Exploring non linearities in Hedge Funds: An application of Particle Filters to Hedge Fund Replication

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Abstract

Of the three main challenges of hedge fund replication, only replication of the well-known nonlinearities of their returns remains undisputed. Recent advances in hedge fund replication using factor models have shown that the use of Bayesian filters helps greatly in capturing the dynamic allocation of assets of hedge fund managers, particularly in the case of aggregates of hedge funds [33, 35]. Furthermore, from a practitioner's perspective, access to the alpha of the funds can be provided on top of capturing the dynamic exposures by adopting a core/satellite approach to building the replication portfolio [35]. In this working paper, we explore tentatively the solutions that Bayesian filters could provide to the replication of hedge fund nonlinearities. Although, not entirely successful, our results show promises and open new grounds for the field.

Keywords: Tracking problem, hedge fund replication, tactical asset allocation, Bayes filter, particle filter, non-linear exposure.

JEL classification: G11, C60.

HF Replication nonlinear case — Tentative Plan

1 Introduction

Over the past decade, hedge-fund replication has encountered a growing interest both from an academic and a practitioner perspective. Recently, Della Casa, Rechsteiner and Lehmann [10] reported the results of an industry survey showing that, even though only 7% of the surveyed institutions had invested in hedge fund replication products in 2007, three times as many were considering investing in 2008. Despite this surge in interest, the practice still faces many critics. If the launch of numerous products (indexes and mutual funds) by several investment banks in the past year can be taken as proof of the attraction of the "clones" of hedge funds (HF) as investment vehicles, there remain nonetheless several shortcomings which need to be addressed. For instance, according to the same survey cited above, 13% of the potential investors do not invest for they do not believe that replicating Hedge Funds'returns was possible; 16% deplore the lack of track record of the products; another 16% consider the products as black boxes. Finally, 25% of the same investors do not invest for a lack of understanding of the methodologies

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employed, while 31% of them were not interested for they see the practice as only replicating an average performance, thus failing to give access to one of the main attractive features of investing in one hedge fund, namely its strategy of management.

As a whole, the reasons put forth by these institutions compound different fundamental questions left unanswered by the literature. Since the seminal work of Fung and Hsieh [14], most of the literature [1, 3, 4, 16, *inter alia*] has focused on assessing and explaining the characteristics of HF returns in terms of their (possibly time-varying) exposures to some underlying factors. Using linear factor models, these authors report the incremental progress in the explanatory power of the different models proposed. Yet, for now, the standard rolling-windows OLS regression methodology, used to capture the dynamic exposures of the underlying HF's portfolio has failed to show consistent out-of-sample results, stressing the difficulty of capturing the tactical asset allocation (TAA) of HF's managers. More recently, more advanced methodologies, in particular Markov-Switching models and Kalman Filter (KF), have been introduced [5, 33] and show superior results to the standard rolling-windows OLS approach. Using the Kalman Filter methodology, Roncalli and Weisang [35] further argue and demonstrate that replicating aggregates of hedge funds, e.g. funds of hedge funds, provides very satisfying results. Especially, when combined with a core/satellite approach, with a core replicating portfolio providing the alternative beta of the target., and satellites of illiquid portfolios providing access to some alpha.

Nonetheless, despite superior dynamic procedures and an ever expanding set of explanatory factors, some nonlinear features of HF returns [11] as well as a substantial part of their performance remain unexplainable, unless surmising ultrahigh frequency trading and investments in illiquid assets or in derivative instruments by HF managers. To our knowledge, while commonly accepted by most authors, because of practical difficulties, these explanations have not led to a systematic assessment nor have been subject to systematic replication procedures.

In this paper, we take further the approach taken in [35]. We apply more advanced Bayesian filters' algorithms, known collectively as particle filters, to go beyond the linear Gaussian framework of the Kalman filter. Our goal is to capture the nonlinearities documented on HF returns. We consider what type of nonlinearities can arise, and what models can be used to explain them.

This paper is thus divided into three main sections. In section two, we provide the framework in which this paper is inscribed, providing a formal definition of a tracking problem, and casting Hedge Fund Replication (HFR) into a tracking problem. We also provide a brief review of Bayesian filters and the methodologies which we use here to solve tracking problems. In section three, we extend previous work to a non-Gaussian non-linear framework. Finally, in section four, we provide our conclusions and the future directions we think should be pursued.

2 Framework

2.1 Hedge Fund Replication: Factor Models and the Gaussian Linear Case

Starting with the work of Fung and Hsieh [14] as an extension of Sharpe's style regression analysis [37] to the world of hedge funds, factor-based models were first introduced as tools for performance analysis. The underlying assumption of Sharpe's style regression is that there exist, as in standard Arbitrage Pricing Theory (APT), a Return-Based Style (RBS) factor structure for the returns of all the assets that compose the investment world of the fund's manager [14, 37]. Factor-based models for hedge fund replication make a similar assumption but use Asset-Based Style (ABS) factors. While RBS factors describe risk factors, and are used to assess performance, ABS factors are directly selected with the purpose of being directly transposable into investment strategies. ABS factors have been used to take into account dynamic trading strategies with possibly nonlinear pay-off profiles [1, 16]. The idea of replicating a hedge fund's portfolio is therefore to take long and short positions in a set of ABS factors suitably selected so as to minimize the error with respect to the individual hedge fund or the hedge fund index.

A generic procedure for HF replication using factor models can therefore be decomposed in two steps. At step 1, one estimates a model of the HF returns as

$$r_k^{\mathrm{HF}} = \sum_{i=1}^{m} w^{(i)} r_k^{(i)} + \varepsilon_k$$

Given the estimated positions $\hat{w}^{(i)}$ on ABS factor $r^{(i)}$ resulting from step 1, step 2 simply constructs the "clone" of the hedge fund by

$$r_k^{\mathrm{Clone}} = \sum_{i=1}^{m} \hat{w}^{(i)} r_k^{(i)}$$

The factor-based approach is thus very intuitive and natural. There are however several caveats to this exercise.

Contrary to the passive replication of equity indices, the replication of hedge funds returns must take into account key unobservable determinants of hedge funds investment strategies such as the returns from the assets in the manager's portfolio; dynamic trading strategies; or the use of leverage [14, 16]. Recall that hedge funds returns do not share the characteristics of more classic investment vehicles, e.g. mutual funds, and are relatively uncorrelated to the main asset classes — see, e.g., [14, Figures 1 and 2, page 280] and [15]. Broadly, the factor models approach is subject to two types of difficulties. One is the in-sample explanatory power of step 1 described above being extremely low. Possible explanations for this are either an absence of systematic risk exposure of the HF industry, or the occurrence of model specification risk due to a faulty selection of the set of factors. Two, the out-of-sample replication is of poor quality. This last difficulty can result, for example, from the presence of noise in the calibration process in step 1, or from a violation of the implicit stationarity assumption of the time series in the model.

One avenue which has been extensively illustrated along the past decade [14, 4, 20, *inter alia*] was to work on the set of factors to include in the model. The underlying tenet of this stream of literature is that both in-sample and out-of-sample poor performances of the factor model are linked to the choice of the factors. Several rationales for different factor selections, including economic arguments and statistical methodologies, have been tested throughout the literature. For example, to different types of HF strategies (Convertible Arbitrage, Fixed Income Arbitrage, Event Driven, Long/Short Equity, etc.) different sets of factors have been proposed. Arguably, one possible reason behind the poor performance of linear factor models is the presence of non-observable dynamic trading strategies producing nonlinear HF return profiles which will not be captured in a linear framework. Thus, in complement to observable factors[1], generally corresponding to asset classes, one proposed methodology [1, 16] is to build synthetic factors corresponding to known trading strategies, including for example the writing of options on equity indices. By construction, these synthetic factors can exhibit nonlinear returns. This methodology thus attempts to render, by means of linear factor models, nonlinearities in the HF returns by modeling the nonlinearities in the synthetic factors. The use of these synthetic factors has been shown to improve the performance of the replicating factor models. There are, however, practical and theoretical difficulties to the use of such a methodology, as explained by Amin and Kat [6]:

> *"First, it is not clear how many options and which strike prices should be included [...] Second, since only a small number of ordinary puts and calls can be included, there is a definite limit to the range and type of non-linearities that can be captured."*

Nonlinearities in HF returns have also been assessed in a more direct approach. Recently, factor models including an option factor have been used to assess the nature and the extent of the presence of nonlinearities in hedge fund returns [11]:

$$r_k^{\mathrm{HF}} = \sum_{i=1}^{m} w^{(i)} r_k^{(i)} + \delta \max\left(r_k^{(1)} - s,\, 0\right) + \varepsilon_k$$

where $r_k^{(1)}$ is an equity index, and s represents the strike (moneyness) of the option. As noticed in [5], this methodology has so far not been implemented as a direct replication process, but rather as an assessment tool for HF investors.

The results in [11] are interesting. For the global index, they cannot reject the null hypothesis of linear returns, while at the category index level, they reject the hypothesis of linearity of returns only for the event-driven and managed futures categories at the 5% level, and for fixed-income arbitrage at the 10% level. Furthermore, testing fund by fund and correcting for possible data snooping, they demonstrate that the hypothesis of linear returns can be rejected for only one fifth of the whole universe of hedge funds reported in the Lipper/TASS database. Breaking down the universe into hedge-fund following arbitrage-based strategies (convertible arbitrage, fixed-income arbitrage, and event-driven), equity-market neutral and long-short strategies, and finally directional strategies (global macro, emerging markets, and managed futures), their results indicate that respectively only 20%, $10-15\%$, and 20% of these three groups exhibit significant nonlinearity with respect to the market return. As the whole, these results suggest that looking at the indexes can be misleading. Moreover, theoretically, dynamic trading of standard assets results in nonlinear return profiles for perfect market timers [26] suggesting that rather than nonlinear factors, one should focus on models capable of capturing the dynamic allocation of HF managers. And, while not ruling out the necessity to model nonlinearities in factor models in some cases, they underline the fact that linear models are most of the time appropriate.

Besides the work on the set of factors, the literature also examined other issues whose results can be summarized in the following way. Overall, on a general basis, linear factor models fail the test of robustness — for a good review see [4] — giving poor out-of-sample results. It

[1] One set of factors often used can be found in [20].

seems, however, that an economic selection of the factors provides significant improvement of the out-of-sample tracking error of the clone hedge fund over other statistical methodologies.

Fairly recently, attempts to capture the dynamic nature of the HF portfolio allocation have been explored in the literature. One method, used extensively [17, 20, 25, 21, *inter alia*], is to use rolling-windows OLS where the coefficients $\left\{w_k^{(i)}\right\}$ at time t_k are estimated by running the OLS regressions of $\left\{r_\ell^{\text{HF}}\right\}_{\ell=k-L}^{k-1}$ on the set of factors $\left\{r_\ell^{(i)}\right\}_{\ell=k-L}^{k-1}$ for $i = 1, \ldots, m$. A common choice for the window length L is 24 months, even though one could consider a longer time-span trading-off the dynamic character of the coefficients for more stable and more robust estimates.

By means of an example, Roncalli and Teiletche [33] have demonstrated however that the OLS-rolling window methodology captures poorly the dynamic allocation in comparison with the Kalman filter (KF). The use of KF estimation however requires caution in its implementation, making the estimation of the positions $\left\{w_k^{(i)}\right\}$ a non-trivial affair. These issues have been explored *inter alia* by Roncalli and Weisang [35]. They further argue that hedge fund replication using the Kalman filter is a viable and practical investment alternative to aggregates of hedge funds, like funds of hedge funds, if one take a core/satellite approach to the construction of the replicating portfolio.

Markov Regime-Switching models have also been considered — see, e.g. [5]. The idea therein is that HF managers switch from a type of portfolio exposure to another depending on some state of the world, assumed to be discrete in nature. One possible interpretation is to consider that the active management consists of changing the asset allocation depending on two states of the economy (high and low). Justifying the number of states or their interpretation is however tricky.

2.2 Method

This section exposes the framework developed in [35], and then provides a brief exposé of the tools that we will use to explore the non Gaussian and non linear cases of hedge fund replication.

2.2.1 Definition of the tracking problem

We follow [7] and [31] in their definition of the general tracking problem. We note $\mathbf{x}_k \in \mathbb{R}^{n_x}$ the vector of states and $\mathbf{z}_k \in \mathbb{R}^{n_z}$ the measurement vector at time index k. In our setting, we assume that the evolution of $\mathbf{x}_k$ is given by a first-order Markov model:

$$\mathbf{x}_k = f\left(t_k, \mathbf{x}_{k-1}, \nu_k\right)$$

where f is a non-linear function and ν_k a noise process. In general, the state $\mathbf{x}_k$ is not observed directly, but partially through the measurement vector $\mathbf{z}_k$. Thus, it is further assumed that the measurement vector is linked to the target state vector through the following measurement equation:

$$\mathbf{z}_k = h\left(t_k, \mathbf{x}_k, \eta_k\right)$$

where h is a non-linear function, and η_k is a second noise process independent from ν_k. Our goal is thus to estimate $\mathbf{x}_k$ from the set of all available measurements $\mathbf{z}_{1:k} = \{\mathbf{z}_i, i = 1, \ldots, k\}$. The goal in a tracking problem is to estimate the state variable $\mathbf{x}_k$, the current state of the system at time t_k, using all available measurement $\mathbf{z}_{1:k} = \{\mathbf{z}_\ell\}_{\ell=1:k}$.

Remark 1 *In the rest of the paper, the following system will be referred to as a tracking problem (henceforth TP):*

$$\begin{cases} \mathbf{x}_k = f\left(t_k, \mathbf{x}_{k-1}, \nu_k\right) \\ \mathbf{z}_k = h\left(t_k, \mathbf{x}_k, \eta_k\right) \end{cases} \tag{1}$$

2.2.2 Link Between HF replication and Tracking Problems

Similarly to [35], we decompose the return of a hedge fund into two components

$$r_k^{(\mathrm{HF})} = \underbrace{\sum_{i=1}^{m} w_k^{(i)} r_k^{(i)}}_{\text{TAA ABS factors}} + \underbrace{\sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}}_{\text{HF ABS factors}} \tag{2}$$

where TAA stands for Tactical Asset Allocation. TAA is a type of investment strategies that attempt to exploit short-term market inefficiencies by establishing positions in an assortment of markets with a goal to profit from relative movements across those markets. These top-down strategies focus on general movements in the market rather than on performance of individual securities.
However, besides TAA, hedge fund managers may invest in a larger universe that encompass TAA but also includes other alternative investment assets and strategies:

- stock picking strategies (which may be found in equity market neutral, long/short event driven hedge funds);
- high frequency trading;
- non-linear exposures using derivatives;
- illiquid assets (corresponding to distressed securities, real estate or private equity).

The idea of HF replication, in particular to create investment vehicles, is to replicate the first term on the RHS of (2). If we note $\eta_k = \sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}$, then HF replication can be described as a TP:

$$\begin{cases} \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ r_k^{(\mathrm{HF})} = \mathbf{r}_k^\top \mathbf{w}_k + \eta_k \end{cases} \tag{3}$$

2.2.3 Capturing Tactical Allocation with Bayesian Filters

The prior density of the weight vector (state vector) at time k is given by the Chapman-Kolmogorov equation

$$p\left(\mathbf{w}_k \mid r_{1:k-1}^{\mathrm{HF}}\right) = \int p\left(\mathbf{w}_k \mid \mathbf{w}_{1:k-1}\right) p\left(\mathbf{w}_{k-1} \mid r_{1:k-1}^{\mathrm{HF}}\right) \mathrm{d}\mathbf{w}_{k-1} \tag{4}$$

where we used the fact that our model is a first-order Markov model to write $p\left(\mathbf{w}_k \mid \mathbf{w}_{1:k-1}, \mathbf{r}^{\mathrm{HF}}{}_{1:k-1}\right) = p\left(\mathbf{w}_k \mid \mathbf{w}_{1:k-1}\right)$. This equation is known as the *Bayes prediction step*. It gives an estimate of the probability density function of $\mathbf{w}_k$ given all available information until $k-1$. At time k, as a new measurement value $r_k^{(\mathrm{HF})}$ becomes available, one can update the probability density of $\mathbf{w}_k$

$$p\left(\mathbf{w}_k \mid r_{1:k}^{(\mathrm{HF})}\right) \propto p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k\right) p\left(\mathbf{w}_k \mid r_{1:k-1}^{(\mathrm{HF})}\right) \tag{5}$$

This equation is known as the *Bayes update step*. The Bayesian filter corresponds to the system of the two recursive equations (4) and (5). In order to initialize the recurrence algorithm, we assume the probability distribution of the initial state vector $p(\mathbf{w}_0)$ to be known.

Using Bayesian filters, we do not only derive the probability distributions $p\left(\mathbf{w}_k \mid r_{1:k-1}^{(\mathrm{HF})}\right)$ and $p\left(\mathbf{w}_k \mid r_{1:k}^{(\mathrm{HF})}\right)$, but we may also compute the best estimates $\hat{\mathbf{w}}_{k|k-1}$ and $\hat{\mathbf{w}}_{k|k}$ which are given by

$$\hat{\mathbf{w}}_{k|k-1} = \mathbb{E}\left[\mathbf{w}_k \mid r_{1:k-1}^{(\mathrm{HF})}\right] = \int \mathbf{w}_k p\left(\mathbf{w}_k \mid r_{1:k-1}^{(\mathrm{HF})}\right) \mathrm{d}\mathbf{w}_k$$

and

$$\hat{\mathbf{w}}_{k|k} = \mathbb{E}\left[\mathbf{w}_k \mid r_{1:k}^{(\mathrm{HF})}\right] = \int \mathbf{w}_k p\left(\mathbf{w}_k \mid r_{1:k}^{(\mathrm{HF})}\right) \mathrm{d}\mathbf{w}_k$$

Remark 2 *In this paper, we used two type of Bayesian filters to solve* (3)*: Kalman filters and particle filters to conduct our research. All the computations done in this paper on particle filters have been done using the public domain Gauss library PF [34] with 50000 particles whereas we have used the Gauss library TSM [32] for Kalman filter.*

Particle filters Particle filtering methods are techniques to implement recursive Bayesian filters using Monte-Carlo simulations. The key idea is to represent the posterior density function by a set of random samples with associated weights and to compute estimates based on these samples and weights [7, 22, 28, 29, 30, 31]. As the samples become very large $N_s \gg 1$, this Monte-Carlo approximation becomes an equivalent representation on the functional description of the posterior pdf. To clarify ideas[2], let $\{\mathbf{w}_k^s, \omega_k^s\}_{s=1}^{N_s}$ denotes a set of support points $\{\mathbf{w}_k^s, s = 1, \ldots, N_s\}$ and their associated weights $\{\omega_k^s, s = 1, \ldots, N_s\}$ characterizing the posterior density $p\left(\mathbf{w}_k \mid r_{0:k}^{(\mathrm{HF})}\right)$. The posterior density at time k can then be approximated as:

$$p\left(\mathbf{w}_k \mid r_k^{(\mathrm{HF})}\right) \approx \sum_{s=1}^{N_s} \omega_k^s \delta\left(\mathbf{w}_k - \mathbf{w}_k^s\right) \tag{6}$$

We have thus a discrete weighted approximation to the true posterior distribution. One common way of choosing the weights is by way of *importance sampling* — see for example [7, 22, 28, 31]. This principle relies on the following idea. In the general case, the probability density $p\left(\mathbf{w}_k \mid r_k^{(\mathrm{HF})}\right)$ is such that it is difficult to draw samples from it. Assume for a moment that $p(\mathbf{w}) \propto \pi(\mathbf{w})$ is a probability density from which it is difficult to draw sample from, but for which $\pi(\mathbf{w})$ is easy to evaluate. Hence, up to proportionality, so is $p(\mathbf{w})$. Also, let $\mathbf{w}^s \sim q(\mathbf{w})$ be samples that are easily drawn from a proposal $q(\cdot)$, called an *importance density*. Then, similarly to 6, a weighted approximation of the density $p(\cdot)$ can be obtained by using:

$$p(\mathbf{w}) \approx \sum_{s=1}^{N_s} \omega^s \delta\left(\mathbf{w} - \mathbf{w}^s\right)$$

where:

$$\omega^s \propto \frac{\pi(\mathbf{w}^s)}{q(\mathbf{w}^s)}$$

[2]Note that the succinct presentation given here of particle filters is adapted to our first-order Markovian framework.

is the normalized weight of the s-th particle. Thus, if the samples $\{\mathbf{w}_k^s\}$ were drawn from a proposal density $q\left(\mathbf{w}_k \mid r_k^{(\mathrm{HF})}\right)$, then the weights in (6) are defined to be:

$$\omega_k^s \propto \frac{p\left(\mathbf{w}_k^s \mid r_k^{(\mathrm{HF})}\right)}{q\left(\mathbf{w}_k^s \mid r_k^{(\mathrm{HF})}\right)} \tag{7}$$

The PF sequential algorithm can thus be subsumed in the following steps. At each iteration, one has samples constituting an approximation of $p\left(\mathbf{w}_{k-1}^s \mid r_{k-1}^{(\mathrm{HF})}\right)$ and wants to approximate $p\left(\mathbf{w}_k^s \mid r_k^{(\mathrm{HF})}\right)$ with a new set of samples. If the importance density can be chosen so as to factorize in the following way:

$$q\left(\mathbf{w}_k \mid r_k^{(\mathrm{HF})}\right) = q\left(\mathbf{w}_k \mid \mathbf{w}_{k-1}, r_k^{(\mathrm{HF})}\right) \times q\left(\mathbf{w}_{k-1} \mid r_{k-1}^{(\mathrm{HF})}\right) \tag{8}$$

then one can obtain samples $\{\mathbf{w}_k^s\}$ by drawing samples from $q\left(\mathbf{w}_k^s \mid r_k^{(\mathrm{HF})}\right)$. To derive the weight update equation:

$$\begin{aligned}
p\left(\mathbf{w}_k | r_k^{(\mathrm{HF})}\right) &= \frac{p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k, r_{k-1}^{(\mathrm{HF})}\right) \times p\left(\mathbf{w}_k \mid r_{k-1}^{(\mathrm{HF})}\right)}{p\left(r_k^{(\mathrm{HF})} \mid r_{k-1}^{(\mathrm{HF})}\right)} \\
&= \frac{p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k, r_{k-1}^{(\mathrm{HF})}\right) \times p\left(\mathbf{w}_k \mid \mathbf{w}_{k-1}, r_{k-1}^{(\mathrm{HF})}\right)}{p\left(r_k^{(\mathrm{HF})} \mid r_{k-1}^{(\mathrm{HF})}\right)} \times p\left(\mathbf{w}_{k-1} \mid r_{k-1}^{(\mathrm{HF})}\right) \\
&= \frac{p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k\right) \times p\left(\mathbf{w}_k \mid \mathbf{w}_{k-1}\right)}{p\left(r_k^{(\mathrm{HF})} \mid r_{k-1}^{(\mathrm{HF})}\right)} \times p\left(\mathbf{w}_{k-1} \mid r_{k-1}^{(\mathrm{HF})}\right) \\
&\propto p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k\right) \times p\left(\mathbf{w}_k \mid \mathbf{w}_{k-1}\right) \times p\left(\mathbf{w}_{k-1} \mid r_{k-1}^{(\mathrm{HF})}\right)
\end{aligned} \tag{9}$$

By substituting (8) and (9) into (7), the weight equation can be derived to be:

$$\omega_k^s \propto \omega_{k-1}^s \frac{p\left(r_k^{(\mathrm{HF})} \mid \mathbf{w}_k^s\right) \times p\left(\mathbf{w}_k^s \mid \mathbf{w}_{k-1}^s\right)}{q\left(\mathbf{w}_k^s \mid \mathbf{w}_{k-1}^s, r_k^{(\mathrm{HF})}\right)} \tag{10}$$

and the posterior density $p\left(\mathbf{w}_k | r_k^{(\mathrm{HF})}\right)$ can be approximated using (6). We refer the reader to [7] for a more detailed but concise exposé of the differences between the different PF algorithms: sequential importance sampling (SIS), generic particle filter, sampling importance resampling (SIR), auxiliary particle filter (APF), and regularized particle filter (RPF). We provide a succinct exposé of the SIS, SIR algorithms as well as the generic particle filter's and the regularized particle filter's in Appendix A. One important feature of PF is that not one implementation is better than all the others. In different contexts, different PFs may have wildly different performances.

3 Hedge Fund Replication: the Non-Gaussian and Non-Linear Case

As seen in [33, 35], HF replication using the KF can provide good results, making it possibly the best method so far to estimate and implement HF clones. However, one may question the wisdom of using a Gaussian linear framework. Indeed, the distributions of HF returns are well known to exhibit skewness and excess kurtosis, and nonlinear effects have been documented in HF returns ever since the seminal paper of Fung and Hsieh in 1997 [14]. In the following section, we relax the Gaussian and linear assumptions. Our goal here is less to provide an "off-the-shelf" solution to the problem of replication than to examine the impact of each assumptions on the quality of the replication. Note that some of the methods (especially those requiring particle filters) used in the following section require some careful implementation, as well as time to be carried out. The plan of the section is the following. We start by looking at the Gaussian distribution assumption. In a second time, we look at the problem of nonlinear assets. Our approach can then be decomposed into three main angles: replicating nonlinear assets; the use of option factors that are determined in a manner exogenous to the replication procedure; and finally, a very general and inclusive approach to the replication procedure using nonlinear assets.

3.1 A fundamental example

We consider the example of replicating the HFRI index as in [33]. The base model considered (6F) is

$$\left\{\begin{array}{l} r_k^{(F)} = \sum_{i=1}^{6} w_k^{(i)} r_k^{(i)} + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ Q_k = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_m^2\right) \end{array}\right. \tag{11}$$

The set of factors that served as a basis for this exercise is: an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a bond position in the 10-years US Treasury (UST) and a FX position in the EUR/USD.

3.2 The Gaussian distribution assumption

The Gaussian distribution is a fundamental assumption to the optimality of the use of KF for HF replication (or for rolling OLS regression as it is). It is however well known that return distributions of hedge funds exhibit negative skewness and positive kurtosis, rendering the use of a Gaussian framework faltering, and requiring at least inquiring into its adequacy. Moreover, one of the attractive features of the approach advocated by Kat [23, for a recent exposé] is to take into account in the replication process stylized facts — such as higher moments of the returns distribution, in particular skewness and kurtosis — in order to provide investors a more accurate exposition to the risk-return profile of the HF industry. Given the relative success of replicating hedge funds using the KF, it may not be necessary to introduce nonlinearities in the factors or in the model's structure to obtain a better replication process. A simple relaxation of the Gaussian assumption, particularly by taking into account the third and fourth moments of the distributions, may be enough to improve the results significantly.

To illustrate the departure from the Gaussian assumption, we reported in Figure 1 three comparative graphics using the results of the 6F model presented in the paragraph above. The top graphic compares the probability density function of the tracking errors $\hat{\mathbf{e}}_k$ obtained using KF (blue line) against a Gaussian approximation of the same density function (dashed green line). The bottom-left graphic compares the probability density function of the HFRI index returns

$r_k^{(\mathrm{RF})}$ (blue line) against the probability density function of the returns of the replicating clone r_k^{Clone} (dashed green line). The bottom-right graphic compares the probability density function of the clone's returns (blue line) against a Gaussian approximation of the same distribution (dashed green line). One can make several comments on Figure 1. First, as illustrated, there is a clear violation of the Gaussian assumption for all three of the estimated distributions. However, not all departures are of the same magnitude. It is obvious that the Clone's distribution is the closest to a Gaussian distribution, probably as a consequence of the KF procedure. Most of the departure of the HFRI returns distribution seems to remain in the tracking error. Thus, in the following, we relax the Gaussian assumption on the distribution of the tracking errors, while keeping the rest of the model's structure (Gaussian innovations of the state variables and linear evolution equation).

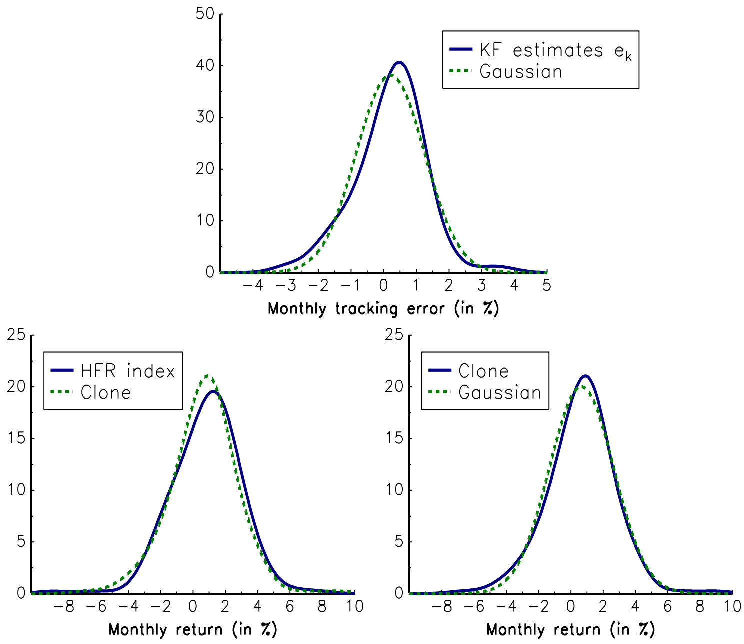


Figure 1: Departure from the Gaussian distribution assumption

Top: kernel estimate of the density function of the tracking error $\hat{\mathbf{e}}_k$ (blue line) vs. Gaussian approximation (dashed green line);

Bottom-left: kernel estimate of the density function of the HFRI index returns $r_k^{(\mathrm{RF})}$ (blue line) vs. kernel estimate of the density function of the replicating clone's returns r_k^{Clone} (dashed green line);

Bottom-right: kernel estimate of the density function of the replicating clone's returns r_k^{Clone} (blue line) vs. Gaussian approximation (dashed green line).

This extended tracking problem can be formalized as:

$$\begin{cases} r_k^{(F)} = \sum_{i=1}^m w_k^{(i)} r_k^{(i)} + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \end{cases}$$

with η_k a general noise process with distribution $\mathcal{H}$. In the following, we assume that $\mathcal{H}$ is a Skew t distribution $\mathcal{ST}(\mu_\eta, \sigma_\eta, \alpha_\eta, \nu_\eta)$, obtained by perturbing a Student t distribution (for further details, cf. [8]). We hope to better capture the higher moments of the HFRI returns distribution. One could consider this methodology as one possible step toward incorporating some of the "sexiest" features of Kat's approach to the robust factor models approach. Note however that, since the returns are not normally distributed, we must resort to using particle filters to obtain the estimates $\hat{\mathbf{w}}_{k|k-1}$. The unknown parameters to estimate are $\theta = \{\sigma_1, \ldots, \sigma_m, \sigma_\eta, \alpha_\eta, \nu_\eta\}$. We consider two estimation methods.

(PF #1) A two-steps procedure consisting of a run of the KF algorithm to obtain ML estimates of $\{\hat{\sigma}_i, i = 1, \ldots, m\}$, followed by an ML estimation of the parameters $\sigma_\eta,\ \alpha_\eta,\ \nu_\eta$[3] of the Skew t distribution based on the tracking errors of the KF run.

(PF #2) A generalized method of moments (GMM) estimation procedure where the $m+3$ parameters are estimated together. The $m+3$ moments conditions are given by:

- the first moment considered is $m_{k,1} = e_k$ because we favor smaller tracking errors;
- the next moments are chosen to impose an orthogonality condition between the tracking error e_k and the return $r_k^{(j)}$ of the j^{th} asset: $m_{k,j+1} = e_k r_k^{(j)},\ \ j = 1, \ldots, m$;
- The last two moments take into account the skewness and kurtosis of the hedge fund returns. They are defined as $m_{k,m+n-1} = \left(r_k^{\text{Clone}} - \bar{r}^{\text{Clone}}\right)^n - \mu_n$ $(n = 3, 4)$ where μ_n is the empirical n^{th} central moment of $r_k^{(\text{HF})}$.

The statistics of the resulting clones obtained[4] are given in Table 1. Note that the estimation procedure using the GMM method is unfortunately extremely long and does not always converge to a solution. Compared to the KF results (LKF), notice that we obtain better results for the performance ($\hat{\mu}_{1\text{Y}}$), but the volatility of the trackers' returns ($\hat{\sigma}_{1\text{Y}}$) and the standard deviation of tracking errors (σ_{TE}) are also higher, providing only a small improvement in terms of Sharpe ratios. The results on skewness and kurtosis are clearly disappointing as the sample values are comparable to those obtained by the KF estimation. One possible explanation for these poor results is that GMM makes a trade-off between the first-moment condition (maximizing π_{AB}) and the last two moment conditions (matching skewness and kurtosis). It does not mean however that building clones with more kurtosis and negative skewness is not possible. Let's consider for example a third set of estimates for the parameters of the Skew t distribution

(PF #3) The estimates are those of (PF #2) except for the parameter $\hat{\alpha}_\eta$ which is forced to -10.

As reported in Table 1, in this case, the tracker's returns present higher kurtosis but the tracking error's volatility is higher too. Other possible explanations for the poor success of these

[3] By assumption, $\mu_\eta = 0$.

[4] The estimated values for the parameters are reported in the following table:

	PF #1	PF #2	PF #3
$\hat{\sigma}_1$	0.023	0.037	0.037
$\hat{\sigma}_2$	0.015	0.016	0.016
$\hat{\sigma}_3$	0.040	0.044	0.044
$\hat{\sigma}_4$	0.021	0.014	0.014
$\hat{\sigma}_5$	0.027	0.022	0.022
$\hat{\sigma}_6$	0.025	0.026	0.026
$\hat{\sigma}_\eta$	0.009	0.003	0.003
$\hat{\alpha}_\eta$	-1.131	-1.130	-10.00
$\hat{\nu}_\eta$	3.738	3.757	3.757

methods are the Gaussian dynamics of the state variables or the lack of non-linear exposures in the tracker. It is for now difficult to test for the first hypothesis as the number of parameters to estimate would grow significantly — it would be a 6-variate Skew t distribution on the state vector — and the execution time of the procedure would be absurdly long. As for the second hypothesis, we address it in the rest of this section on HF replication in the non-Gaussian non-linear case.

Table 1: Results with a Skew t distribution $\mathcal{ST}(0, \sigma_\eta, \alpha_\eta, \nu_\eta)$

The different statistics reported are $\hat{\mu}_{1\mathrm{Y}}$ the annualized performance; $\hat{\sigma}_{1\mathrm{Y}}$ the yearly volatility; s the sharpe ratio; γ_1 the skewness; γ_2 the excess kurtosis; π_{AB} the proportion of the HF index performance explained by the tracker and σ_{TE} the yearly tracking error. ρ, τ and ϱ are respectively the linear correlation, the Kendall tau and the Spearman rho between the monthly returns of the HF index and the tracker. All statistics are expressed in percents.

	$\hat{\mu}_{1\mathrm{Y}}$	$\hat{\sigma}_{1\mathrm{Y}}$	s	γ_1	γ_2
HF	9.94	7.06	0.77	-0.57	2.76
LKF	7.55	6.91	0.45	-0.02	2.25
PF #1	7.76	7.44	0.45	-0.03	2.02
PF #2	7.57	7.28	0.43	-0.11	1.93
PF #3	6.90	7.99	0.31	-0.57	2.88
	π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF	75.93	3.52	87.35	67.10	84.96
PF #1	78.09	4.03	84.71	63.49	81.94
PF #2	76.13	4.25	82.51	61.60	80.20
PF #3	69.43	5.11	77.62	54.75	73.55

3.3 Taking into account non-linear assets

Considering non-linear assets as factors in the replication model does not change the structure of the TP system. It suffices to notice that by considering a universe of factors composed of respectively m_1 and m_2 linear and nonlinear assets, the TP can still be written as:

$$\begin{cases} r_k^{(F)} = \sum_{i=1}^{m_1} w_k^{(i)} r_k^{(i)} + \sum_{i=m_1+1}^{m_1+m_2} w_k^{(i)} r_k^{(i)} + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k, \end{cases}$$

and even though some factors are "nonlinear" assets, the exposures $w_k^{(i)}$ are still linear and the TP system may be solved in the same way as in the previous section. The difficulty however with non-linear assets is to price the corresponding strategy. There are often only two possibilities:

1. Build ourselves the non-linear strategy. In this case, we have to calibrate the different parameters of the model, compute the backtest and use the backtest of the strategy as the non-linear factor.

2. Use custom indexes provided by investment banks like JP Morgan, Goldman Sachs, etc.

The second solution is often easier to implement because the first method assumes that we have the capacities to trade the strategy. It may however introduce biases because the performance of the index taken as factor depends on the proprietary strategy and on the market data of the index provider.

One must say that this methodology is certainly not new, and has been used, under one form or another by various authors [1, 2, 16, *inter alia*], with a relative success in increasing the explanatory power of the replication model. However, considering the difficulty of pricing such nonlinear assets, the question of whether the inclusion of a nonlinear asset can susceptibly provide a better replication methodology is of particular value. Indeed, the argument has been made that the component of HF returns due to non-linear assets in their portfolios can be partially replicated using the alternative beta methodology presented above since options may be replicated by delta hedging, i.e. taking linear positions in standard assets. We examine this claim in the following section before considering the introduction of option factors in the model.

3.3.1 Replicating non-linear assets

As argued above, since options may be replicated by delta hedging, the component of HF returns due to non-linear assets could theoretically be partially replicated by alternative beta. The argument however is more relevant as a marketing strategy for brokers of HF replicators than truly robust. To illustrate our claim, we provide below an example of the replication of a non-linear asset whose underlying strategy is well known using a Kalman filter and the methodology presented above. We consider the replication of the CBOE S&P 500 BuyWrite index more commonly known under the name BXM. The description of the BXM is the following[5]:

> *The BXM is a passive total return index based on buying an S&P 500 stock index portfolio, and selling the near-term S&P 500 Index call option, generally on the third Friday of each month. The SPX call written will have about one month remaining to expiration, with an exercise price just above the prevailing index level (i.e., slightly out of the money). The SPX call is held until expiration and cash settled, at which time a new one-month, near-the-money call is written.*

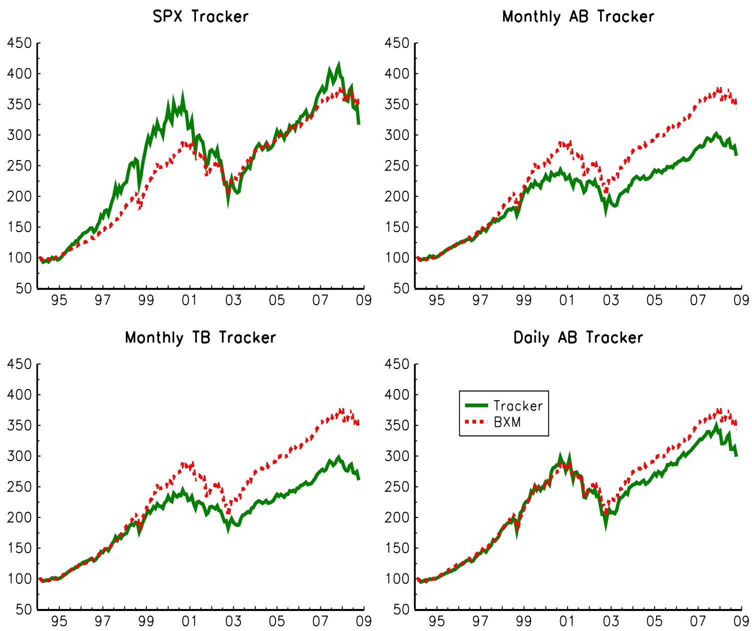
One may wonder if the replication of this non-linear asset with linear exposures on the S&P 500 index provides satisfying results, which would support the alternative beta argument for nonlinear asset replication. We consider 4 replicating portfolios (trackers):

1. A long position on the SPX index.
2. An alternative beta (AB) tracker using a monthly rebalacing method.
3. A portfolio consisting of a position of 57.9% on SPX futures and a position of 100% in cash. The 57.9% figure corresponds to the average value of the dynamic alternative beta. This tracker provides the traditional beta (TB) tracker as a benchmark for the AB trackers.
4. Finally, we consider an alternative beta (AB) tracker using a daily rebalacing method. This replicating portfolio uses the same methodology as the second tracker presented but at a higher frequency for purpose of comparing the two.

[5] This definition is taken from the website of the Chicago Board Options Exchange (CBOE) at http://www.cboe.com/micro/bxm/introduction.aspx.

The results of the replicating portfolios are reported in Figure 2 and Table 2. Notice that the use of a monthly rebalancing dynamic portfolio does not provide better results than using a constant beta portfolio. In order to improve the results, we have to use a higher frequency, a daily rebalancing period in our example. What happens? Let's call δ_k the *delta* of the hedging portfolio of the written call. The estimated weight $\hat{w}_k$ at time k may be approximated by $1-\delta_{k-1}$ where δ_{k-1} is the delta at time $k-1$. When one writes the call ATM option with one month of remaining to expiration, $\hat{w}_k$ is close to 50%. During the life of the option, the change in δ_k, and thus in $\hat{w}_k$, is relatively smooth and at the expiration date of the option, δ_k is equal to 1 or 0 depending if one exercises the option or not. If the frequency is daily, the estimated weight $\hat{w}_k$ will reflect this behavior varying smoothly everyday. If the frequency is monthly, δ_k is independent from δ_{k-1}. In this case, it is more difficult to replicate the BXM index and that explains that the monthly AB tracker has a comparable volatility of tracking errors than the TB tracker (with fixed weights $\hat{w}_k = 0.579$).

Figure 2: Tracking the BXM index



Now, there are several lessons to learn from this example. First, and it seems to provide support to the claim hereby tested, it is true that one can replicate option-based strategy by means of a dynamic alternative beta replication procedure. However, one must not forget that at best, in most cases, HF returns are only available on a monthly basis. As such, as we have seen, nonlinearities are not amenable to be replicated using only linear positions in standard assets. This explains why, even if its implementation is sometimes questionable because of its inherent difficulty, we believe that the presence of nonlinearities, when attested, often calls for the use of nonlinear asset factors. Before exploring this avenue in the rest of this section, there is one more comment to make which will be useful in the last section of this paper when we consider the access to the alpha of the HF strategy. Recall that we defined in section **??** the alternative alpha as the unexplained residual of the replicated strategy. Recall also that when we examined above the Gaussian assumption, most of the departure was captured by the distribution of the

Table 2: Statistics of BXM trackers

	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
BXM	8.69	9.92	0.43	-1.17	3.53
SPX	8.18	14.29	0.26	-0.58	0.62
AB (monthly)	6.90	8.55	0.29	-0.49	1.09
TB	6.75	8.15	0.29	-0.57	0.59
AB (daily)	7.75	9.91	0.34	-0.75	2.11
	π_{AB}	σ_{TE}	ρ	τ	ϱ
SPX	94.09	7.44	87.21	64.86	82.91
AB (monthly)	79.40	5.10	85.81	63.79	82.06
TB	77.69	4.88	87.20	64.96	83.02
AB (daily)	89.22	3.40	94.13	69.29	85.46

tracking error. The point here is simply that, it appears if there are non-linear assets in the HF portfolio, a substantial part of the introduced nonlinearities will remain uncaptured and will appear in the alternative alpha. It thus prompts the thought that some of the alpha's performance is not accessible because its replication requires trading at high frequencies.

3.3.2 Using option factors with exogenous strikes

We now consider the linear factors (henceforth LF) model to which we add one non-linear asset factor. The tracking problem becomes:

$$\begin{cases} r_k^{(\mathrm{HF})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} + w_k^{(m+1)} r_k^{(m+1)}(s_k) + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \end{cases}$$

where $r_k^{(m+1)}(s_k)$ is the return of a systematic one-month option selling[6] strategy on S&P 500 and s_k is the (exogenous) strike of the option at time k. Different values of the strike (moneyness) were implemented by taking the arbitrary values 95%, 100%, 105%. Note that, in this case, the TP system remains linear with respect to the state vector and we may solve it using Kalman filter. To price the option strategy, we used the Bloomberg's implied volatility data[7]. Results are reported in Table 3. It is interesting to note that even if we find non-linear factors (henceforth NLF) trackers with higher performance and Sharpe ratios than LF's, we do not obtain better results in terms of the volatility of the tracking error and the correlation between the HF index and the clone is not higher than the LF tracker.

Agarwal and Naik [2] find evidence that some hedge fund strategies exhibit the non-linear payoff structure described above. Diez de los Rios and Garcia [11] further find that there is statistical support for rejecting linearity only for a few categories (emerging markets, short bias and managed futures). If we consider the HFRI Macro and Relative Value indices, considering the results and the improvement in the proportion of HF returns explained, one may consider, in a first approach, that they exhibit this non-linear structure. However, these results are

[6]It is not necessary to consider an option buying strategy because we do not constrain the weights $w_k^{(m+1)}$ to be positive. So, a negative weight on the selling strategy is equivalent to a long position on the buying strategy.

[7]The corresponding Bloomberg's functions are HIST_CALL_IMP_VOL and HIST_PUT_IMP_VOL.

Table 3: Results of replicating the HFRI index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		9.94	7.06	0.77	-0.57	2.76
LKF		7.55	6.91	0.45	-0.02	2.25
	95%	7.61	6.93	0.46	-0.21	2.91
Call	100%	7.92	6.94	0.50	-0.22	2.92
	105%	8.14	6.88	0.54	-0.06	2.40
	95%	7.77	6.98	0.48	-0.22	3.35
Put	100%	8.15	6.97	0.53	-0.20	3.29
	105%	8.27	6.92	0.55	-0.04	2.60
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		75.93	3.52	87.35	67.10	84.96
	95%	76.62	3.55	87.12	65.48	83.56
Call	100%	79.74	3.58	86.95	65.53	83.61
	105%	81.90	3.55	87.07	66.81	84.57
	95%	78.20	3.49	87.68	66.61	84.33
Put	100%	81.98	3.53	87.40	66.60	84.22
	105%	83.21	3.48	87.62	67.68	85.21

Table 4: Results of replicating the HFRI Macro index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		9.53	6.95	0.72	0.10	0.95
LKF		6.67	5.97	0.38	0.36	2.06
	95%	6.70	5.39	0.43	0.03	1.07
Call	100%	6.85	5.58	0.44	0.20	1.06
	105%	7.47	5.68	0.54	0.03	1.45
	95%	6.81	5.77	0.42	-0.13	2.17
Put	100%	7.22	5.81	0.48	-0.11	1.83
	105%	7.99	5.95	0.60	-0.06	1.43
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		69.97	5.71	61.82	44.21	62.28
	95%	70.26	5.69	60.13	43.00	60.35
Call	100%	71.86	5.69	60.71	43.35	60.57
	105%	78.31	5.55	63.14	43.95	61.89
	95%	71.42	5.61	62.54	43.74	61.65
Put	100%	75.73	5.59	63.07	43.86	61.63
	105%	83.79	5.45	65.40	45.47	63.30

Table 5: Results of replicating the HFRI Relative Value index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		8.50	3.62	1.11	-2.76	15.22
LKF		5.77	2.75	0.52	0.16	1.18
	95%	6.44	2.97	0.69	-0.48	3.29
Call	100%	6.71	3.02	0.77	-0.46	3.69
	105%	7.13	2.97	0.92	0.12	1.97
	95%	6.75	4.28	0.55	-6.49	66.57
Put	100%	6.97	3.07	0.84	-0.84	6.06
	105%	6.81	2.85	0.85	0.24	1.80
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		67.94	3.07	56.46	40.48	55.74
	95%	75.76	3.01	59.68	38.38	53.10
Call	100%	78.99	3.05	58.74	37.10	51.05
	105%	83.88	3.20	54.06	38.58	53.30
	95%	79.49	3.38	64.42	38.94	54.77
Put	100%	82.00	3.03	59.74	37.60	52.40
	105%	80.13	3.15	54.63	39.26	54.44

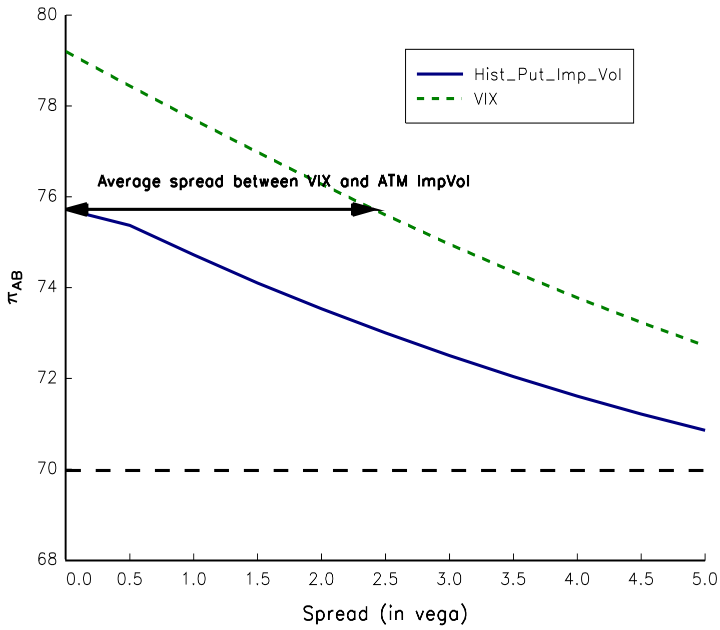
highly dependent on the quality of the data. If we take into account a spread (explained by the volatility skew and the bid/ask effect) between the published implied volatility and the traded volatility, the results are less favorable. Figure 5 presents the impact of the trading spread on the replication performance of the HFRI Macro index with selling put options at 105%. It is obvious that at this strike, the impact of the skew is very high and that we have to consider a high spread. In this case, the difference between the LKF model and the NLF model is not so important. This suggests that HF replication including nonlinear assets as an investment vehicle still remains difficult to implement as any replicating portfolio would be confronted to implementation noise and distortions.

So far, we have considered that the non-linear assets included in the replication model were priced independently of the strategy followed by the HF fund manager. In the next section, we relax this assumption by considering that the parameters of the nonlinear strategy are dependent on the HF manager's decisions. One word of caution must be given here. We do necessarily hope that the results of the following section will be directly applicable in a systematic replication procedure, but we do hope however that it will give us further insight into the risk structure of the replicating strategy as well as the manager's outlook on available investments.

3.3.3 Using option factors with endogenous strikes

One of the drawbacks of the previous model is that the strikes of the options are exogenous, and thus do not depend on the decisions of the manager. From the point of view of pure replication of the performance and the strategy of a HF manager, it could be seen as an obvious deficiency. Indeed, a HF manager can adapt option strikes to his or her tactical bets and macroeconomic views. To our knowledge, there exist few academic works dealing with this problem directly, and when they do, strikes are considered constant over the period of study. In the spirit of the

Figure 3: Impact of the trading spread on the replication performance of the HFRI Macro index with selling ATM put options



alternative beta methodology presented previously, it is more realistic to assume that the option strikes are time-varying, and are part of the manager's general strategy. Thus, we propose in this section to consider option factors with endogenous strikes.

Before considering this difficult exercise, we must point out one necessary assumption that we need to make in our context. One could content that the argument above in favor of the use of endogenous strikes is dubious when applied to an aggregate of HF, as in the case of indices for example, for the idiosyncratic tactical bets of a particular manager are lost in the aggregation process. We must therefore make the assumption that the aggregation process results in an average strike reflective of the entire position of the underlying aggregate. This is not dissimilar to the argument we made earlier about the better fitness of replication methods to replicate aggregates over single HF. While the nonlinear character of the underlying panel of option exposures with different parameters (e.g. call and put, selling or buying positions, in-the-money or out-of-the-money strikes) does not easily lead to aggregation within the same class of nonlinear factors, for liquidity and trading reasons, we think it plausible that the time-varying strikes of a limited number of option factors on general asset classes can provide a good proxy for replication.

They are two possible ways to estimate these strikes. One possibility is to consider an econometric method to estimate the option strikes separately from the tracking problem. For example, we may first consider a macroeconomic model to estimate the strikes, then build the option factors using the time varying estimated strikes and finally estimate the linear exposures using the Kalman filter. In this case, the option strikes are endogenous in the sense that they have been estimated, but they are also exogenous in the sense that their estimation is independent from the TP system. Another possibility is to consider that the option strike belongs to the state vector of a nonlinear TP system. Thus, we obtain:

$$\begin{cases} \begin{pmatrix} \mathbf{w}_k \\ s_k \end{pmatrix} = \begin{pmatrix} \mathbf{w}_{k-1} \\ s_{k-1} \end{pmatrix} + \begin{pmatrix} \nu_k \\ \varepsilon_k \end{pmatrix} \\ r_k^{(\mathrm{HF})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} + w_k^{(m+1)} r_k^{(m+1)} (s_k) + \eta_k \end{cases} \tag{12}$$

where the measurement equation is nonlinear with respect to the strike state variable s_k. As in the non-Gaussian case, the nonlinearity of the system prevents us to use the KF, and we must resort to using PF. One of the difficulties is to estimate the unknown parameters of (12). As in the previous sections, we assume that $\eta_k \sim \mathcal{N}\left(0, \sigma_\eta^2\right)$ and:

$$\begin{pmatrix} \nu_k \\ \varepsilon_k \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & \sigma_s^2 \end{pmatrix} \right)$$

with $Q = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_m, \sigma_{m+1}^2\right)$. The vector of unkown parameters to estimate is then $\theta = \{\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_s, \sigma_\eta\}$.

The estimation of parameters in the context of nonlinear TP is not a trivial problem, and the literature while expanding is still relatively scarce (see for example [12, 9, 36, 27, 38]). Because of the nonlinearities included in the TP, general methods are overwhelmingly based on an extension of the Expectation-Maximization principle using discrete approximations of the different densities by means of particles. Moreover, the examples generally considered consist of a small number of parameters to estimate and small number of particles. For example, Wills, Schön and Ninness [38] consider estimating the parameters of a stochastic volatility model, comprising only 3 parameters and using 50 particles over 10000 time periods. In our case however, we have $m + 3$ parameters (ie. 9 parameters if we use six linear factors) and a period of 177 observations. Moreover, the number of particles we need to use to satisfyingly duplicate the results of the KF in a linear framework is very high (generally more than 10000). Thus, with one additional nonlinear factor and three more parameters we cannot expect satisfying results for a lesser number of particles. We tried the ML methods described in the cited papers above. However, the task is still extremely difficult. The optimization step remains time-consuming and sensitive to the specified initial values and the number of particles. For this reason, we preferred the use a grid-based method to estimate the parameters, although it didn't solve all the implementation problems. For instance, let's note d_i the number of discretized values used for the parameter θ_i. A grid-based estimation requires that we run the PF algorithm d times with $d = \prod_{i=1}^{m+3} d_i$. Running a PF with 50000 particles takes about 30 seconds on our computers. So, with $d_i = 5$ and $m = 6$, it would take a little less than 2 years (678 days) to run until the end. This curse of dimensionality required of us to proceed yet again differently, and we decided to follow a two-step procedure:

1. the parameters σ_η and σ_i for $i = 1, \ldots, m$ are estimated using the method of maximum likelihood considering the linear factor model;

2. the last two parameters σ_{m+1} and σ_s are estimated using the grid-based method conditionally on the previous estimates. If f denotes the statistic of interest in the maximization (or minimization) of and if Ω denotes the set of grid points, we have:

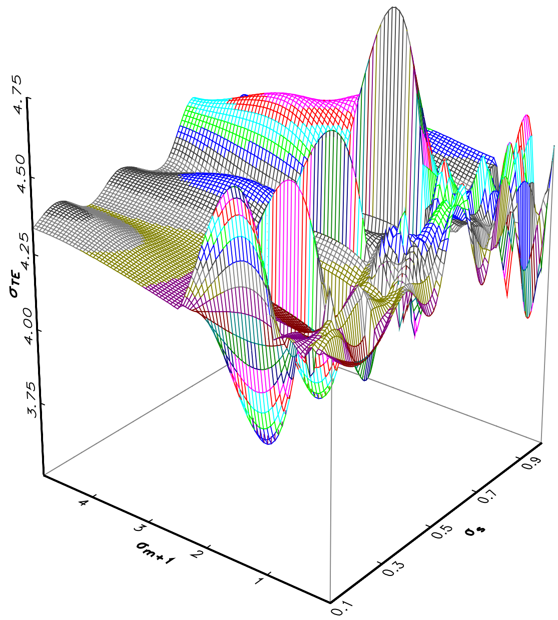
$$\{\hat{\sigma}_{m+1}, \hat{\sigma}_s\} = \arg\max f\left(\sigma_{m+1}, \sigma_s \mid \hat{\sigma}_1, \ldots, \hat{\sigma}_m, \hat{\sigma}_\eta\right) \quad \text{u.c.} \quad (\sigma_{m+1}, \sigma_s) \in \Omega.$$

This approach is obviously biased compared to a full estimation of all the parameters. However, with respect to the linear factor model, it provides consistent results which could help explain the remaining alpha.

In what follows, we present two examples using the HFRI index and the HFRI Relative Value index. Both examples considered a put option on the S&P500 index. In the second step of our procedure, we chose to minimize the volatility σ_{TE} of the tracking errors. In Figure 4, we report the statistic of interest for the HFRI index with respect to σ_{m+1} and σ_s. Notice that the surface does not present an obvious minimum. Moreover, we remark that the volatility of the tracking error is above 3.52% which is the corresponding statistic for the linear model. Thus, using endogenous strikes does not improve the volatility of the tracking errors. For the HFRI Relative Value index, we obtain more convincing results. First, notice that the surface in Figure 5 presents a more convex function profile. And we estimate that the minimum is reached for $\sigma_{m+1} = 2.5\%$ and $\sigma_s = 1\%$.

We reported the exposures $w_k^{(i)}$ and the strike s_k of the put option in Figure 6. In the case of a fixed strike, notice that the exposure on the put option is very volatile. This is not the case when the strike is endogenous. The results suggest that HF managers are globally selling ITM put options. However, one major difficulty which is not taken into account here is the effect of the volatility's smile, and possible liquidity limitations.

Figure 4: Grid approach applied to the HFRI index



4 Conclusion

In this paper, we presented a formal framework for hedge fund replication by introducing the notion of tracking problems which may be solved using Bayesian filters. We extended the

Figure 5: Grid approach applied to the HFRI RV index

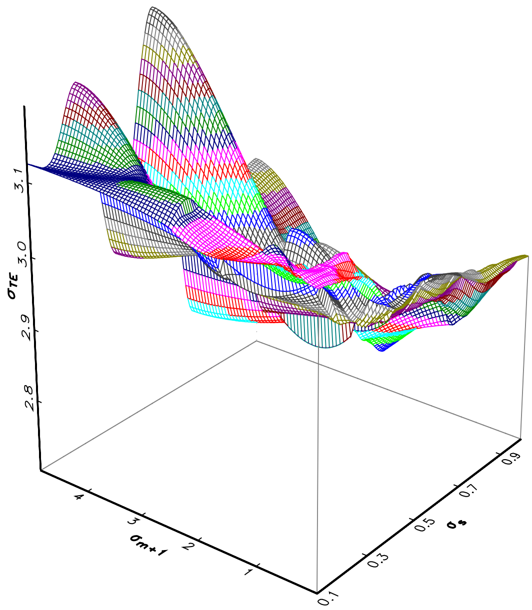
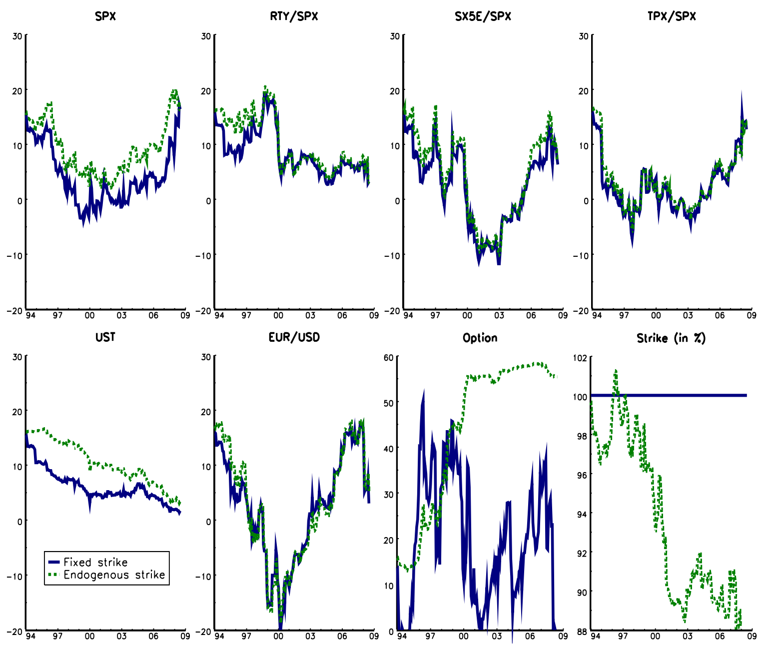


Figure 6: Exposures and option strikes for the HFRI RV index



methodology of [35] to non-Gaussian nonlinear cases using particle filters. These advanced tracking techniques were used first to capture some stylized facts of HF returns, like negative skewness and excess kurtosis. They further enabled us to estimate endogenous option strikes in an attempt to capture non-linear exposures. The results obtained using particle filters are to some extent disappointing. First, it seems that matching higher moments of HF returns implies a necessary trade-off with higher volatility of the tracking errors of the HF clone. Second, consistent with some recent findings in the literature, we found little evidence of the presence of nonlinearities in the distribution of the returns of the overall hedge fund strategies.

Nevertheless, we believe these results to be very interesting. From the academics' point of view, introducing particle filters opens a door for a better understanding of HF returns and the underlying risks of the HF strategies. If it already has direct implications from a risk management perspective, we also surmise that particles filters are one of the main avenues toward a better monitoring of for now unaccounted risks, as they are contained in the higher moments of the returns' distribution — we have yet to explore the use of ML estimation procedures for particle filters in the nonlinear context.

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A Numerical Algorithms for implementation of Particle Filters

In this appendix, we provide, in pseudo code, the algorithms for the particle filters implemented for the purpose of this study. In Appendix **??**, we presented the algorithm, known under the name Sequential Importance Sampling (SIS), which forms the basis for most sequential Monte Carlo filters developed over the past decade [7]. We start by providing its pseudo code in Algorithm 1, before exposing the more advanced algorithms we used: a generic Particle Filter

(GPF), a Sampling Importance Resampling (SIR) algorithm, and a regularized Particle Filter (RPF).

Algorithm 1 SIS Particle Filter

---

procedure SIS_PARTICLE_FILTER($\mathbf{z}_{1:T}$,N_s) ▷ Runs a SIS Particle Filter
  $\{\mathbf{x}_0^i, w_0^i\}_{i=1:N_s} \sim p_0(.)$ ▷ Initialization
  $k \leftarrow 1$
  while $k < T$ **do**
    $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s} \leftarrow$ SIS_STEP($\mathbf{x}_{k-1}^i$, w_{k-1}^i, $\mathbf{z}_k$)
    $k \leftarrow k + 1$
  end while
  return $\{\mathbf{x}_{1:T}^i, w_{1:T}^i\}_{i=1:N_s}$
end procedure

procedure SIS_STEP($\mathbf{x}_{k-1}^i$, w_{k-1}^i, $\mathbf{z}_k$) ▷ Propagates the sample from state $k-1$ to state k
  for $i = 1 : N_s$ **do**
    Draw $\mathbf{x}_k^i \sim q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$
    Assign the particle a weight, w_k^i, according to 10
  end for
  return $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$
end procedure

The SIS algorithm is thus a very simple algorithm, easy to implement. However, it commonly suffers from a degeneracy phenomenon, where after only a few iterations, all but one particle will have negligible weights. This degeneracy problems implies that a large computational effort will be devoted to updating particles whose contribution to the approximation of the filtering density $p(\mathbf{x}_k \mid \mathbf{z}_{1:k})$ is quasi null. In order to alleviate this problem, more advanced algorithm have been devised. One way to deal with degeneracy is to carefully choose the importance density function $q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$. We leave to the reader to consult [7] for a discussion of the importance of the choice of the importance density. Another simple idea is to resample the particles when a certain measure of degeneracy becomes too large (or too small). For example, one could calculate the effective sample size N_{eff} defined as:

$$N_{\text{eff}} = \frac{N_s}{1 + \sigma\left(w_k^{*i}\right)^2}$$

where $w_k^{*i} = p\left(\mathbf{x}_k^i \mid \mathbf{z}_{1:k}\right) / q\left(\mathbf{x}_k^i \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$ is referred to as the "true weight." As this cannot be valued exactly, this quantity can be estimated using:

$$\hat{N}_{\text{eff}} = \frac{1}{\sum_{i=1}^{N_s}\left(w_k^i\right)^2} \tag{A-1}$$

We provide in Algorithm 2 and in Algorithm 3 respectively the resampling algorithm we used and the generic Particle Filter which is deduced from the SIS algorithm by adding this resampling step to avoid degeneracy.

Algorithm 2 Resampling Algorithm

procedure RESAMPLE($\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$)
- $c_1 \leftarrow 0$ ▷ Initialise the CDF
- **for** $i = 2 : N_s$ **do** ▷ Construct the CDF
  - $c_i \leftarrow c_{i-1} + w_k^i$
- **end for**

- $i \leftarrow 1$ ▷ Start at the bottom of the CDF
- $u_1 \sim \mathbb{U}\left[0, N_s^{-1}\right]$ ▷ Draw a starting point
- **for** $j = 1 : N_s$ **do**
  - $u_j \leftarrow u_1 + N_s^{-1}(j-1)$ ▷ Move along the CDF
  - **while** $u_j > c_i$ **do**
    - $i \leftarrow i + 1$
  - **end while**
  - $\mathbf{x}_k^{j*} = \mathbf{x}_k^i$ ▷ Assign sample
  - $w_k^j = N_s^{-1}$ ▷ Assign weight
  - $\text{parent}_j \leftarrow i$ ▷ Assign parent
- **end for**
- **return** $\left\{\mathbf{x}_k^{j*}, w_k^j, \text{parent}_j\right\}_{j=1:N_s}$

end procedure

In many particle filters implementations, one uses the prior density $p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i\right)$ as the importance density $q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$ for even though it is often suboptimal, it simplifies the weights update equation 10 into:

$$w_k^i \propto w_{k-1}^i \times p\left(\mathbf{z}_k \mid \mathbf{x}_k^i\right)$$

Furthermore, if resampling is applied at every step — this particular implementation is called the Sampling Importance Resampling (SIR) of which we give the algorithm in pseudo code in Algorithm 4 — then we have $w_{k-1}^i = 1/N_s \, \forall i$, and so:

$$w_k^i \propto p\left(\mathbf{z}_k \mid \mathbf{x}_k^i\right) \tag{A-2}$$

The weights given in A-2 are normalized before the resampling stage.

The regularized Particle Filter is based on the same idea as the Generic Particle Filter, with the same resampling condition, but the resampling step provides an entirely new sample based on a continuous approximation of the posterior filtering density $p\left(\mathbf{x}_k \mid \mathbf{z}_k\right)$, such that we have the following approximation:

$$\hat{p}\left(\mathbf{x}_k \mid \mathbf{z}_k\right) = \sum_{i=1}^{N_s} w_k^i K_h\left(\mathbf{x}_k - \mathbf{x}_k^i\right) \tag{A-3}$$

where:

$$K_h\left(\mathbf{x}\right) = \frac{1}{h^{n_x}} K\left(\frac{\mathbf{x}}{h}\right)$$

is the re-scaled Kernel density $K\left(\cdot\right)$, $h > 0$ is the Kernel bandwidth, n_x is the dimension of the state vector $\mathbf{x}$, and w_k^i, $i = 1, \ldots, N_s$ are normalized weights. The Kernel $K\left(\cdot\right)$ and bandwidth

Algorithm 3 Generic Particle Filter

```
procedure GENERIC_PARTICLE_FILTER(z_{1:T}, N_s)          ▷ Runs a Generic Particle Filter
    {x_0^i, w_0^i}_{i=1:N_s} ~ p_0(.)                     ▷ Initialization
    k ← 1
    while k < T do
        {x_k^i, w_k^i}_{i=1:N_s} ← PF_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
        k ← k + 1
    end while
    return {x_{1:T}^i, w_{1:T}^i}_{i=1:N_s}
end procedure

procedure PF_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
    for i = 1 : N_s do
        Draw x_k^i ~ q(x_k | x_{k-1}^i, z_k)
        Assign the particle a weight, w_k^i, according to 10
    end for
    t ← Σ_{i=1}^{N_s} w_k^i                               ▷ Calculate total weight
    for i = 1 : N_s do
        w_k^i ← t^{-1} w_k^i
    end for
    Calculate N̂_eff using A-1
    if N̂_eff < N_s then
        {x_k^i, w_k^i, −}_{i=1:N_s} ← RESAMPLE({x_k^i, w_k^i}_{i=1:N_s})
    end if
end procedure
```

h should be chosen to minimize the Mean Integrated Square Error (MISE), between the true posterior density and the corresponding regularized empirical representation in A-3, defined as:

$$\text{MISE}(\hat{p}) = \mathbb{E}\left[\int \left[\hat{p}\left(\mathbf{x}_k \mid \mathbf{z}_k\right) - p\left(\mathbf{x}_k \mid \mathbf{z}_k\right)\right]^2 \mathrm{d}\mathbf{x}_k\right]$$

One can show that in the case where all the samples have the same weight, the optimal choice of the Kernel is the Epanechnikov Kernel:

$$K_{\text{opt}} = \begin{cases} \frac{n_x+2}{2c_{n_x}}\left(1 - \|x\|^2\right) & \text{if } \|x\| < 1, \\ 0 & \text{otherwise} \end{cases}$$

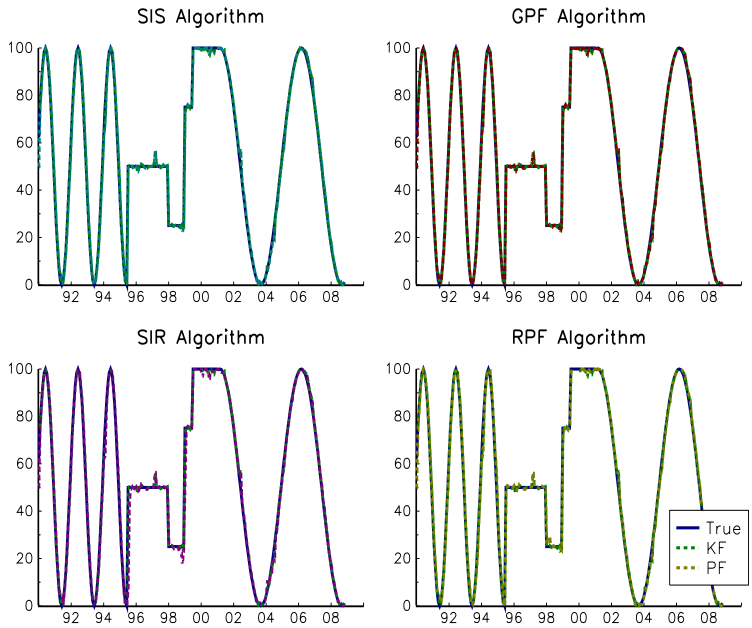
where c_{n_x} is the volume of the unit hypersphere in $\mathbb{R}^{n_x}$. Furthermore, when the underlying density is Gaussian with a unit covariance matrix, the optimal choice for the bandwidth is:

$$h_{\text{opt}} = AN_s^{-\frac{1}{n_x+4}}$$
$$A = \left[8c_{n_x}^{-1}\left(n_x+4\right)\left(2\sqrt{\pi}\right)^{n_x}\right]^{-\frac{1}{n_x+4}}$$

We can now provide the algorithm for the regularized Particle Filter in Algorithm 5.

We also illustrate these algorithms by reproducing the example given in Appendix **??** using $N_s = 1000$ particles. Note that the parameters of the distributions in the particle filters were estimated using the Kalman filter. The results (sample means) are reported in Figure 7 with, from top to bottom and left to right, the SIS, the generic PF, the SIR and the RPF runs.

Figure 7: Solving example **??** using particle filers — $N_s = 1000$.



Algorithm 4 SIR Particle Filter

procedure SIR_PARTICLE_FILTER($\mathbf{z}_{1:T}$,N_s) ▷ Runs a SIR Particle Filter
  $\{\mathbf{x}_0^i,\, w_0^i\}_{i=1:N_s} \sim p_0(.)$ ▷ Initialization
  $k \leftarrow 1$
  while $k < T$ **do**
    $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s} \leftarrow$ SIR_STEP($\mathbf{x}_{k-1}^i,\, w_{k-1}^i,\, \mathbf{z}_k$)
    $k \leftarrow k+1$
  end while
  return $\{\mathbf{x}_{1:T}^i, w_{1:T}^i\}_{i=1:N_s}$
end procedure

procedure SIR_STEP($\mathbf{x}_{k-1}^i,\, w_{k-1}^i,\, \mathbf{z}_k$)
  for $i = 1 : N_s$ **do**
    Draw $\mathbf{x}_k^i \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i)$
    $w_k^i \leftarrow p(\mathbf{z}_k \mid \mathbf{x}_k^i)$
  end for
  $t \leftarrow \sum_{i=1}^{N_s} w_k^i$ ▷ Calculate total weight
  for $i = 1 : N_s$ **do**
    $w_k^i \leftarrow t^{-1} w_k^i$
  end for
  $\{\mathbf{x}_k^i, w_k^i, -\}_{i=1:N_s} \leftarrow$ RESAMPLE($\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$) ▷ Systematic resampling
end procedure

Algorithm 5 Regularized Particle Filter

procedure REGULARIZED_PARTICLE_FILTER($\mathbf{z}_{1:T}$,N_s) ▷ Runs a Regularized Particle Filter

  $\{\mathbf{x}_0^i,\, w_0^i\}_{i=1:N_s} \sim p_0(.)$ ▷ Initialization

  $k \leftarrow 1$

  while $k < T$ **do**

    $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s} \leftarrow$ RPF_STEP($\mathbf{x}_{k-1}^i,\, w_{k-1}^i,\, \mathbf{z}_k$)

    $k \leftarrow k + 1$

  end while

  return $\{\mathbf{x}_{1:T}^i, w_{1:T}^i\}_{i=1:N_s}$

end procedure

procedure RPF_STEP($\mathbf{x}_{k-1}^i,\, w_{k-1}^i,\, \mathbf{z}_k$)

  for $i = 1 : N_s$ **do**

    Draw $\mathbf{x}_k^i \sim q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$

    Assign the particle a weight, w_k^i, according to 10

  end for

  $t \leftarrow \sum_{i=1}^{N_s} w_k^i$ ▷ Calculate total weight

  for $i = 1 : N_s$ **do**

    $w_k^i \leftarrow t^{-1} w_k^i$

  end for

  Calculate $\widehat{N_{\text{eff}}}$ using A-1

  if $\widehat{N_{\text{eff}}} < N_s$ **then**

    Compute the empirical covariance matrix S_k of $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$

    Compute $\mathbf{D}_k \leftarrow \text{Chol}(S_k)$ ▷ Cholesky decomposition of S_k: $\mathbf{D}_k\mathbf{D}_k^\top = S_k$

    $\{\mathbf{x}_k^i, w_k^i, -\}_{i=1:N_s} \leftarrow$ RESAMPLE($\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$)

    for $i = 1 : N_s$ **do**

      Draw $\epsilon^i \sim K_{\text{opt}}$ from the Epanechnikov Kernel

      $\mathbf{x}_k^{i*} \leftarrow \mathbf{x}_k^i + h_{\text{opt}}\mathbf{D}_k\epsilon^i$

    end for

    return $\{\mathbf{x}_k^{i*}, w_k^i\}_{i=1:N_s}$

  else

    return $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$

  end if

end procedure