.83	42.30
3	−0.71	99.29	−43.22	−39.98	−38.89	−43.37
4	−0.73	99.27	−44.28	−41.11	−39.96	−44.43
5	1.22	101.22	67.46	68.71	71.93	67.46
6	1.01	101.01	54.64	56.88	59.09	54.61
7	1.04	101.04	56.46	58.57	60.91	56.44
8	1.08	101.08	58.89	60.82	63.35	58.87
9	−1.61	98.39	−89.22	−90.67	−85.05	−89.53
$\text{VaR}_{99\%}$ (w; one day)			154.79	171.20	151.16	155.64
$\text{ES}_{97.5\%}$ (w; one day)			150.04	165.10	146.37	150.84

TABLE 2.15: Calculation of the P&L using the vega coefficient

s	S_{t+h}	Σ_{t+h}	Π_s	$\Pi_s^{\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}+\boldsymbol{\upsilon}}$
1	98.07	15.58	−182.25	−79.09	−187.78	−179.71	−184.19
2	99.31	18.68	−65.61	−23.62	−62.48	−61.45	−65.92
3	99.29	16.96	−97.23	−54.40	−94.38	−93.29	−97.77
4	99.27	22.88	6.87	51.54	10.43	11.58	7.10
5	101.22	19.87	65.20	−2.33	66.38	69.61	65.13
6	101.01	19.92	53.24	−1.43	55.45	57.66	53.18
7	101.04	21.29	79.03	23.08	81.65	84.00	79.52
8	101.08	22.93	110.21	52.43	113.25	115.78	111.30
9	98.39	20.85	−74.21	15.21	−75.46	−69.84	−74.32
$\text{VaR}_{99\%}$ (w; one day)			181.70	77.57	190.77	179.29	183.76
$\text{ES}_{97.5\%}$ (w; one day)			172.09	73.90	184.90	169.34	173.81

Remark 14 *We do not present here the non-linear quadratic VaR, which consists in computing the VaR of option portfolios with the Cornish-Fisher expansion (Zangari, 1996; Britten-Jones and Schaefer, 1999). It is called 'quadratic' because it uses the delta-gamma approximation and requires calculating the moments of the quadratic form* $(S_{t+h} - S_t)^2$. *The treatment of this approach is left as Exercise 2.4.8 on page 123.*

[86] We found previously that the VaR was equal to \$154.79 with the full pricing method.

The hybrid method On the one hand, the full pricing method has the advantage to be accurate, but also the drawback to be time-consuming because it performs a complete revaluation of the portfolio for each scenario. On the other hand, the method based on the sensitivities is less accurate, but also faster than the re-pricing approach. Indeed, the Greek coefficients are calculated once and for all, and their values do not depend on the scenario. The hybrid method consists of combining the two approaches:

1. we first calculate the P&L for each (historical or simulated) scenario with the method based on the sensitivities;
2. we then identify the worst scenarios;
3. we finally revalue these worst scenarios by using the full pricing method.

The underlying idea is to consider the faster approach to locate the value-at-risk, and then to use the most accurate approach to calculate the right value.

TABLE 2.16: The 10 worst scenarios identified by the hybrid method

i	Full pricing		Greeks					
			$\mathbf{\Delta}-\mathbf{\Gamma}-\mathbf{\Theta}-\boldsymbol{\upsilon}$		$\mathbf{\Delta}-\mathbf{\Theta}$		$\mathbf{\Delta}-\mathbf{\Theta}-\boldsymbol{\upsilon}$	
	s	Π_s	s	Π_s	s	Π_s	s	Π_s
1	100	−183.86	100	−186.15	182	−187.50	134	−202.08
2	1	−182.25	1	−184.19	169	−176.80	100	−198.22
3	134	−181.15	134	−183.34	27	−174.55	1	−192.26
4	27	−163.01	27	−164.26	134	−170.05	169	−184.32
5	169	−162.82	169	−164.02	69	−157.66	27	−184.04
6	194	−159.46	194	−160.93	108	−150.90	194	−175.36
7	49	−150.25	49	−151.43	194	−149.77	49	−165.41
8	245	−145.43	245	−146.57	49	−147.52	182	−164.96
9	182	−142.21	182	−142.06	186	−145.27	245	−153.37
10	79	−135.55	79	−136.52	100	−137.38	69	−150.68

In Table 2.16, we consider the previous example with the implied volatility as a risk factor. We have reported the worst scenarios corresponding to the order statistic $i : n_S$ with $i \leq 10$. In the case of the full pricing method, the five worst scenarios are the 100^{th}, 1^{st}, 134^{th}, 27^{th} and 169^{th}. This implies that the hybrid method will give the right result if it is able to select the 100^{th}, 1^{st} and 134^{th} scenarios to compute the value-at-risk which corresponds to the average of the second and third order statistics. If we consider the $\mathbf{\Delta}-\mathbf{\Gamma}-\mathbf{\Theta}-\boldsymbol{\upsilon}$ approximation, we identify the same ten worst scenarios. It is perfectly normal, as it is easy to price an European call option. It will not be the case with exotic options, because the approximation may not be accurate. For instance, if we consider our example with the $\mathbf{\Delta}-\mathbf{\Theta}$ approximation, the five worst scenarios becomes the 182^{th}, 169^{st}, 27^{th}, 134^{th} and 69^{th}. If we revaluate these 5 worst scenarios, the 99% value-at-risk is equal to:

$$\mathrm{VaR}_{99\%}\left(w;\text{one day}\right) = \frac{1}{2}\left(163.01 + 162.82\right) = \$162.92$$

which is a result far from the value of \$180.70 found with the full pricing method. With the 10 worst scenarios, we obtain:

$$\mathrm{VaR}_{99\%}\left(w;\text{one day}\right) = \frac{1}{2}\left(181.15 + 163.01\right) = \$172.08$$

Once again, we do not find the exact value, because the $\boldsymbol{\Delta} - \boldsymbol{\Theta}$ approximation fails to detect the first scenario among the 10 worst scenarios. This problem vanishes with the $\boldsymbol{\Delta} - \boldsymbol{\Theta} - \boldsymbol{\upsilon}$ approximation, even if it gives a ranking different than this obtained with the full pricing method. In practice, the hybrid approach is widespread and professionals generally use the identification method with 10 worst scenarios[87].

2.2.5.3 Backtesting

When we consider a model to price a product, the valuation is known as '*mark-to-model*' and requires more attention than the mark-to-market approach. In this last case, the simulated P&L is the difference between the mark-to-model value at time $t+1$ and the current mark-to-market value:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

At time $t+1$, the realized P&L is the difference between two mark-to-market values:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-market}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

For exotic options and OTC derivatives, we don't have market prices and the portfolio is valuated using the mark-to-model approach. This means that the simulated P&L is the difference between two mark-to-model values:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

and the realized P&L is also the difference between two mark-to-model values:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

In the case of the mark-to-model valuation, we see the relevance of the pricing model in terms of risk management. Indeed, if the pricing model is wrong, the value-at-risk is wrong too and this cannot be detected by the backtesting procedure, which has little signification. This is why the supervisory authority places great importance on model risk.

2.2.5.4 Model risk

Model risk cannot be summarized in a unique definition due to its complexity. For instance, Derman (1996, 2001) considers six types of model risk (inapplicability of modeling, incorrect model, incorrect solutions, badly approximated solution, bugs and unstable data). Rebonato (2001) defines model risk as "*the risk of a significant difference between the mark-to-model value of an instrument, and the price at which the same instrument is revealed to have traded in the market*". According to Morini (2001), these two approaches are different. For Riccardo Rebonato, there is not a true value of an instrument before it will be traded on the market. Model risk can therefore be measured by selling the instrument in the market. For Emanuel Derman, an instrument has an intrinsic true value, but it is unknown. The proposition of Rebonato is certainly the right way to define model risk, but it does not help to measure model risk from an ex-ante point of view. Moreover, this approach does

[87] Its application is less frequent than in the past because computational times have dramatically decreased with the evolution of technology, in particular the development of parallel computing.

not distinguish between model risk and liquidity risk. The conception of Derman is more adapted to manage model risk and calibrate the associated provisions. This is the approach that has been adopted by banks and regulators. Nevertheless, the multifaceted nature of this approach induces very different implementations across banks, because it appears as a catalogue with an infinite number of rules.

We consider a classification with four main types of model risk:

1. the operational risk;
2. the parameter risk;
3. the risk of mis-specification;
4. the hedging risk.

The operational risk is the risk associated to the implementation of the pricer. It concerns programming mistakes or bugs, but also mathematical errors in closed-form formulas, approximations or numerical methods. A typical example is the use of a numerical scheme for solving a partial differential equation. The accuracy of the option price and the Greek coefficients will depend on the specification of the numerical algorithm (explicit, implicit or mixed scheme) and the discretization parameters (time and space steps). Another example is the choice of the Monte Carlo method and the number of simulations.

The parameter risk is the risk associated to the input parameters, in particular those which are difficult to estimate. A wrong value of one parameter can lead to a mis-pricing, even though the model is right and well implemented. In this context, the question of available and reliable data is a key issue. It is particularly true when the parameters are unobservable and are based on an expert's opinion. A typical example concerns the value of correlations in multi-asset options. Even if there is no problem with data, some parameters are indirectly related to market data via a calibration set. In this case, they may change with the specification of the calibration set. For instance, the pricing of exotic interest rate options is generally based on parameters calibrated from prices of plain vanilla instruments (caplets and swaptions). The analysis of parameter risk consists then of measuring the impact of parameter changes on the price and the hedging portfolio of the exotic option.

The risk of mis-specification is the risk associated to the mathematical model, because it may not include all risk factors, the dynamics of the risk factors is not adequate or the dependence between them is not well defined. It is generally easy to highlight this risk, because various models calibrated with the same set of instruments can produce different prices for the same exotic option. The big issue is to define what is the least bad model. For example, in the case of equity options, we have the choice between many models: Black-Scholes, local volatility, Heston model, other stochastic volatility models, jump-diffusion, etc. In practice, the frontier between the risk of parameters and the risk of mis-specification may be unclear as shown by the seminal work of uncertainty on pricing and hedging by Avellaneda *et al.* (1995). Moreover, a model which appears to be good for pricing may not be well adapted for risk management. This explains that the trader and the risk manager can use sometimes two different models for the same option payoff.

The hedging risk is the risk associated to the trading management of the option portfolio. The sales margin corresponds to the difference between the transaction price and the mark-to-model price. The sales margin is calculated at the inception date of the transaction. To freeze the margin, we have to hedge the option. The mark-to-model value is then transferred to the option trader and represents the hedging cost. We face here the risk that the realized hedging cost will be larger than the mark-to-model price. A typical example is a put option, which has a negative delta. The hedging portfolio corresponds then to a short selling on the underlying asset. Sometimes, this short position may be difficult to implement (e.g. a ban on short selling) or may be very costly (e.g. due to a change in the bank funding condition). Some events may also generate a rebalancing risk. The most famous example is certainly the hedge fund crisis in October 2008, which has imposed redemption restrictions or gates. This caused difficulties to traders, who managed call options on hedge funds and were unable to reduce their deltas at this time. The hedging risk does not only concern the feasibility of the hedging implementation, but also its adequacy with the model. As an illustration, we suppose that we use a stochastic volatility model for an option, which is sensitive to the vanna coefficient. The risk manager can then decide to use this model for measuring the value-at-risk, but the trader can also prefer to implement a Black-Scholes hedging portfolio[88]. This is not a problem that the risk manager uses a different model than the trader if the model risk only includes the first three categories. However, it will be a problem if it also concerns hedging risk.

In the Basel III framework, the Basel Committee highlights the role of the model validation team:

> "*A distinct unit of the bank that is separate from the unit that designs and implements the internal models must conduct the initial and ongoing validation of all internal models used to determine market risk capital requirements. The model validation unit must validate all internal models used for purposes of the IMA on at least an annual basis. [...] Banks must maintain a process to ensure that their internal models have been adequately validated by suitably qualified parties independent of the model development process to ensure that each model is conceptually sound and adequately reflects all material risks. Model validation must be conducted both when the model is initially developed and when any significant changes are made to the model*" (BCBS, 2019, pages 68-69).

Therefore, model risk justifies that model validation is an integral part of the risk management process for exotic options. The tasks of a model validation team are multiple and concern reviewing the programming code, checking mathematical formulas and numerical approximations, validating market data, testing the calibration stability, challenging the pricer with alternative models, proposing provision buffers, etc. This team generally operates at the earliest stages of the pricer development (or when the pricer changes), whereas the risk manager is involved to follow the product on a daily basis. In Chapter 9, we present the different tools available for the model validation unit in order to assess the robustness of risk measures that are based on mark-to-model prices.

Remark 15 *It is a mistake to think that model risk is an operational risk. Model risk is intrinsically a market risk. Indeed, it exists because exotic options are difficult to price and hedge, implying that commercial risk is high. This explains that sales margins are larger than for vanilla options and implicitly include model risk, which is therefore inherent to the business of exotic derivatives.*

2.3 Risk allocation

Measuring the risk of a portfolio is a first step to manage it. In particular, a risk measure is a single number that is not very helpful for understanding the sources of the portfolio risk.

[88] There may be many reasons for implementing more simple hedging portfolios: the trader may be more confident in the robustness, there is no market instrument to replicate the vanna position, etc.

To go further, we must define precisely the notion of risk contribution in order to propose risk allocation principles.

Let us consider two trading desks A and B, whose risk measure is respectively $\mathcal{R}(w_A)$ and $\mathcal{R}(w_B)$. At the global level, the risk measure is equal to $\mathcal{R}(w_{A+B})$. The question is then how to allocate $\mathcal{R}(w_{A+B})$ to the trading desks A and B:

$$\mathcal{R}(w_{A+B}) = \mathcal{RC}_A(w_{A+B}) + \mathcal{RC}_B(w_{A+B})$$

There is no reason that $\mathcal{RC}_A(w_{A+B}) = \mathcal{R}(w_A)$ and $\mathcal{RC}_B(w_{A+B}) = \mathcal{R}(w_B)$ except if there is no diversification. This question is an important issue for the bank because risk allocation means capital allocation:

$$\mathcal{K}(w_{A+B}) = \mathcal{K}_A(w_{A+B}) + \mathcal{K}_B(w_{A+B})$$

Capital allocation is not neutral, because it will impact the profitability of business units that compose the bank.

Remark 16 *This section is based on Chapter 2 of the book of Roncalli (2013).*

2.3.1 Euler allocation principle

According to Litterman (1996), risk allocation consists in decomposing the risk portfolio into a sum of risk contributions by sub-portfolios (assets, trading desks, etc.). The concept of risk contribution is key in identifying concentrations and understanding the risk profile of the portfolio, and there are different methods for defining them. As illustrated by Denault (2001), some methods are more pertinent than others and the Euler principle is certainly the most used and accepted one.

We decompose the P&L as follows:

$$\Pi = \sum_{i=1}^{n} \Pi_i$$

where Π_i is the P&L of the i^{th} sub-portfolio. We note $\mathcal{R}(\Pi)$ the risk measure associated with the P&L[89]. Let us consider the risk-adjusted performance measure (RAPM) defined by[90]:

$$\mathrm{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathcal{R}(\Pi)}$$

Tasche (2008) considers the portfolio-related RAPM of the i^{th} sub-portfolio defined by:

$$\mathrm{RAPM}(\Pi_i \mid \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathcal{R}(\Pi_i \mid \Pi)}$$

Based on the notion of RAPM, Tasche (2008) states two properties of risk contributions that are desirable from an economic point of view:

1. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ to portfolio-wide risk $\mathcal{R}(\Pi)$ satisfy the full allocation property if:
$$\sum_{i=1}^{n} \mathcal{R}(\Pi_i \mid \Pi) = \mathcal{R}(\Pi) \tag{2.11}$$

[89] We recall that $\mathcal{R}(\Pi) = \mathcal{R}(-L)$.

[90] This concept is close to the RAROC measure introduced by Banker Trust (see page 2).

2. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ are RAPM compatible if there are some $\varepsilon_i > 0$ such that[91]:

$$\text{RAPM}(\Pi_i \mid \Pi) > \text{RAPM}(\Pi) \Rightarrow \text{RAPM}(\Pi + h\Pi_i) > \text{RAPM}(\Pi) \tag{2.12}$$

for all $0 < h < \varepsilon_i$.

Tasche (2008) shows therefore that if there are risk contributions that are RAPM compatible in the sense of the two previous properties (2.11) and (2.12), then $\mathcal{R}(\Pi_i \mid \Pi)$ is uniquely determined as:

$$\mathcal{R}(\Pi_i \mid \Pi) = \frac{\mathrm{d}}{\mathrm{d}h}\mathcal{R}(\Pi + h\Pi_i)\bigg|_{h=0} \tag{2.13}$$

and the risk measure is homogeneous of degree 1. In the case of a subadditive risk measure, one can also show that:

$$\mathcal{R}(\Pi_i \mid \Pi) \leq \mathcal{R}(\Pi_i) \tag{2.14}$$

This means that the risk contribution of the sub-portfolio i is always smaller than its stand-alone risk measure. The difference is related to the risk diversification.

Let us return to risk measure $\mathcal{R}(w)$ defined in terms of weights. The previous framework implies that the risk contribution of sub-portfolio i is uniquely defined as:

$$\mathcal{RC}_i = w_i \frac{\partial\,\mathcal{R}(w)}{\partial\, w_i} \tag{2.15}$$

and the risk measure satisfies the Euler decomposition:

$$\mathcal{R}(w) = \sum_{i=1}^{n} w_i \frac{\partial\,\mathcal{R}(w)}{\partial\, w_i} = \sum_{i=1}^{n} \mathcal{RC}_i \tag{2.16}$$

This relationship is also called the Euler allocation principle.

Remark 17 *We can always define the risk contributions of a risk measure by using Equation (2.15). However, this does not mean that the risk measure satisfies the Euler decomposition (2.16).*

Remark 18 *Kalkbrener (2005) develops an axiomatic approach to risk contribution. In particular, he shows that the Euler allocation principle is the only risk allocation method compatible with diversification principle (2.14) if the risk measure is subadditive.*

If we assume that the portfolio return $R(w)$ is a linear function of the weights w, the expression of the standard deviation-based risk measure becomes:

$$\begin{aligned} \mathcal{R}(w) &= -\mu(w) + c \cdot \sigma(w) \\ &= -w^\top \mu + c \cdot \sqrt{w^\top \Sigma w} \end{aligned}$$

where μ and Σ are the mean vector and the covariance matrix of sub-portfolios. It follows that the vector of marginal risks is:

$$\begin{aligned} \frac{\partial\,\mathcal{R}(w)}{\partial\, w} &= -\mu + c \cdot \frac{1}{2}\left(w^\top \Sigma w\right)^{-1}(2\Sigma w) \\ &= -\mu + c \cdot \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

[91] This property means that assets with a better risk-adjusted performance than the portfolio continue to have a better RAPM if their allocation increases in a small proportion.

The risk contribution of the i^{th} sub-portfolio is then:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + c \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right)$$

We verify that the standard deviation-based risk measure satisfies the full allocation property:

$$\begin{aligned}
\sum_{i=1}^n \mathcal{RC}_i &= \sum_{i=1}^n w_i \cdot \left(-\mu_i + c \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \\
&= w^\top \left(-\mu + c \cdot \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \right) \\
&= -w^\top \mu + c \cdot \sqrt{w^\top \Sigma w} \\
&= \mathcal{R}(w)
\end{aligned}$$

Because Gaussian value-at-risk and expected shortfall are two special cases of the standard deviation-based risk measure, we conclude that they also satisfy the Euler allocation principle. In the case of the value-at-risk, the risk contribution becomes:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + \Phi^{-1}(\alpha) \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \tag{2.17}$$

whereas in the case of the expected shortfall, it is equal to:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)} \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \tag{2.18}$$

Remark 19 *Even if the risk measure is convex, it does not necessarily satisfy the Euler allocation principle. The most famous example is the variance of the portfolio return. We have* $\operatorname{var}(w) = w^\top \Sigma w$ *and* $\partial_w \operatorname{var}(w) = 2\Sigma w$. *It follows that* $\sum_{i=1}^n w_i \cdot \partial_{w_i} \operatorname{var}(w) = \sum_{i=1}^n w_i \cdot (2\Sigma w)_i = 2w^\top \Sigma w = 2\operatorname{var}(w) > \operatorname{var}(w)$. *In the case of the variance, the sum of the risk contributions is then always larger than the risk measure itself, because the variance does not satisfy the homogeneity property.*

Example 19 *We consider the Apple/Coca-Cola portfolio that has been used for calculating the Gaussian VaR on page 68. We recall that the nominal exposures were* \$1 093.3 *(Apple) and* \$842.8 *(Coca-Cola), the estimated standard deviation of daily returns was equal to* 1.3611% *for Apple and* 0.9468% *for Coca-Cola and the cross-correlation of stock returns was equal to* 12.0787%.

In the two-asset case, the expression of the value-at-risk or the expected shortfall is:

$$\mathcal{R}(w) = -w_1\mu_1 - w_2\mu_2 + c\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}$$

It follows that the marginal risk of the first asset is:

$$\mathcal{MR}_1 = -\mu_1 + c\frac{w_1\sigma_1^2 + w_2\rho\sigma_1\sigma_2}{\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}}$$

We then deduce that the risk contribution of the first asset is:

$$\mathcal{RC}_1 = -w_1\mu_1 + c\frac{w_1^2\sigma_1^2 + w_1w_2\rho\sigma_1\sigma_2}{\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}}$$

By using the numerical values[92] of Example 19, we obtain the results given in Tables 2.17 and 2.18. We verify that the sum of risk contributions is equal to the risk measure. We notice that the stock Apple explains 75.14% of the risk whereas it represents 56.47% of the allocation.

TABLE 2.17: Risk decomposition of the 99% Gaussian value-at-risk

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	1093.3	2.83%	30.96	75.14%
Coca-Cola	842.8	1.22%	10.25	24.86%
$\mathcal{R}(w)$			41.21	

TABLE 2.18: Risk decomposition of the 99% Gaussian expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	1093.3	3.24%	35.47	75.14%
Coca-Cola	842.8	1.39%	11.74	24.86%
$\mathcal{R}(w)$			47.21	

2.3.2 Application to non-normal risk measures

2.3.2.1 Main results

In the previous section, we provided formulas for when asset returns are normally distributed. However, the previous expressions can be extended in the general case. For the value-at-risk, Gouriéroux *et al.* (2000) show that the risk contribution is equal to[93]:

$$\begin{aligned} \mathcal{RC}_i &= \mathcal{R}(\Pi_i \mid \Pi) \\ &= -\mathbb{E}[\Pi_i \mid \Pi = -\operatorname{VaR}_\alpha(\Pi)] \\ &= \mathbb{E}[L_i \mid L(w) = \operatorname{VaR}_\alpha(L)] \end{aligned} \tag{2.19}$$

Formula (2.19) is more general than Equation (2.17) obtained in the Gaussian case. Indeed, we can retrieve the latter if we assume that the returns are Gaussian. We recall that the portfolio return is $R(w) = \sum_{i=1}^n w_i R_i = w^\top R$. The portfolio loss is defined by $L(w) = -R(w)$. We deduce that:

$$\begin{aligned} \mathcal{RC}_i &= \mathbb{E}[-w_i R_i \mid -R(w) = \operatorname{VaR}_\alpha(w;h)] \\ &= -w_i \mathbb{E}[R_i \mid R(w) = -\operatorname{VaR}_{\alpha;h}(w)] \end{aligned}$$

Because $R(w)$ is a linear combination of R, the random vector $(R, R(w))$ is Gaussian and we have:

$$\begin{pmatrix} R \\ R(w) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ w^\top \mu \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma w \\ w^\top \Sigma & w^\top \Sigma w \end{pmatrix} \right)$$

[92] We set $\mu_1 = \mu_2 = 0$.

[93] See also Hallerbach (2003).

We know that $\text{VaR}_\alpha(w;h) = -w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}$. It follows that[94]:

$$\begin{aligned}
\mathbb{E}\left[R \mid R(w) = -\text{VaR}_\alpha(w;h)\right] &= \mathbb{E}\left[R \mid R(w) = w^\top \mu - \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}\right] \\
&= \mu + \Sigma w \left(w^\top \Sigma w\right)^{-1} \cdot \\
&\quad \left(w^\top \mu - \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w} - w^\top \mu\right)
\end{aligned}$$

and:

$$\begin{aligned}
\mathbb{E}\left[R \mid R(w) = -\text{VaR}_\alpha(w;h)\right] &= \mu - \Phi^{-1}(\alpha)\Sigma w \frac{\sqrt{w^\top \Sigma w}}{\left(w^\top \Sigma w\right)^{-1}} \\
&= \mu - \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}}
\end{aligned}$$

We finally obtain the same expression as Equation (2.17):

$$\begin{aligned}
\mathcal{RC}_i &= -w_i \left(\mu - \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}}\right)_i \\
&= -w_i \mu_i + \Phi^{-1}(\alpha)\frac{w_i \cdot (\Sigma w)_i}{\sqrt{w^\top \Sigma w}}
\end{aligned}$$

In the same way, Tasche (2002) shows that the general expression of the risk contributions for the expected shortfall is:

$$\begin{aligned}
\mathcal{RC}_i &= \mathcal{R}(\Pi_i \mid \Pi) \\
&= -\mathbb{E}\left[\Pi_i \mid \Pi \leq -\text{VaR}_\alpha(\Pi)\right] \\
&= \mathbb{E}\left[L_i \mid L(w) \geq \text{VaR}_\alpha(L)\right]
\end{aligned} \tag{2.20}$$

Using Bayes' theorem, it follows that:

$$\mathcal{RC}_i = \frac{\mathbb{E}\left[L_i \cdot \mathbb{1}\left\{L(w) \geq \text{VaR}_\alpha(L)\right\}\right]}{1-\alpha}$$

If we apply the previous formula to the Gaussian case, we obtain:

$$\mathcal{RC}_i = -\frac{w_i}{1-\alpha}\mathbb{E}\left[R_i \cdot \mathbb{1}\left\{R(w) \leq -\text{VaR}_\alpha(L)\right\}\right]$$

After some tedious computations, we retrieve the same expression as found previously[95].

2.3.2.2 Calculating risk contributions with historical and simulated scenarios

The case of value-at-risk When using historical or simulated scenarios, the VaR is calculated as follows:

$$\text{VaR}_\alpha(w;h) = -\Pi_{((1-\alpha)n_S:n_S)} = L_{(\alpha n_S:n_S)}$$

Let $\mathfrak{R}_\Pi(s)$ be the rank of the P&L associated to the s^{th} observation meaning that:

$$\mathfrak{R}_\Pi(s) = \sum_{j=1}^{n_S} \mathbb{1}\left\{\Pi_j \leq \Pi_s\right\}$$

---

[94]We use the formula of the conditional expectation presented in Appendix A.2.2.4 on page 1062.

[95]The derivation of the formula is left as an exercise (Section 2.4.9 on page 123).

We deduce that:

$$\Pi_s = \Pi_{(\mathfrak{R}_\Pi(s):n_S)}$$

Formula (2.19) is then equivalent to decompose $\Pi_{((1-\alpha)n_S:n_S)}$ into individual P&Ls. We have $\Pi_s = \sum_{i=1}^{n} \Pi_{i,s}$ where $\Pi_{i,s}$ is the P&L of the i^{th} sub-portfolio for the s^{th} scenario. It follows that:

$$\begin{aligned}
\mathrm{VaR}_\alpha(w;h) &= -\Pi_{((1-\alpha)n_S:n_S)} \\
&= -\Pi_{\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)} \\
&= -\sum_{i=1}^{n} \Pi_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)}
\end{aligned}$$

where $\mathfrak{R}_\Pi^{-1}$ is the inverse function of the rank. We finally deduce that:

$$\begin{aligned}
\mathcal{RC}_i &= -\Pi_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)} \\
&= L_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)}
\end{aligned}$$

The risk contribution of the i^{th} sub-portfolio is the loss of the i^{th} sub-portfolio corresponding to the scenario $\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)$. If $(1-\alpha)n_S$ is not an integer, we have:

$$\mathcal{RC}_i = -\left(\Pi_{i,\mathfrak{R}_\Pi^{-1}(q)} + ((1-\alpha)n_S - q)\left(\Pi_{i,\mathfrak{R}_\Pi^{-1}(q+1)} - \Pi_{i,\mathfrak{R}_\Pi^{-1}(q)}\right)\right)$$

where $q = q_\alpha(n_S)$ is the integer part of $(1-\alpha)n_S$.

Let us consider Example 13 on page 68. We have found that the historical value-at-risk is \$47.39. It corresponds to the linear interpolation between the second and third largest loss. Using results in Table 2.7 on page 70, we notice that $\mathfrak{R}_\Pi^{-1}(1) = 236$, $\mathfrak{R}_\Pi^{-1}(2) = 69$, $\mathfrak{R}_\Pi^{-1}(3) = 85$, $\mathfrak{R}_\Pi^{-1}(4) = 23$ and $\mathfrak{R}_\Pi^{-1}(5) = 242$. We deduce that the second and third order statistics correspond to the 69^{th} and 85^{th} historical scenarios. The risk decomposition is reported in Table 2.19. Therefore, we calculate the risk contribution of the Apple stock as follows:

$$\begin{aligned}
\mathcal{RC}_1 &= -\frac{1}{2}(\Pi_{1,69} + \Pi_{1,85}) \\
&= -\frac{1}{2}(10 \times (105.16 - 109.33) + 10 \times (104.72 - 109.33)) \\
&= \$43.9
\end{aligned}$$

For the Coca-Cola stock, we obtain:

$$\begin{aligned}
\mathcal{RC}_2 &= -\frac{1}{2}(\Pi_{2,69} + \Pi_{2,85}) \\
&= -\frac{1}{2}(20 \times (41.65 - 42.14) + 20 \times (42.28 - 42.14)) \\
&= \$3.5
\end{aligned}$$

If we compare these results with those obtained with the Gaussian VaR, we observe that the risk decomposition is more concentrated for the historical VaR. Indeed, the exposure on Apple represents 96.68% whereas it was previously equal to 75.14%. The problem is that the estimator of the risk contribution only uses two observations, implying that its variance is very high.

TABLE 2.19: Risk decomposition of the 99% historical value-at-risk

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	77.77	43.92	92.68%
Coca-Cola	43.53%	7.97	3.47	7.32%
$\mathcal{R}(w)$			47.39	

We can consider three techniques to improve the efficiency of the estimator $\mathcal{RC}_i = L_{i,\mathfrak{R}_\Pi^{-1}(n_S(1-a))}$. The first approach is to use a regularization method (Scaillet, 2004). The idea is to estimate the value-at-risk by weighting the order statistics:

$$\begin{aligned}\mathrm{VaR}_\alpha(w;h) &= -\sum_{s=1}^{n_S} \varpi_\alpha(s;n_S)\,\Pi_{(s:n_S)} \\ &= -\sum_{s=1}^{n_S} \varpi_\alpha(s;n_S)\,\Pi_{\mathfrak{R}_\Pi^{-1}(s)}\end{aligned}$$

where $\varpi_\alpha(s;n_S)$ is a weight function dependent on the confidence level α. The expression of the risk contribution then becomes:

$$\mathcal{RC}_i = -\sum_{s=1}^{n_S} \varpi_\alpha(s;n_S)\,\Pi_{i,\mathfrak{R}_\Pi^{-1}(s)}$$

Of course, this naive method can be improved by using more sophisticated approaches such as importance sampling (Glasserman, 2005).

In the second approach, asset returns are assumed to be elliptically distributed. In this case, Carroll *et al.* (2001) show that[96]:

$$\mathcal{RC}_i = \mathbb{E}[L_i] + \frac{\operatorname{cov}(L,L_i)}{\sigma^2(L)}\left(\mathrm{VaR}_\alpha(L) - \mathbb{E}[L]\right) \tag{2.21}$$

Estimating the risk contributions with historical scenarios is then straightforward. It suffices to apply Formula (2.21) by replacing the statistical moments by their sample statistics:

$$\mathcal{RC}_i = \bar{L}_i + \frac{\sum_{s=1}^{n_S}\left(L_s-\bar{L}\right)\left(L_{i,s}-\bar{L}_i\right)}{\sum_{s=1}^{n_S}\left(L_s-\bar{L}\right)^2}\left(\mathrm{VaR}_\alpha(L)-\bar{L}\right)$$

where $\bar{L}_i = n_S^{-1}\sum_{s=1}^{n_S} L_{i,s}$ and $\bar{L} = n_S^{-1}\sum_{s=1}^{n_S} L_s$. Equation (2.21) can be viewed as the estimation of the conditional expectation $\mathbb{E}\left[L_i \mid L = \mathrm{VaR}_\alpha(L)\right]$ in a linear regression framework:

$$L_i = \beta L + \varepsilon_i$$

[96] We verify that the sum of the risk contributions is equal to the value-at-risk:

$$\begin{aligned}\sum_{i=1}^n \mathcal{RC}_i &= \sum_{i=1}^n \mathbb{E}[L_i] + \left(\mathrm{VaR}_\alpha(L)-\mathbb{E}[L]\right)\sum_{i=1}^n \frac{\operatorname{cov}(L,L_i)}{\sigma^2(L)} \\ &= \mathbb{E}[L] + \left(\mathrm{VaR}_\alpha(L)-\mathbb{E}[L]\right) \\ &= \mathrm{VaR}_\alpha(L)\end{aligned}$$

Because the least squares estimator is $\hat{\beta} = \operatorname{cov}(L, L_i) / \sigma^2(L)$, we deduce that:

$$\begin{aligned} \mathbb{E}[L_i | L = \mathrm{VaR}_\alpha(L)] &= \hat{\beta}\,\mathrm{VaR}_\alpha(L) + \mathbb{E}[\varepsilon_i] \\ &= \hat{\beta}\,\mathrm{VaR}_\alpha(L) + \left(\mathbb{E}[L_i] - \hat{\beta}\mathbb{E}[L]\right) \\ &= \mathbb{E}[L_i] + \hat{\beta}(\mathrm{VaR}_\alpha(L) - \mathbb{E}[L]) \end{aligned}$$

Epperlein and Smillie (2006) extend Formula (2.21) in the case of non-elliptical distributions. If we consider the generalized conditional expectation $\mathbb{E}[L_i | L = x] = f(x)$ where the function f is unknown, the estimator is given by the kernel regression[97]:

$$\hat{f}(x) = \frac{\sum_{s=1}^{n_S} \mathcal{K}(L_s - x) L_{i,s}}{\sum_{s=1}^{n_S} \mathcal{K}(L_s - x)}$$

where $\mathcal{K}(u)$ is the kernel function. We deduce that:

$$\mathcal{RC}_i = \hat{f}(\mathrm{VaR}_\alpha(L))$$

Epperlein and Smillie (2006) note however that this risk decomposition does not satisfy the Euler allocation principle. This is why they propose the following correction:

$$\begin{aligned} \mathcal{RC}_i &= \frac{\mathrm{VaR}_\alpha(L)}{\sum_{i=1}^{n} \mathcal{RC}_i} \hat{f}(\mathrm{VaR}_\alpha(L)) \\ &= \mathrm{VaR}_\alpha(L) \frac{\sum_{s=1}^{n_S} \mathcal{K}(L_s - \mathrm{VaR}_\alpha(L)) L_{i,s}}{\sum_{i=1}^{n}\sum_{s=1}^{n_S} \mathcal{K}(L_s - \mathrm{VaR}_\alpha(L)) L_{i,s}} \\ &= \mathrm{VaR}_\alpha(L) \frac{\sum_{s=1}^{n_S} \mathcal{K}(L_s - \mathrm{VaR}_\alpha(L)) L_{i,s}}{\sum_{s=1}^{n_S} \mathcal{K}(L_s - \mathrm{VaR}_\alpha(L)) L_s} \end{aligned}$$

In Table 2.20, we have reported the risk contributions of the 99% value-at-risk for Apple and Coca-Cola stocks. The case **G** corresponds to the Gaussian value-at-risk whereas all the other cases correspond to the historical value-at-risk. For the case **R1**, the regularization weights are $\varpi_{99\%}(2; 250) = \varpi_{99\%}(3; 250) = \frac{1}{2}$ and $\varpi_{99\%}(s; 250) = 0$ when $s \neq 2$ or $s \neq 3$. It corresponds to the classical interpolation method between the second and third order statistics. For the case **R2**, we have $\varpi_{99\%}(s; 250) = \frac{1}{4}$ when $s \leq 4$ and $\varpi_{99\%}(s; 250) = 0$ when $s > 4$. The value-at-risk is therefore estimated by averaging the first four order statistics. The cases **E** and **K** correspond to the methods based on the elliptical and kernel approaches. For these two cases, we obtain a risk decomposition, which is closer to this obtained with the Gaussian method. This is quite logical as the Gaussian distribution is a special case of elliptical distributions and the kernel function is also Gaussian.

TABLE 2.20: Risk contributions calculated with regularization techniques

Asset	**G**	**R1**	**R2**	**E**	**K**
Apple	30.97	43.92	52.68	35.35	39.21
Coca-Cola	10.25	3.47	2.29	12.03	8.17
$\mathcal{R}(w)$	41.21	47.39	54.96	47.39	47.39

[97] $\hat{f}(x)$ is called the Nadaraya-Watson estimator (see Section 10.1.4.2 on page 641).

Example 20 *Let $L = L_1 + L_2$ be the portfolio loss where L_i $(i = 1, 2)$ is defined as follows:*

$$L_i = w_i \left(\mu_i + \sigma_i T_i\right)$$

and T_i has a Student's t distribution with the number of degrees of freedom ν_i. The dependence function between the losses (L_1, L_2) is given by the Clayton copula:

$$\mathbf{C}\left(u_1, u_2\right) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

For the numerical illustration, we consider the following values: $w_1 = 100$, $\mu_1 = 10\%$, $\sigma_1 = 20\%$, $\nu_1 = 6$, $w_2 = 200$, $\mu_2 = 10\%$, $\sigma_2 = 25\%$, $\nu_2 = 4$ and $\theta = 2$. The confidence level α of the value-at-risk is set to 90%.

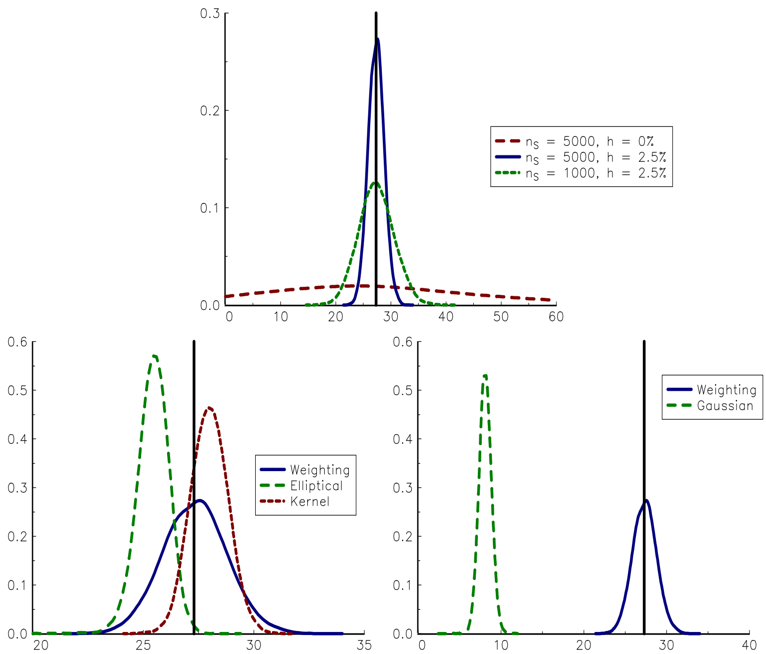

FIGURE 2.19: Density function of the different risk contribution estimators

In Figure 2.19, we compare the different statistical estimators of the risk contribution $\mathcal{RC}_1$ when we use $n_S = 5\,000$ simulations. Concerning the regularization method, we consider the following weight function applied to the order statistics of losses[98]:

$$\varpi_\alpha^L\left(s; n_S\right) = \frac{1}{2hn_S + 1} \cdot \mathbb{1}\left\{\frac{\left|s - q_\alpha\left(n_S\right)\right|}{n_S} \leq h\right\}$$

It corresponds to a uniform kernel on the range $\left[q_\alpha\left(n_S\right) - hn_S, q_\alpha\left(n_S\right) + hn_S\right]$. In the first panel, we report the probability density function of $\mathcal{RC}_1$ when h is equal to 0% and 2.5%. The case $h = 0\%$ is the estimator based on only one observation. We verify that the variance

[98] This is equivalent to use this weight function applied to the order statistics of P&Ls:

$$\varpi_\alpha\left(s; n_S\right) = \frac{1}{2hn_S + 1} \cdot \mathbb{1}\left\{\frac{\left|s - q_{\bar{\alpha}}\left(n_S\right)\right|}{n_S} \leq h\right\}$$

of this estimator is larger for $h = 0\%$ than for $h = 2.5\%$. However, we notice that this last estimator is a little biased, because we estimate the quantile 90% by averaging the order statistics corresponding to the range $[87.5\%, 92.5\%]$. In the second panel, we compare the weighting method with the elliptical and kernel approaches. These two estimators have a smaller variance, but a larger bias because they assume that the loss distribution is elliptical or may be estimated using a Gaussian kernel. Finally, the third panel shows the probability density function of $\mathcal{RC}_1$ estimated with the Gaussian value-at-risk.

The case of expected shortfall On page 70, we have shown that the expected shortfall is estimated as follows:

$$\mathrm{ES}_\alpha(L) = \frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{L_s \geq \mathrm{VaR}_\alpha(L)\} \cdot L_s$$

or:

$$\mathrm{ES}_\alpha(L) = -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{\Pi_s \leq -\mathrm{VaR}_\alpha(L)\} \cdot \Pi_s$$

It corresponds to the average of the losses larger or equal than the value-at-risk. It follows that:

$$\begin{aligned}
\mathrm{ES}_\alpha(L) &= -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{q_\alpha(n_S)} \Pi_{(s:n_S)} \\
&= -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{q_\alpha(n_S)} \Pi_{\mathfrak{R}_\Pi^{-1}(s)} \\
&= -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{q_\alpha(n_S)} \sum_{i=1}^{n} \Pi_{i,\mathfrak{R}_\Pi^{-1}(s)}
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathcal{RC}_i &= -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{q_\alpha(n_S)} \Pi_{i,\mathfrak{R}_\Pi^{-1}(s)} \\
&= \frac{1}{q_\alpha(n_S)} \sum_{s=1}^{q_\alpha(n_S)} L_{i,\mathfrak{R}_\Pi^{-1}(s)}
\end{aligned}$$

In the Apple/Coca-Cola example, we recall that the 99% daily value-at-risk is equal to \$47.39. The corresponding expected shortfall is then the average of the two largest losses:

$$\mathrm{ES}_\alpha(w; \text{one day}) = \frac{84.34 + 51.46}{2} = \$67.90$$

For the risk contribution, we obtain[99]:

$$\mathcal{RC}_1 = \frac{87.39 + 41.69}{2} = \$64.54$$

[99] Because we have:

$$\Pi_{(1:250)} = -87.39 + 3.05 = -84.34$$

and:

$$\Pi_{(2:250)} = -41.69 - 9.77 = -51.46$$

and:

$$\mathcal{RC}_2 = \frac{-3.05 + 9.77}{2} = \$3.36$$

The corresponding risk decomposition is given in Tables 2.21 and 2.22 for $\alpha = 99\%$ and $\alpha = 97.5\%$. With the new rules of Basel III, the capital is higher for this example.

TABLE 2.21: Risk decomposition of the 99% historical expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	114.29	64.54	95.05%
Coca-Cola	43.53%	7.72	3.36	4.95%
$\mathcal{R}(w)$			67.90	

TABLE 2.22: Risk decomposition of the 97.5% historical expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	78.48	44.32	91.31%
Coca-Cola	43.53%	9.69	4.22	8.69%
$\mathcal{R}(w)$			48.53	

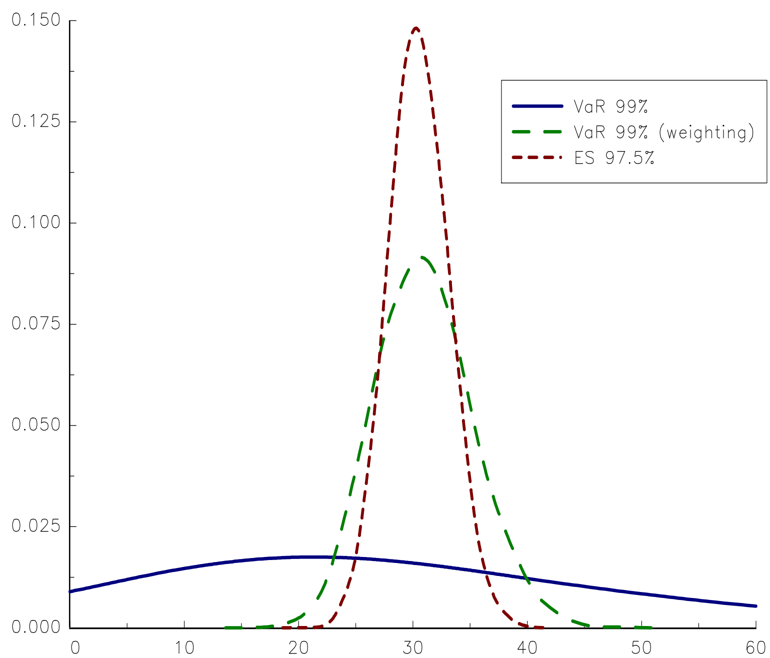

FIGURE 2.20: Probability density function of the $\mathcal{RC}_1$ estimator for the 99% VaR and 97.5% ES

In Figure 2.20, we report the probability density function of the $\mathcal{RC}_1$ estimator in the case of Example 20. We consider the 99% value-at-risk and the 97.5% expected shortfall with $n_S = 5\,000$ simulated scenarios. For the VaR risk measure, the risk contribution is estimated using respectively only one single observation and a weighting function corresponding to a uniform window[100]. We notice that the estimator has a smaller variance with the expected shortfall risk measure. Of course, we can always reduce the variance of ES risk contributions by using the previous smoothing techniques (Scaillet, 2004), but this is less of an issue than for the value-at-risk measure.

2.4 Exercises

2.4.1 Calculating regulatory capital with the Basel I standardized measurement method

1. We consider an interest rate portfolio with the following exposures: a long position of \$100 mn on four-month instruments, a short position of \$50 mn on five-month instruments, a long position of \$10 mn on fifteen-year instruments and a short position of \$50 mn on twelve-year instruments.

   (a) Using BCBS (1996a), explain the maturity approach for computing the capital requirement due to the interest rate risk.

   (b) By assuming that the instruments correspond to bonds with coupons larger than 3%, calculate the capital requirement of the trading portfolio.

2. We consider the following portfolio of stocks:

Stock	3M	Exxon	IBM	Pfizer	AT&T	Cisco	Oracle
$\mathcal{L}_i$	100	100	10	50	60	90	
$\mathcal{S}_i$		50					80

   where $\mathcal{L}_i$ and $\mathcal{S}_i$ indicate the long and short exposures on stock i expressed in \$ mn.

   (a) Calculate the capital charge for the specific risk.

   (b) Calculate the capital charge for the general market risk.

   (c) How can the investor hedge the market risk of his portfolio by using S&P 500 futures contracts? What is the corresponding capital charge? Verify that the investor minimizes the total capital charge in this case.

3. We consider a net exposure $\mathcal{N}_w$ on an equity portfolio w. We note $\sigma(w)$ the annualized volatility of the portfolio return.

   (a) Calculate the required capital under the standardized measurement method.

   (b) Calculate the required capital under the internal model method if we assume that the bank uses a Gaussian value-at-risk[101].

   (c) Deduce an upper bound $\sigma(w) \leq \sigma^+$ under which the required capital under SMM is higher than the required capital under IMA.

   (d) Comment on these results.

[100] We set $h = 0.5\%$ meaning that the risk contribution is estimated with 51 observations for the 99% value-at-risk.

[101] We consider the Basel II capital requirement rules.

4. We consider the portfolio with the following long and short positions expressed in \$ mn:

Asset	EUR	JPY	CAD	Gold	Sugar	Corn	Cocoa
$\mathcal{L}_i$	100	50		50	50	60	90
$\mathcal{S}_i$	100	100	50			80	110

   (a) How do you explain that some assets present both long and short positions?

   (b) Calculate the required capital under the simplified approach.

5. We consider the following positions (in \$) of the commodity i:

Time band	0−1M	1M−3M	6M−1Y	1Y−2Y	2Y−3Y	3Y+
$\mathcal{L}_i(t)$	500	0	1 800	300	0	0
$\mathcal{S}_i(t)$	300	900	100	600	100	200

   (a) Using BCBS (1996a), explain the maturity ladder approach for commodities.

   (b) Compute the capital requirement.

2.4.2 Covariance matrix

We consider a universe of there stocks A, B and C.

1. The covariance matrix of stock returns is:

$$\Sigma = \begin{pmatrix} 4\% & & \\ 3\% & 5\% & \\ 2\% & -1\% & 6\% \end{pmatrix}$$

   (a) Calculate the volatility of stock returns.

   (b) Deduce the correlation matrix.

2. We assume that the volatilities are 10%, 20% and 30%. whereas the correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 0\% & 100\% \end{pmatrix}$$

   (a) Write the covariance matrix.

   (b) Calculate the volatility of the portfolio $(50\%, 50\%, 0)$.

   (c) Calculate the volatility of the portfolio $(60\%, -40\%, 0)$. Comment on this result.

   (d) We assume that the portfolio is long \$150 on stock A, long \$500 on stock B and short \$200 on stock C. Find the volatility of this long/short portfolio.

3. We consider that the vector of stock returns follows a one-factor model:

$$R = \beta\mathcal{F} + \varepsilon$$

   We assume that $\mathcal{F}$ and ε are independent. We note $\sigma_{\mathcal{F}}^2$ the variance of $\mathcal{F}$ and $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \tilde{\sigma}_3^2\right)$ the covariance matrix of idiosyncratic risks ε_t. We use the following numerical values: $\sigma_{\mathcal{F}} = 50\%$, $\beta_1 = 0.9$, $\beta_2 = 1.3$, $\beta_3 = 0.1$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$ and $\tilde{\sigma}_3 = 15\%$.

(a) Calculate the volatility of stock returns.

(b) Calculate the correlation between stock returns.

4. Let X and Y be two independent random vectors. We note $\mu(X)$ and $\mu(Y)$ the vector of means and $\Sigma(X)$ and $\Sigma(Y)$ the covariance matrices. We define the random vector $Z = (Z_1, Z_2, Z_3)$ where Z_i is equal to the product $X_i Y_i$.

   (a) Calculate $\mu(Z)$ and $\operatorname{cov}(Z)$.

   (b) We consider that $\mu(X)$ is equal to zero and $\Sigma(X)$ corresponds to the covariance matrix of Question 2. We assume that Y_1, Y_2 and Y_3 are three independent uniform random variables $\mathcal{U}_{[0,1]}$. Calculate the 99% Gaussian value-at-risk of the portfolio corresponding to Question 2(d) when Z is the random vector of asset returns. Compare this value with the Monte Carlo VaR.

2.4.3 Risk measure

1. We denote $\mathbf{F}$ the cumulative distribution function of the loss L.

   (a) Give the mathematical definition of the value-at-risk and expected shortfall risk measures.

   (b) Show that:

$$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathbf{F}^{-1}(t)\,\mathrm{d}t$$

   (c) We assume that L follows a Pareto distribution $\mathcal{P}(\theta, x_-)$ defined by:

$$\Pr\{L \le x\} = 1 - \left(\frac{x}{x_-}\right)^{-\theta}$$

   where $x \ge x_-$ and $\theta > 1$. Calculate the moments of order one and two. Interpret the parameters x_- and θ. Calculate $\mathrm{ES}_\alpha(L)$ and show that:

$$\mathrm{ES}_\alpha(L) > \mathrm{VaR}_\alpha(L)$$

   (d) Calculate the expected shortfall when L is a Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$. Show that:

$$\Phi(x) = -\frac{\phi(x)}{x^1} + \frac{\phi(x)}{x^3} + \ldots$$

   Deduce that:

$$\mathrm{ES}_\alpha(L) \to \mathrm{VaR}_\alpha(L) \text{ when } \alpha \to 1$$

   (e) Comment on these results in a risk management perspective.

2. Let $\mathcal{R}(L)$ be a risk measure of the loss L.

   (a) Is $\mathcal{R}(L) = \mathbb{E}[L]$ a coherent risk measure?

   (b) Same question if $\mathcal{R}(L) = \mathbb{E}[L] + \sigma(L)$.

3. We assume that the probability distribution $\mathbf{F}$ of the loss L is defined by:

$$\Pr\{L = \ell_i\} = \begin{cases} 20\% & \text{if } \ell_i = 0 \\ 10\% & \text{if } \ell_i \in \{1,2,3,4,5,6,7,8\} \end{cases}$$

   (a) Calculate ES_α for $\alpha = 50\%$, $\alpha = 75\%$ and $\alpha = 90\%$.

   (b) Let us consider two losses L_1 and L_2 with the same distribution $\mathbf{F}$. Build a joint distribution of (L_1, L_2) which does not satisfy the subadditivity property when the risk measure is the value-at-risk.

2.4.4 Value-at-risk of a long/short portfolio

We consider a long/short portfolio composed of a long position on asset A and a short position on asset B. The long exposure is equal to \$2 mn whereas the short exposure is equal to \$1 mn. Using the historical prices of the last 250 trading days of assets A and B, we estimate that the asset volatilities σ_A and σ_B are both equal to 20% per year and that the correlation $\rho_{A,B}$ between asset returns is equal to 50%. In what follows, we ignore the mean effect.

1. Calculate the Gaussian VaR of the long/short portfolio for a one-day holding period and a 99% confidence level.

2. How do you calculate the historical VaR? Using the historical returns of the last 250 trading days, the five worst scenarios of the 250 simulated daily P&L of the portfolio are $-58\,700$, $-56\,850$, $-54\,270$, $-52\,170$ and $-49\,231$. Calculate the historical VaR for a one-day holding period and a 99% confidence level.

3. We assume that the multiplication factor m_c is 3. Deduce the required capital if the bank uses an internal model based on the Gaussian value-at-risk. Same question when the bank uses the historical VaR. Compare these figures with those calculated with the standardized measurement method.

4. Show that the Gaussian VaR is multiplied by a factor equal to $\sqrt{7/3}$ if the correlation $\rho_{A,B}$ is equal to -50%. How do you explain this result?

5. The portfolio manager sells a call option on the stock A. The delta of the option is equal to 50%. What does the Gaussian value-at-risk of the long/short portfolio become if the nominal of the option is equal to \$2 mn? Same question when the nominal of the option is equal to \$4 mn. How do you explain this result?

6. The portfolio manager replaces the short position on the stock B by selling a call option on the stock B. The delta of the option is equal to 50%. Show that the Gaussian value-at-risk is minimum when the nominal is equal to four times the correlation $\rho_{A,B}$. Deduce then an expression of the lowest Gaussian VaR. Comment on these results.

2.4.5 Value-at-risk of an equity portfolio hedged with put options

We consider two stocks A and B and an equity index I. We assume that the risk model corresponds to the CAPM and we have:

$$R_j = \beta_j R_I + \varepsilon_j$$

where R_j and R_I are the returns of stock j and the index. We assume that R_I and ε_j are independent. The covariance matrix of idiosyncratic risks is diagonal and we note $\tilde{\sigma}_j$ the volatility of ε_j.

1. The parameters are the following: $\sigma^2(R_I) = 4\%$, $\beta_A = 0.5$, $\beta_B = 1.5$, $\tilde{\sigma}_A^2 = 3\%$ and $\tilde{\sigma}_B^2 = 7\%$.

   (a) Calculate the volatility of stocks A and B and the cross-correlation.

   (b) Find the correlation between the stocks and the index.

   (c) Deduce the covariance matrix.

2. The current price of stocks A and B is equal to \$100 and \$50 whereas the value of the index is equal to \$50. The composition of the portfolio is 4 shares of A, 10 shares of B and 5 shares of I.

   (a) Determine the Gaussian value-at-risk for a confidence level of 99% and a 10-day holding period.

   (b) Using the historical returns of the last 260 trading days, the five lowest simulated daily P&Ls of the portfolio are -62.39, -55.23, -52.06, -51.52 and -42.83. Calculate the historical VaR for a confidence level of 99% and a 10-day holding period.

   (c) What is the regulatory capital[102] if the bank uses an internal model based on the Gaussian value-at-risk? Same question when the bank uses the historical value-at-risk. Compare these figures with those calculated with the standardized measurement method.

3. The portfolio manager would like to hedge the directional risk of the portfolio. For that, he purchases put options on the index I at a strike of \$45 with a delta equal to -25%. Write the expression of the P&L using the delta approach.

   (a) How many options should the portfolio manager purchase for hedging 50% of the index exposure? Deduce the Gaussian value-at-risk of the corresponding portfolio?

   (b) The portfolio manager believes that the purchase of 96 put options minimizes the value-at-risk. What is the basis for his reasoning? Do you think that it is justified? Calculate then the Gaussian VaR of this new portfolio.

2.4.6 Risk management of exotic options

Let us consider a short position on an exotic option, whose its current value $\mathcal{C}_t$ is equal to \$6.78. We assume that the price S_t of the underlying asset is \$100 and the implied volatility Σ_t is equal to 20%.

1. At time $t+1$, the value of the underlying asset is \$97 and the implied volatility remains constant. We find that the P&L of the trader between t and $t+1$ is equal to \$1.37. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_t$, gamma $\boldsymbol{\Gamma}_t$ and vega[103] $\boldsymbol{\upsilon}_t$ are respectively equal to 49%, 2% and 40%?

2. At time $t+2$, the price of the underlying asset is \$97 while the implied volatility increases from 20% to 22%. The value of the option $\mathcal{C}_{t+2}$ is now equal to \$6.17. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+1}$, gamma $\boldsymbol{\Gamma}_{t+1}$ and vega $\boldsymbol{\upsilon}_{t+1}$ are respectively equal to 43%, 2% and 38%?

3. At time $t+3$, the price of the underlying asset is \$95 and the value of the implied volatility is 19%. We find that the P&L of the trader between $t+2$ and $t+3$ is equal to \$0.58. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+2}$, gamma $\boldsymbol{\Gamma}_{t+2}$ and vega $\boldsymbol{\upsilon}_{t+2}$ are respectively equal to 44%, 1.8% and 38%.

4. What can we conclude in terms of model risk?

[102] We assume that the multiplication factor m_c is equal to 3.
[103] Measured in volatility points.

2.4.7 P&L approximation with Greek sensitivities

1. Let $\mathcal{C}_t$ be the value of an option at time t. Define the delta, gamma, theta and vega coefficients of the option.

2. We consider an European call option with strike K. Give the value of option in the case of the Black-Scholes model. Deduce then the Greek coefficients.

3. We assume that the underlying asset is a non-dividend stock, the residual maturity of the call option is equal to one year, the current price of the stock is equal to \$100 and the interest rate is equal to 5%. We also assume that the implied volatility is constant and equal to 20%. In the table below, we give the value of the call option $\mathcal{C}_0$ and the Greek coefficients $\mathbf{\Delta}_0$, $\mathbf{\Gamma}_0$ and $\mathbf{\Theta}_0$ for different values of K:

K	80	95	100	105	120
$\mathcal{C}_0$	24.589	13.346	10.451	8.021	3.247
$\mathbf{\Delta}_0$	0.929	0.728	0.637	0.542	0.287
$\mathbf{\Gamma}_0$	0.007	0.017	0.019	0.020	0.017
$\mathbf{\Theta}_0$	−4.776	−6.291	−6.414	−6.277	−4.681

   (a) Explain how these values have been calculated. Comment on these numerical results.

   (b) One day later, the value of the underlying asset is \$102. Using the Black-Scholes formula, we obtain:

K	80	95	100	105	120
$\mathcal{C}_1$	26.441	14.810	11.736	9.120	3.837

   Explain how the option premium $\mathcal{C}_1$ is calculated. Deduce then the P&L of a long position on this option for each strike K.

   (c) For each strike price, calculate an approximation of the P&L by considering the sensitivities $\mathbf{\Delta}$, $\mathbf{\Delta} - \mathbf{\Gamma}$, $\mathbf{\Delta} - \mathbf{\Theta}$ and $\mathbf{\Delta} - \mathbf{\Gamma} - \mathbf{\Theta}$. Comment on these results.

   (d) Six months later, the value of the underlying asset is \$148. Repeat Questions 3(b) and 3(c) with these new parameters. Comment on these results.

2.4.8 Calculating the non-linear quadratic value-at-risk

1. Let $X \sim \mathcal{N}(0,1)$. Show that the even moments of X are given by the following relationship:
$$\mathbb{E}\left[X^{2n}\right] = (2n-1)\,\mathbb{E}\left[X^{2n-2}\right]$$
with $n \in \mathbb{N}$. Calculate the odd moments of X.

2. We consider a long position on a call option. The current price S_t of the underlying asset is equal to \$100, whereas the delta and the gamma of the option are respectively equal to 50% and 2%. We assume that the annual return of the asset follows a Gaussian distribution with an annual volatility equal to 32.25%.

   (a) Calculate the daily Gaussian value-at-risk using the delta approximation with a 99% confidence level.

   (b) Calculate the daily Gaussian value-at-risk by considering the delta-gamma approximation.

   (c) Deduce the daily Cornish-Fisher value-at-risk.

3. Let $X \sim \mathcal{N}(\mu, I)$ and $Y = X^\top AX$ with A a symmetric square matrix.

   (a) We recall that:

$$\begin{array}{rcl}\mathbb{E}[Y] & = & \mu^\top A\mu + \operatorname{tr}(A) \\ \mathbb{E}\left[Y^2\right] & = & \mathbb{E}^2[Y] + 4\mu^\top A^2\mu + 2\operatorname{tr}\left(A^2\right)\end{array}$$

   Deduce the moments of $Y = X^\top AX$ when $X \sim \mathcal{N}(\mu, \Sigma)$.

   (b) We suppose that $\mu = \mathbf{0}$. We recall that:

$$\begin{array}{rcl}\mathbb{E}\left[Y^3\right] & = & (\operatorname{tr}(A))^3 + 6\operatorname{tr}(A)\operatorname{tr}\left(A^2\right) + 8\operatorname{tr}\left(A^3\right) \\ \mathbb{E}\left[Y^4\right] & = & (\operatorname{tr}(A))^4 + 32\operatorname{tr}(A)\operatorname{tr}\left(A^3\right) + 12\left(\operatorname{tr}\left(A^2\right)\right)^2 + \\ & & 12(\operatorname{tr}(A))^2\operatorname{tr}\left(A^2\right) + 48\operatorname{tr}\left(A^4\right)\end{array}$$

   Compute the moments, the skewness and the excess kurtosis of $Y = X^\top AX$ when $X \sim \mathcal{N}(\mathbf{0}, \Sigma)$.

4. We consider a portfolio $w = (w_1, \ldots, w_n)$ of options. We assume that the vector of daily asset returns is distributed according to the Gaussian distribution $\mathcal{N}(\mathbf{0}, \Sigma)$. We note $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ the vector of deltas and the matrix of gammas.

   (a) Calculate the daily Gaussian value-at-risk using the delta approximation. Define the analytical expression of the risk contributions.

   (b) Calculate the daily Gaussian value-at-risk by considering the delta-gamma approximation.

   (c) Calculate the daily Cornish-Fisher value-at-risk when assuming that the portfolio is delta neutral.

   (d) Calculate the daily Cornish-Fisher value-at-risk in the general case by only considering the skewness.

5. We consider a portfolio composed of 50 options in a first asset, 20 options in a second asset and 20 options in a third asset. We assume that the gamma matrix is:

$$\boldsymbol{\Gamma} = \begin{pmatrix} 4.0\% & & \\ 1.0\% & 1.0\% & \\ 0.0\% & -0.5\% & 1.0\% \end{pmatrix}$$

   The actual price of the assets is normalized and is equal to 100. The daily volatility levels of the assets are respectively equal to 1%, 1.5% and 2% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 15\% & 100\% \end{pmatrix}$$

   (a) Compare the different methods to compute the daily value-at-risk with a 99% confidence level if the portfolio is delta neutral.

   (b) Same question if we now consider that the deltas are equal to 50%, 40% and 60%. Compute the risk decomposition in the case of the delta and delta-gamma approximations. What do you notice?

2.4.9 Risk decomposition of the expected shortfall

We consider a portfolio composed of n assets. We assume that asset returns $R = (R_1, \ldots, R_n)$ are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$. We note $L(w)$ the loss of the portfolio.

1. Find the distribution of $L(w)$.
2. Define the expected shortfall $\mathrm{ES}_\alpha(w)$. Calculate its expression in the present case.
3. Calculate the risk contribution $\mathcal{RC}_i$ of asset i. Deduce that the expected shortfall verifies the Euler allocation principle.
4. Give the expression of $\mathcal{RC}_i$ in terms of conditional loss. Retrieve the formula of $\mathcal{RC}_i$ found in Question 3. What is the interest of the conditional representation?

2.4.10 Expected shortfall of an equity portfolio

We consider an investment universe, which is composed of two stocks A and B. The current price of the two stocks is respectively equal to \$100 and \$200, their volatilities are equal to 25% and 20% whereas the cross-correlation is equal to −20%. The portfolio is long on 4 stocks A and 3 stocks B.

1. Calculate the Gaussian expected shortfall at the 97.5% confidence level for a ten-day time horizon.
2. The eight worst scenarios of daily stock returns among the last 250 historical scenarios are the following:

s	1	2	3	4	5	6	7	8
R_A	−3%	−4%	−3%	−5%	−6%	+3%	+1%	−1%
R_B	−4%	+1%	−2%	−1%	+2%	−7%	−3%	−2%

   Calculate then the historical expected shortfall at the 97.5% confidence level for a ten-day time horizon.

2.4.11 Risk measure of a long/short portfolio

We consider an investment universe, which is composed of two stocks A and B. The current prices of the two stocks are respectively equal to \$50 and \$20. Their volatilities are equal to 25% and 20% whereas the cross-correlation is equal to +12.5%. The portfolio is long on 2 stocks A and short on 5 stocks B.

1. Gaussian risk measure
   (a) Calculate the Gaussian value-at-risk at the 99% confidence level for a ten-day time horizon.
   (b) Calculate the Gaussian expected shortfall at the 97.5% confidence level for a ten-day time horizon.
2. Historical risk measure
   The ten worst scenarios of daily stock returns (expressed in %) among the last 250 historical scenarios are the following:

s	1	2	3	4	5	6	7	8	9	10
R_A	−0.6	−3.7	−5.8	−4.2	−3.7	0.0	−5.7	−4.3	−1.7	−4.1
R_B	5.7	2.3	−0.7	0.6	0.9	4.5	−1.4	0.0	2.3	−0.2
D	−6.3	−6.0	−5.1	−4.8	−4.6	−4.5	−4.3	−4.3	−4.0	−3.9

   where $D = R_A - R_B$ is the difference of the returns.

(a) Calculate the historical value-at-risk at the 99% confidence level for a ten-day time horizon.

(b) Calculate the historical expected shortfall at the 97.5% confidence level for a ten-day time horizon.

(c) Give an approximation of the capital charge under Basel II, Basel 2.5 and Basel III standards by considering the historical risk measure[104].

2.4.12 Kernel estimation of the expected shortfall

1. We consider a random variable X. We note $\mathcal{K}(u)$ the kernel function associated to the sample $\{x_1, \ldots, x_n\}$. Show that:

$$\begin{aligned} \mathbb{E}\left[X \cdot \mathbb{1}\{X \leq x\}\right] &= \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\frac{x-x_i}{\boldsymbol{h}}} x_i \mathcal{K}(u)\,\mathrm{d}u + \\ & \quad \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\frac{x-x_i}{\boldsymbol{h}}} \boldsymbol{h}\, u \mathcal{K}(u)\,\mathrm{d}u \end{aligned}$$

2. Find the expression of the first term by considering the integrated kernel function $\mathcal{I}(u)$.

3. Show that the second term tends to zero when $\boldsymbol{h} \rightarrow 0$.

4. Deduce an approximation of the expected shortfall $\mathrm{ES}_\alpha(w; h)$.

[104] We assume that the multiplicative factor is equal to 3 (Basel II), and the '*stressed*' risk measure is 2 times the '*normal*' risk measure (Basel 2.5).