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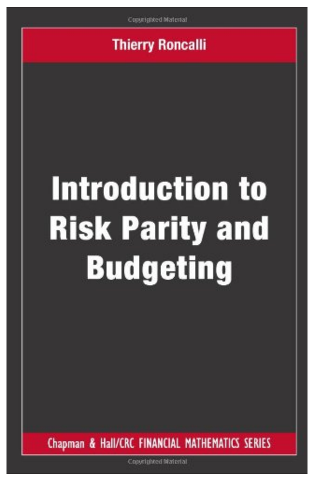
Introduction to Risk Parity and Budgeting

This book contains solutions of the tutorial exercises which are provided in Appendix B of TR-RPB:

(TR-RPB) RONCALLI T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series, 410 pages.



Description and materials of *Introduction to Risk Parity and Budgeting* are available on the author's website:

`http://www.thierry-roncalli.com/riskparitybook.html`

or on the Chapman & Hall website:

`http://www.crcpress.com/product/isbn/9781482207156`

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Contents

Chapter 1

Exercises related to modern portfolio theory

1.1 Markowitz optimized portfolios

1. The weights of the minimum variance portfolio are: $x_1^\star = 3.05\%$, $x_2^\star = 3.05\%$ and $x_3^\star = 93.89\%$. We have $\sigma(x^\star) = 4.94\%$.

2. We have to solve a σ-problem (TR-RPB, page 5). The optimal value of ϕ is 49.99 and the optimized portfolio is: $x_1^\star = 6.11\%$, $x_2^\star = 6.11\%$ and $x_3^\star = 87.79\%$.

3. If the ex-ante volatility is equal to 10%, the optimal value of ϕ becomes 4.49 and the optimized portfolio is: $x_1^\star = 37.03\%$, $x_2^\star = 37.03\%$ and $x_3^\star = 25.94\%$.

4. We notice that $x_1^\star = x_2^\star$. This is normal because the first and second assets present the same characteristics in terms of expected return, volatility and correlation with the third asset.

5. (a) We obtain the following results:

i	MV	$\sigma(x) = 5\%$	$\sigma(x) = 10\%$
1	8.00%	8.00%	37.03%
2	0.64%	3.66%	37.03%
3	91.36%	88.34%	25.94%
ϕ	$+\infty$	75.19	4.49

For the MV portfolio, we have $\sigma(x^\star) = 4.98\%$.

(b) We consider the γ-formulation (TR-RPB, page 7). The corresponding dual program is (TR-RPB, page 302):

$$\begin{aligned} \lambda^\star &= \arg\min \frac{1}{2}\lambda^\top \bar{Q}\lambda - \lambda^\top \bar{R} \\ &\text{u.c.} \quad \lambda \geq 0 \end{aligned}$$

with[1] $\bar{Q} = S\Sigma^{-1}S^\top$, $\bar{R} = \gamma S\Sigma^{-1}\mu - T$, $\gamma = \phi^{-1}$,

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -8\% \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1^\star$ is the Lagrange coefficient associated to the 8% minimum exposure for the first asset ($x_1 \geq 8\%$ in the primal program and first row of the S matrix in the dual program). $\max(\lambda_2^\star, \lambda_3^\star)$ is the Lagrange coefficient associated to the fully invested portfolio constraint ($\sum_{i=1}^3 x_i = 100\%$ in the primal program and second and third rows of the S matrix in the dual program). Finally, the Lagrange coefficients $\lambda_4^\star$, $\lambda_5^\star$ and $\lambda_6^\star$ are associated to the positivity constraints of the weights x_1, x_2 and x_3.

(c) We have to solve the previous quadratic programming problem by considering the value of ϕ corresponding to the results of Question 5(a). We obtain $\lambda_1^\star = 0.0828\%$ for the minimum variance portfolio, $\lambda_1^\star = 0.0488\%$ for the optimized portfolio with a 5% ex-ante volatility and $\lambda_1^\star = 0$ for the optimized portfolio with a 10% ex-ante volatility.

(d) We verify that the Lagrange coefficient is zero for the optimized portfolio with a 10% ex-ante volatility, because the constraint $x_1 \geq 8\%$ is not reached. The cost of this constraint is larger for the minimum variance portfolio. Indeed, a relaxation ε of this constraint permits to reduce the variance by a factor equal to $2 \cdot 0.0828\% \cdot \varepsilon$.

6. If we solve the minimum variance problem with $x_1 \geq 20\%$, we obtain a portfolio which has an ex-ante volatility equal to 5.46%. There isn't a portfolio whose volatility is smaller than this lower bound. We know that the constraints $x_i \geq 0$ are not reached for the minimum variance problem regardless of the constraint $x_1 \geq 20\%$. Let ξ be the lower bound of x_1. Because of the previous results, we have $0\% \leq \xi \leq 20\%$. We would like to find the minimum variance portfolio $x^\star$ such that the constraint $x_1 \geq \xi$ is reached and $\sigma(x^\star) = \sigma^\star = 5\%$. In this case, the optimization problem with three variables reduces to a minimum variance problem with two variables with the constraint $x_2 + x_3 = 1 - \xi$ because $x_1^\star = \xi$. We then have:

$$\begin{aligned} x^\top \Sigma x &= x_2^2\sigma_2^2 + 2x_2x_3\rho_{2,3}\sigma_2\sigma_3 + x_3^2\sigma_3^2 + \\ &\quad \xi^2\sigma_1^2 + 2\xi x_2\rho_{1,2}\sigma_1\sigma_2 + 2\xi x_3\rho_{1,3}\sigma_1\sigma_3 \end{aligned}$$

[1] We recall that μ and Σ are the vector of expected returns and the covariance matrix of asset returns.

The objective function becomes:

$$\begin{aligned} x^\top \Sigma x &= (1-\xi-x_3)^2 \sigma_2^2 + 2(1-\xi-x_3) x_3 \rho_{2,3}\sigma_2\sigma_3 + x_3^2\sigma_3^2 + \\ &\quad \xi^2\sigma_1^2 + 2\xi(1-\xi-x_3)\rho_{1,2}\sigma_1\sigma_2 + 2\xi x_3 \rho_{1,3}\sigma_1\sigma_3 \\ &= x_3^2\left(\sigma_2^2 - 2\rho_{2,3}\sigma_2\sigma_3 + \sigma_3^2\right) + \\ &\quad 2x_3\left((1-\xi)\left(\rho_{2,3}\sigma_2\sigma_3 - \sigma_2^2\right) - \xi\rho_{1,2}\sigma_1\sigma_2 + \xi\rho_{1,3}\sigma_1\sigma_3\right) + \\ &\quad (1-\xi)^2\sigma_2^2 + \xi^2\sigma_1^2 + 2\xi(1-\xi)\rho_{1,2}\sigma_1\sigma_2 \end{aligned}$$

We deduce that:

$$\frac{\partial\, x^\top \Sigma x}{\partial\, x_3} = 0 \Leftrightarrow x_3^\star = \frac{(1-\xi)\left(\sigma_2^2 - \rho_{2,3}\sigma_2\sigma_3\right) + \xi\sigma_1\left(\rho_{1,2}\sigma_2 - \rho_{1,3}\sigma_3\right)}{\sigma_2^2 - 2\rho_{2,3}\sigma_2\sigma_3 + \sigma_3^2}$$

The minimum variance portfolio is then:

$$\left\{ \begin{array}{l} x_1^\star = \xi \\ x_2^\star = a - (a+c)\,\xi \\ x_3^\star = b - (b-c)\,\xi \end{array} \right.$$

with $a = \left(\sigma_3^2 - \rho_{2,3}\sigma_2\sigma_3\right)/d$, $b = \left(\sigma_2^2 - \rho_{2,3}\sigma_2\sigma_3\right)/d$, $c = \sigma_1\left(\rho_{1,2}\sigma_2 - \rho_{1,3}\sigma_3\right)/d$ and $d = \sigma_2^2 - 2\rho_{2,3}\sigma_2\sigma_3 + \sigma_3^2$. We also have:

$$\begin{aligned} \sigma^2(x) &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + x_3^2\sigma_3^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2 + 2x_1x_3\rho_{1,3}\sigma_1\sigma_3 + \\ &\quad 2x_2x_3\rho_{2,3}\sigma_2\sigma_3 \\ &= \xi^2\sigma_1^2 + (a-(a+c)\,\xi)^2\sigma_2^2 + (b-(b-c)\,\xi)^2\sigma_3^2 + \\ &\quad 2\xi\,(a-(a+c)\,\xi)\,\rho_{1,2}\sigma_1\sigma_2 + \\ &\quad 2\xi\,(b-(b-c)\,\xi)\,\rho_{1,3}\sigma_1\sigma_3 + \\ &\quad 2\,(a-(a+c)\,\xi)\,(b-(b-c)\,\xi)\,\rho_{2,3}\sigma_2\sigma_3 \end{aligned}$$

We deduce that the optimal value $\xi^\star$ such that $\sigma(x^\star) = \sigma^\star$ satisfies the polynomial equation of the second degree:

$$\alpha\xi^2 + 2\beta\xi + \left(\gamma - {\sigma^\star}^2\right) = 0$$

with:

$$\left\{ \begin{array}{lcl} \alpha &=& \sigma_1^2 + (a+c)^2\sigma_2^2 + (b-c)^2\sigma_3^2 - 2(a+c)\,\rho_{1,2}\sigma_1\sigma_2 - \\ && 2(b-c)\,\rho_{1,3}\sigma_1\sigma_3 + 2(a+c)(b-c)\,\rho_{2,3}\sigma_2\sigma_3 \\ \beta &=& -a(a+c)\,\sigma_2^2 - b(b-c)\,\sigma_3^2 + a\rho_{1,2}\sigma_1\sigma_2 + b\rho_{1,3}\sigma_1\sigma_3 - \\ && (a(b-c) + b(a+c))\,\rho_{2,3}\sigma_2\sigma_3 \\ \gamma &=& a^2\sigma_2^2 + b^2\sigma_3^2 + 2ab\rho_{2,3}\sigma_2\sigma_3 \end{array} \right.$$

By using the numerical values, the solutions of the quadratic equation are $\xi_1 = 9.09207\%$ and $\xi_2 = -2.98520\%$. The optimal solution is then $\xi^\star = 9.09207\%$. In order to check this result, we report in Figure 1.1 the volatility of the minimum variance portfolio when we impose the constraint $x_1 \geq x_1^-$. We verify that the volatility is larger than 5% when $x_1 \geq \xi^\star$.

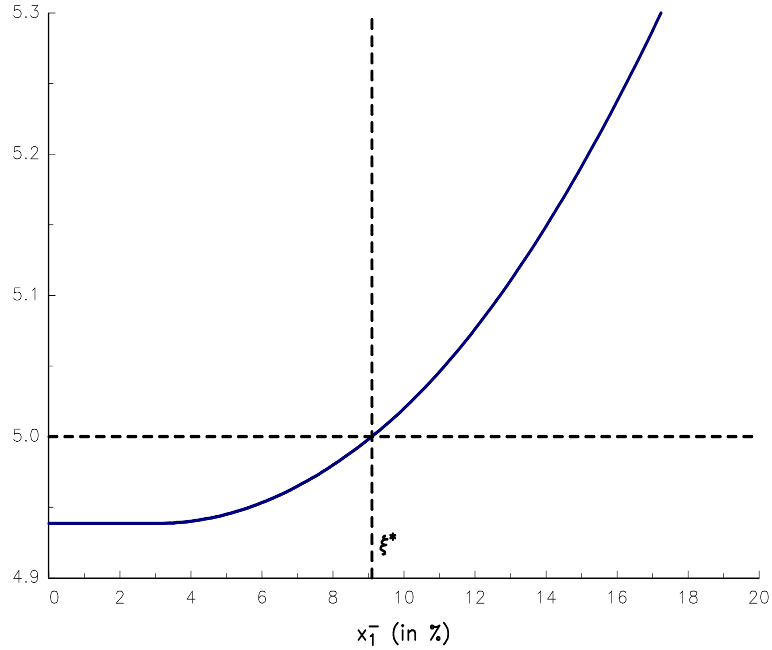

FIGURE 1.1: Volatility of the minimum variance portfolio (in %)

1.2 Variations on the efficient frontier

1. We deduce that the covariance matrix is:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 \\ 0.300 & 4.000 & 3.500 & 2.400 \\ 1.500 & 3.500 & 6.250 & 6.000 \\ 2.250 & 2.400 & 6.000 & 9.000 \end{pmatrix} \times 10^{-2}$$

   We then have to solve the γ-formulation of the Markowitz problem (TR-RPB, page 7). We obtain the results[2] given in Table 1.1. We represent the efficient frontier in Figure 1.2.

2. We solve the γ-problem with $\gamma = 0$. The minimum variance portfolio is then $x_1^\star = 72.74\%$, $x_2^\star = 49.46\%$, $x_3^\star = -20.45\%$ and $x_4^\star = -1.75\%$. We deduce that $\mu(x^\star) = 4.86\%$ and $\sigma(x^\star) = 12.00\%$.

3. There is no solution when the target volatility $\sigma^\star$ is equal to 10% because the minimum variance portfolio has a volatility larger than 10%. Finding

[2] The weights, expected returns and volatilities are expressed in %.

TABLE 1.1: Solution of Question 1

γ	−1.00	−0.50	−0.25	0.00	0.25	0.50	1.00	2.00
$x_1^\star$	94.04	83.39	78.07	72.74	67.42	62.09	51.44	30.15
$x_2^\star$	120.05	84.76	67.11	49.46	31.82	14.17	−21.13	−91.72
$x_3^\star$	−185.79	−103.12	−61.79	−20.45	20.88	62.21	144.88	310.22
$x_4^\star$	71.69	34.97	16.61	−1.75	−20.12	−38.48	−75.20	−148.65
$\mu(x^\star)$	1.34	3.10	3.98	4.86	5.74	6.62	8.38	11.90
$\sigma(x^\star)$	22.27	15.23	12.88	12.00	12.88	15.23	22.27	39.39

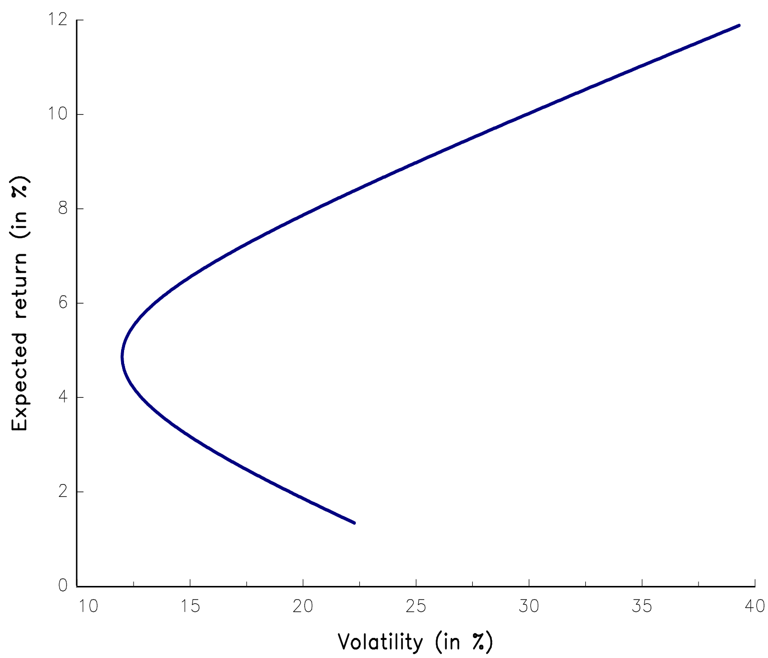

FIGURE 1.2: Markowitz efficient frontier

the optimized portfolio for $\sigma^\star = 15\%$ or $\sigma^\star = 20\%$ is equivalent to solving a σ-problem (TR-RPB, page 5). If $\sigma^\star = 15\%$ (resp. $\sigma^\star = 20\%$), we obtain an implied value of γ equal to 0.48 (resp. 0.85). Results are given in the following Table:

$\sigma^\star$	15.00	20.00
$x_1^\star$	62.52	54.57
$x_2^\star$	15.58	−10.75
$x_3^\star$	58.92	120.58
$x_4^\star$	−37.01	−64.41
$\mu(x^\star)$	6.55	7.87
γ	0.48	0.85

4. Let $x^{(\alpha)}$ be the portfolio defined by the relationship $x^{(\alpha)} = (1-\alpha)\, x^{(1)} + \alpha x^{(2)}$ where $x^{(1)}$ is the minium variance portfolio and $x^{(2)}$ is the optimized portfolio with a 20% ex-ante volatility. We obtain the following results:

α	$\sigma\left(x^{(\alpha)}\right)$	$\mu\left(x^{(\alpha)}\right)$
−0.50	14.42	3.36
−0.25	12.64	4.11
0.00	12.00	4.86
0.10	12.10	5.16
0.20	12.41	5.46
0.50	14.42	6.36
0.70	16.41	6.97
1.00	20.00	7.87

We have reported these portfolios in Figure 1.3. We notice that they are located on the efficient frontier. This is perfectly normal because we know that a combination of two optimal portfolios corresponds to another optimal portfolio.

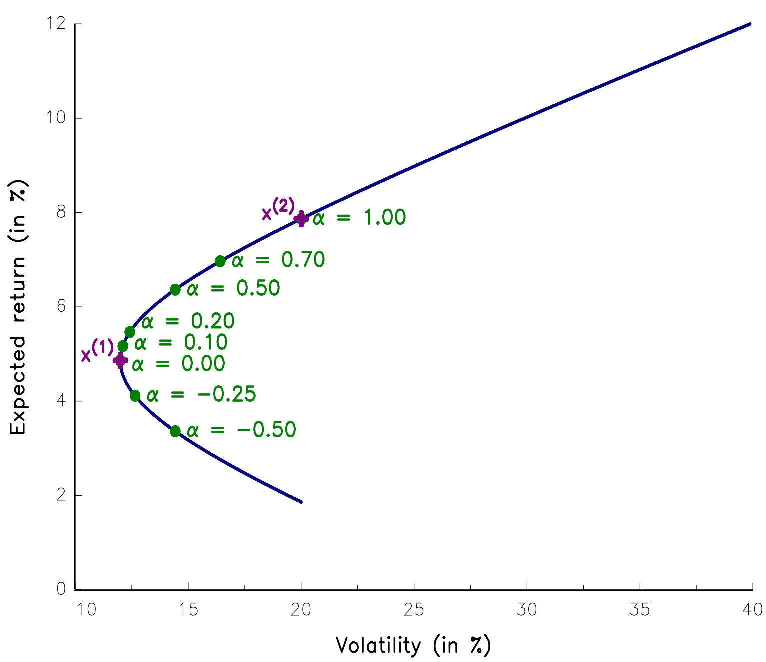

FIGURE 1.3: Mean-variance diagram of portfolios $x^{(\alpha)}$

5. If we consider the constraint $0 \leq x_i \leq 1$, we obtain the following results:

$\sigma^\star$	MV	12.00	15.00	20.00
$x_1^\star$	65.49	✓	45.59	24.88
$x_2^\star$	34.51	✓	24.74	4.96
$x_3^\star$	0.00	✓	29.67	70.15
$x_4^\star$	0.00	✓	0.00	0.00
$\mu(x^\star)$	5.35	✓	6.14	7.15
$\sigma(x^\star)$	12.56	✓	15.00	20.00
γ	0.00	✓	0.62	1.10

6. (a) We have:

$$\mu = \begin{pmatrix} 5.0 \\ 6.0 \\ 8.0 \\ 6.0 \\ 3.0 \end{pmatrix} \times 10^{-2}$$

and:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 & 0.000 \\ 0.300 & 4.000 & 3.500 & 2.400 & 0.000 \\ 1.500 & 3.500 & 6.250 & 6.000 & 0.000 \\ 2.250 & 2.400 & 6.000 & 9.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix} \times 10^{-2}$$

(b) We solve the γ-problem and obtain the efficient frontier given in Figure 1.4.

(c) This efficient frontier is a straight line. This line passes through the risk-free asset and is tangent to the efficient frontier of Figure 1.2. This exercise is a direct application of the *Separation Theorem* of Tobin.

(d) We consider two optimized portfolios of this efficient frontier. They corresponds to $\gamma = 0.25$ and $\gamma = 0.50$. We obtain the following results:

γ	0.25	0.50
$x_1^\star$	18.23	36.46
$x_2^\star$	−1.63	−3.26
$x_3^\star$	34.71	69.42
$x_4^\star$	−18.93	−37.86
$x_5^\star$	67.62	35.24
$\mu(x^\star)$	4.48	5.97
$\sigma(x^\star)$	6.09	12.18

The first portfolio has an expected return equal to 4.48% and a volatility equal to 6.09%. The weight of the risk-free asset is 67.62%.

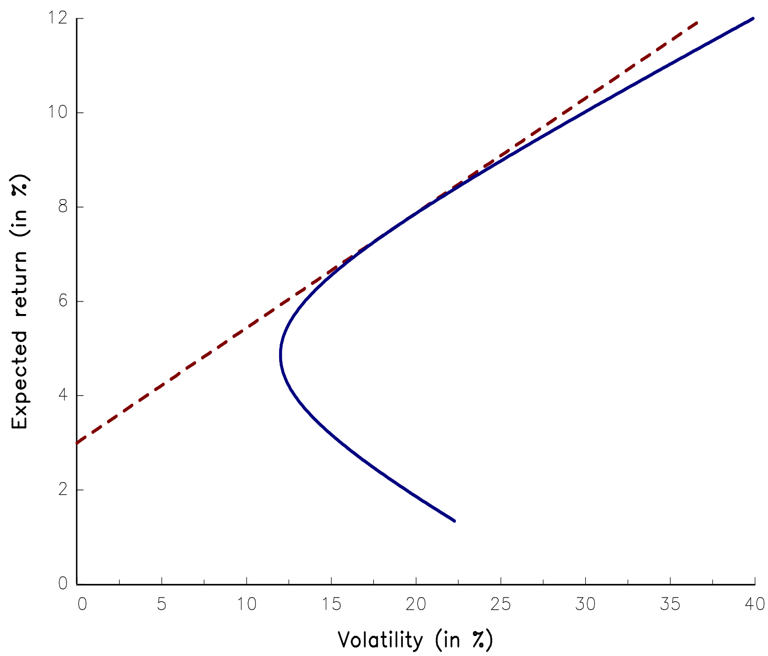

FIGURE 1.4: Efficient frontier when the risk-free asset is introduced

This explains the low volatility of this portfolio. For the second portfolio, the weight of the risk-free asset is lower and equal to 35.24%. The expected return and the volatility are then equal to 5.97% and 12.18%. We note $x^{(1)}$ and $x^{(2)}$ these two portfolios. By definition, the Sharpe ratio of the market portfolio $x^{\star}$ is the tangency of the line. We deduce that:

$$\begin{aligned} \mathrm{SR}\left(x^{\star} \mid r\right) &= \frac{\mu\left(x^{(2)}\right) - \mu\left(x^{(1)}\right)}{\sigma\left(x^{(2)}\right) - \sigma\left(x^{(1)}\right)} \\ &= \frac{5.97 - 4.48}{12.18 - 6.09} \\ &= 0.2436 \end{aligned}$$

The Sharpe ratio of the market portfolio $x^{\star}$ is then equal to 0.2436.

(e) By construction, every portfolio $x^{(\alpha)}$ which belongs to the tangency line is a linear combination of two portfolios $x^{(1)}$ and $x^{(2)}$ of this efficient frontier:

$$x^{(\alpha)} = (1 - \alpha)\, x^{(1)} + \alpha x^{(2)}$$

The market portfolio $x^{\star}$ is the portfolio $x^{(\alpha)}$ which has a zero weight in the risk-free asset. We deduce that the value $\alpha^\star$ which corresponds to the market portfolio satisfies the following relationship:

$$(1-\alpha^\star)\, x_5^{(1)} + \alpha^\star x_5^{(2)} = 0$$

because the risk-free asset is the fifth asset of the portfolio. It follows that:

$$\begin{aligned} \alpha^\star &= \frac{x_5^{(1)}}{x_5^{(1)} - x_5^{(2)}} \\ &= \frac{67.62}{67.62 - 35.24} \\ &= 2.09 \end{aligned}$$

We deduce that the market portfolio is:

$$x^\star = (1-2.09)\cdot \begin{pmatrix} 18.23 \\ -1.63 \\ 34.71 \\ -18.93 \\ 67.62 \end{pmatrix} + 2.09\cdot \begin{pmatrix} 36.46 \\ -3.26 \\ 69.42 \\ -37.86 \\ 35.24 \end{pmatrix} = \begin{pmatrix} 56.30 \\ -5.04 \\ 107.21 \\ -58.46 \\ 0.00 \end{pmatrix}$$

We check that the Sharpe ratio of this portfolio is 0.2436.

(a) We have:

$$\tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

and:

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}$$

(b) This problem is entirely solved in TR-RPB on page 13.

1.3 Sharpe ratio

1. (a) We have (TR-RPB, page 12):

$$\mathrm{SR}_i = \frac{\mu_i - r}{\sigma_i}$$

(b) We have:

$$\mathrm{SR}(x \mid r) = \frac{x_1\mu_1 + x_2\mu_2 - r}{\sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}}$$

(c) If the second asset corresponds to the risk-free asset, its volatility σ_2 and its correlation ρ with the first asset are equal to zero. We deduce that:

$$\begin{aligned}\mathrm{SR}(x \mid r) &= \frac{x_1\mu_1 + (1-x_1)\,r - r}{\sqrt{x_1^2\sigma_1^2}} \\ &= \frac{x_1(\mu_1 - r)}{|x_1|\,\sigma_1} \\ &= \operatorname{sgn}(x_1)\cdot \mathrm{SR}_1\end{aligned}$$

We finally obtain that:

$$\mathrm{SR}(x \mid r) = \begin{cases} -\mathrm{SR}_1 & \text{if } x_1 < 0 \\ +\mathrm{SR}_1 & \text{if } x_1 > 0\end{cases}$$

2. (a) Let $R(x)$ be the return of the portfolio x. We have:

$$\mathbb{E}[R(x)] = \sum_{i=1}^{n} n^{-1}\mu_i = n^{-1}\sum_{i=1}^{n}\mu_i$$

and:

$$\sigma(R(x)) = \sqrt{\sum_{i=1}^{n}\left(n^{-1}\sigma_i\right)^2} = n^{-1}\sqrt{\sum_{i=1}^{n}\sigma_i^2}$$

We deduce that the Sharpe ratio of the portfolio x is:

$$\begin{aligned}\mathrm{SR}(x \mid r) &= \frac{n^{-1}\sum_{i=1}^{n}\mu_i - r}{n^{-1}\sqrt{\sum_{i=1}^{n}\sigma_i^2}} \\ &= \frac{\sum_{i=1}^{n}(\mu_i - r)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}}\end{aligned}$$

because $r = n^{-1}\sum_{i=1}^{n} r$.

(b) Another expression of the Sharpe ratio is:

$$\begin{aligned}\mathrm{SR}(x \mid r) &= \sum_{i=1}^{n}\frac{\sigma_i}{\sqrt{\sum_{j=1}^{n}\sigma_i^2}}\cdot\frac{(\mu_i - r)}{\sigma_i} \\ &= \sum_{i=1}^{n} w_i\,\mathrm{SR}_i\end{aligned}$$

with:

$$w_i = \frac{\sigma_i}{\sqrt{\sum_{j=1}^{n}\sigma_i^2}}$$

(c) Because $0 < \sigma_i < \sqrt{\sum_{j=1}^{n} \sigma_i^2}$, we deduce that:

$$0 < w_i < 1$$

(d) We obtain the following results:

	w_1	w_2	w_3	w_4	w_5	$\sum_{i=1}^{n} w_i$	$\mathrm{SR}(x \mid r)$
$\mathcal{A}_1$	38.5%	38.5%	57.7%	19.2%	57.7%	211.7%	0.828
$\mathcal{A}_2$	25.5%	25.5%	34.1%	17.0%	85.1%	187.3%	0.856

It may be surprising that the portfolio based on the set $\mathcal{A}_2$ has a larger Sharpe ratio than the portfolio based on the set $\mathcal{A}_1$, because four assets of $\mathcal{A}_2$ are all dominated by the assets of $\mathcal{A}_1$. Only the fifth asset of $\mathcal{A}_2$ has a higher Sharpe ratio. However, we easily understand this result if we consider the previous decomposition. Indeed, this fifth asset has a higher volatility than the other assets. It follows that its contribution w_5 to the Sharpe ratio is then much greater.

3. (a) We have:

$$\begin{aligned} \sigma(R(x)) &= \sqrt{\sum_{i=1}^{n} \left(n^{-1}\sigma\right)^2 + 2\sum_{i>j}^{n} \rho \left(n^{-1}\sigma\right)^2} \\ &= \sigma\sqrt{\rho + n^{-1}(1-\rho)} \end{aligned}$$

We deduce that the Sharpe ratio is:

$$\mathrm{SR}(x \mid r) = \frac{n^{-1}\sum_{i=1}^{n} \mu_i - r}{\sigma\sqrt{\rho + n^{-1}(1-\rho)}}$$

(b) It follows that:

$$\begin{aligned} \mathrm{SR}(x \mid r) &= \frac{1}{\sqrt{\rho + n^{-1}(1-\rho)}} n^{-1} \sum_{i=1}^{n} \frac{(\mu_i - r)}{\sigma} \\ &= w \cdot \left(\frac{1}{n}\sum_{i=1}^{n} \mathrm{SR}_i\right) \end{aligned}$$

with:

$$w = \frac{1}{\sqrt{\rho + n^{-1}(1-\rho)}}$$

(c) One seeks n such that:

$$\frac{1}{\sqrt{\rho + n^{-1}(1-\rho)}} = w$$

We deduce that:

$$n^\star = w^2 \frac{1-\rho}{1-\rho w^2}$$

If $\rho = 50\%$ and $w = 1.25$, we obtain:

$$\begin{aligned} n^\star &= 1.25^2 \frac{1-0.5}{1-0.5 \cdot 1.25^2} \\ &= 3.57 \end{aligned}$$

Four assets are sufficient to improve the Sharpe ratio by a factor of 25%.

(d) We notice that:

$$w = \frac{1}{\sqrt{\rho + n^{-1}(1-\rho)}} < \frac{1}{\sqrt{\rho}}$$

If $\rho = 80\%$, then $w < 1.12$. We cannot improve the Sharpe ratio by 25% when the correlation is equal to 80%.

(e) The most important parameter is the correlation ρ. The lower this correlation, the larger the increase of the Sharpe ratio. If the correlation is high, the gain in terms of Sharpe ratio is negligible. For instance, if $\rho \geq 80\%$, the gain cannot exceed 12%.

4. (a) Let $R^g(x)$ be the gross performance of the portfolio. We note m and p the management and performance fees. The net performance $R^n(x)$ is equal to:

$$R^n(x) = (R^g(x) - m) - p\,(R^g(x) - m - \text{Libor})_+$$

If we assume that $R^g(x) - m - \text{Libor} > 0$, we obtain:

$$\begin{aligned} R^n(x) &= (R^g(x) - m) - p\,(R^g(x) - m - \text{Libor}) \\ &= (1-p)(R^g(x) - m) + p\,\text{Libor} \end{aligned}$$

We deduce that:

$$R^g(x) = m + \frac{(R^n(x) - p\,\text{Libor})}{1-p}$$

Using the numerical values, we obtain:

$$\begin{aligned} R^g(x) &= 1\% + \frac{(\text{Libor} + 4\% - 10\% \cdot \text{Libor})}{(1-10\%)} \\ &= \text{Libor} + 544 \text{ bps} \end{aligned}$$

Moreover, if we assume that the performance fees have little influence on the volatility of the portfolio[3], the Sharpe ratio of the

[3]This is not true in practice.

hedge funds portfolio is equal to:

$$\begin{aligned} \mathrm{SR}(x \mid r) &= \frac{\text{Libor} + 544 \text{ bps} - \text{Libor}}{4\%} \\ &= 1.36 \end{aligned}$$

(b) We obtain the following results:

	ρ						
	0.00	0.10	0.20	0.30	0.50	0.75	0.90
$n = 10$	3.16	2.29	1.89	1.64	1.35	1.14	1.05
$n = 20$	4.47	2.63	2.04	1.73	1.38	1.15	1.05
$n = 30$	5.48	2.77	2.10	1.76	1.39	1.15	1.05
$n = 50$	7.07	2.91	2.15	1.78	1.40	1.15	1.05
$+\infty$	$+\infty$	3.16	2.24	1.83	1.41	1.15	1.05

This means for instance that if the correlation among the hedge funds is equal to 20%, the Sharpe ratio of a portfolio of 30 hedge funds is multiplied by a factor of 2.10 with respect to the average Sharpe ratio.

(c) If we assume that the average Sharpe ratio of single hedge funds is 0.5 and if we target a Sharpe ratio equal to 1.36 gross of fees, the multiplication factor w must satisfy the following inequality:

$$\begin{aligned} w &\geq \frac{\mathrm{SR}(x \mid r)}{n^{-1}\sum_{i=1}^{n} \mathrm{SR}_i} \\ &= \frac{1.36}{0.50} \\ &= 2.72 \end{aligned}$$

It is then not possible to achieve a net performance of Libor + 400 bps with a volatility of 4% if the correlation between these hedge funds is larger than 20%.

1.4 Beta coefficient

1. (a) The beta of an asset is the ratio between its covariance with the market portfolio return and the variance of the market portfolio return (TR-RPB, page 16). In the CAPM theory, we have:

$$\mathbb{E}[R_i] = r + \beta_i (\mathbb{E}[R(b)] - r)$$

where R_i is the return of asset i, $R(b)$ is the return of the market portfolio and r is the risk-free rate. The beta β_i of asset i is:

$$\beta_i = \frac{\operatorname{cov}(R_i, R(b))}{\operatorname{var}(R(b))}$$

Let Σ be the covariance matrix of asset returns. We have $\operatorname{cov}(R, R(b)) = \Sigma b$ and $\operatorname{var}(R(b)) = b^\top \Sigma b$. We deduce that:

$$\beta_i = \frac{(\Sigma b)_i}{b^\top \Sigma b}$$

(b) We recall that the mathematical operator $\mathbb{E}$ is bilinear. Let c be the covariance $\operatorname{cov}(c_1 X_1 + c_2 X_2, X_3)$. We then have:

$$\begin{aligned} c &= \mathbb{E}\left[\left(c_1 X_1 + c_2 X_2 - \mathbb{E}\left[c_1 X_1 + c_2 X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\ &= \mathbb{E}\left[\left(c_1\left(X_1 - \mathbb{E}\left[X_1\right]\right) + c_2\left(X_2 - \mathbb{E}\left[X_2\right]\right)\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\ &= c_1 \mathbb{E}\left[\left(X_1 - \mathbb{E}\left[X_1\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] + \\ &\quad\ c_2 \mathbb{E}\left[\left(X_2 - \mathbb{E}\left[X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\ &= c_1 \operatorname{cov}\left(X_1, X_3\right) + c_2 \operatorname{cov}\left(X_2, X_3\right) \end{aligned}$$

(c) We have:

$$\begin{aligned} \beta(x \mid b) &= \frac{\operatorname{cov}(R(x), R(b))}{\operatorname{var}(R(b))} \\ &= \frac{\operatorname{cov}\left(x^\top R, b^\top R\right)}{\operatorname{var}\left(b^\top R\right)} \\ &= \frac{x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b}{b^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b} \\ &= \frac{x^\top \Sigma b}{b^\top \Sigma b} \\ &= x^\top \frac{\Sigma b}{b^\top \Sigma b} \\ &= x^\top \beta \\ &= \sum_{i=1}^n x_i \beta_i \end{aligned}$$

with $\beta = (\beta_1, \ldots, \beta_n)$. The beta of portfolio x is then the weighted mean of asset betas. Another way to show this result is to exploit the result of Question 1(b). We have:

$$\begin{aligned} \beta(x \mid b) &= \frac{\operatorname{cov}\left(\sum_{i=1}^n x_i R_i, R(b)\right)}{\operatorname{var}(R(b))} \\ &= \sum_{i=1}^n x_i \frac{\operatorname{cov}(R_i, R(b))}{\operatorname{var}(R(b))} \\ &= \sum_{i=1}^n x_i \beta_i \end{aligned}$$

(d) We obtain $\beta\left(x^{(1)} \mid b\right) = 0.80$ and $\beta\left(x^{(2)} \mid b\right) = 0.85$.

2. The weights of the market portfolio are then $b = n^{-1}\mathbf{1}$.

(a) We have:

$$\begin{aligned}
\beta &= \frac{\operatorname{cov}\left(R, R\left(b\right)\right)}{\operatorname{var}\left(R\left(b\right)\right)} \\
&= \frac{\Sigma b}{b^\top \Sigma b} \\
&= \frac{n^{-1}\Sigma\mathbf{1}}{n^{-2}\left(\mathbf{1}^\top\Sigma\mathbf{1}\right)} \\
&= n\frac{\Sigma\mathbf{1}}{\left(\mathbf{1}^\top\Sigma\mathbf{1}\right)}
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\sum_{i=1}^{n}\beta_i &= \mathbf{1}^\top\beta \\
&= \mathbf{1}^\top n\frac{\Sigma\mathbf{1}}{\left(\mathbf{1}^\top\Sigma\mathbf{1}\right)} \\
&= n\frac{\mathbf{1}^\top\Sigma\mathbf{1}}{\left(\mathbf{1}^\top\Sigma\mathbf{1}\right)} \\
&= n
\end{aligned}$$

(b) If $\rho_{i,j} = 0$, we have:

$$\beta_i = n\frac{\sigma_i^2}{\sum_{j=1}^{n}\sigma_j^2}$$

We deduce that:

$$\begin{aligned}
\beta_1 \geq \beta_2 \geq \beta_3 &\Rightarrow n\frac{\sigma_1^2}{\sum_{j=1}^{3}\sigma_j^2} \geq n\frac{\sigma_2^2}{\sum_{j=1}^{3}\sigma_j^2} \geq n\frac{\sigma_3^2}{\sum_{j=1}^{3}\sigma_j^2} \\
&\Rightarrow \sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \\
&\Rightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3
\end{aligned}$$

(c) If $\rho_{i,j} = \rho$, it follows that:

$$\begin{aligned}
\beta_i &\propto \sigma_i^2 + \sum_{j\neq i}\rho\sigma_i\sigma_j \\
&= \sigma_i^2 + \rho\sigma_i\sum_{j\neq i}\sigma_j + \rho\sigma_i^2 - \rho\sigma_i^2 \\
&= \left(1-\rho\right)\sigma_i^2 + \rho\sigma_i\sum_{j=1}^{n}\sigma_j \\
&= f\left(\sigma_i\right)
\end{aligned}$$

with:

$$f(z) = (1-\rho) z^2 + \rho z \sum_{j=1}^{n} \sigma_j$$

The first derivative of $f(z)$ is:

$$f'(z) = 2(1-\rho) z + \rho \sum_{j=1}^{n} \sigma_j$$

If $\rho \geq 0$, then $f(z)$ is an increasing function for $z \geq 0$ because $(1-\rho) \geq 0$ and $\rho \sum_{j=1}^{n} \sigma_j \geq 0$. This explains why the previous result remains valid:

$$\beta_1 \geq \beta_2 \geq \beta_3 \Rightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho \geq 0$$

If $-(n-1)^{-1} \leq \rho < 0$, then f' is decreasing if $z < -2^{-1}\rho(1-\rho)^{-1} \sum_{j=1}^{n} \sigma_j$ and increasing otherwise. We then have:

$$\beta_1 \geq \beta_2 \geq \beta_3 \nRightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho < 0$$

In fact, the result remains valid in most cases. To obtain a counter-example, we must have large differences between the volatilities and a correlation close to $-(n-1)^{-1}$. For example, if $\sigma_1 = 5\%$, $\sigma_2 = 6\%$, $\sigma_3 = 80\%$ and $\rho = -49\%$, we have $\beta_1 = -0.100$, $\beta_2 = -0.115$ and $\beta_3 = 3.215$.

(d) We assume that $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 22\%$, $\rho_{1,2} = 70\%$, $\rho_{1,3} = 20\%$ and $\rho_{2,3} = -50\%$. It follows that $\beta_1 = 1.231$, $\beta_2 = 0.958$ and $\beta_3 = 0.811$. We thus have found an example such that $\beta_1 > \beta_2 > \beta_3$ and $\sigma_1 < \sigma_2 < \sigma_3$.

(e) There is no reason that we have either $\sum_{i=1}^{n} \beta_i < n$ or $\sum_{i=1}^{n} \beta_i > n$. Let us consider the previous numerical example. If $b = (5\%, 25\%, 70\%)$, we obtain $\sum_{i=1}^{3} \beta_i = 1.808$ whereas if $b = (20\%, 40\%, 40\%)$, we have $\sum_{i=1}^{3} \beta_i = 3.126$.

3. (a) We have:

$$\begin{aligned} \sum_{i=1}^{n} b_i \beta_i &= \sum_{i=1}^{n} b_i \frac{(\Sigma b)_i}{b^\top \Sigma b} \\ &= b^\top \frac{\Sigma b}{b^\top \Sigma b} \\ &= 1 \end{aligned}$$

If $\beta_i = \beta_j = \beta$, then $\beta = 1$ is an obvious solution because the previous relationship is satisfied:

$$\sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i = 1$$

(b) If $\beta_i = \beta_j = \beta$, then we have:

$$\sum_{i=1}^{n} b_i \beta = 1 \Leftrightarrow \beta = \frac{1}{\sum_{i=1}^{n} b_i} = 1$$

β can only take one value, the solution is then unique. We know that the marginal volatilities are the same in the case of the minimum variance portfolio x (TR-RPB, page 173):

$$\frac{\partial\, \sigma(x)}{\partial\, x_i} = \frac{\partial\, \sigma(x)}{\partial\, x_j}$$

with $\sigma(x) = \sqrt{x^\top \Sigma x}$ the volatility of the portfolio x. It follows that:

$$\frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{(\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

By dividing the two terms by $\sqrt{x^\top \Sigma x}$, we obtain:

$$\frac{(\Sigma x)_i}{x^\top \Sigma x} = \frac{(\Sigma x)_j}{x^\top \Sigma x}$$

The asset betas are then the same in the minimum variance portfolio. Because we have:

$$\begin{cases} \beta_i = \beta_j \\ \sum_{i=1}^{n} x_i \beta_i = 1 \end{cases}$$

we deduce that:

$$\beta_i = 1$$

4. (a) We have:

$$\begin{aligned} & \sum_{i=1}^{n} b_i \beta_i = 1 \\ \Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i \\ \Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i - \sum_{i=1}^{n} b_i = 0 \\ \Leftrightarrow \quad & \sum_{i=1}^{n} b_i (\beta_i - 1) = 0 \end{aligned}$$

We obtain the following system of equations:

$$\begin{cases} \sum_{i=1}^{n} b_i (\beta_i - 1) = 0 \\ b_i \geq 0 \end{cases}$$

Let us assume that the asset j has a beta greater than 1. We then have:

$$\begin{cases} b_j (\beta_j - 1) + \sum_{i \neq j} b_i (\beta_i - 1) = 0 \\ b_i \geq 0 \end{cases}$$

It follows that $b_j (\beta_j - 1) > 0$ because $b_j > 0$ (otherwise the beta is zero). We must therefore have $\sum_{i \neq j} x_i (\beta_i - 1) < 0$. Because $b_i \geq 0$, it is necessary that at least one asset has a beta smaller than 1.

(b) We use standard notations to represent Σ. We seek a portfolio such that $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 < 0$. To simplify this problem, we assume that the three assets have the same volatility. We also obtain the following system of inequalities:

$$\begin{cases} b_1 + b_2 \rho_{1,2} + b_3 \rho_{1,3} > 0 \\ b_1 \rho_{1,2} + b_2 + b_3 \rho_{2,3} > 0 \\ b_1 \rho_{1,3} + b_2 \rho_{2,3} + b_3 < 0 \end{cases}$$

It is sufficient that $b_1 \rho_{1,3} + b_2 \rho_{2,3}$ is negative and b_3 is small. For example, we may consider $b_1 = 50\%$, $b_2 = 45\%$, $b_3 = 5\%$, $\rho_{1,2} = 50\%$, $\rho_{1,3} = 0\%$ and $\rho_{2,3} = -50\%$. We obtain $\beta_1 = 1.10$, $\beta_2 = 1.03$ and $\beta_3 = -0.27$.

5. (a) We perform the linear regression $R_{i,t} = \beta_i R_t (b) + \varepsilon_{i,t}$ (TR-RPB, page 16) and we obtain $\hat{\beta}_i = 1.06$.

(b) We deduce that the contribution c_i of the market factor is (TR-RPB, page 16):

$$c_i = \frac{\beta_i^2 \operatorname{var}(R(b))}{\operatorname{var}(R_i)} = 90.62\%$$

1.5 Tangency portfolio

1. To find the tangency portfolio, we can maximize the Sharpe ratio or determine the efficient frontier by including the risk-free asset in the asset universe (see Exercise 1.2 on page 4). We obtain the following result:

r	2%	3%	4%
x_1	10.72%	13.25%	17.43%
x_2	12.06%	12.34%	12.80%
x_3	28.92%	29.23%	29.73%
x_4	48.30%	45.19%	40.04%
$\mu(x)$	8.03%	8.27%	8.68%
$\sigma(x)$	4.26%	4.45%	4.84%
$\mathrm{SR}(x \mid r)$	141.29%	118.30%	96.65%

(a) The tangency portfolio is $x = (10.72\%, 12.06\%, 28.92\%, 48.30\%)$ if the return of the risk-free asset is equal to 2%. Its Sharpe ratio is 1.41.

(b) The tangency portfolio becomes:

$$x = (13.25\%, 12.34\%, 29.23\%, 45.19\%)$$

and $\mathrm{SR}(x \mid r)$ is equal to 1.18.

(c) The tangency portfolio becomes

$$x = (17.43\%, 12.80\%, 29.73\%, 40.04\%)$$

and $\mathrm{SR}(x \mid r)$ is equal to 0.97.

(d) When r rises, the weight of the first asset increases whereas the weight of the fourth asset decreases. This is because the tangency portfolio must have a higher expected return, that is a higher volatility when r increases. The tangency portfolio will then be more exposed to high volatility assets (typically, the first asset) and less exposed to low volatility assets (typically, the fourth asset).

2. We recall that the optimization problem is (TR-RPB, page 19):

$$x^{\star} = \arg\max x^{\top}(\mu + \phi\Sigma b) - \frac{\phi}{2}x^{\top}\Sigma x - \left(\frac{\phi}{2}b^{\top}\Sigma b + b^{\top}\mu\right)$$

We write it as a QP program:

$$x^{\star} = \arg\min \frac{1}{2}x^{\top}\Sigma x - x^{\top}(\gamma\mu + \Sigma b)$$

with $\gamma = \phi^{-1}$. With the long-only constraint, we obtain the results given in Table 1.2.

(a) The portfolio which minimizes the tracking error volatility is the benchmark. The portfolio which maximizes the tracking error volatility is the solution of the optimization problem:

$$\begin{aligned} x^{\star} &= \arg\max (x-b)^{\top}\Sigma(x-b) \\ &= \arg\min -\frac{1}{2}x^{\top}\Sigma x + x^{\top}\Sigma b \end{aligned}$$

We obtain $x = (0\%, 0\%, 0\%, 100\%)$.

(b) There are an infinite number of solutions. In Figure 1.5, we report the relationship between the excess performance $\mu(x \mid b)$ and the tracking error volatility $\sigma(x \mid b)$. We notice that the first part of this relationship is a straight line. In the second panel, we verify that

TABLE 1.2: Solution of Question 2

	b	$\min \sigma(e)$	$\max \sigma(e)$	$\sigma(e) = 3\%$	$\max \mathrm{IR}(x \mid b)$
x_1	60.00%	60.00%	0.00%	83.01%	60.33%
x_2	30.00%	30.00%	0.00%	16.99%	29.92%
x_3	10.00%	10.00%	0.00%	0.00%	9.75%
x_4	0.00%	0.00%	100.00%	0.00%	0.00%
$\mu(x)$	12.80%	12.80%	6.00%	14.15%	12.82%
$\sigma(x)$	10.99%	10.99%	5.00%	13.38%	11.03%
$\mathrm{SR}(x \mid 3\%)$	89.15%	89.15%	60.00%	83.32%	89.04%
$\mu(x \mid b)$	0.00%	0.00%	−6.80%	1.35%	0.02%
$\sigma(x \mid b)$	0.00%	0.00%	12.08%	3.00%	0.05%
$\mathrm{IR}(x \mid b)$	0.00%	0.00%	−56.31%	45.01%	46.54%

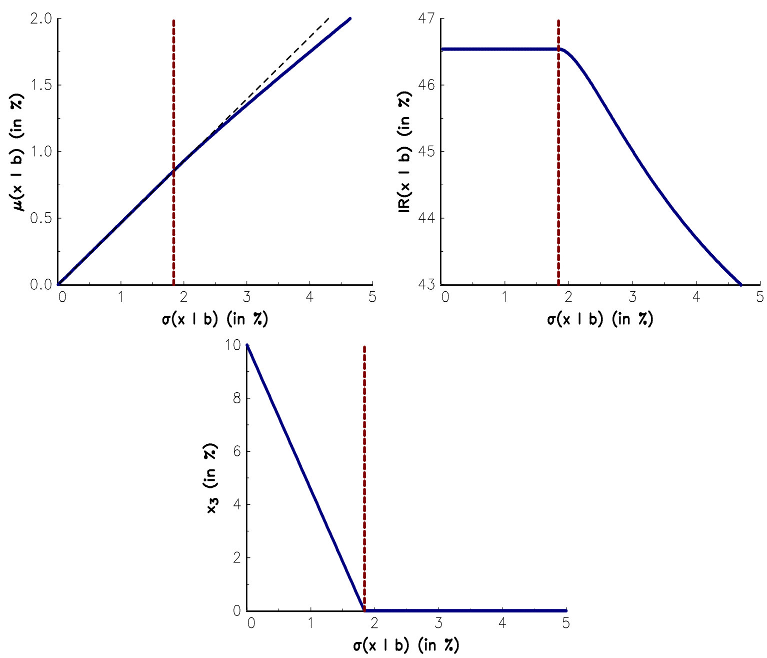

FIGURE 1.5: Maximizing the information ratio

the information ratio is constant and is equal to 46.5419%. In fact, the solutions which maximize the information ratio correspond to optimized portfolios such that the weight of the third asset remains positive (third panel). This implies that $\sigma(x \mid b) \leq 1.8384\%$. For instance, one possible solution is $x = (60.33\%, 29.92\%, 9.75\%, 0.00\%)$. Another solution is $x = (66.47\%, 28.46\%, 5.06\%, 0.00\%)$.

(c) With the constraint $x_i \in [10\%, 50\%]$, the portfolio with the lowest tracking error volatility is $x = (50\%, 30\%, 10\%, 10\%)$. Its information ratio is negative and is equal to -0.57. This means that the portfolio has a negative excess return. The portfolio with the highest tracking error volatility is $x = (10\%, 10\%, 30\%, 50\%)$ and $\sigma(e)$ is equal to 8.84%. In fact, there is no portfolio which satisfies the constraint $x_i \in [10\%, 50\%]$ and has a positive information ratio.

(d) When $r = 3\%$, the tangency portfolio is:

$$x = (13.25\%, 12.34\%, 29.23\%, 45.19\%)$$

and has an information ratio equal to -0.55. This implies that there is no equivalence between the Sharpe ratio ordering and the information ratio ordering.

1.6 Information ratio

1. (a) We have $R(b) = b^\top R$ and $R(x) = x^\top R$. The tracking error is then:

$$e = R(x) - R(b) = (x - b)^\top R$$

It follows that the volatility of the tracking error is:

$$\sigma(x \mid b) = \sigma(r) = \sqrt{(x - b)^\top \Sigma (x - b)}$$

(b) The definition of $\rho(x, b)$ is:

$$\rho(x, b) = \frac{\mathbb{E}\left[(R(x) - \mu(x))(R(b) - \mu(b))\right]}{\sigma(x)\,\sigma(b)}$$

We obtain:

$$\begin{aligned}
\rho(x,b) &= \frac{\mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(b^\top R - b^\top \mu\right)\right]}{\sigma(x)\,\sigma(b)} \\
&= \frac{\mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(R^\top b - \mu^\top b\right)\right]}{\sigma(x)\,\sigma(b)} \\
&= \frac{x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b}{\sigma(x)\,\sigma(b)} \\
&= \frac{x^\top \Sigma b}{\sqrt{x^\top \Sigma x}\sqrt{b^\top \Sigma b}}
\end{aligned}$$

(c) We have:

$$\begin{aligned}
\sigma^2(x \mid b) &= (x-b)^\top \Sigma (x-b) \\
&= x^\top \Sigma x + b^\top \Sigma b - 2x^\top \Sigma b \\
&= \sigma^2(x) + \sigma^2(b) - 2\rho(x,b)\,\sigma(x)\,\sigma(b) \qquad (1.1)
\end{aligned}$$

We deduce that the correlation between portfolio x and benchmark b is:

$$\rho(x,b) = \frac{\sigma^2(x) + \sigma^2(b) - \sigma^2(x \mid b)}{2\sigma(x)\,\sigma(b)} \qquad (1.2)$$

(d) Using Equation (1.1), we deduce that:

$$\sigma^2(x \mid b) \leq \sigma^2(x) + \sigma^2(b) + 2\sigma(x)\,\sigma(b)$$

because $\rho(x,b) \geq -1$. We then have:

$$\begin{aligned}
\sigma(x \mid b) &\leq \sqrt{\sigma^2(x) + \sigma^2(b) + 2\sigma(x)\,\sigma(b)} \\
&\leq \sigma(x) + \sigma(b)
\end{aligned}$$

Using Equation (1.2), we obtain:

$$\frac{\sigma^2(x) + \sigma^2(b) - \sigma^2(x \mid b)}{2\sigma(x)\,\sigma(b)} \leq 1$$

It follows that:

$$\sigma^2(x) + \sigma^2(b) - 2\sigma(x)\,\sigma(b) \leq \sigma^2(x \mid b)$$

and:

$$\begin{aligned}
\sigma(x \mid b) &\geq \sqrt{(\sigma(x) - \sigma(b))^2} \\
&\geq |\sigma(x) - \sigma(b)|
\end{aligned}$$

(e) The lower bound is $|\sigma(x) - \sigma(b)|$. Even if the correlation is close to one, the volatility of the tracking error may be high because portfolio x and benchmark b don't have the same level of volatility. This happens when the portfolio is leveraged with respect to the benchmark.

2. (a) If $\sigma(x \mid b) = \sigma(y \mid b)$, then:

$$\mathrm{IR}(x \mid b) \geq \mathrm{IR}(y \mid b) \Leftrightarrow \mu(x \mid b) \geq \mu(y \mid b)$$

The two portfolios have the same tracking error volatility, but one portfolio has a greater excess return. In this case, it is obvious that x is preferred to y.

(b) If $\sigma(x \mid b) \neq \sigma(y \mid b)$ and $\mathrm{IR}(x \mid b) \geq \mathrm{IR}(y \mid b)$, we consider a combination of benchmark b and portfolio x:

$$z = (1 - \alpha) b + \alpha x$$

with $\alpha \geq 0$. It follows that:

$$z - b = \alpha (x - b)$$

We deduce that:

$$\mu(z \mid b) = (z - b)^\top \mu = \alpha \mu(x \mid b)$$

and:

$$\sigma^2(z \mid b) = (z - b)^\top \Sigma (z - b) = \alpha^2 \sigma^2(x \mid b)$$

We finally obtain that:

$$\mu(z \mid b) = \mathrm{IR}(x \mid b) \cdot \sigma(z \mid b)$$

Every combination of benchmark b and portfolio x has then the same information ratio than portfolio x. In particular, we can take:

$$\alpha = \frac{\sigma(y \mid b)}{\sigma(x \mid b)}$$

In this case, portfolio z has the same tracking error volatility than portfolio y:

$$\begin{aligned} \sigma(z \mid b) &= \alpha \sigma(x \mid b) \\ &= \sigma(y \mid b) \end{aligned}$$

but a higher excess return:

$$\begin{aligned} \mu(z \mid b) &= \mathrm{IR}(x \mid b) \cdot \sigma(z \mid b) \\ &= \mathrm{IR}(x \mid b) \cdot \sigma(y \mid b) \\ &\geq \mathrm{IR}(y \mid b) \cdot \sigma(y \mid b) \\ &\geq \mu(y \mid b) \end{aligned}$$

So, we prefer portfolio x to portfolio y.

(c) We have:

$$\alpha = \frac{3\%}{5\%} = 60\%$$

Portfolio z which is defined by:

$$z = 0.4 \cdot b + 0.6 \cdot x$$

has then the same tracking error volatility than portfolio y, but a higher excess return:

$$\begin{array}{rcl} \mu (z \mid b) & = & 0.6 \cdot 5\% \\ & = & 3\% \end{array}$$

In Figure 1.6, we have represented portfolios x, y and z. We verify that $z \succ y$ implying that $x \succ y$.

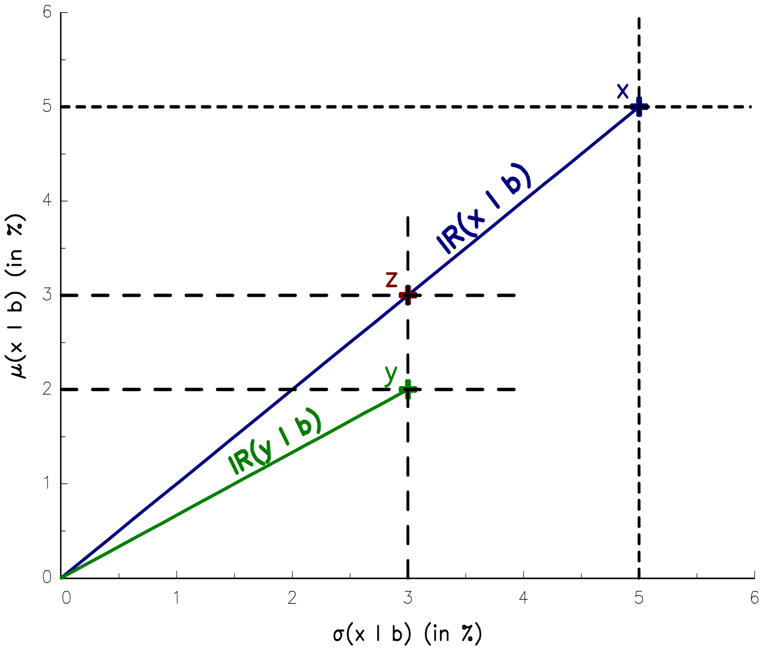

FIGURE 1.6: Information ratio of portfolio z

3. (a) Let $z(x_0)$ be the combination of the tracker x_0 and the portfolio x. We have:

$$z(x_0) = (1 - \alpha) x_0 + \alpha x$$

and:

$$z(x_0) - b = (1 - \alpha)(x_0 - b) + \alpha (x - b)$$

It follows that:

$$\mu\left(z\left(x_0\right) \mid b\right) = (1-\alpha)\,\mu\left(x_0 \mid b\right) + \alpha\mu\left(x \mid b\right)$$

and:

$$\begin{aligned}
\sigma^2\left(z\left(x_0\right) \mid b\right) &= \left(z\left(x_0\right) - b\right)^\top \Sigma \left(z\left(x_0\right) - b\right) \\
&= (1-\alpha)^2 \left(x_0 - b\right)^\top \Sigma \left(x_0 - b\right) + \\
&\quad \alpha^2 (x-b)^\top \Sigma (x-b) + \\
&\quad 2\alpha(1-\alpha)\left(x_0 - b\right)^\top \Sigma (x-b) \\
&= (1-\alpha)^2 \sigma^2\left(x_0 \mid b\right) + \alpha^2\sigma^2\left(x \mid b\right) + \\
&\quad \alpha(1-\alpha)\left(\sigma^2\left(x_0 \mid b\right) + \sigma^2\left(x \mid b\right) - \sigma^2\left(x \mid x_0\right)\right) \\
&= (1-\alpha)\,\sigma^2\left(x_0 \mid b\right) + \alpha\sigma^2\left(x \mid b\right) + \\
&\quad \left(\alpha^2 - \alpha\right)\sigma^2\left(x \mid x_0\right)
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathrm{IR}\left(z\left(x_0\right) \mid b\right) &= \frac{\mu\left(z\left(x_0\right) \mid b\right)}{\sigma\left(z\left(x_0\right) \mid b\right)} \\
&= \frac{(1-\alpha)\,\mu\left(x_0 \mid b\right) + \alpha\mu\left(x \mid b\right)}{\sqrt{\begin{array}{c}(1-\alpha)\,\sigma^2\left(x_0 \mid b\right) + \alpha\sigma^2\left(x \mid b\right) + \\ \left(\alpha^2-\alpha\right)\sigma^2\left(x \mid x_0\right)\end{array}}}
\end{aligned}$$

(b) We have to find α such that $\sigma\left(z\left(x_0\right) \mid b\right) = \sigma\left(y \mid b\right)$. The equation is:

$$(1-\alpha)\,\sigma^2\left(x_0 \mid b\right) + \alpha\sigma^2\left(x \mid b\right) + \left(\alpha^2-\alpha\right)\sigma^2\left(x \mid x_0\right) = \sigma^2\left(y \mid b\right)$$

It is a second-order polynomial equation:

$$A\alpha^2 + B\alpha + C = 0$$

with $A = \sigma^2\left(x \mid x_0\right)$, $B = \sigma^2\left(x \mid b\right) - \sigma^2\left(x \mid x_0\right) - \sigma^2\left(x_0 \mid b\right)$ and $C = \sigma^2\left(x_0 \mid b\right) - \sigma^2\left(y \mid b\right)$. Using the numerical values, we obtain $\alpha = 42.4\%$. We deduce that $\mu\left(z\left(x_0\right) \mid b\right) = 97$ bps and $\mathrm{IR}\left(z\left(x_0\right) \mid b\right) = 0.32$.

(c) In Figure 1.7, we have represented portfolios x_0, x, y, z and $z\left(x_0\right)$. In this case, we have $y \succ z\left(x_0\right)$. We conclude that the preference ordering based on the information ratio is not valid when it is difficult to replicate the benchmark b.

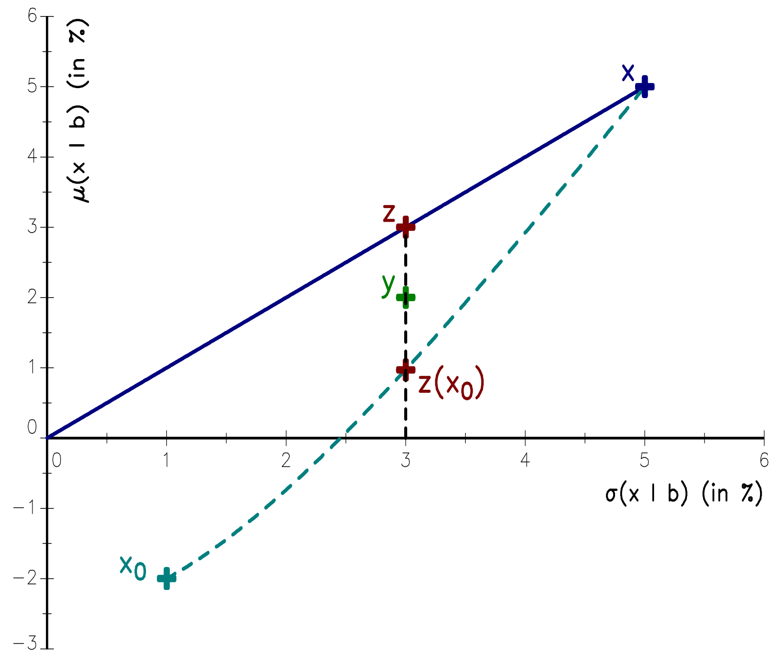

FIGURE 1.7: Information ratio of portfolio $z(x_0)$

1.7 Building a tilted portfolio

1. The ERC portfolio is defined in TR-RPB page 119. We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	32.47%	10.83%	3.52%	25.00%
2	25.41%	13.84%	3.52%	25.00%
3	21.09%	16.67%	3.52%	25.00%
4	21.04%	16.71%	3.52%	25.00%

2. The benchmark b is the ERC portfolio. Using the tracking-error optimization problem (TR-RPB, page 19), we obtain the optimized portfolios given in Table 1.3.

   (a) If the tracking error volatility is set to 1%, the optimal portfolio is $(38.50\%, 20.16\%, 20.18\%, 21.16\%)$. The excess return is equal to 1.13%, which implies an information ratio equal to 1.13.

   (b) If the tracking error is equal to 10%, the information ratio of the optimal portfolio decreases to 0.81.

TABLE 1.3: Solution of Question 2

$\sigma(e)$	0%	1%	5%	10%	max
x_1	32.47%	38.50%	63.48%	96.26%	0.00%
x_2	25.41%	20.16%	0.00%	0.00%	0.00%
x_3	21.09%	20.18%	15.15%	0.00%	0.00%
x_4	21.04%	21.16%	21.37%	3.74%	100.00%
$\mu(x \mid b)$		1.13%	5.66%	8.05%	3.24%
$\sigma(x \mid b)$		1.00%	5.00%	10.00%	25.05%
$\mathrm{IR}(x \mid b)$		1.13	1.13	0.81	0.13
$\sigma(x)$	14.06%	13.89%	13.86%	14.59%	30.00%
$\rho(x \mid b)$		99.75%	93.60%	75.70%	55.71%

(c) We have[4]:

$$\sigma(x \mid b) = \sqrt{\sigma^2(x) - 2\rho(x \mid b)\,\sigma(x)\,\sigma(b) + \sigma^2(b)}$$

We suppose that $\rho(x \mid b) \in [\rho_{\min}, \rho_{\max}]$. Because x may be equal to b, $\rho_{\max}$ is equal to 1. We deduce that:

$$0 \leq \sigma(x \mid b) \leq \sqrt{\sigma^2(x) - 2\rho_{\min}\sigma(x)\,\sigma(b) + \sigma^2(b)}$$

If $\rho_{\min} = -1$, the upper bound of the tracking error volatility is:

$$\sigma(x \mid b) \leq \sigma(x) + \sigma(b)$$

If $\rho_{\min} = 0$, the upper bound becomes:

$$\sigma(x \mid b) \leq \sqrt{\sigma^2(x) + \sigma^2(b)}$$

If $\rho_{\min} = 50\%$, we use the Cauchy-Schwarz inequality and we obtain:

$$\begin{aligned}\sigma(x \mid b) &\leq \sqrt{\sigma^2(x) - \sigma(x)\,\sigma(b) + \sigma^2(b)} \\ &\leq \sqrt{(\sigma(x) - \sigma(b))^2 + \sigma(x)\,\sigma(b)} \\ &\leq |\sigma(x) - \sigma(b)| + \sqrt{\sigma(x)\,\sigma(b)}\end{aligned}$$

Because we have imposed a long-only constraint, it is difficult to find a portfolio which has a negative correlation. For instance, if we consider the previous results, we observe that the correlation

[4] We recall that the correlation between portfolio x and benchmark b is equal to:

$$\rho(x \mid b) = \frac{x^\top \Sigma b}{\sqrt{x^\top \Sigma x}\sqrt{b^\top \Sigma b}}$$

TABLE 1.4: Solution of Question 3

$\sigma(e)$	0%	1%	5%	10%	35%
x_1	32.47%	38.50%	62.65%	92.82%	243.72%
x_2	25.41%	20.16%	−0.83%	−27.07%	−158.28%
x_3	21.09%	20.18%	16.54%	11.99%	−10.77%
x_4	21.04%	21.16%	21.65%	22.27%	25.34%
$\mu(x \mid b)$		1.13%	5.67%	11.34%	39.71%
$\sigma(x \mid b)$		1.00%	5.00%	10.00%	35.00%
$\mathrm{IR}(x \mid b)$		1.13	1.13	1.13	1.13
$\sigma(x)$	14.06%	13.89%	13.93%	15.50%	34.96%
$\rho(x \mid b)$		99.75%	93.62%	77.55%	19.81%

is larger than 50%. In this case, $\sigma(x) \simeq \sigma(b)$ and the order of magnitude of $\sigma(x \mid b)$ is $\sigma(b)$. Because $\sigma(b)$ is equal to 14.06%, it is not possible to find a portfolio which has a tracking error volatility equal to 35%. Even if we consider that $\rho(x \mid b) = 0$, the order of magnitude of $\sigma(x \mid b)$ is $\sqrt{2}\sigma(b)$, that is 28%. We are far from the target value which is equal to 35%. In fact, the portfolio which maximizes the tracking error volatility is $(0\%, 0\%, 0\%, 100\%)$ and the maximum tracking error volatility is 25.05%. We conclude that there is no solution to this question.

3. We obtain the results given in Table 1.4. The deletion of the long-only constraint permits now to find a portfolio with a tracking error volatility which is equal to 35%. We notice that optimal portfolios have the same information ratio. This is perfectly normal because the efficient frontier $\{\sigma(x^\star \mid b), \mu(x^\star \mid b)\}$ is a straight line when there is no constraint[5] (TR-RPB, page 21). It follows that:

$$\mathrm{IR}(x^\star \mid b) = \frac{\mu(x^\star \mid b)}{\sigma(x^\star \mid b)} = \text{constant}$$

Let x_0 be one optimized portfolio corresponding to a given tracking error volatility. Without any constraints, the optimized portfolios may be written as:

$$x^\star = b + \ell \cdot (x_0 - b)$$

We then decompose the optimized portfolio $x^\star$ as the sum of the benchmark b and a leveraged long-short portfolio $x_0 - b$. Let us consider the previous results with x_0 corresponding to the optimal portfolio for a 1% tracking error volatility. We verify that the optimal portfolio which has a tracking error volatility equal to 5% (resp. 10% and 35%) is a portfolio

[5]For instance, we have reported the constrained and unconstrained efficient frontiers in Figure 1.8.

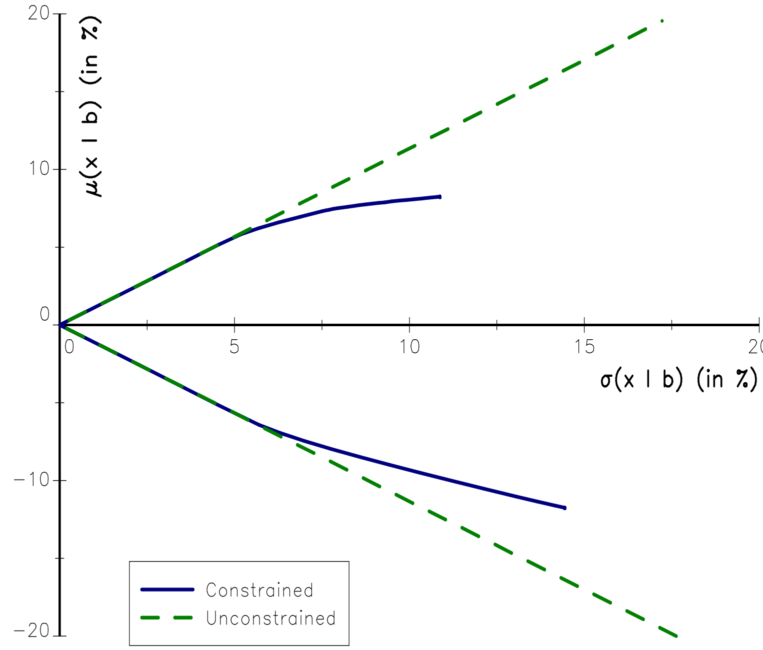

FIGURE 1.8: Constrained and unconstrained efficient frontier

leveraged 5 times (resp. 10 and 35 times) with respect to x_0. Indeed, we have:

$$\begin{aligned}\sigma\left(x^\star \mid b\right) &= \sigma\left(b + \ell \cdot \left(x_0 - b\right) \mid b\right) \\ &= \ell \cdot \sigma\left(x_0 - b \mid b\right) \\ &= \ell \cdot \sigma\left(x_0 \mid b\right)\end{aligned}$$

We deduce that the leverage is the ratio of tracking error volatilities:

$$\ell = \frac{\sigma\left(x^\star \mid b\right)}{\sigma\left(x_0 \mid b\right)}$$

In this case, we verify that:

$$\begin{aligned}\mathrm{IR}\left(x^\star \mid b\right) &= \frac{\mu\left(b + \ell \cdot \left(x_0 - b\right) \mid b\right)}{\ell \cdot \sigma\left(x_0 \mid b\right)} \\ &= \frac{\ell \cdot \mu\left(x_0 \mid b\right)}{\ell \cdot \sigma\left(x_0 \mid b\right)} \\ &= \frac{\mu\left(x_0 \mid b\right)}{\sigma\left(x_0 \mid b\right)}\end{aligned}$$

1.8 Implied risk premium

1. (a) The optimal portfolio is the solution of the following optimization problem:

$$x^\star = \arg\max \mathcal{U}(x)$$

The first-order condition $\partial_x \mathcal{U}(x) = 0$ is:

$$(\mu - r) - \phi\Sigma x^\star = 0$$

We deduce that:

$$\begin{aligned} x^\star &= \frac{1}{\phi}\Sigma^{-1}(\mu - r\mathbf{1}) \\ &= \frac{1}{\phi}\Sigma^{-1}\pi \end{aligned}$$

We verify that the optimal portfolio is a linear function of the risk premium $\pi = \mu - r\mathbf{1}$.

(b) If the investor holds the portfolio x_0, he thinks that it is an optimal investment. We then have:

$$\pi - \phi\Sigma x_0 = 0$$

We deduce that the implied risk premium is:

$$\pi = \phi\Sigma x_0$$

The risk premium is related to three parameters which depend on the investor (the risk aversion ϕ and the composition of the portfolio x_0) and a market parameter (the covariance matrix Σ).

(c) Because $\pi = \phi\Sigma x_0$, we have:

$$x_0^\top \pi = \phi x_0^\top \Sigma x_0$$

We deduce that:

$$\begin{aligned} \phi &= \frac{x_0^\top \pi}{x_0^\top \Sigma x_0} \\ &= \frac{1}{\sqrt{x_0^\top \Sigma x_0}} \cdot \frac{x_0^\top \pi}{\sqrt{x_0^\top \Sigma x_0}} \\ &= \frac{\mathrm{SR}(x_0 \mid r)}{\sqrt{x_0^\top \Sigma x_0}} \end{aligned}$$

(d) It follows that:

$$\begin{aligned}\pi &= \phi\Sigma x_0 \\ &= \frac{\mathrm{SR}\left(x_0 \mid r\right)}{\sqrt{x_0^\top \Sigma x_0}} \Sigma x_0 \\ &= \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}\end{aligned}$$

We know that:

$$\frac{\partial\, \sigma\left(x_0\right)}{\partial\, x} = \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}$$

We deduce that:

$$\pi_i = \mathrm{SR}\left(x_0 \mid r\right) \cdot \mathcal{MR}_i$$

The implied risk premium of asset i is then a linear function of its marginal volatility and the proportionality factor is the Sharpe ratio of the portfolio.

(e) In microeconomics, the price of a good is equal to its marginal cost at the equilibrium. We retrieve this marginalism principle in the relationship between the asset price π_i and the asset risk, which is equal to the product of the Sharpe ratio and the marginal volatility of the asset.

(f) We have:

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \mathrm{SR}\left(x_0 \mid r\right) \cdot \mathcal{MR}_i$$

Another expression of the Sharpe ratio is then:

$$\mathrm{SR}\left(x_0 \mid r\right) = \frac{\sum_{i=1}^{n} \pi_i}{\sum_{i=1}^{n} \mathcal{MR}_i}$$

It is the ratio of the sum of implied risk premia divided by the sum of marginal volatilities. We also notice that:

$$x_i \pi_i = \mathrm{SR}\left(x_0 \mid r\right) \cdot \left(x_i \cdot \mathcal{MR}_i\right)$$

We deduce that:

$$\mathrm{SR}\left(x_0 \mid r\right) = \frac{\sum_{i=1}^{n} x_i \pi_i}{\sum_{i=1}^{n} \mathcal{RC}_i}$$

In this case, the Sharpe ratio is the weighted sum of implied risk premia divided by the sum of risk contributions. In fact, it is the definition of the Sharpe ratio:

$$\mathrm{SR}\left(x_0 \mid r\right) = \frac{\sum_{i=1}^{n} x_i \pi_i}{\mathcal{R}\left(x_0\right)}$$

with $\mathcal{R}\left(x_0\right) = \sum_{i=1}^{n} \mathcal{RC}_i = \sqrt{x_0^\top \Sigma x_0}$.

2. (a) Let $x^\star$ be the market portfolio. The implied risk premium is:

$$\pi = \mathrm{SR}\left(x^\star \mid r\right) \frac{\Sigma x^\star}{\sigma\left(x^\star\right)}$$

The vector of asset betas is:

$$\begin{aligned}\beta &= \frac{\mathrm{cov}\left(R, R\left(x^\star\right)\right)}{\mathrm{var}\left(R\left(x^\star\right)\right)} \\ &= \frac{\Sigma x^\star}{\sigma^2\left(x^\star\right)}\end{aligned}$$

We deduce that:

$$\mu - r = \left(\frac{\mu\left(x^\star\right) - r}{\sigma\left(x^\star\right)}\right) \frac{\beta \sigma^2\left(x^\star\right)}{\sigma\left(x^\star\right)}$$

or:

$$\mu - r = \beta\left(\mu\left(x^\star\right) - r\right)$$

For asset i, we obtain:

$$\mu_i - r = \beta_i\left(\mu\left(x^\star\right) - r\right)$$

or equivalently:

$$\mathbb{E}\left[R_i\right] - r = \beta_i\left(\mathbb{E}\left[R\left(x^\star\right)\right] - r\right)$$

We retrieve the CAPM relationship.

(b) The beta is generally defined in terms of risk:

$$\beta_i = \frac{\mathrm{cov}\left(R_i, R\left(x^\star\right)\right)}{\mathrm{var}\left(R\left(x^\star\right)\right)}$$

We sometimes forget that it is also equal to:

$$\beta_i = \frac{\mathbb{E}\left[R_i\right] - r}{\mathbb{E}\left[R\left(x^\star\right)\right] - r}$$

It is the ratio between the risk premium of the asset and the excess return of the market portfolio.

3. (a) As the volatility of the portfolio $\sigma\left(x\right)$ is a convex risk measure, we have (TR-RPB, page 78):

$$\mathcal{RC}_i \leq x_i \sigma_i$$

We deduce that $\mathcal{MR}_i \leq \sigma_i$. Moreover, we have $\mathcal{MR}_i \geq 0$ because $\rho_{i,j} \geq 0$. The marginal volatility is then bounded:

$$0 \leq \mathcal{MR}_i \leq \sigma_i$$

Using the fact that $\pi_i = \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{MR}_i$, we deduce that:

$$0 \leq \pi_i \leq \mathrm{SR}\left(x \mid r\right) \cdot \sigma_i$$

(b) π_i is equal to the upper bound when $\mathcal{MR}_i = \sigma_i$, that is when the portfolio is fully invested in the i^{th} asset:

$$x_j = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases}$$

(c) We have (TR-RPB, page 101):

$$\mathcal{MR}_i = \frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma(x)}$$

If $\rho_{i,j} = 0$ and $x_i = 0$, then $\mathcal{MR}_i = 0$ and $\pi_i = 0$. The risk premium of the asset reaches then the lower bound when this asset is not correlated to the other assets and when it is not invested.

(d) Negative correlations do not change the upper bound, but the lower bound may be negative because the marginal volatility may be negative.

4. (a) Results are given in the following table:

i	x_i	$\mathcal{MR}_i$	π_i	β_i	$\pi_i/\pi(x)$
1	25.00%	20.08%	10.04%	1.52	1.52
2	25.00%	12.28%	6.14%	0.93	0.93
3	50.00%	10.28%	5.14%	0.78	0.78
$\pi(x)$			6.61%		

(b) Results are given in the following table:

i	x_i	$\mathcal{MR}_i$	π_i	β_i	$\pi_i/\pi(x)$
1	5.00%	9.19%	4.59%	0.66	0.66
2	5.00%	2.33%	1.17%	0.17	0.17
3	90.00%	14.86%	7.43%	1.07	1.07
$\pi(x)$			6.97%		

(c) Results are given in the following table:

i	x_i	$\mathcal{MR}_i$	π_i	β_i	$\pi_i/\pi(x)$
1	100.00%	25.00%	12.50%	1.00	1.00
2	0.00%	10.00%	5.00%	0.40	0.40
3	0.00%	3.75%	1.88%	0.15	0.15
$\pi(x)$			12.50%		

(d) If we compare the results of the second portfolio with respect to the results of the first portfolio, we notice that the risk premium of the third asset increases whereas the risk premium of the first and second assets decreases. The second investor is then overweighted in the third asset, because he implicitly considers that the third asset is very well rewarded. If we consider the results of the third portfolio, we verify that the risk premium may be strictly positive even if the weight of the asset is equal to zero.

1.9 Black-Litterman model

1. (a) We consider the portfolio optimization problem in the presence of a benchmark (TR-RPB, page 17). We obtain the following results (expressed in %):

$\sigma(x^\star \mid b)$	1.00	2.00	3.00	4.00	5.00
$x_1^\star$	35.15	36.97	38.78	40.60	42.42
$x_2^\star$	26.32	19.30	12.28	5.26	−1.76
$x_3^\star$	38.53	43.74	48.94	54.14	59.34
$\mu(x^\star \mid b)$	1.31	2.63	3.94	5.25	6.56

2. (a) Let b be the benchmark (that is the equally weighted portfolio). We recall that the implied risk aversion parameter is:

$$\phi = \frac{\operatorname{SR}(b \mid r)}{\sqrt{b^\top \Sigma b}}$$

and the implied risk premium is:

$$\tilde{\mu} = r + \operatorname{SR}(b \mid r) \frac{\Sigma b}{\sqrt{b^\top \Sigma b}}$$

We obtain $\phi = 3.4367$, $\tilde{\mu}_1 = 7.56\%$, $\tilde{\mu}_2 = 8.94\%$ and $\tilde{\mu}_3 = 5.33\%$.

(b) In this case, the views of the portfolio manager corresponds to the trends observed in the market. We then have $P = I_n$, $Q = \hat{\mu}$ and[6] $\Omega = \operatorname{diag}\left(\sigma^2(\hat{\mu}_1), \ldots, \sigma^2(\hat{\mu}_n)\right)$. The views $P\mu = Q + \varepsilon$ become:

$$\mu = \hat{\mu} + \varepsilon$$

with $\varepsilon \sim \mathcal{N}(\mathbf{0}, \Omega)$.

(c) We have (TR-RPB, page 25):

$$\begin{aligned}
\bar{\mu} &= E[\mu \mid P\mu = Q + \varepsilon] \\
&= \tilde{\mu} + \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} (Q - P\tilde{\mu}) \\
&= \tilde{\mu} + \tau\Sigma (\tau\Sigma + \Omega)^{-1} (\hat{\mu} - \tilde{\mu})
\end{aligned}$$

We obtain $\bar{\mu}_1 = 5.16\%$, $\bar{\mu}_2 = 2.38\%$ and $\bar{\mu}_3 = 2.47\%$.

(d) We optimize the quadratic utility function with $\phi = 3.4367$. The Black-Litterman portfolio is then $x_1 = 56.81\%$, $x_2 = -23.61\%$ and $x_3 = 66.80\%$. Its volatility tracking error is $\sigma(x \mid b) = 8.02\%$ and its alpha is $\mu(x \mid b) = 10.21\%$.

[6] If we suppose that the trends are not correlated.

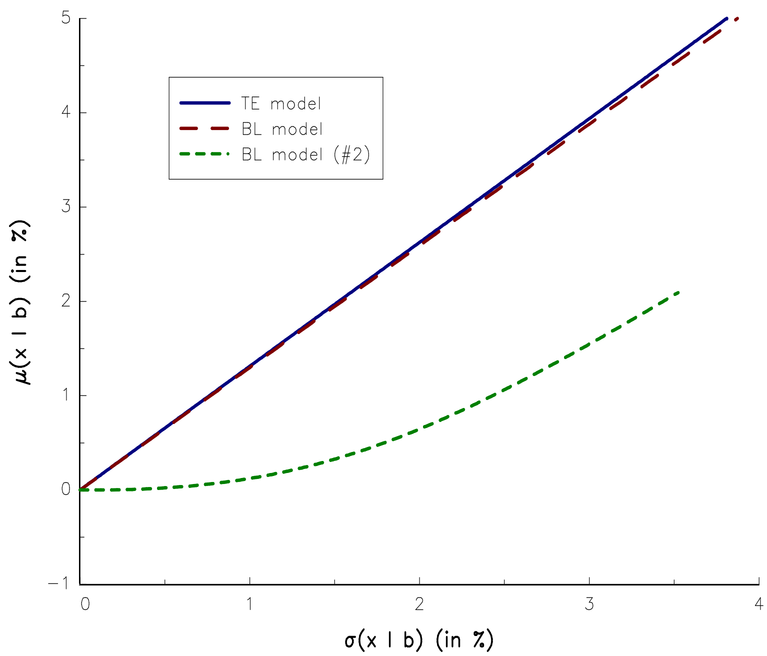

FIGURE 1.9: Efficient frontier of TE and BL portfolios

3. (a) If $\tau = 0$, $\bar{\mu} = \tilde{\mu}$. The BL portfolio x is then equal to the neutral portfolio b. We also have:

$$\begin{aligned}\lim_{\tau\to\infty} \bar{\mu} &= \tilde{\mu} + \lim_{\tau\to\infty} \tau\Sigma^{\top} \left(\tau\Sigma + \Omega\right)^{-1} \left(\hat{\mu} - \tilde{\mu}\right) \\ &= \tilde{\mu} + \left(\hat{\mu} - \tilde{\mu}\right) \\ &= \hat{\mu}\end{aligned}$$

In this case, $\bar{\mu}$ is independent from the implied risk premium $\hat{\mu}$ and is exactly equal to the estimated trends $\hat{\mu}$. The BL portfolio x is then the Markowitz optimized portfolio with the given value of ϕ.

(b) We would like to find the BL portfolio such that $\sigma(x \mid b) = 3\%$. We know that $\sigma(x \mid b) = 0$ if $\tau = 0$. Thanks to Question 2(d), we also know that $\sigma(x \mid b) = 8.02\%$ if $\tau = 1\%$. It implies that the optimal portfolio corresponds to a specific value of τ which is between 0 and 1%. If we apply the bi-section algorithm, we find that $\tau^{\star} = 0.242\%$. The composition of the optimal portfolio is then $x_1^{\star} = 41.18\%$, $x_2^{\star} = 11.96\%$ and $x_3^{\star} = 46.85\%$. We obtain an alpha equal to 3.88%, which is a little bit smaller than the alpha of 3.94% obtained for the TE portfolio.

(c) We have reported the relationship between $\sigma(x \mid b)$ and $\mu(x \mid b)$ in Figure 1.9. We notice that the information ratio of BL portfolios is very close to the information ratio of TE portfolios. We may explain that because of the homogeneity of the estimated trends $\hat{\mu}_i$ and the volatilities $\sigma(\hat{\mu}_i)$. If we suppose that $\sigma(\hat{\mu}_1) = 1\%$, $\sigma(\hat{\mu}_2) = 5\%$ and $\sigma(\hat{\mu}_3) = 15\%$, we obtain the relationship #2. In this case, the BL model produces a smaller information ratio than the TE model. We explain this because $\bar{\mu}$ is the right measure of expected return for the BL model whereas it is $\hat{\mu}$ for the TE model. We deduce that the ratios $\bar{\mu}_i/\hat{\mu}_i$ are more volatile for the parameter set #2, in particular when τ is small.

1.10 Portfolio optimization with transaction costs

1. (a) The turnover is defined in TR-RPB on page 58. Results are given in Table 1.5.

   (b) The relationship is reported in Figure 1.10. We notice that the turnover is not an increasing function of the tracking error volatility. Controlling the last one does not then permit to control the turnover.

   (c) We consider the optimization program given in TR-RPB on page 59. Results are reported in Figure 1.11. We note that the turnover constraint reduces the risk/return tradeoff of MVO portfolios.

   (d) We obtain the results reported in Table 1.6. We notice that there is no solution if $\tau^+ = 10\%$. if $\tau^+ = 80\%$, we retrieve the unconstrained optimized portfolio.

   (e) Results are reported in Figure 1.12. After having rebalanced the allocation seven times, we obtain a portfolio which is located on the efficient frontier.

TABLE 1.5: Solution of Question 1(a)

$\sigma^\star$	EW	4.00	4.50	5.00	5.50	6.00
$x_1^\star$	16.67	28.00	14.44	4.60	0.00	0.00
$x_2^\star$	16.67	41.44	40.11	39.14	34.34	26.18
$x_3^\star$	16.67	11.99	14.86	16.94	17.99	18.13
$x_4^\star$	16.67	17.24	24.89	30.44	35.38	39.79
$x_5^\star$	16.67	0.00	0.00	0.00	0.00	0.00
$x_6^\star$	16.67	1.33	5.70	8.88	12.29	15.91
$\mu(x^\star)$	6.33	6.26	6.84	7.26	7.62	7.93
$\sigma(x^\star)$	5.63	4.00	4.50	5.00	5.50	6.00
$\tau\left(x \mid x^{(0)}\right)$	0.00	73.36	63.32	73.04	75.42	68.17

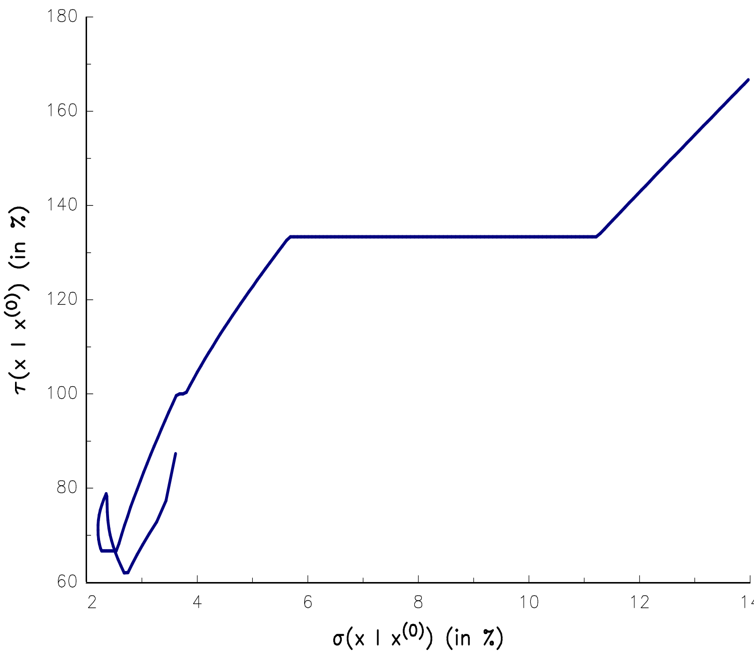

FIGURE 1.10: Relationship between tracking error volatility and turnover

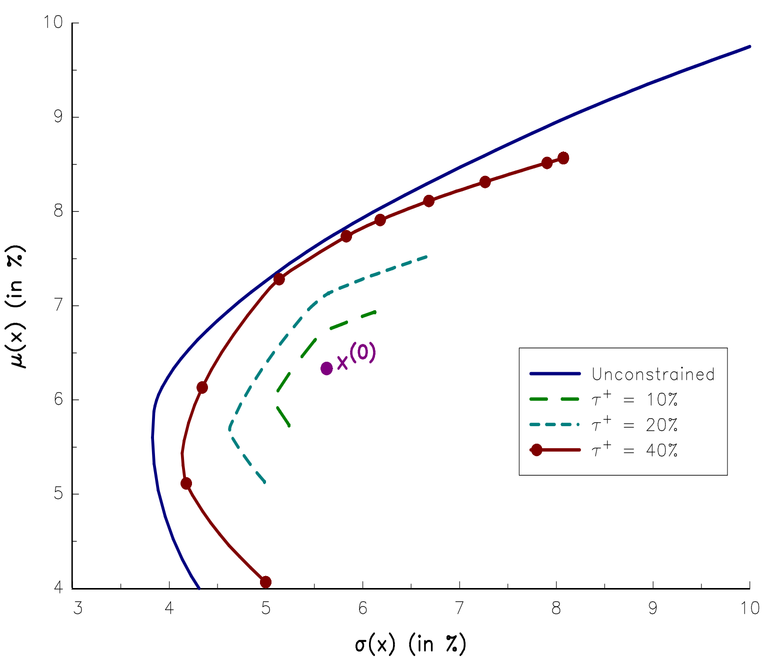

FIGURE 1.11: Efficient frontier with turnover constraints

TABLE 1.6: Solution of Question 1(d)

τ^+	EW	10.00	20.00	40.00	80.00
$x_1^\star$	16.67		16.67	16.67	4.60
$x_2^\star$	16.67		26.67	34.82	39.14
$x_3^\star$	16.67		16.67	16.67	16.94
$x_4^\star$	16.67		16.67	18.51	30.44
$x_5^\star$	16.67		11.15	0.26	0.00
$x_6^\star$	16.67		12.18	13.07	8.88
$\mu(x^\star)$	6.33		6.39	7.14	7.26
$\sigma(x^\star)$	5.63		5.00	5.00	5.00
$\tau\left(x \mid x^{(0)}\right)$	0.00		20.00	40.00	73.04

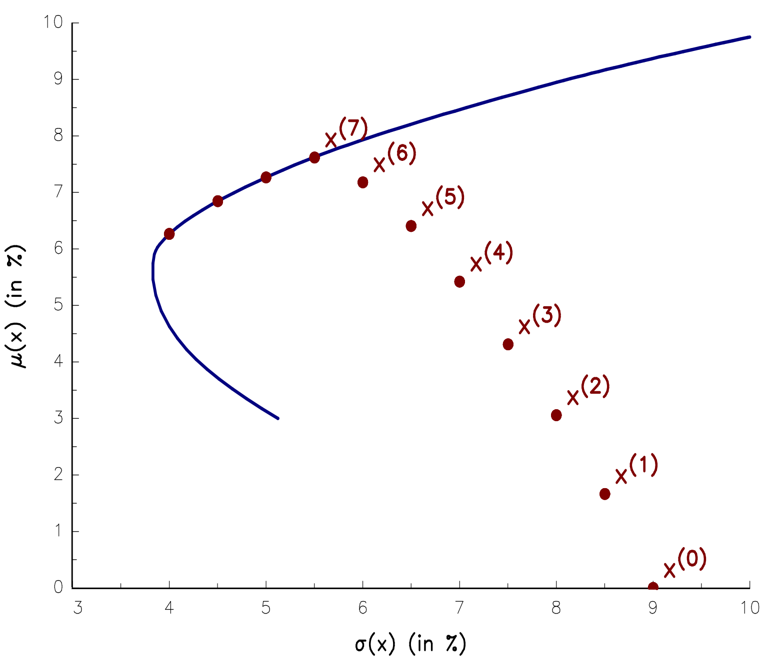

FIGURE 1.12: Path of rebalanced portfolios

2. (a) The weight x_i of asset i is equal to the actual weight $x_i^{(0)}$ plus the positive change x_i^+ minus the negative change x_i^-:

$$x_i = x_i^0 + x_i^+ - x_i^-$$

The transactions costs are equal to:

$$\mathcal{C} = \sum_{i=1}^n x_i^- c_i^- + \sum_{i=1}^n x_i^+ c_i^+$$

Their financing are done by considering a part of the actual wealth:

$$\sum_{i=1}^n x_i + \mathcal{C} = 1$$

Moreover, the expected return of the portfolio is equal to $\mu(x) - \mathcal{C}$. We deduce that the γ-problem of Markowitz becomes:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma \left(x^\top \mu - \mathcal{C} \right) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} x = x^0 + x^+ - x^- \\ \mathbf{1}^\top x + \mathcal{C} = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \\ \mathbf{0} \leq x^- \leq \mathbf{1} \\ \mathbf{0} \leq x^+ \leq \mathbf{1} \end{array} \right. \end{aligned}$$

The associated QP problem is:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} X^\top Q X - x^\top R \\ & \text{u.c.} \left\{ \begin{array}{l} AX = B \\ \mathbf{0} \leq X \leq \mathbf{1} \end{array} \right. \end{aligned}$$

with:

$$\begin{aligned} Q &= \begin{pmatrix} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad R = \gamma \begin{pmatrix} \mu \\ -c^- \\ -c^+ \end{pmatrix}, \\ A &= \begin{pmatrix} \mathbf{1}^\top & (c^-)^\top & (c^+)^\top \\ I_n & I_n & -I_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ x^0 \end{pmatrix} \end{aligned}$$

(b) We obtain the following results:

	EW	#1	#2
$x_1^\star$	16.67	4.60	16.67
$x_2^\star$	16.67	39.14	30.81
$x_3^\star$	16.67	16.94	16.67
$x_4^\star$	16.67	30.44	22.77
$x_5^\star$	16.67	0.00	0.00
$x_6^\star$	16.67	8.88	12.46
$\mu(x^\star)$	6.33	7.26	7.17
$\sigma(x^\star)$	5.63	5.00	5.00
$\mathcal{C}$		1.10	0.62
$\mu(x^\star) - \mathcal{C}$		6.17	6.55

The portfolio #1 is optimized without taking into account the transaction costs. We obtain an expected return equal to 7.26%. However, the trading costs $\mathcal{C}$ are equal to 1.10% and reduce the net expected return to 6.17%. By taking into account the transaction costs, it is possible to find an optimized portfolio #2 which has a net expected return equal to 6.55%.

(c) In the case of a long-only portfolio, the financing of transaction costs is done by the long positions:

$$\sum_{i=1}^{n} x_i + \mathcal{C} = 1$$

In a long-short portfolio, the cost $\mathcal{C}$ may be financed by both the long and short positions. We then have to choose how to finance it. For instance, if we suppose that 50% (resp. 50%) of the cost is financed by the short (resp. long) positions, we obtain:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma \left(x^\top \mu - \mathcal{C} \right) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} x = x^0 + x^+ - x^- \\ \mathbf{1}^\top x + \mathcal{C} = 1 \\ \sum_{i=1}^n \left(x_i + \frac{1}{2}\mathcal{C} \right) \mathbb{1}_i = \sum_{i=1}^n \left(x_i - \frac{1}{2}\mathcal{C} \right) \left(1 - \mathbb{1}_i \right) \\ -\left(1 - \mathbb{1}_i\right) x^S \leq x_i \leq \mathbb{1}_i x^L \\ 0 \leq x_i^- \leq 1 \\ 0 \leq x_i^+ \leq 1 \end{array} \right. \end{aligned}$$

with $\mathbb{1}_i$ an indicator function which takes the value 1 if we want to be long in the asset i or 0 if we want to be short. x^S and x^L indicate the maximum short and long exposures by asset. As previously, it is then easy to write the corresponding QP program.

1.11 Impact of constraints on the CAPM theory

1. (a) At the equilibrium, we have:

$$\mathbb{E}[R_i] = r + \beta_i (\mathbb{E}[R(x^\star)] - r)$$

We introduce the notation $\beta(\mathbf{e}_i \mid x)$ to design the beta of asset i with respect to portfolio x. The previous relationship can be written as follows:

$$\mu_i - r = \beta(\mathbf{e}_i \mid x^\star)(\mu(x^\star) - r)$$

(b) We have:

$$\begin{aligned} \mu_i - r &= \beta(\mathbf{e}_i \mid x^\star)(\mu(x^\star) - r) + \\ &\quad \beta(\mathbf{e}_i \mid x)(\mu(x) - r) - \beta(\mathbf{e}_i \mid x)(\mu(x) - r) \\ &= \pi(\mathbf{e}_i \mid x) + \beta(\mathbf{e}_i \mid x^\star)(\mu(x^\star) - r) - \beta(\mathbf{e}_i \mid x)(\mu(x) - r) \end{aligned}$$

We recall that:

$$\beta(\mathbf{e}_i \mid x^\star)(\mu(x^\star) - r) = \mathcal{MR}_i(x^\star)\,\mathrm{SR}(x^\star \mid r)$$

We deduce that:

$$\mu_i - r = \pi(\mathbf{e}_i \mid x) + \delta_i(x^\star, x)$$

with:

$$\delta_i(x^\star, x) = \mathcal{MR}_i(x^\star)\,\mathrm{SR}(x^\star \mid r) - \mathcal{MR}_i(x)\,\mathrm{SR}(x \mid r)$$

The risk premium of asset i can be decomposed as the sum of the beta return $\pi(\mathbf{e}_i \mid x)$ and the deviation $\delta_i(x^\star, x)$, which depends on the marginal volatilities and the Sharpe ratios.

(c) The beta return overestimates the risk premium of asset i if $\pi(\mathbf{e}_i \mid x) > \mu_i - r$, that is when $\delta_i(x^\star, x) < 0$. We then have:

$$\mathcal{MR}_i(x)\,\mathrm{SR}(x \mid r) > \mathcal{MR}_i(x^\star)\,\mathrm{SR}(x^\star \mid r)$$

or:

$$\mathcal{MR}_i(x) > \mathcal{MR}_i(x^\star)\frac{\mathrm{SR}(x^\star \mid r)}{\mathrm{SR}(x \mid r)}$$

Because $\mathrm{SR}(x^\star \mid r) > \mathrm{SR}(x \mid r)$, it follows that $\mathcal{MR}_i(x) \gg \mathcal{MR}_i(x^\star)$. We conclude that the beta return overestimates the risk premium of asset i if its marginal volatility $\mathcal{MR}_i(x)$ in the portfolio x is large enough compared to its marginal volatility $\mathcal{MR}_i(x^\star)$ in the market portfolio $x^\star$.

(d) We have:

$$\begin{aligned} \mathrm{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\ &= \frac{\mu(x) - x^\top r}{\sigma(x)} \end{aligned}$$

because $\mathbf{1}^\top x = 1$. The optimization program is then:

$$x^\star = \arg\max \frac{\mu(x) - x^\top r}{\sigma(x)}$$

We deduce that the first-order condition is:

$$\frac{\left(\partial_x \mu(x^\star) - r\mathbf{1}\right)\sigma(x^\star) - \left(\mu(x^\star) - r\right)\partial_x \sigma(x^\star)}{\sigma^2(x^\star)} = \mathbf{0}$$

or:

$$\frac{\partial_x \mu(x^\star) - r\mathbf{1}}{\mu(x^\star) - r} = \frac{\partial_x \sigma(x^\star)}{\sigma(x^\star)}$$

We have:

$$\mu(x^\star) = \mu^\top x^\star = \frac{\mu^\top \Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})}$$

and:

$$\mu(x^\star) - r = \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})}$$

The return variance of $x^\star$ is also:

$$\begin{aligned} \sigma^2(x^\star) &= \frac{1}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})} \cdot \frac{(\mu - r\mathbf{1})^\top \Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})} \\ &= \frac{1}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})}\left(\mu(x^\star) - r\right) \end{aligned}$$

It follows that:

$$\begin{aligned} \frac{\partial_x \sigma(x^\star)}{\sigma(x^\star)} &= \frac{\Sigma x^\star}{\sigma^2(x^\star)} \\ &= \frac{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})\Sigma x^\star}{\left(\mu(x^\star) - r\right)} \\ &= \frac{\mathbf{1}^\top \Sigma^{-1}\Sigma x^\star}{\left(\mu(x^\star) - r\right)}(\mu - r\mathbf{1}) \\ &= \frac{\mu - r\mathbf{1}}{\mu(x^\star) - r} \\ &= \frac{\partial_x \mu(x^\star) - r\mathbf{1}}{\mu(x^\star) - r} \end{aligned}$$

$x^\star$ satisfies then the first-order condition.

(e) The first-order condition is $(\mu - r\mathbf{1}) - \phi\Sigma x = \mathbf{0}$. The solution is then:

$$x^{\star} = \frac{1}{\phi}\Sigma^{-1}(\mu - r\mathbf{1})$$

The value of the utility function at the optimum is:

$$\begin{aligned}\mathcal{U}(x^{\star}) &= \frac{1}{\phi}(\mu - r\mathbf{1})^{\top}\Sigma^{-1}(\mu - r\mathbf{1}) - \\ &\quad \frac{\phi}{2}\frac{(\mu - r\mathbf{1})^{\top}\Sigma^{-1}\Sigma\Sigma^{-1}(\mu - r\mathbf{1})}{\phi^2} \\ &= \frac{1}{2\phi}(\mu - r\mathbf{1})^{\top}\Sigma^{-1}(\mu - r\mathbf{1})\end{aligned}$$

We also have:

$$\begin{aligned}\mathrm{SR}(x^{\star} \mid r) &= \frac{(\mu - r\mathbf{1})^{\top}x^{\star}}{\sqrt{x^{\star\top}\Sigma x^{\star}}} \\ &= \sqrt{(\mu - r\mathbf{1})^{\top}\Sigma^{-1}(\mu - r\mathbf{1})}\end{aligned}$$

We obtain:

$$\mathcal{U}(x^{\star}) = \frac{1}{2\phi}\mathrm{SR}^2(x^{\star} \mid r)$$

Maximizing the utility function is then equivalent to maximizing the Sharpe ratio. In fact, the tangency portfolio corresponds to the value of ϕ such that $\mathbf{1}^{\top}x^{\star} = 1$ (no cash in the portfolio). We have:

$$\mathbf{1}^{\top}x^{\star} = \frac{1}{\phi}\mathbf{1}^{\top}\Sigma^{-1}(\mu - r\mathbf{1})$$

It follows that:

$$\phi = \frac{1}{\mathbf{1}^{\top}\Sigma^{-1}(\mu - r\mathbf{1})}$$

We deduce that the tangency portfolio is equal to:

$$x^{\star} = \frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^{\top}\Sigma^{-1}(\mu - r\mathbf{1})}$$

2. (a) We obtain the following results:

i	$x_i^{\star}$	$\beta(\mathbf{e}_i \mid x^{\star})$	$\mathcal{MR}_i(x^{\star})$	$\pi(\mathbf{e}_i \mid x^{\star})$
1	−13.27%	1.77%	6.46%	5.00%
2	21.27%	1.77%	6.46%	5.00%
3	62.84%	0.71%	2.58%	2.00%
4	29.16%	1.42%	5.17%	4.00%

We verify that $\pi(\mathbf{e}_i \mid x) = \mu_i - r$.

(b) We obtain the following results:

i	x_i	$\beta(\mathbf{e}_i \mid x)$	$\mathcal{MR}_i(x)$	$\pi(\mathbf{e}_i \mid x)$	$\delta_i(x^\star, x)$
1	0.00%	2.96%	11.79%	8.38%	−3.38%
2	9.08%	1.77%	7.04%	5.00%	0.00%
3	63.24%	0.71%	2.82%	2.00%	0.00%
4	27.68%	1.42%	5.63%	4.00%	0.00%

Even if x is a tangency portfolio, the beta return differs from the risk premium because of the constraints. By imposing that $x_i \geq 0$, we overestimate the beta of the first asset and its beta return. This explains that $\delta_1(x^\star, x) < 0$.

(c) We obtain the following results:

i	x_i	$\beta(\mathbf{e}_i \mid x)$	$\mathcal{MR}_i(x)$	$\pi(\mathbf{e}_i \mid x)$	$\delta_i(x^\star, x)$
1	10.00%	3.04	16.26%	9.82%	−4.82%
2	10.00%	1.83	9.79%	5.91%	−0.91%
3	48.55%	0.40	2.13%	1.28%	0.72%
4	31.45%	1.02	5.44%	3.28%	0.72%

(d) We consider the portfolio $x = (0\%, 0\%, 50\%, 50\%)$. We obtain the following results:

i	x_i	$\beta(\mathbf{e}_i \mid x)$	$\mathcal{MR}_i(x)$	$\pi(\mathbf{e}_i \mid x)$	$\delta_i(x^\star, x)$
1	0.00%	1.24	6.09%	3.71%	1.29%
2	0.00%	0.25	1.22%	0.74%	4.26%
3	50.00%	0.33	1.62%	0.99%	1.01%
4	50.00%	1.67	8.22%	5.01%	−1.01%

1.12 Generalization of the Jagannathan-Ma shrinkage approach

1. (a) Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem (TR-RPB, page 66):

$$\tilde{x} = x^\star\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{\begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + (\lambda^+ - \lambda^-)\,\mathbf{1}^\top + \mathbf{1}\,(\lambda^+ - \lambda^-)^\top \end{array}\right.$$

where λ^- and λ^+ are the vectors of Lagrange coefficients associated to the lower and upper bounds.

(b) The unconstrained MV portfolio is:

$$x^{\star} = \begin{pmatrix} 50.581\% \\ 1.193\% \\ -6.299\% \\ -3.054\% \\ 57.579\% \end{pmatrix}$$

(c) The constrained MV portfolio is:

$$\tilde{x} = \begin{pmatrix} 40.000\% \\ 16.364\% \\ 3.636\% \\ 0.000\% \\ 40.000\% \end{pmatrix}$$

The Lagrange coefficients are:

$$\lambda^{-} = \begin{pmatrix} 0.000\% \\ 0.000\% \\ 0.000\% \\ 0.118\% \\ 0.000\% \end{pmatrix} \quad \text{and} \quad \lambda^{+} = \begin{pmatrix} 0.345\% \\ 0.000\% \\ 0.000\% \\ 0.000\% \\ 0.290\% \end{pmatrix}$$

The implied volatilities are 17.14%, 20.00%, 25.00%, 24.52% and 16.82%. For the implied shrinkage correlation matrix, we obtain:

$$\tilde{\rho} = \begin{pmatrix} 100.00\% & & & & \\ 53.80\% & 100.00\% & & & \\ 34.29\% & 20.00\% & 100.00\% & & \\ 49.98\% & 38.37\% & 79.63\% & 100.00\% & \\ 53.21\% & 53.20\% & 69.31\% & 49.62\% & 100.00\% \end{pmatrix}$$

(d) If we impose that $3\% \leq x_i \leq 40\%$, the optimal solution becomes:

$$\tilde{x} = \begin{pmatrix} 40.000\% \\ 14.000\% \\ 3.000\% \\ 3.000\% \\ 40.000\% \end{pmatrix}$$

The Lagrange function of the optimization problem is (TR-RPB, page 66):

$$\begin{aligned} \mathcal{L}\left(x; \lambda_0, \lambda_1, \lambda^{-}, \lambda^{+}\right) &= \frac{1}{2} x^{\top} \Sigma x - \\ &\quad \lambda_0 \left(\mathbf{1}^{\top} x - 1\right) - \lambda_1 \left(\mu^{\top} x - \mu^{\star}\right) - \\ &\quad \lambda^{-^{\top}} \left(x - x^{-}\right) - \lambda^{+^{\top}} \left(x^{+} - x\right) \end{aligned}$$

The first-order condition is:

$$\sigma(x)\frac{\partial\,\sigma(x)}{\partial\,x}-\lambda_0\mathbf{1}-\lambda_1\mu-\lambda^-+\lambda^+=\mathbf{0}$$

It follows that:

$$\frac{\partial\,\sigma(x)}{\partial\,x}=\frac{\lambda_0\mathbf{1}+\lambda_1\mu+\lambda^--\lambda^+}{\sigma(x)}$$

We deduce that:

$$\sigma(x+\Delta x)\simeq\sigma(x)+\Delta x^\top\frac{\partial\,\sigma(x)}{\partial\,x}$$

Using the portfolio obtained in Question 1(c), we have $\sigma(x)=12.79\%$ whereas the marginal volatilities $\partial_x\,\sigma(x)$ are equal to:

$$\frac{1}{12.79}\left(1.891+\begin{pmatrix}0.000\\0.000\\0.000\\0.118\\0.000\end{pmatrix}-\begin{pmatrix}0.345\\0.000\\0.000\\0.000\\0.290\end{pmatrix}\right)=\begin{pmatrix}12.09\%\\14.78\%\\14.78\%\\15.70\%\\12.51\%\end{pmatrix}$$

It follows that the approximated value of the portfolio volatility is 12.82% whereas the exact value is 12.84%.

2. (a) The Lagrange function of the unconstrained problem is:

$$\mathcal{L}(x;\lambda_0,\lambda_1)=\frac{1}{2}x^\top\Sigma x-\lambda_0\left(\mathbf{1}^\top x-1\right)-\lambda_1\left(\mu^\top x-\mu^\star\right)$$

with $\lambda_0\geq 0$ and $\lambda_1\geq 0$. The unconstrained solution $x^\star$ satisfies the following first-order conditions:

$$\left\{\begin{array}{l}\Sigma x^\star-\lambda_0\mathbf{1}-\lambda_1\mu=\mathbf{0}\\ \mathbf{1}^\top x^\star-1=0\\ \mu^\top x^\star-\mu^\star=0\end{array}\right.$$

If we now consider the constraints $Cx\geq D$, we have:

$$\begin{aligned}\mathcal{L}\left(x;\lambda_0,\lambda_1,\lambda^-,\lambda^+\right) &= \frac{1}{2}x^\top\Sigma x-\lambda_0\left(\mathbf{1}^\top x-1\right)-\\ &\quad \lambda_1\left(\mu^\top x-\mu^\star\right)-\lambda^\top(Cx-D)\end{aligned}$$

with $\lambda_0\geq 0$, $\lambda_1\geq 0$ and $\lambda\geq\mathbf{0}$. In this case, the constrained solution $\tilde{x}$ satisfies the following Kuhn-Tucker conditions:

$$\left\{\begin{array}{l}\Sigma\tilde{x}-\lambda_0\mathbf{1}-\lambda_1\mu-C^\top\lambda=\mathbf{0}\\ \mathbf{1}^\top\tilde{x}-1=0\\ \mu^\top\tilde{x}-\mu^\star=0\\ \min(\lambda,C\tilde{x}-D)=\mathbf{0}\end{array}\right.$$

To show that $x^{\star}\left(\mu,\tilde{\Sigma}\right)$ is the solution of the constrained problem, we follow the same approach used in the case of lower and upper bounds (TR-RPB, page 67). We have:

$$\begin{aligned}\tilde{\Sigma}\tilde{x} &= \Sigma\tilde{x}-\left(C^{\top}\lambda\mathbf{1}^{\top}+\mathbf{1}\lambda^{\top}C\right)\tilde{x} \\ &= \lambda_0\mathbf{1}+\lambda_1\mu+C^{\top}\lambda-C^{\top}\lambda\mathbf{1}^{\top}\tilde{x}-\mathbf{1}\lambda^{\top}C\tilde{x}\end{aligned}$$

The Kuhn-Tucker condition $\min\left(\lambda, C\tilde{x}-D\right)=\mathbf{0}$ implies that $\lambda^{\top}\left(C\tilde{x}-D\right)=0$. We deduce that:

$$\begin{aligned}\tilde{\Sigma}\tilde{x} &= \lambda_0\mathbf{1}+\lambda_1\mu+C^{\top}\lambda-C^{\top}\lambda-\mathbf{1}\lambda^{\top}D \\ &= \left(\lambda_0-\lambda^{\top}D\right)\mathbf{1}+\lambda_1\mu\end{aligned}$$

It proves that $\tilde{x}$ is the solution of the unconstrained optimization problem with the following unconstrained Lagrange coefficients $\lambda_0^{\star}=\lambda_0-\lambda^{\top}D$ and $\lambda_0^{\star}=\lambda_1$.

(b) Because $\left(AB\right)^{\top}=B^{\top}A^{\top}$, we have:

$$\begin{aligned}\tilde{\Sigma}^{\top} &= \Sigma^{\top}-\left(C^{\top}\lambda\mathbf{1}^{\top}+\mathbf{1}\lambda^{\top}C\right)^{\top} \\ &= \Sigma-\left(\mathbf{1}\lambda^{\top}C+C^{\top}\lambda\mathbf{1}^{\top}\right) \\ &= \tilde{\Sigma}\end{aligned}$$

This proves that $\tilde{\Sigma}$ is a symmetric matrix. For any vector x, we have:

$$\begin{aligned}x^{\top}\tilde{\Sigma}x &= x^{\top}\left(\Sigma-\left(C^{\top}\lambda\mathbf{1}^{\top}+\mathbf{1}\lambda^{\top}C\right)\right)x \\ &= x^{\top}\Sigma x-x^{\top}C^{\top}\lambda\mathbf{1}^{\top}x-x^{\top}\mathbf{1}\lambda^{\top}Cx \\ &= x^{\top}\Sigma x-2x^{\top}C^{\top}\lambda\mathbf{1}^{\top}x\end{aligned}$$

We first consider the case where the constraint $\mu^{\top}x\geq\mu^{\star}$ vanishes and the optimization program corresponds to the minimum variance problem. The first-order condition is:

$$C^{\top}\lambda=\Sigma\tilde{x}-\lambda_0\mathbf{1}$$

It follows that:

$$\left(x^{\top}\mathbf{1}\right)\cdot\left(x^{\top}C^{\top}\lambda\right)=\left(x^{\top}\mathbf{1}\right)\cdot\left(x^{\top}\Sigma\tilde{x}\right)-\lambda_0\left(x^{\top}\mathbf{1}\right)^2$$

Due to Cauchy-Schwarz inequality, we also have:

$$\begin{aligned}\left|\left(x^{\top}\mathbf{1}\right)\cdot\left(x^{\top}\Sigma\tilde{x}\right)\right| &= \left|\left(x^{\top}\mathbf{1}\right)\cdot\left(x^{\top}\Sigma^{1/2}\Sigma^{1/2}\tilde{x}\right)\right| \\ &\leq \left|\left(x^{\top}\mathbf{1}\right)\right|\cdot\left(x^{\top}\Sigma x\right)^{1/2}\cdot\left(\tilde{x}^{\top}\Sigma\tilde{x}\right)^{1/2}\end{aligned}$$

Using the Kuhn-Tucker condition[7], we obtain:

$$\begin{aligned} x^\top \tilde{\Sigma} x &= x^\top \Sigma x - 2\left(x^\top \mathbf{1}\right)\cdot\left(x^\top \Sigma \tilde{x}\right) + 2\lambda_0\left(x^\top \mathbf{1}\right)^2 \\ &\geq x^\top \Sigma x - 2\left|\left(x^\top \mathbf{1}\right)\cdot\left(x^\top \Sigma \tilde{x}\right)\right| + 2\lambda_0\left(x^\top \mathbf{1}\right)^2 \\ &\geq x^\top \Sigma x - 2\left|\left(x^\top \mathbf{1}\right)\right|\cdot\left(x^\top \Sigma x\right)^{1/2}\cdot\left(\tilde{x}^\top \Sigma \tilde{x}\right)^{1/2} + \\ &\quad 2\lambda_0\left(x^\top \mathbf{1}\right)^2 \\ &= x^\top \Sigma x - 2\left|\left(x^\top \mathbf{1}\right)\right|\cdot\left(x^\top \Sigma x\right)^{1/2}\cdot\left(\lambda^\top D + \lambda_0\right)^{1/2} + \\ &\quad 2\lambda_0\left(x^\top \mathbf{1}\right)^2 \end{aligned}$$

We deduce that:

$$\begin{aligned} x^\top \tilde{\Sigma} x &\geq x^\top \Sigma x - 2\left|\left(x^\top \mathbf{1}\right)\right|\cdot\left(x^\top \Sigma x\right)^{1/2}\cdot\left(\lambda^\top D + \lambda_0\right)^{1/2} + \\ &\quad \lambda_0\left(x^\top \mathbf{1}\right)^2 + \left(\lambda^\top D + \lambda_0\right)\left(x^\top \mathbf{1}\right)^2 - \left(\lambda^\top D\right)\left(x^\top \mathbf{1}\right)^2 \\ &= (a-b)^2 + \left(\lambda_0 - \lambda^\top D\right)\left(x^\top \mathbf{1}\right)^2 \end{aligned}$$

where $a = \sqrt{x^\top \Sigma x}$ and $b = \left|\left(x^\top \mathbf{1}\right)\right|\cdot\left(\lambda^\top D + \lambda_0\right)^{1/2}$. If $\lambda_0 \geq \lambda^\top D$, then $x^\top \tilde{\Sigma} x \geq 0$ and $\tilde{\Sigma}$ is a positive semi-definite matrix. If $\lambda_0 < \lambda^\top D$, the matrix $\tilde{\Sigma}$ may be indefinite. Let us consider a universe of three assets. Their volatilities are equal to 15%, 15% and 5% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 20\% & 20\% & 100\% \end{pmatrix}$$

If $\mathcal{C} = \{20\% \leq x_i \leq 80\%\}$, the minimum variance portfolio is $(20\%, 20\%, 60\%)$ and the implied covariance matrix $\tilde{\Sigma}$ is not positive semi-definite. If $\mathcal{C} = \{20\% \leq x_i \leq 50\%\}$, the minimum variance portfolio is $(25\%, 25\%, 50\%)$ and the implied covariance matrix $\tilde{\Sigma}$ is positive semi-definite. The extension to the case $\mu^\top x \geq \mu^\star$ is straightforward because this constraint may be encompassed in the restriction set $\mathcal{C} = \{x \in \mathbb{R}^n : Cx \geq D\}$.

(c) We have $x \geq x^-$ and $x \leq x^+$. Imposing lower and upper bounds is then equivalent to:

$$\begin{pmatrix} I_n \\ -I_n \end{pmatrix} x \geq \begin{pmatrix} x^- \\ -x^+ \end{pmatrix}$$

[7] We have:

$$\begin{aligned} \tilde{x}^\top \Sigma \tilde{x} &= \tilde{x}^\top\left(C^\top \lambda + \lambda_0 \mathbf{1}\right) \\ &= \tilde{x}^\top C^\top \lambda + \lambda_0 \tilde{x}^\top \mathbf{1} \\ &= \lambda^\top\left(C\tilde{x}\right) + \lambda_0 \\ &= \lambda^\top D + \lambda_0 \end{aligned}$$

Let $\lambda = (\lambda^-, \lambda^+)$ be the lagrange coefficients associated with the constraint $Cx \geq D$. We have:

$$\begin{aligned} C^\top \lambda &= \begin{pmatrix} I_n & -I_n \end{pmatrix} \begin{pmatrix} \lambda^- \\ \lambda^+ \end{pmatrix} \\ &= \lambda^- - \lambda^+ \end{aligned}$$

We deduce that the implied shrinkage covariance matrix is:

$$\begin{aligned} \tilde{\Sigma} &= \Sigma - \left(C^\top \lambda \mathbf{1}^\top + \mathbf{1} \lambda^\top C \right) \\ &= \Sigma - \left(\lambda^- - \lambda^+ \right) \mathbf{1}^\top - \mathbf{1} \left(\lambda^- - \lambda^+ \right)^\top \\ &= \Sigma + \left(\lambda^+ - \lambda^- \right) \mathbf{1}^\top + \mathbf{1} \left(\lambda^+ - \lambda^- \right)^\top \end{aligned}$$

We retrieve the results of Jagannathan and Ma (2003).

(d) We write the constraints as follows:

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} x \geq \begin{pmatrix} -0.40 \\ 0.10 \end{pmatrix}$$

We obtain the following composition for the minimum variance portfolio:

$$\tilde{x} = \begin{pmatrix} 44.667\% \\ -4.667\% \\ -19.195\% \\ 10.000\% \\ 69.195\% \end{pmatrix}$$

The lagrange coefficients are 0.043% and 0.134%. The implied volatilities are 15.29%, 20.21%, 25.00%, 24.46% and 15.00%. For the implied shrinkage correlation matrix, we obtain:

$$\tilde{\rho} = \begin{pmatrix} 100.00\% & & & & \\ 51.34\% & 100.00\% & & & \\ 30.57\% & 20.64\% & 100.00\% & & \\ 47.72\% & 38.61\% & 79.58\% & 100.00\% & \\ 41.14\% & 50.89\% & 70.00\% & 47.45\% & 100.00\% \end{pmatrix}$$

3. (a) We consider the same technique used in QP problems (TR-RPB, page 302):

$$\begin{aligned} Ax = B &\Leftrightarrow \left\{ \begin{array}{l} Ax \geq B \\ Ax \leq B \end{array} \right. \\ &\Leftrightarrow \begin{pmatrix} A \\ -A \end{pmatrix} x \geq \begin{pmatrix} B \\ -B \end{pmatrix} \end{aligned}$$

We can then use the previous framework with:

$$C = \begin{pmatrix} A \\ -A \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} B \\ -B \end{pmatrix}$$

(b) We write the constraints as follows:

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} -0.50 \\ 0.00 \\ 0.00 \end{pmatrix}$$

We obtain the following composition for the minimum variance portfolio:

$$\tilde{x} = \begin{pmatrix} 46.033\% \\ 3.967\% \\ -13.298\% \\ 31.649\% \\ 31.649\% \end{pmatrix}$$

The lagrange coefficients are 0.316%, 0.709% and 0. The implied volatilities are 16.97%, 21.52%, 25.00%, 21.98% and 19.15%. For the implied shrinkage correlation matrix, we obtain:

$$\tilde{\rho} = \begin{pmatrix} 100.00\% & & & & \\ 58.35\% & 100.00\% & & & \\ 33.95\% & 24.46\% & 100.00\% & & \\ 39.70\% & 33.96\% & 78.08\% & 100.00\% & \\ 59.21\% & 61.26\% & 69.63\% & 44.54\% & 100.00\% \end{pmatrix}$$

Remark 1 *The original model of Jagannathan and Ma (2003) concerns the minimum variance portfolio. Extension to mean-variance portfolios is straightforward if we consider the Markowitz constraint $\mu(x) \geq \mu^{\star}$ as a special case of the general constraint $Cx \geq D$ treated in this exercise.*

Chapter 2

Exercises related to the risk budgeting approach

2.1 Risk measures

1. (a) We have (TR-RPB, page 74):

$$\text{VaR}(\alpha) = \inf\{\ell : \Pr\{L \geq \ell\} \geq \alpha\}$$

and:

$$\text{ES}(\alpha) = \mathbb{E}[L \mid L \geq \text{VaR}(\alpha)]$$

(b) We assume that $\mathbf{F}$ is continuous. It follows that $\text{VaR}(\alpha) = \mathbf{F}^{-1}(\alpha)$. We deduce that:

$$\begin{aligned}
\text{ES}(\alpha) &= \mathbb{E}\left[L \mid L \geq \mathbf{F}^{-1}(\alpha)\right] \\
&= \int_{\mathbf{F}^{-1}(\alpha)}^{\infty} x \frac{f(x)}{1 - \mathbf{F}(\mathbf{F}^{-1}(\alpha))} \, \mathrm{d}x \\
&= \frac{1}{1-\alpha} \int_{\mathbf{F}^{-1}(\alpha)}^{\infty} x f(x) \, \mathrm{d}x
\end{aligned}$$

We consider the change of variable $t = \mathbf{F}(x)$. Because $\mathrm{d}t = f(x)\,\mathrm{d}x$ and $\mathbf{F}(\infty) = 1$, we obtain:

$$\text{ES}(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} \mathbf{F}^{-1}(t) \, \mathrm{d}t$$

(c) We have:

$$f(x) = \theta \frac{x^{-(\theta+1)}}{x_-^{-\theta}}$$

The non-centered moment of order n is[1]:

$$\begin{aligned}
\mathbb{E}\left[L^n\right] &= \int_{x_-}^{\infty} x^n \theta \frac{x^{-(\theta+1)}}{x_-^{-\theta}} \,\mathrm{d}x \\
&= \frac{\theta}{x_-^{-\theta}} \int_{x_-}^{\infty} x^{n-\theta-1} \,\mathrm{d}x \\
&= \frac{\theta}{x_-^{-\theta}} \left[\frac{x^{n-\theta}}{n-\theta}\right]_{x_-}^{\infty} \\
&= \frac{\theta}{\theta-n} x_-^n
\end{aligned}$$

We deduce that:

$$\mathbb{E}\left[L\right] = \frac{\theta}{\theta-1} x_-$$

and:

$$\mathbb{E}\left[L^2\right] = \frac{\theta}{\theta-2} x_-^2$$

The variance of the loss is then:

$$\operatorname{var}\left(L\right) = \mathbb{E}\left[L^2\right] - \mathbb{E}^2\left[L\right] = \frac{\theta}{\left(\theta-1\right)^2\left(\theta-2\right)} x_-^2$$

x_- is a scale parameter whereas θ is a parameter to control the distribution tail. We have:

$$1 - \left(\frac{\mathbf{F}^{-1}\left(\alpha\right)}{x_-}\right)^{-\theta} = \alpha$$

We deduce that:

$$\operatorname{VaR}\left(\alpha\right) = \mathbf{F}^{-1}\left(\alpha\right) = x_-\left(1-\alpha\right)^{-\theta^{-1}}$$

We also obtain:

$$\begin{aligned}
\operatorname{ES}\left(\alpha\right) &= \frac{1}{1-\alpha} \int_{\alpha}^{1} x_-\left(1-t\right)^{-\theta^{-1}} \,\mathrm{d}t \\
&= \frac{x_-}{1-\alpha} \left[-\frac{1}{1-\theta^{-1}} \left(1-t\right)^{1-\theta^{-1}}\right]_{\alpha}^{1} \\
&= \frac{\theta}{\theta-1} x_-\left(1-\alpha\right)^{-\theta^{-1}} \\
&= \frac{\theta}{\theta-1} \operatorname{VaR}\left(\alpha\right)
\end{aligned}$$

Because $\theta > 1$, we have $\frac{\theta}{\theta-1} > 1$ and:

$$\operatorname{ES}\left(\alpha\right) > \operatorname{VaR}\left(\alpha\right)$$

[1]The moment exists if $n \neq \theta$.

(d) We have:

$$\mathrm{ES}(\alpha) = \frac{1}{1-\alpha}\int_{\mu+\sigma\Phi^{-1}(\alpha)}^{\infty} x\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)\,\mathrm{d}x$$

By considering the change of variable $t = \sigma^{-1}(x-\mu)$, we obtain (TR-RPB, page 75):

$$\begin{aligned}
\mathrm{ES}(\alpha) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty}(\mu+\sigma t)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\,\mathrm{d}t \\
&= \frac{\mu}{1-\alpha}\left[\Phi(t)\right]_{\Phi^{-1}(\alpha)}^{\infty} + \\
&\quad \frac{\sigma}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} t\exp\left(-\frac{1}{2}t^2\right)\,\mathrm{d}t \\
&= \mu + \frac{\sigma}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^2\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \mu + \frac{\sigma}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^2\right) \\
&= \mu + \frac{\sigma}{(1-\alpha)}\phi\left(\Phi^{-1}(\alpha)\right)
\end{aligned}$$

Because $\phi'(x) = -x\phi(x)$, we have:

$$\begin{aligned}
1-\Phi(x) &= \int_x^{\infty}\phi(t)\,\mathrm{d}t \\
&= \int_x^{\infty}\left(-\frac{1}{t}\right)(-t\phi(t))\,\mathrm{d}t \\
&= \int_x^{\infty}\left(-\frac{1}{t}\right)\phi'(t)\,\mathrm{d}t
\end{aligned}$$

We consider the integration by parts with $u(t) = -t^{-1}$ and $v'(t) = \phi(t)$:

$$\begin{aligned}
1-\Phi(x) &= \left[-\frac{\phi(t)}{t}\right]_x^{\infty} - \int_x^{\infty}\frac{1}{t^2}\phi(t)\,\mathrm{d}t \\
&= \frac{\phi(x)}{x} + \int_x^{\infty}\frac{1}{t^3}(-t\phi(t))\,\mathrm{d}t \\
&= \frac{\phi(x)}{x} + \int_x^{\infty}\frac{1}{t^3}\phi'(t)\,\mathrm{d}t
\end{aligned}$$

We consider another integration by parts with $u(t) = t^{-3}$ and $v'(t) = \phi(t)$:

$$\begin{aligned}
1-\Phi(x) &= \frac{\phi(x)}{x} + \left[\frac{\phi(t)}{t^3}\right]_x^{\infty} - \int_x^{\infty} -\frac{3}{t^4}\phi(t)\,\mathrm{d}t \\
&= \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} - \int_x^{\infty}\frac{3}{t^5}\phi'(t)\,\mathrm{d}t
\end{aligned}$$

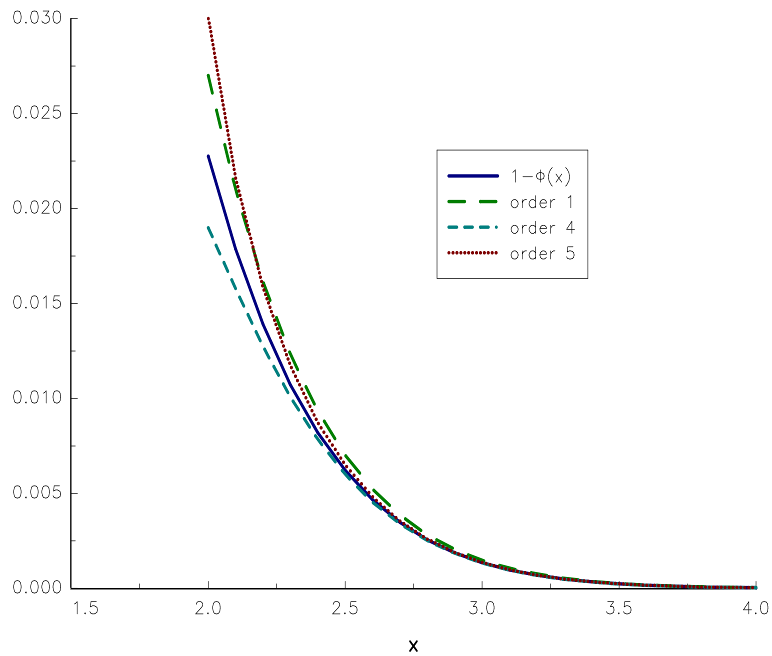

FIGURE 2.1: Approximation of $1 - \Phi(x)$

We continue to use the integration by parts with $v'(t) = \phi(t)$. At the end, we obtain:

$$\begin{aligned} 1 - \Phi(x) &= \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} + 3\frac{\phi(x)}{x^5} - 3 \cdot 5 \frac{\phi(x)}{x^7} + \\ & \quad 3 \cdot 5 \cdot 7 \frac{\phi(x)}{x^9} - \ldots \\ &= \frac{\phi(x)}{x} + \frac{1}{x^2} \sum_{n=1}^{\infty} (-1)^n \left(\prod_{i=1}^{n} (2i-1) \right) \frac{\phi(x)}{x^{2n-1}} \\ &= \frac{\phi(x)}{x} + \frac{\Psi(x)}{x^2} \end{aligned}$$

We have represented the approximation in Figure 2.1. We finally deduce that:

$$\phi(x) = x(1 - \Phi(x)) - \frac{\Psi(x)}{x}$$

By using the previous expression of $\mathrm{ES}(\alpha)$, we obtain with $x =$

$\Phi^{-1}(\alpha)$:

$$\begin{aligned}
\mathrm{ES}(\alpha) &= \mu + \frac{\sigma}{(1-\alpha)}\phi\left(\Phi^{-1}(\alpha)\right) \\
&= \mu + \frac{\sigma}{(1-\alpha)}\phi(x) \\
&= \mu + \frac{\sigma}{(1-\alpha)}\left(\Phi^{-1}(\alpha)(1-\alpha) - \frac{\Psi\left(\Phi^{-1}(\alpha)\right)}{\Phi^{-1}(\alpha)}\right) \\
&= \mu + \sigma\Phi^{-1}(\alpha) - \sigma\frac{\Psi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)\Phi^{-1}(\alpha)} \\
&= \mathrm{VaR}(\alpha) - \sigma\frac{\Psi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)\Phi^{-1}(\alpha)}
\end{aligned}$$

We deduce that $\mathrm{ES}(\alpha) \to \mathrm{VaR}(\alpha)$ because:

$$\lim_{\alpha\to 1}\frac{\Psi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)\Phi^{-1}(\alpha)} = 0$$

(e) For the Gaussian distribution, the expected shortfall and the value-at-risk coincide for high confidence level α. It is not the case with the Pareto distribution, which has a fat tail. The use of the Pareto distribution can then produce risk measures which may be much higher than those based on the Gaussian distribution.

2. (a) We have (TR-RPB, page 73):

$$\begin{aligned}
\mathcal{R}(L_1+L_2) &= \mathbb{E}[L_1+L_2] = \mathbb{E}[L_1] + \mathbb{E}[L_2] = \mathcal{R}(L_1) + \mathcal{R}(L_2) \\
\mathcal{R}(\lambda L) &= \mathbb{E}[\lambda L] = \lambda\mathbb{E}[L] = \lambda\mathcal{R}(L) \\
\mathcal{R}(L+m) &= \mathbb{E}[L-m] = \mathbb{E}[L] - m = \mathcal{R}(L) - m
\end{aligned}$$

We notice that:

$$\mathbb{E}[L] = \int_{-\infty}^{\infty} x\,\mathrm{d}\mathbf{F}(x) = \int_0^1 \mathbf{F}^{-1}(t)\,\mathrm{d}t$$

We deduce that if $\mathbf{F}_1(x) \geq \mathbf{F}_2(x)$, then $\mathbf{F}_1^{-1}(t) \leq \mathbf{F}_2^{-1}(t)$ and $\mathbb{E}[L_1] \leq \mathbb{E}[L_2]$. We conclude that $\mathcal{R}$ is a coherent risk measure.

(b) We have:

$$\begin{aligned}
\mathcal{R}(L_1+L_2) &= \mathbb{E}[L_1+L_2] + \sigma(L_1+L_2) \\
&= \mathbb{E}[L_1] + \mathbb{E}[L_2] + \\
&\quad \sqrt{\sigma^2(L_1) + \sigma^2(L_2) + 2\rho(L_1,L_2)\sigma(L_1)\sigma(L_2)}
\end{aligned}$$

Because $\rho(L_1, L_2) \leq 1$, we deduce that:

$$\begin{aligned}
\mathcal{R}(L_1 + L_2) &\leq \mathbb{E}[L_1] + \mathbb{E}[L_2] + \\
&\quad \sqrt{\sigma^2(L_1) + \sigma^2(L_2) + 2\sigma(L_1)\sigma(L_2)} \\
&\leq \mathbb{E}[L_1] + \mathbb{E}[L_2] + \sigma(L_1) + \sigma(L_2) \\
&\leq \mathcal{R}(L_1) + \mathcal{R}(L_2)
\end{aligned}$$

We have:

$$\begin{aligned}
\mathcal{R}(\lambda L) &= \mathbb{E}[\lambda L] + \sigma(\lambda L) \\
&= \lambda \mathbb{E}[L] + \lambda \sigma(L) \\
&= \lambda \mathcal{R}(L)
\end{aligned}$$

and:

$$\begin{aligned}
\mathcal{R}(L + m) &= \mathbb{E}[L - m] + \sigma(L - m) \\
&= \mathbb{E}[L] - m + \sigma(L) \\
&= \mathcal{R}(L) - m
\end{aligned}$$

If we consider the convexity property, we notice that (TR-RPB, page 73):

$$\begin{aligned}
\mathcal{R}(\lambda L_1 + (1-\lambda) L_2) &\leq \mathcal{R}(\lambda L_1) + \mathcal{R}((1-\lambda) L_2) \\
&\leq \lambda \mathcal{R}(L_1) + (1-\lambda) \mathcal{R}(L_2)
\end{aligned}$$

We conclude that $\mathcal{R}$ is a convex risk measure.

3. We have:

ℓ_i	0	1	2	3	4	5	6	7	8
$\Pr\{L = \ell_i\}$	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$\Pr\{L \leq \ell_i\}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

(a) We have $\text{VaR}(50\%) = 3$, $\text{VaR}(75\%) = 6$, $\text{VaR}(90\%) = 7$ and:

$$\begin{aligned}
\text{ES}(50\%) &= \frac{3 \times 10\% + \ldots + 8 \times 10\%}{60\%} = 5.5 \\
\text{ES}(75\%) &= \frac{6 \times 10\% + \ldots + 8 \times 10\%}{30\%} = 7.0 \\
\text{ES}(90\%) &= \frac{7 \times 10\% + 8 \times 10\%}{20\%} = 7.5
\end{aligned}$$

(b) We have to build a bivariate distribution such that (TR-RPB, page 73):

$$\mathbf{F}_1^{-1}(\alpha) + \mathbf{F}_2^{-1}(\alpha) < \mathbf{F}_{1+2}^{-1}(\alpha)$$

To this end, we may use the Makarov inequalities. For instance, we may consider an ordinal sum of the copula $\mathbf{C}^+$ for $(u_1, u_2) \leq (\alpha, \alpha)$ and another copula $\mathbf{C}_\alpha$ for $(u_1, u_2) > (\alpha, \alpha)$ to produce a bivariate distribution which does not satisfy the subadditivity property. By taking for example $\alpha = 70\%$ and $\mathbf{C}_\alpha = \mathbf{C}^-$, we obtain the following bivariate distribution[2]:

ℓ_i	0	1	2	3	4	5	6	7	8	$p_{2,i}$
0	0.2									0.2
1		0.1								0.1
2			0.1							0.1
3				0.1						0.1
4					0.1					0.1
5						0.1				0.1
6									**0,1**	0.1
7								**0,1**		0.1
8							**0,1**			0.1
$p_{1,i}$	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	

We then have:

ℓ_i	0	2	4	6	8	10	14
$\Pr\{L_1 + L_2 = \ell_i\}$	0.2	0.1	0.1	0.1	0.1	0.1	0.3
$\Pr\{L_1 + L_2 \leq \ell_i\}$	0.2	0.3	0.4	0.5	0.6	0.7	1.0

Because $\mathbf{F}_1^{-1}(80\%) = \mathbf{F}_2^{-1}(80\%) = 6$ and $\mathbf{F}_{1+2}^{-1}(80\%) = 14$, we obtain:

$$\mathbf{F}_1^{-1}(80\%) + \mathbf{F}_2^{-1}(80\%) < \mathbf{F}_{1+2}^{-1}(80\%)$$

2.2 Weight concentration of a portfolio

1. (a) We have represented the function $y = \mathcal{L}(x)$ in Figure 2.2. It verifies $\mathcal{L}(x) \geq x$ and $\mathcal{L}(x) \leq 1$. The Gini coefficient is defined as follows (TR-RPB, page 127):

$$\begin{aligned} G &= \frac{A}{A+B} \\ &= \left(\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - \frac{1}{2}\right) \Big/ \frac{1}{2} \\ &= 2\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1 \end{aligned}$$

[2]We have $p_{1,i} = \Pr\{L_1 = \ell_i\}$ and $p_{2,i} = \Pr\{L_2 = \ell_i\}$.

(b) If $\alpha \geq 0$, the function $\mathcal{L}_\alpha(x) = x^\alpha$ is increasing. We have $\mathcal{L}_\alpha(1) = 1$, $\mathcal{L}_\alpha(x) \leq 1$ and $\mathcal{L}_\alpha(x) \geq x$. We deduce that $\mathcal{L}_\alpha$ is a Lorenz curve. For the Gini index, we have:

$$\begin{aligned} \mathcal{G}(\alpha) &= 2\int_0^1 x^\alpha \,\mathrm{d}x - 1 \\ &= 2\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 - 1 \\ &= \frac{1-\alpha}{1+\alpha} \end{aligned}$$

We deduce that $\mathcal{G}(0) = 1$, $\mathcal{G}\left(\frac{1}{2}\right) = 1/3$ et $\mathcal{G}(1) = 0$. $\alpha = 0$ corresponds to the perfect concentration whereas $\alpha = 1$ corresponds to the perfect equality.

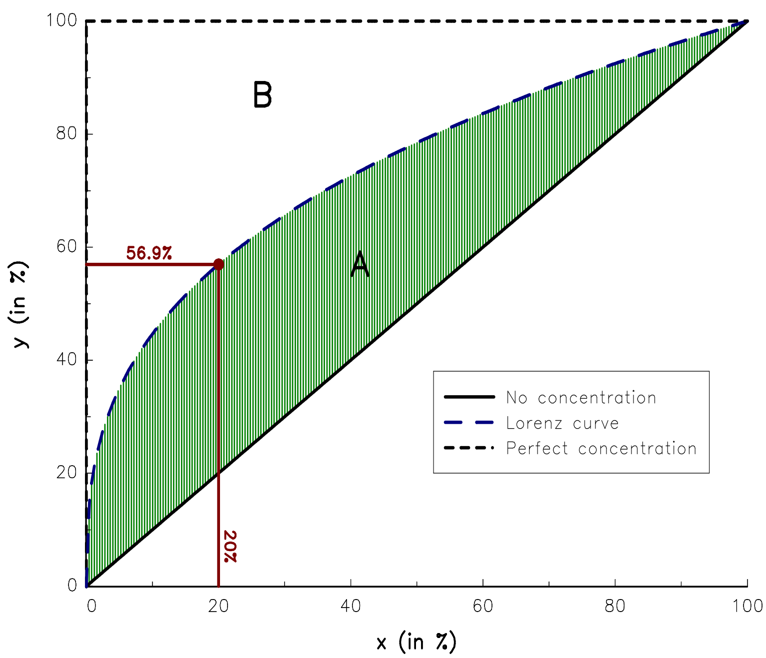

FIGURE 2.2: Lorenz curve

2. (a) We have $\mathcal{L}_w(0) = 0$ and $\mathcal{L}_w(1) = \sum_{j=1}^n w_j = 1$. If $x_2 \geq x_1$, we have $\mathcal{L}_w(x_2) \geq \mathcal{L}_w(x_2)$. $\mathcal{L}_w$ is then a Lorenz curve. The Gini coefficient

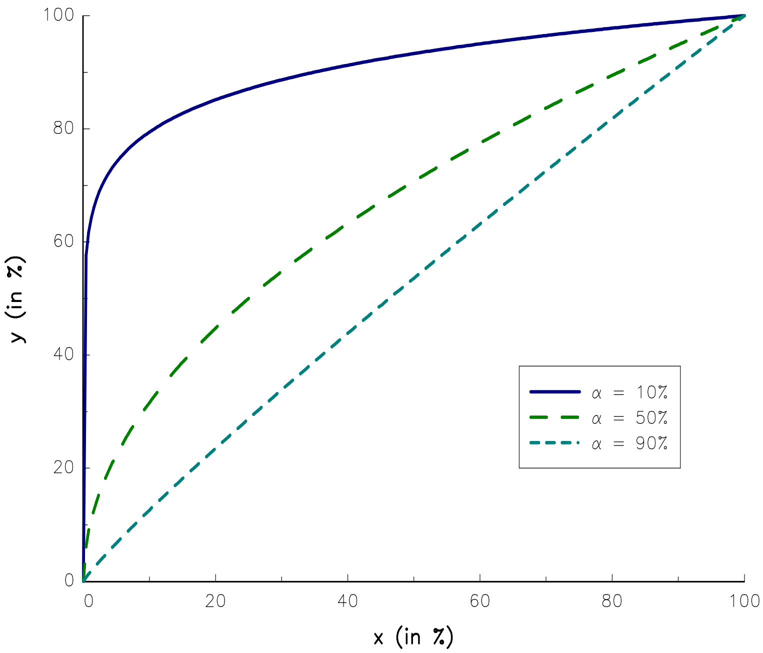

FIGURE 2.3: Function $y = x^\alpha$

is equal to:

$$\begin{aligned} \mathcal{G} &= 2\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1 \\ &= \frac{2}{n}\sum_{i=1}^n \sum_{j=1}^i w_j - 1 \end{aligned}$$

If $w_j = n^{-1}$, we have:

$$\begin{aligned} \lim_{n\to\infty} \mathcal{G} &= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^n \frac{i}{n} - 1 \\ &= \lim_{n\to\infty} \frac{2}{n} \cdot \frac{n(n+1)}{2n} - 1 \\ &= \lim_{n\to\infty} \frac{1}{n} = 0 \end{aligned}$$

If $w_1 = 1$, we have:

$$\begin{aligned} \lim_{n\to\infty} \mathcal{G} &= \lim_{n\to\infty} 1 - \frac{1}{n} \\ &= 1 \end{aligned}$$

We note that that the perfect equality does not correspond to the case $\mathcal{G} = 0$ except in the asymptotic case. This is why we may slightly modify the definition of $\mathcal{L}_w(x)$:

$$\mathcal{L}_w(x) = \begin{cases} \sum_{j=1}^{i} w_j & \text{if} \quad x = n^{-1}i \\ \sum_{j=1}^{i} w_j + w_{i+1}(nx - i) & \text{if} \quad n^{-1}i < x < n^{-1}(i+1) \end{cases}$$

While the previous definition corresponds to a constant piecewise function, this one defines an affine piecewise function. In this case, the computation of the Gini index is done using a trapezoidal integration:

$$\mathcal{G} = \frac{2}{n}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{i} w_j + \frac{1}{2}\right) - 1$$

(b) The Herfindahl index is equal to 1 if the portfolio is concentrated in only one asset. We seek to minimize $\mathcal{H} = \sum_{i=1}^{n} w_i^2$ under the constraint $\sum_{i=1}^{n} w_i = 1$. The Lagrange function is then:

$$f(w_1, \ldots, w_n; \lambda) = \sum_{i=1}^{n} w_i^2 - \lambda\left(\sum_{i=1}^{n} w_i - 1\right)$$

The first-order conditions are $2w_i - \lambda = 0$. We deduce that $w_i = w_j$. $\mathcal{H}$ reaches its minimum when $w_i = n^{-1}$. It corresponds to the equally weighted portfolio. In this case, we have:

$$\mathcal{H} = \frac{1}{n}$$

(c) The statistic $\mathcal{N}$ is the degree of freedom or the equivalent number of equally weighted assets. For instance, if $\mathcal{H} = 0.5$, then $\mathcal{N} = 2$. It is a portfolio equivalent to two equally weighted assets.

3. (a) The minimum variance portfolio is $w_1^{(4)} = 82.342\%$, $w_2^{(4)} = 13.175\%$, $w_3^{(4)} = 3.294\%$, $w_4^{(4)} = 0.823\%$ and $w_5^{(4)} = 0.366\%$.

(b) For each portfolio, we sort the weights in descending order. For the portfolio $w^{(1)}$, we have $w_1^{(1)} = 40\%$, $w_2^{(1)} = 30\%$, $w_3^{(1)} = 20\%$, $w_4^{(1)} = 10\%$ and $w_5^{(1)} = 0\%$. It follows that:

$$\begin{aligned} \mathcal{H}\left(w^{(1)}\right) &= \sum_{i=1}^{5}\left(w_i^{(1)}\right)^2 \\ &= 0.10^2 + 0.20^2 + 0.30^2 + 0.40^2 \\ &= 0.30 \end{aligned}$$

We also have:

$$\begin{aligned}\mathcal{G}\left(w^{(1)}\right) &= \frac{2}{5}\left(\sum_{i=1}^{4}\sum_{j=1}^{i}\tilde{w}_j^{(1)}+\frac{1}{2}\right)-1 \\ &= \frac{2}{5}\left(0.40+0.70+0.90+1.00+\frac{1}{2}\right)-1 \\ &= 0.40\end{aligned}$$

For the portfolios $w^{(2)}$, $w^{(3)}$ and $w^{(4)}$, we obtain $\mathcal{H}\left(w^{(2)}\right)=0.30$, $\mathcal{H}\left(w^{(3)}\right)=0.25$, $\mathcal{H}\left(w^{(4)}\right)=0.70$, $\mathcal{G}\left(w^{(2)}\right)=0.40$, $\mathcal{G}\left(w^{(3)}\right)=0.28$ and $\mathcal{G}\left(w^{(4)}\right)=0.71$. We have $\mathcal{N}\left(w^{(2)}\right)=\mathcal{N}\left(w^{(1)}\right)=3.33$, $\mathcal{N}\left(w^{(3)}\right)=4.00$ and $\mathcal{N}\left(w^{(4)}\right)=1.44$.

(c) All the statistics show that the least concentrated portfolio is $w^{(3)}$. The most concentrated portfolio is paradoxically the minimum variance portfolio $w^{(4)}$. We generally assimilate variance optimization to diversification optimization. We show in this example that diversifying in the Markowitz sense does not permit to minimize the concentration.

2.3 ERC portfolio

1. We note Σ the covariance matrix of asset returns.

   (a) Let $\mathcal{R}(x)$ be a risk measure of the portfolio x. If this risk measure satisfies the Euler principle, we have (TR-RPB, page 78):

   $$\mathcal{R}(x)=\sum_{i=1}^{n}x_i\frac{\partial\,\mathcal{R}(x)}{\partial\,x_i}$$

   We can then decompose the risk measure as a sum of asset contributions. This is why we define the risk contribution $\mathcal{RC}_i$ of asset i as the product of the weight by the marginal risk:

   $$\mathcal{RC}_i=x_i\frac{\partial\,\mathcal{R}(x)}{\partial\,x_i}$$

   When the risk measure is the volatility $\sigma(x)$, it follows that:

   $$\begin{aligned}\mathcal{RC}_i &= x_i\frac{(\Sigma x)_i}{\sqrt{x^\top\Sigma x}} \\ &= \frac{x_i\left(\sum_{k=1}^{n}\rho_{i,k}\sigma_i\sigma_k x_k\right)}{\sigma(x)}\end{aligned}$$

(b) The ERC portfolio corresponds to the risk budgeting portfolio when the risk measure is the return volatility $\sigma(x)$ and when the risk budgets are the same for all the assets (TR-RPB, page 119). It means that $\mathcal{RC}_i = \mathcal{RC}_j$, that is:

$$x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = x_j \frac{\partial\, \sigma(x)}{\partial\, x_j}$$

(c) We have:

$$\begin{aligned} \overline{\mathcal{RC}} &= \frac{1}{n} \sum_{i=1}^{n} \mathcal{RC}_i \\ &= \frac{1}{n} \sigma(x) \end{aligned}$$

It follows that:

$$\begin{aligned} \operatorname{var}(\mathcal{RC}) &= \frac{1}{n} \sum_{i=1}^{n} \left(\mathcal{RC}_i - \overline{\mathcal{RC}}\right)^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(\mathcal{RC}_i - \frac{1}{n}\sigma(x)\right)^2 \\ &= \frac{1}{n^2 \sigma(x)} \sum_{i=1}^{n} \left(n x_i (\Sigma x)_i - \sigma^2(x)\right)^2 \end{aligned}$$

To compute the ERC portfolio, we may consider the following optimization program:

$$x^\star = \arg\min \sum_{i=1}^{n} \left(n x_i (\Sigma x)_i - \sigma^2(x)\right)^2$$

Because we know that the ERC portfolio always exists (TR-RPB, page 108), the objective function at the optimum $x^\star$ is necessarily equal to 0. Another equivalent optimization program is to consider the L^2 norm. In this case, we have (TR-RPB, page 102):

$$x^\star = \arg\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i \cdot (\Sigma x)_i - x_j \cdot (\Sigma x)_j\right)^2$$

(d) We have:

$$\begin{aligned} \beta_i(x) &= \frac{(\Sigma x)_i}{x^\top \Sigma x} \\ &= \frac{\mathcal{MR}_i}{\sigma(x)} \end{aligned}$$

We deduce that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \mathcal{MR}_i \\ &= x_i \beta_i(x)\,\sigma(x)\end{aligned}$$

The relationship $\mathcal{RC}_i = \mathcal{RC}_j$ becomes:

$$x_i \beta_i(x) = x_j \beta_j(x)$$

It means that the weight is inversely proportional to the beta:

$$x_i \propto \frac{1}{\beta_i(x)}$$

We can use the Jacobi power algorithm (TR-RPB, page 308). Let $x^{(k)}$ be the portfolio at iteration k. We define the portfolio $x^{(k+1)}$ as follows:

$$x^{(k+1)} = \frac{\beta_i^{-1}\left(x^{(k)}\right)}{\sum_{j=1}^{n} \beta_j^{-1}\left(x^{(k)}\right)}$$

Starting from an initial portfolio $x^{(0)}$, the limit portfolio is the ERC portfolio if the algorithm converges:

$$\lim_{k\to\infty} x^{(k)} = x_{\text{erc}}$$

(e) Starting from the EW portfolio, we obtain for the first five iterations:

k	0	1	2	3	4	5
$x_1^{(k)}$ (in %)	33.3333	43.1487	40.4122	41.2314	40.9771	41.0617
$x_2^{(k)}$ (in %)	33.3333	32.3615	31.9164	32.3529	32.1104	32.2274
$x_3^{(k)}$ (in %)	33.3333	24.4898	27.6714	26.4157	26.9125	26.7109
$\beta_1\left(x^{(k)}\right)$	0.7326	0.8341	0.8046	0.8147	0.8113	0.8126
$\beta_2\left(x^{(k)}\right)$	0.9767	1.0561	1.0255	1.0397	1.0337	1.0363
$\beta_3\left(x^{(k)}\right)$	1.2907	1.2181	1.2559	1.2405	1.2472	1.2444

The next iterations give the following results:

k	6	7	8	9	10	11
$x_1^{(k)}$ (in %)	41.0321	41.0430	41.0388	41.0405	41.0398	41.0401
$x_2^{(k)}$ (in %)	32.1746	32.1977	32.1878	32.1920	32.1902	32.1909
$x_3^{(k)}$ (in %)	26.7933	26.7593	26.7734	26.7676	26.7700	26.7690
$\beta_1\left(x^{(k)}\right)$	0.8121	0.8123	0.8122	0.8122	0.8122	0.8122
$\beta_2\left(x^{(k)}\right)$	1.0352	1.0356	1.0354	1.0355	1.0355	1.0355
$\beta_3\left(x^{(k)}\right)$	1.2456	1.2451	1.2453	1.2452	1.2452	1.2452

Finally, the algorithm converges after 14 iterations with the following stopping criteria:

$$\sup_i \left| x_i^{(k+1)} - x_i^{(k)} \right| \le 10^{-6}$$

and we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	41.04%	12.12%	4.97%	33.33%
2	32.19%	15.45%	4.97%	33.33%
3	26.77%	18.58%	4.97%	33.33%

2. (a) We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$$

We deduce that:

$$\begin{aligned} \mathcal{RC}_i &= \frac{x_i \left(\sum_{k=1}^n \beta_i \beta_k \sigma_m^2 x_k + \tilde{\sigma}_i^2 x_i\right)}{\tilde{\sigma}(x)} \\ &= \frac{x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2}{\sigma(x)} \end{aligned}$$

with:

$$B = \sum_{k=1}^n x_k \beta_k \sigma_m^2$$

The ERC portfolio satisfies then:

$$x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2 = x_j \beta_j B + x_j^2 \tilde{\sigma}_j^2$$

or:

$$\left(x_i \beta_i - x_j \beta_j\right) B = \left(x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2\right)$$

(b) If $\beta_i = \beta_j = \beta$, we have:

$$\left(x_i - x_j\right) \beta B = \left(x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2\right)$$

Because $\beta > 0$, we deduce that:

$$\begin{aligned} x_i > x_j &\Leftrightarrow x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2 > 0 \\ &\Leftrightarrow x_j \tilde{\sigma}_j > x_i \tilde{\sigma}_i \\ &\Leftrightarrow \tilde{\sigma}_i < \tilde{\sigma}_j \end{aligned}$$

We conclude that the weight x_i is a decreasing function of the specific volatility $\tilde{\sigma}_i$.

(c) If $\tilde{\sigma}_i = \tilde{\sigma}_j = \tilde{\sigma}$, we have:

$$\left(x_i \beta_i - x_j \beta_j\right) B = \left(x_j^2 - x_i^2\right) \tilde{\sigma}^2$$

We deduce that:

$$\begin{aligned} x_i > x_j &\Leftrightarrow \left(x_i \beta_i - x_j \beta_j\right) B < 0 \\ &\Leftrightarrow x_i \beta_i < x_j \beta_j \\ &\Leftrightarrow \beta_i < \beta_j \end{aligned}$$

We conclude that the weight x_i is a decreasing function of the sensitivity β_i.

(d) We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	21.92%	19.73%	4.32%	25.00%
2	24.26%	17.83%	4.32%	25.00%
3	25.43%	17.00%	4.32%	25.00%
4	28.39%	15.23%	4.32%	25.00%

2.4 Computing the Cornish-Fisher value-at-risk

1. We have:

$$\begin{aligned}\mathbb{E}\left[X^{2n}\right] &= \int_{-\infty}^{+\infty} x^{2n}\phi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} x^{2n-1}x\phi(x)\,\mathrm{d}x\end{aligned}$$

Using the integration by parts formula, we obtain[3]:

$$\begin{aligned}\mathbb{E}\left[X^{2n}\right] &= \left[-x^{2n-1}\phi(x)\right]_{-\infty}^{+\infty} + (2n-1)\int_{-\infty}^{+\infty} x^{2n-2}\phi(x)\,\mathrm{d}x \\ &= (2n-1)\int_{-\infty}^{+\infty} x^{2n-2}\phi(x)\,\mathrm{d}x \\ &= (2n-1)\,\mathbb{E}\left[X^{2n-2}\right]\end{aligned}$$

We deduce that $\mathbb{E}\left[X^2\right]=1$, $\mathbb{E}\left[X^4\right]=(2\times2-1)\,\mathbb{E}\left[X^2\right]=3$, $\mathbb{E}\left[X^6\right]=(2\times3-1)\,\mathbb{E}\left[X^4\right]=15$ and $\mathbb{E}\left[X^8\right]=(2\times4-1)\,\mathbb{E}\left[X^4\right]=105$. For the odd moments, we obtain:

$$\begin{aligned}\mathbb{E}\left[X^{2n+1}\right] &= \int_{-\infty}^{+\infty} x^{2n+1}\phi(x)\,\mathrm{d}x \\ &= 0\end{aligned}$$

because $x^{2n+1}\phi(x)$ is an odd function.

2. Let C_t be the value of the call option at time t. The PnL is equal to:

$$\Pi = C_{t+1} - C_t$$

We also have $S_{t+1} = (1+R_{t+1})\,S_t$ with R_{t+1} the daily asset return. We notice that the daily volatility is equal to:

$$\sigma = \frac{32.25\%}{\sqrt{260}} = 2\%$$

[3]because $\phi'(x) = -x\phi(x)$.

We deduce that $R_{t+1} \sim \mathcal{N}(0, 2\%)$.

(a) We have:

$$\begin{aligned} \Pi &\simeq \Delta (S_{t+1} - S_t) \\ &= \Delta R_{t+1} S_t \end{aligned}$$

It follows that $\Pi \sim \mathcal{N}(0, \Delta\sigma S_t)$ and:

$$\mathrm{VaR}_\alpha = \Phi^{-1}(\alpha)\, \Delta\sigma S_t$$

The numerical application gives $\mathrm{VaR}_\alpha = 2.33$ dollars.

(b) In the case of the delta-gamma approximation, we obtain:

$$\begin{aligned} \Pi &\simeq \Delta (S_{t+1} - S_t) + \frac{1}{2}\Gamma (S_{t+1} - S_t)^2 \\ &= \Delta R_{t+1} S_t + \frac{1}{2}\Gamma R_{t+1}^2 S_t^2 \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathbb{E}[\Pi] &= \mathbb{E}\left[\Delta R_{t+1} S_t + \frac{1}{2}\Gamma R_{t+1}^2 S_t^2\right] \\ &= \frac{1}{2}\Gamma S_t^2 \mathbb{E}\left[R_{t+1}^2\right] \\ &= \frac{1}{2}\Gamma \sigma^2 S_t^2 \end{aligned}$$

and:

$$\begin{aligned} \mathbb{E}\left[\Pi^2\right] &= \mathbb{E}\left[\left(\Delta R_{t+1} S_t + \frac{1}{2}\Gamma R_{t+1}^2 S_t^2\right)^2\right] \\ &= \mathbb{E}\left[\Delta^2 R_{t+1}^2 S_t^2 + \Delta\Gamma R_{t+1}^3 S_t^3 + \frac{1}{4}\Gamma^2 R_{t+1}^4 S_t^4\right] \end{aligned}$$

We have $R_{t+1} = \sigma X$ with $X \sim \mathcal{N}(0,1)$. It follows that:

$$\mathbb{E}\left[\Pi^2\right] = \Delta^2 \sigma^2 S_t^2 + \frac{3}{4}\Gamma^2 \sigma^4 S_t^4$$

because $\mathbb{E}[X] = 0$, $\mathbb{E}\left[X^2\right] = 1$, $\mathbb{E}\left[X^3\right] = 0$ and $\mathbb{E}\left[X^4\right] = 3$. The standard deviation of the PnL is then:

$$\begin{aligned} \sigma(\Pi) &= \sqrt{\Delta^2 \sigma^2 S_t^2 + \frac{3}{4}\Gamma^2 \sigma^4 S_t^4 - \left(\frac{1}{2}\Gamma \sigma^2 S_t^2\right)^2} \\ &= \sqrt{\Delta^2 \sigma^2 S_t^2 + \frac{1}{2}\Gamma^2 \sigma^4 S_t^4} \end{aligned}$$

Therefore, the Gaussian approximation of the PnL is:

$$\Pi \sim \mathcal{N}\left(\frac{1}{2}\Gamma\sigma^2 S_t^2, \sqrt{\Delta^2\sigma^2 S_t^2 + \frac{1}{2}\Gamma^2\sigma^4 S_t^4}\right)$$

We deduce that the Gaussian value-at-risk is:

$$\mathrm{VaR}_\alpha = -\frac{1}{2}\Gamma\sigma^2 S_t^2 + \Phi^{-1}(\alpha)\sqrt{\Delta^2\sigma^2 S_t^2 + \frac{1}{2}\Gamma^2\sigma^4 S_t^4}$$

The numerical application gives $\mathrm{VaR}_\alpha = 2.29$ dollars.

(c) Let $L = -\Pi$ be the loss. We recall that the Cornish-Fisher value-at-risk is equal to (TR-RPB, page 94):

$$\mathrm{VaR}(\alpha) = \mu(L) + z_\alpha(\gamma_1, \gamma_2) \cdot \sigma(L)$$

with:

$$\begin{aligned} z_\alpha(\gamma_1, \gamma_2) &= z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2 - \\ &\quad \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)\gamma_1^2 + \cdots \end{aligned}$$

and $z_\alpha = \Phi^{-1}(\alpha)$. γ_1 et γ_2 are the skewness and excess kurtosis of the loss L. We have seen that:

$$\Pi = \Delta\sigma S_t X + \frac{1}{2}\Gamma\sigma^2 S_t^2 X^2$$

with $X \sim \mathcal{N}(0,1)$. Using the results in Question 1, we have $\mathbb{E}[X] = \mathbb{E}\left[X^3\right] = \mathbb{E}\left[X^5\right] = \mathbb{E}\left[X^7\right] = 0$, $\mathbb{E}\left[X^2\right] = 1$, $\mathbb{E}\left[X^4\right] = 3$, $\mathbb{E}\left[X^6\right] = 15$ and $\mathbb{E}\left[X^8\right] = 105$. We deduce that:

$$\begin{aligned} \mathbb{E}\left[\Pi^3\right] &= \mathbb{E}\left[\Delta^3\sigma^3 S_t^3 X^3 + \frac{3}{2}\Delta^2\Gamma\sigma^4 S_t^4 X^4\right] + \\ &\quad \mathbb{E}\left[\frac{3}{4}\Delta\Gamma^2\sigma^5 S_t^5 X^5 + \frac{1}{8}\Gamma^3\sigma^6 S_t^6 X^6\right] \\ &= \frac{9}{2}\Delta^2\Gamma\sigma^4 S_t^4 + \frac{15}{8}\Gamma^3\sigma^6 S_t^6 \end{aligned}$$

and:

$$\begin{aligned} \mathbb{E}\left[\Pi^4\right] &= \mathbb{E}\left[\left(\Delta\sigma S_t X + \frac{1}{2}\Gamma\sigma^2 S_t^2 X^2\right)^4\right] \\ &= 3\Delta^4\sigma^4 S_t^4 + \frac{45}{2}\Delta^2\Gamma^2\sigma^6 S_t^6 + \frac{105}{16}\Gamma^4\sigma^8 S_t^8 \end{aligned}$$

The centered moments are then:

$$\begin{aligned}
\mathbb{E}\left[(\Pi - \mathbb{E}[\Pi])^3\right] &= \mathbb{E}\left[\Pi^3\right] - 3\mathbb{E}[\Pi]\,\mathbb{E}\left[\Pi^2\right] + 2\mathbb{E}^3[\Pi] \\
&= \frac{9}{2}\Delta^2\Gamma\sigma^4 S_t^4 + \frac{15}{8}\Gamma^3\sigma^6 S_t^6 - \frac{3}{2}\Delta^2\Gamma\sigma^4 S_t^4 - \\
&\quad \frac{9}{8}\Gamma^3\sigma^6 S_t^6 + \frac{2}{8}\Gamma^3\sigma^6 S_t^6 \\
&= 3\Delta^2\Gamma\sigma^4 S_t^4 + \Gamma^3\sigma^6 S_t^6
\end{aligned}$$

and:

$$\begin{aligned}
\mathbb{E}\left[(\Pi - \mathbb{E}[\Pi])^4\right] &= \mathbb{E}\left[\Pi^4\right] - 4\mathbb{E}[\Pi]\,\mathbb{E}\left[\Pi^3\right] + 6\mathbb{E}^2[\Pi]\,\mathbb{E}\left[\Pi^2\right] - \\
&\quad 3\mathbb{E}^4[\Pi] \\
&= 3\Delta^4\sigma^4 S_t^4 + \frac{45}{2}\Delta^2\Gamma^2\sigma^6 S_t^6 + \frac{105}{16}\Gamma^4\sigma^8 S_t^8 - \\
&\quad 9\Delta^2\Gamma^2\sigma^6 S_t^6 - \frac{15}{4}\Gamma^4\sigma^8 S_t^8 + \\
&\quad \frac{3}{2}\Delta^2\Gamma^2\sigma^6 S_t^6 + \frac{9}{8}\Gamma^4\sigma^8 S_t^8 - \frac{3}{16}\Gamma^4\sigma^8 S_t^8 \\
&= 3\Delta^4\sigma^4 S_t^4 + 15\Delta^2\Gamma^2\sigma^6 S_t^6 + \frac{15}{4}\Gamma^4\sigma^8 S_t^8
\end{aligned}$$

It follows that the skewness is:

$$\begin{aligned}
\gamma_1(L) &= -\gamma_1(\Pi) \\
&= -\frac{\mathbb{E}\left[(\Pi - \mathbb{E}[\Pi])^3\right]}{\sigma^3(\Pi)} \\
&= -\frac{3\Delta^2\Gamma\sigma^4 S_t^4 + \Gamma^3\sigma^6 S_t^6}{\left(\Delta^2\sigma^2 S_t^2 + \frac{1}{2}\Gamma^2\sigma^4 S_t^4\right)^{3/2}} \\
&= -\frac{6\sqrt{2}\Delta^2\Gamma\sigma^4 S_t^4 + 2\sqrt{2}\Gamma^3\sigma^6 S_t^6}{\left(2\Delta^2\sigma^2 S_t^2 + \Gamma^2\sigma^4 S_t^4\right)^{3/2}}
\end{aligned}$$

whereas the excess kurtosis is:

$$\begin{aligned}
\gamma_2(L) &= \gamma_2(\Pi) \\
&= \frac{\mathbb{E}\left[(\Pi - \mathbb{E}[\Pi])^4\right]}{\sigma^4(\Pi)} - 3 \\
&= \frac{3\Delta^4\sigma^4 S_t^4 + 15\Delta^2\Gamma^2\sigma^6 S_t^6 + \frac{15}{4}\Gamma^4\sigma^8 S_t^8}{\left(\Delta^2\sigma^2 S_t^2 + \frac{1}{2}\Gamma^2\sigma^4 S_t^4\right)^2} - 3 \\
&= \frac{12\Delta^2\Gamma^2\sigma^6 S_t^6 + 3\Gamma^4\sigma^8 S_t^8}{\left(\Delta^2\sigma^2 S_t^2 + \frac{1}{2}\Gamma^2\sigma^4 S_t^4\right)^2}
\end{aligned}$$

Using the numerical values, we obtain $\mu(L) = -0.0400$, $\sigma(L) = 1.0016$, $\gamma_1(L) = -0.2394$, $\gamma_2(L) = 0.0764$, $z_\alpha(\gamma_1, \gamma_2) = 2.1466$ and $\mathrm{VaR}_\alpha = 2.11$ dollars. The value-at-risk is reduced with the Cornish-Fisher approximation because the skewness is negative whereas the excess kurtosis is very small.

3. (a) We have:

$$\begin{aligned} Y &= X^\top A X \\ &= \left(\Sigma^{-1/2} X\right)^\top \Sigma^{1/2} A \Sigma^{1/2} \left(\Sigma^{-1/2} X\right) \\ &= \tilde{X}^\top \tilde{A} \tilde{X} \end{aligned}$$

with $\tilde{A} = \Sigma^{1/2} A \Sigma^{1/2}$, $\tilde{X} \sim \mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$, $\tilde{\mu} = \Sigma^{-1/2}\mu$ and $\tilde{\Sigma} = I$. We deduce that:

$$\begin{aligned} \mathbb{E}[Y] &= \tilde{\mu}^\top \tilde{A} \tilde{\mu} + \operatorname{tr}\left(\tilde{A}\right) \\ &= \mu^\top A \mu + \operatorname{tr}\left(\Sigma^{1/2} A \Sigma^{1/2}\right) \\ &= \mu^\top A \mu + \operatorname{tr}(A\Sigma) \end{aligned}$$

and:

$$\begin{aligned} \operatorname{var}(Y) &= \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] \\ &= 4\tilde{\mu}^\top \tilde{A}^2 \tilde{\mu} + 2\operatorname{tr}\left(\tilde{A}^2\right) \\ &= 4\mu^\top A\Sigma A \mu + 2\operatorname{tr}\left(\Sigma^{1/2} A\Sigma A \Sigma^{1/2}\right) \\ &= 4\mu^\top A\Sigma A \mu + 2\operatorname{tr}\left((A\Sigma)^2\right) \end{aligned}$$

(b) For the moments, we obtain:

$$\begin{aligned} \mathbb{E}[Y] &= \operatorname{tr}(A\Sigma) \\ \mathbb{E}\left[Y^2\right] &= (\operatorname{tr}(A\Sigma))^2 + 2\operatorname{tr}\left((A\Sigma)^2\right) \\ \mathbb{E}\left[Y^3\right] &= (\operatorname{tr}(A\Sigma))^3 + 6\operatorname{tr}(A\Sigma)\operatorname{tr}\left((A\Sigma)^2\right) + 8\operatorname{tr}\left((A\Sigma)^3\right) \\ \mathbb{E}\left[Y^4\right] &= (\operatorname{tr}(A\Sigma))^4 + 32\operatorname{tr}(A\Sigma)\operatorname{tr}\left((A\Sigma)^3\right) + \\ & \quad 12\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^2 + 12(\operatorname{tr}(A\Sigma))^2\operatorname{tr}\left((A\Sigma)^2\right) + \\ & \quad 48\operatorname{tr}\left((A\Sigma)^4\right) \end{aligned}$$

It follows that the first and second centered moments are $\mu(Y) =$

$\operatorname{tr}(A\Sigma)$ and $\operatorname{var}(Y) = 2\operatorname{tr}\left((A\Sigma)^2\right)$. For the third centered moment, we have:

$$\begin{aligned}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^3\right] &= \mathbb{E}\left[Y^3\right] - 3\mathbb{E}\left[Y^2\right]\mathbb{E}[Y] + 2\mathbb{E}^3[Y] \\
&= (\operatorname{tr}(A\Sigma))^3 + 6\operatorname{tr}(A\Sigma)\operatorname{tr}\left((A\Sigma)^2\right) + \\
&\quad 8\operatorname{tr}\left((A\Sigma)^3\right) - 3(\operatorname{tr}(A\Sigma))^3 - \\
&\quad 6\operatorname{tr}\left((A\Sigma)^2\right)\operatorname{tr}(A\Sigma) + 2(\operatorname{tr}(A\Sigma))^3 \\
&= 8\operatorname{tr}\left((A\Sigma)^3\right)
\end{aligned}$$

whereas we obtain for the fourth centered moment:

$$\begin{aligned}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^4\right] &= \mathbb{E}\left[Y^4\right] - 4\mathbb{E}\left[Y^3\right]\mathbb{E}[Y] + 6\mathbb{E}\left[Y^2\right]\mathbb{E}^2[Y] - \\
&\quad 3\mathbb{E}^4[Y] \\
&= (\operatorname{tr}(A\Sigma))^4 + 32\operatorname{tr}(A\Sigma)\operatorname{tr}\left((A\Sigma)^3\right) + \\
&\quad 12\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^2 + 48\operatorname{tr}\left((A\Sigma)^4\right) \\
&\quad 12(\operatorname{tr}(A\Sigma))^2\operatorname{tr}\left((A\Sigma)^2\right) - 4(\operatorname{tr}(A\Sigma))^4 - \\
&\quad 24(\operatorname{tr}(A\Sigma))^2\operatorname{tr}\left((A\Sigma)^2\right) - \\
&\quad 32\operatorname{tr}\left((A\Sigma)^3\right)\operatorname{tr}(A\Sigma) + 6(\operatorname{tr}(A\Sigma))^4 + \\
&\quad 12\operatorname{tr}\left((A\Sigma)^2\right)(\operatorname{tr}(A\Sigma))^2 - 3(\operatorname{tr}(A\Sigma))^4 \\
&= 12\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^2 + 48\operatorname{tr}\left((A\Sigma)^4\right)
\end{aligned}$$

The skewness is then equal to:

$$\begin{aligned}
\gamma_1(Y) &= \frac{8\operatorname{tr}\left((A\Sigma)^3\right)}{\left(2\operatorname{tr}\left((A\Sigma)^2\right)\right)^{3/2}} \\
&= \frac{2\sqrt{2}\operatorname{tr}\left((A\Sigma)^3\right)}{\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^{3/2}}
\end{aligned}$$

For the excess kurtosis, we obtain:

$$\begin{aligned}\gamma_2(Y) &= \frac{12\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^2 + 48\operatorname{tr}\left((A\Sigma)^4\right)}{\left(2\operatorname{tr}\left((A\Sigma)^2\right)\right)^2} - 3 \\ &= \frac{12\operatorname{tr}\left((A\Sigma)^4\right)}{\left(\operatorname{tr}\left((A\Sigma)^2\right)\right)^2}\end{aligned}$$

4. We have:

$$\Pi = x^\top (C_{t+1} - C_t)$$

where C_t is the vector of option prices.

(a) The expression of the PnL is:

$$\begin{aligned}\Pi &\simeq x^\top (\Delta \circ (S_{t+1} - S_t)) \\ &= x^\top ((\Delta \circ S_t) \circ R_{t+1}) \\ &= \tilde{\Delta}^\top R_{t+1}\end{aligned}$$

with $\tilde{\Delta}$ the vector of delta exposures in dollars:

$$\tilde{\Delta}_i = x_i \Delta_i S_{i,t}$$

Because $R_{t+1} \sim \mathcal{N}(\mathbf{0}, \Sigma)$, it follows that $\Pi \sim \mathcal{N}\left(0, \sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta}}\right)$. We deduce that the Gaussian value-at-risk is:

$$\mathrm{VaR}_\alpha = \Phi^{-1}(\alpha) \sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta}}$$

The risk contribution of option i is then equal to:

$$\begin{aligned}\mathcal{RC}_i &= x_i \frac{\Phi^{-1}(\alpha)\left(\Sigma\tilde{\Delta}\right)_i \Delta_i S_{i,t}}{\sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta}}} \\ &= \Phi^{-1}(\alpha) \frac{\tilde{\Delta}_i \cdot \left(\Sigma\tilde{\Delta}\right)_i}{\sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta}}}\end{aligned}$$

(b) In the case of the delta-gamma approximation, we obtain:

$$\begin{aligned}\Pi &\simeq x^\top (\Delta \circ (S_{t+1} - S_t)) + \\ &\quad \frac{1}{2} x^\top \left(\Gamma \circ (S_{t+1} - S_t) \circ (S_{t+1} - S_t)^\top\right) x \\ &= \tilde{\Delta}^\top R_{t+1} + \frac{1}{2} R_{t+1}^\top \tilde{\Gamma} R_{t+1}\end{aligned}$$

with $\tilde{\Gamma}$ the matrix of gamma exposures in dollars:

$$\tilde{\Gamma}_{i,j} = x_i x_j \Gamma_{i,j} S_{i,t} S_{j,t}$$

We deduce that:

$$\begin{aligned}
\mathbb{E}\left[\Pi\right] &= \mathbb{E}\left[\tilde{\Delta}^\top R_{t+1} + \frac{1}{2} R_{t+1}^\top \tilde{\Gamma} R_{t+1}\right] \\
&= \frac{1}{2}\mathbb{E}\left[R_{t+1}^\top \tilde{\Gamma} R_{t+1}\right] \\
&= \frac{1}{2}\operatorname{tr}\left(\tilde{\Gamma}\Sigma\right)
\end{aligned}$$

and:

$$\begin{aligned}
\operatorname{var}\left(\Pi\right) &= \mathbb{E}\left[\left(\Pi - \mathbb{E}\left[\Pi\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\tilde{\Delta}^\top R_{t+1} + \frac{1}{2} R_{t+1}^\top \tilde{\Gamma} R_{t+1} - \frac{1}{2}\operatorname{tr}\left(\tilde{\Gamma}\Sigma\right)\right)^2\right] \\
&= \mathbb{E}\left[\left(\tilde{\Delta}^\top R_{t+1}\right)^2\right] + \frac{1}{4}\mathbb{E}\left[\left(R_{t+1}^\top \tilde{\Gamma} R_{t+1} - \operatorname{tr}\left(\tilde{\Gamma}\Sigma\right)\right)^2\right] + \\
&\quad \mathbb{E}\left[\left(\tilde{\Delta}^\top R_{t+1}\right)\left(R_{t+1}^\top \tilde{\Gamma} R_{t+1} - \operatorname{tr}\left(\tilde{\Gamma}\Sigma\right)\right)\right] \\
&= \mathbb{E}\left[\left(\tilde{\Delta}^\top R_{t+1}\right)^2\right] + \frac{1}{4}\operatorname{var}\left(R_{t+1}^\top \tilde{\Gamma} R_{t+1}\right) \\
&= \tilde{\Delta}^\top \Sigma \tilde{\Delta} + \frac{1}{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)
\end{aligned}$$

Therefore, the Gaussian approximation of the PnL is:

$$\Pi \sim \mathcal{N}\left(\frac{1}{2}\operatorname{tr}\left(\tilde{\Gamma}\Sigma\right), \sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta} + \frac{1}{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)}\right)$$

We deduce that the Gaussian value-at-risk is:

$$\mathrm{VaR}_\alpha = -\frac{1}{2}\operatorname{tr}\left(\tilde{\Gamma}\Sigma\right) + \Phi^{-1}\left(\alpha\right)\sqrt{\tilde{\Delta}^\top \Sigma \tilde{\Delta} + \frac{1}{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)}$$

(c) If the portfolio is delta neutral, Δ is equal to zero and we have:

$$\Pi \simeq \frac{1}{2} R_{t+1}^\top \tilde{\Gamma} R_{t+1}$$

Let $L = -\Pi$ be the loss. Using the formulas of Question 3(b), we obtain:

$$\mu\left(L\right) = -\frac{1}{2}\operatorname{tr}\left(\tilde{\Gamma}\Sigma\right)$$

$$\sigma(L) = \sqrt{\frac{1}{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)}$$

$$\gamma_1(L) = -\frac{2\sqrt{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^3\right)}{\left(\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)\right)^{3/2}}$$

$$\gamma_2(L) = \frac{12\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^4\right)}{\left(\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)\right)^2}$$

We have all the statistics to compute the Cornish-Fisher value-at-risk.

(d) We notice that the previous formulas obtained in the multivariate case are perfectly coherent with those obtained in the univariate case. When the portfolio is not delta neutral, we could then postulate that the skewness is[4]:

$$\gamma_1(L) = -\frac{6\sqrt{2}\tilde{\Delta}^\top\Sigma\Gamma\Sigma\tilde{\Delta} + 2\sqrt{2}\operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^3\right)}{\left(2\tilde{\Delta}^\top\Sigma\tilde{\Delta} + \operatorname{tr}\left(\left(\tilde{\Gamma}\Sigma\right)^2\right)\right)^{3/2}}$$

In fact, it is the formula obtained by Britten-Jones and Schaeffer (1999)[5].

5. (a) Using the numerical values, we obtain $\mu(L) = -78.65$, $\sigma(L) = 88.04$, $\gamma_1(L) = -2.5583$ and $\gamma_2(L) = 10.2255$. The value-at-risk is then equal to 0 for the delta approximation, 126.16 for the delta-gamma approximation and -45.85 for the Cornish-Fisher approximation. We notice that we obtain an absurd result in the last case, because the distribution is far from the Gaussian distribution (high skewness and kurtosis). If we consider a smaller order expansion:

$$_\alpha(\gamma_1, \gamma_2) = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2$$

the value-at-risk is equal to 171.01.

[4]You may easily verify that we obtained this formula in the case $n = 2$ by developing the different polynomials.

[5]Britten-Jones M. and Schaeffer S.M. (1999), Non-Linear Value-at-Risk, *European Finance Review*, 2(2), pp. 167-187.

(b) In this case, we obtain 126.24 for the delta approximation, 161.94 for the delta-gamma approximation and −207.84 for the Cornish-Fisher approximation. For the delta approximation, the risk decomposition is:

Option	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	50.00	0.86	42.87	33.96%
2	20.00	0.77	15.38	12.19%
3	30.00	2.27	67.98	53.85%
VaR_α			126.24	100.00%

For the delta-gamma approximation, we have:

Option	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	50.00	4.06	202.92	125.31%
2	20.00	1.18	23.62	14.59%
3	30.00	1.04	31.10	19.21%
VaR_α			161.94	159.10%

We notice that the delta-gamma approximation does not satisfy the Euler decomposition. Finally, we obtain the following ERC portfolio:

Option	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	42.38	0.90	38.35	33.33%
2	37.16	1.03	38.35	33.33%
3	20.46	1.87	38.35	33.33%
VaR_α			115.05	

2.5 Risk budgeting when risk budgets are not strictly positive

1. (a) We obtain the following portfolio:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_1$	x_i	20.29%	15.95%	20.82%	14.88%	9.97%	18.08%	8.55%
	$\mathcal{MR}_i$	16.85%	16.07%	12.31%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i$	3.42%	2.56%	2.56%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

We notice that the last three assets have a positive weight ($x_i > 0$) and a marginal risk equal to zero ($\mathcal{MR}_i = 0$). We deduce that the number of solutions is $2^3 = 8$ (TR-RPB, page 110).

(b) We obtain the following portfolio:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_2$	x_i	33.77%	29.05%	37.18%	0.00%	0.00%	0.00%	16.07%
	$\mathcal{MR}_i$	19.03%	16.59%	12.96%	−4.54%	−1.55%	−2.39%	
	$\mathcal{RC}_i$	6.43%	4.82%	4.82%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

We notice that the marginal risk of the assets that have a nil weight is negative. We confirm that that the number of solutions is $2^3 = 8$ (TR-RPB, page 110).

(c) We obtain the six other solutions reported in Table 2.1.

TABLE 2.1: The six other RB portfolios

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_3$	x_i	25.57%	21.56%	28.38%	24.49%	0.00%	0.00%	11.55%
	$\mathcal{MR}_i$	18.07%	16.08%	12.21%	0.00%	−0.99%	−1.88%	
	$\mathcal{RC}_i$	4.62%	3.47%	3.47%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	
$\mathcal{S}_4$	x_i	27.27%	23.15%	29.60%	0.00%	19.98%	0.00%	12.71%
	$\mathcal{MR}_i$	18.64%	16.47%	12.88%	−4.12%	0.00%	−2.43%	
	$\mathcal{RC}_i$	5.08%	3.81%	3.81%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	
$\mathcal{S}_5$	x_i	25.42%	20.34%	25.95%	0.00%	0.00%	28.29%	11.22%
	$\mathcal{MR}_i$	17.66%	16.55%	12.98%	−3.93%	−1.62%	0.00%	
	$\mathcal{RC}_i$	4.49%	3.37%	3.37%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	
$\mathcal{S}_6$	x_i	23.31%	19.49%	25.61%	20.76%	10.84%	0.00%	10.42%
	$\mathcal{MR}_i$	17.88%	16.03%	12.21%	0.00%	0.00%	−1.94%	
	$\mathcal{RC}_i$	4.17%	3.13%	3.13%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	
$\mathcal{S}_7$	x_i	22.14%	17.61%	23.05%	17.89%	0.00%	19.32%	9.46%
	$\mathcal{MR}_i$	17.09%	16.11%	12.31%	0.00%	−1.11%	0.00%	
	$\mathcal{RC}_i$	3.78%	2.84%	2.84%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	
$\mathcal{S}_8$	x_i	21.71%	17.09%	21.77%	0.00%	15.37%	24.05%	9.36%
	$\mathcal{MR}_i$	17.25%	16.44%	12.90%	−3.49%	0.00%	0.00%	
	$\mathcal{RC}_i$	3.75%	2.81%	2.81%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

2. (a) We obtain the following portfolio:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_1$	x_i	33.77%	29.05%	37.18%	0.00%	0.00%	0.00%	16.07%
	$\mathcal{MR}_i$	19.03%	16.59%	12.96%	4.54%	−1.55%	2.39%	
	$\mathcal{RC}_i$	6.43%	4.82%	4.82%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

There is only one asset such that that the weight is zero ($x_i = 0$) and the marginal risk is negative ($\mathcal{MR}_i < 0$). We deduce that the number of solutions is $2^1 = 2$ (TR-RPB, page 110).

(b) The second solution is:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_2$	x_i	27.27%	23.15%	29.60%	0.00%	19.98%	0.00%	12.71%
	$\mathcal{MR}_i$	18.64%	16.47%	12.88%	4.12%	0.00%	2.43%	
	$\mathcal{RC}_i$	5.08%	3.81%	3.81%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

3. (a) All the correlations are positive. We deduce that there is only one solution (TR-RPB, page 101).

(b) We obtain the following portfolio:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_1$	x_i	33.78%	29.05%	37.17%	0.00%	0.00%	0.00%	16.07%
	$\mathcal{MR}_i$	19.03%	16.59%	12.96%	4.54%	1.55%	2.39%	
	$\mathcal{RC}_i$	6.43%	4.82%	4.82%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

4. We obtain now:

Solution		1	2	3	4	5	6	$\sigma(x)$
$\mathcal{S}_1$	x_i	33.77%	29.05%	37.17%	0.00%	0.00%	0.00%	16.07%
	$\mathcal{MR}_i$	19.03%	16.59%	12.96%	−0.61%	−0.31%	−0.48%	
	$\mathcal{RC}_i$	6.43%	4.82%	4.82%	0.00%	0.00%	0.00%	
	$\mathcal{RC}_i^\star$	40.00%	30.00%	30.00%	0.00%	0.00%	0.00%	

We deduce that there are many solutions.

Remark 2 *This last question has been put in the wrong way. In fact, we wanted to show that the number of solutions depends on the correlation coefficients, but also on the values taken by the volatilities. If one or more assets which are not risk budgeted ($b_i = 0$) present a negative correlation with the assets which are risk budgeted ($b_i > 0$), the solution may not be unique. The number of solutions will depend on the anti-correlation strength and on the volatility level. If the volatilities of the assets which are not risk budgeted are very different from the other volatilities, the solution may be unique, because the diversification effect is small.*

2.6 Risk parity and factor models

1. (a) We have:

$$\begin{aligned}\sigma_i^2 &= \left(A\Omega A^\top + D\right)_{i,i} \\ &= \sum_{j=1}^{3} A_{i,j}^2 \omega_j^2 + \tilde{\sigma}_i^2\end{aligned}$$

The normalized risk decomposition with common and specific factors is then:

$$\sum_{j=1}^{3} c_{i,j} + \tilde{c}_i = 1$$

with $c_{i,j} = A_{i,j}^2 \omega_j^2 / \sigma_i^2$ and $\tilde{c}_i = \tilde{\sigma}_i^2 / \sigma_i^2$. We obtain the following results:

i	σ_i	$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$\sum_{j=1}^{3} c_{i,j}$	$\tilde{c}_i$
1	18.89%	90.76%	0.28%	4.48%	95.52%	4.48%
2	22.83%	92.90%	1.20%	1.11%	95.20%	4.80%
3	25.39%	89.33%	0.35%	0.40%	90.07%	9.93%
4	17.68%	81.89%	0.08%	10.03%	92.00%	8.00%
5	14.91%	44.99%	2.81%	7.20%	55.01%	44.99%
6	28.98%	93.38%	0.48%	0.30%	94.16%	5.84%

We notice that individual risks are mainly concentrated in the first factor. We may then assimilate this factor as a market risk factor. The expression of the correlation is:

$$\begin{aligned}\rho_{i,j} &= \frac{\left(A\Omega A^\top + D\right)_{i,j}}{\sigma_i \sigma_j} \\ &= \frac{\sum_{k=1}^{3} A_{i,k} A_{j,k} \omega_k^2}{\sqrt{\left(\sum_{k=1}^{3} A_{i,k}^2 \omega_k^2 + \tilde{\sigma}_i^2\right)\left(\sum_{k=1}^{3} A_{j,k}^2 \omega_k^2 + \tilde{\sigma}_j^2\right)}}\end{aligned}$$

We obtain:

$$\rho = \begin{pmatrix} 100.0\% & & & & & \\ 90.2\% & 100.0\% & & & & \\ 91.7\% & 91.1\% & 100.0\% & & & \\ 79.4\% & 90.2\% & 83.4\% & 100.0\% & & \\ 68.7\% & 60.0\% & 64.1\% & 52.7\% & 100.0\% & \\ 90.5\% & 93.0\% & 90.6\% & 89.4\% & 64.5\% & 100.0\% \end{pmatrix}$$

(b) We obtain:

$$A^{+} = \begin{pmatrix} 0.152 & 0.150 & 0.188 & 0.112 & 0.113 & 0.234 \\ -0.266 & -0.567 & -0.355 & 0.220 & 0.608 & 0.578 \\ 0.482 & -0.185 & 0.237 & -0.598 & 0.406 & -0.171 \end{pmatrix}$$

The number of assets n is larger than the number of risk factors m. We deduce that the Moore-Penrose inverse A^{+} can be written as the OLS projector:

$$A^{+} = \left(A^{\top} A\right)^{-1} A^{\top}$$

We obtain:

$$B^{+} = \begin{pmatrix} 0.152 & -0.266 & 0.482 \\ 0.150 & -0.567 & -0.185 \\ 0.188 & -0.355 & 0.237 \\ 0.112 & 0.220 & -0.598 \\ 0.113 & 0.608 & 0.406 \\ 0.234 & 0.578 & -0.171 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} 0.579 & -0.385 & -0.074 & 0.624 & -0.045 & -0.347 \\ 0.064 & 0.587 & -0.519 & 0.109 & 0.525 & -0.307 \\ 0.480 & 0.061 & -0.588 & -0.262 & -0.399 & 0.439 \end{pmatrix}$$

$$\tilde{B}^{+} = \begin{pmatrix} 0.579 & 0.064 & 0.480 \\ -0.385 & 0.587 & 0.061 \\ -0.074 & -0.519 & -0.588 \\ 0.624 & 0.109 & -0.262 \\ -0.045 & 0.525 & -0.399 \\ -0.347 & -0.307 & 0.439 \end{pmatrix}$$

By construction, we have:

$$B^{+} = B^{\top} \left(B B^{\top}\right)^{-1}$$

(c) The weights y and $\tilde{y}$ are equal to:

$$\begin{aligned} y &= A^{\top} x \\ &= \begin{pmatrix} 94.000\% \\ 10.000\% \\ 17.500\% \end{pmatrix} \end{aligned}$$

and:

$$\begin{aligned} \tilde{y} &= \tilde{B} x \\ &= \begin{pmatrix} 1.445\% \\ 9.518\% \\ -3.091\% \end{pmatrix} \end{aligned}$$

We have (TR-RPB, page 141):

$$x = x_c + x_s$$

where $x_c = B^+ y$ is the exposure to common factors and $x_s = \tilde{B}^+ \tilde{y}$ is the exposure to specific factors. We then obtain:

i	x_i	$x_{i,c}$	$x_{i,s}$
1	20.000%	20.033%	−0.033%
2	10.000%	5.159%	4.841%
3	15.000%	18.234%	−3.234%
4	5.000%	2.256%	2.744%
5	30.000%	23.839%	6.161%
6	20.000%	24.779%	−4.779%

(d) We recall that (TR-RPB, page 142):

$$\begin{aligned} \mathcal{MR}\left(\mathcal{F}_j\right) &= \left(A^+ \frac{\partial\, \sigma\left(x\right)}{\partial\, x}\right)_j \\ \mathcal{MR}\left(\tilde{\mathcal{F}}_j\right) &= \left(\tilde{B} \frac{\partial\, \sigma\left(x\right)}{\partial\, x}\right)_j \end{aligned}$$

We deduce that:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	94.00%	20.02%	18.81%	97.95%
$\mathcal{F}_2$	10.00%	1.09%	0.11%	0.57%
$\mathcal{F}_3$	17.50%	1.27%	0.22%	1.15%
$\tilde{\mathcal{F}}_1$	1.44%	−0.20%	0.00%	−0.01%
$\tilde{\mathcal{F}}_2$	9.52%	0.50%	0.05%	0.25%
$\tilde{\mathcal{F}}_3$	−3.09%	−0.62%	0.02%	0.10%
$\sigma\left(x\right)$			19.21%	

(e) We have:

$$\begin{aligned} R_t &= A \mathcal{F}_t + \varepsilon_t \\ &= \begin{pmatrix} A & I_n \end{pmatrix} \begin{pmatrix} \mathcal{F}_t \\ \varepsilon_t \end{pmatrix} \\ &= A' \mathcal{F}'_t + \varepsilon'_t \end{aligned}$$

with $D' = \mathbf{0}$ and:

$$\Omega' = \begin{pmatrix} \Omega & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}$$

We obtain:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	94.00%	17.23%	16.19%	84.30%
$\mathcal{F}_2$	10.00%	0.22%	0.02%	0.11%
$\mathcal{F}_3$	17.50%	0.42%	0.07%	0.38%
$\mathcal{F}_4$	20.00%	2.38%	0.48%	2.48%
$\mathcal{F}_5$	10.00%	2.71%	0.27%	1.41%
$\mathcal{F}_6$	15.00%	3.37%	0.51%	2.64%
$\mathcal{F}_7$	5.00%	1.82%	0.09%	0.47%
$\mathcal{F}_8$	30.00%	2.77%	0.83%	4.33%
$\mathcal{F}_9$	20.00%	3.73%	0.75%	3.88%
$\sigma(x)$			19.21%	

We don't find the same results for the risk decomposition with respect to the common factors. This is normal because we face an identification problem. Other parameterizations may induce other results. By considering the specific factors as common factors, we reduce the part explained by the common factors. Indeed, the identification problem becomes less and less important when n/m tends to ∞.

2. (a) If we consider the optimization problem defined in TR-RPB on page 144, we obtain:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	70.11%	18.05%	12.66%	78.72%
$\mathcal{F}_2$	37.06%	3.39%	1.26%	7.81%
$\mathcal{F}_3$	39.44%	3.30%	1.30%	8.10%
$\tilde{\mathcal{F}}_1$	1.11%	−0.26%	0.00%	−0.02%
$\tilde{\mathcal{F}}_2$	32.42%	2.11%	0.68%	4.25%
$\tilde{\mathcal{F}}_3$	−12.88%	−1.42%	0.18%	1.14%
$\sigma(x)$			16.08%	

We see that it is not possible to target a risk contribution of 10% for the second and third risk factors, because the first factor explains most of the risk of long-only portfolios. To have a smaller sensibility to the first risk factor, we need to consider a long-short portfolio.

(b) In terms of risk factors, we obtain:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	27.63%	6.72%	1.86%	10.00%
$\mathcal{F}_2$	107.39%	6.92%	7.43%	40.00%
$\mathcal{F}_3$	107.23%	6.93%	7.43%	40.00%
$\tilde{\mathcal{F}}_1$	24.27%	−0.09%	−0.02%	−0.12%
$\tilde{\mathcal{F}}_2$	38.77%	3.58%	1.39%	7.47%
$\tilde{\mathcal{F}}_3$	−21.45%	−2.29%	0.49%	2.65%
$\sigma(x)$			18.57%	

We deduce the following RB portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.52%	7.21%	2.42%	13.01%
2	−64.47%	3.85%	−2.48%	−13.36%
3	−16.81%	6.87%	−1.16%	−6.22%
4	−12.44%	2.15%	−0.27%	−1.44%
5	139.75%	13.08%	18.27%	98.42%
6	20.45%	8.71%	1.78%	9.60%
$\sigma(x)$			18.57%	

(c) In terms of risk factors, we obtain:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	23.53%	4.87%	1.15%	5.00%
$\mathcal{F}_2$	220.92%	9.33%	20.62%	90.00%
$\mathcal{F}_3$	40.05%	2.86%	1.15%	5.00%
$\tilde{\mathcal{F}}_1$	71.12%	0.27%	0.19%	0.83%
$\tilde{\mathcal{F}}_2$	−12.15%	2.59%	−0.32%	−1.38%
$\tilde{\mathcal{F}}_3$	−12.20%	−1.04%	0.13%	0.56%
$\sigma(x)$			22.91%	

We deduce the following RB portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	−1.43%	3.76%	−0.05%	−0.24%
2	−164.36%	1.18%	−1.95%	−8.50%
3	−56.24%	2.86%	−1.61%	−7.02%
4	73.59%	3.55%	2.61%	11.39%
5	148.43%	10.30%	15.28%	66.69%
6	100.02%	8.63%	8.63%	37.67%
$\sigma(x)$			22.91%	

(d) In terms of risk factors, we obtain:

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	26.73%	4.70%	1.26%	5.00%
$\mathcal{F}_2$	43.70%	2.87%	1.26%	5.00%
$\mathcal{F}_3$	239.40%	9.44%	22.60%	90.00%
$\tilde{\mathcal{F}}_1$	70.84%	0.27%	0.19%	0.77%
$\tilde{\mathcal{F}}_2$	14.03%	1.92%	0.27%	1.07%
$\tilde{\mathcal{F}}_3$	26.81%	−1.72%	−0.46%	−1.84%
$\sigma(x)$			25.12%	

We deduce the following RB portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	162.53%	7.83%	12.73%	50.67%
2	−82.45%	1.81%	−1.50%	−5.96%
3	17.99%	6.66%	1.20%	4.77%
4	−91.93%	−1.74%	1.60%	6.35%
5	120.33%	10.19%	12.26%	48.80%
6	−26.47%	4.40%	−1.16%	−4.64%
$\sigma(x)$			25.12%	

(e) To be exposed to the second factor, we use a portfolio which is long on the first three assets and short on the last three assets. This result is coherent with the matrix A. We obtain a similar result if we want to be exposed to the third asset. Nevertheless, the figures of risk contributions may be confusing. We might have thought that the risk was split between the long leg and the short leg. It is not the case. Nonetheless, this is normal because if the risk is perfectly split between the long leg and the short leg, the risk exposure to this factor vanishes!

2.7 Risk allocation with the expected shortfall risk measure

1. (a) We have:

$$\begin{aligned} L(x) &= -R(x) \\ &= -x^\top R \end{aligned}$$

It follows that:

$$L(x) \sim \mathcal{N}(-\mu(x), \sigma(x))$$

with $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$.

(b) The expected shortfall $\mathrm{ES}_\alpha(L)$ is the average of value-at-risks at level α and higher:

$$\mathrm{ES}_\alpha(L) = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha(L)]$$

We know that the value-at-risk is (TR-RPB, page 74):

$$\mathrm{VaR}_\alpha(x) = -x^\top \mu + \Phi^{-1}(\alpha)\sqrt{x^\top \Sigma x}$$

We deduce that:

$$\mathrm{ES}_{\alpha}(x)=\frac{1}{1-\alpha}\int_{u^{-}}^{\infty}\frac{u}{\sigma(x)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{u+\mu(x)}{\sigma(x)}\right)^{2}\right)\mathrm{d}u$$

where $u^{-}=-\mu(x)+\sigma(x)\Phi^{-1}(\alpha)$. With the change of variable $t=\sigma(x)^{-1}(u+\mu(x))$, we obtain:

$$\begin{aligned}
\mathrm{ES}_{\alpha}(x) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty}(-\mu(x)+\sigma(x)t)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^{2}\right)\mathrm{d}t \\
&= -\frac{\mu(x)}{1-\alpha}\left[\Phi(t)\right]_{\Phi^{-1}(\alpha)}^{\infty}+ \\
&\quad \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty}t\exp\left(-\frac{1}{2}t^{2}\right)\mathrm{d}t \\
&= -\mu(x)+\frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^{2}\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= -\mu(x)+\frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^{2}\right)
\end{aligned}$$

The expected shortfall of portfolio x is then (TR-RPB, page 75):

$$\mathrm{ES}_{\alpha}(x)=-x^{\top}\mu+\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\sqrt{x^{\top}\Sigma x}$$

(c) The vector of marginal risk is defined as follows (TR-RPB, page 80):

$$\begin{aligned}
\mathcal{MR} &= \frac{\partial\,\mathrm{ES}_{\alpha}(x)}{\partial\,x} \\
&= -\mu+\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\frac{\Sigma x}{\sqrt{x^{\top}\Sigma x}}
\end{aligned}$$

We deduce that the risk contribution $\mathcal{RC}_i$ of the asset i is:

$$\begin{aligned}
\mathcal{RC}_i &= x_i\cdot\mathcal{MR}_i \\
&= -x_i\mu_i+\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\frac{x_i\cdot(\Sigma x)_i}{\sqrt{x^{\top}\Sigma x}}
\end{aligned}$$

It follows that:

$$\begin{aligned}
\sum_{i=1}^{n}\mathcal{RC}_i &= \sum_{i=1}^{n}-x_i\mu_i+\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\frac{x_i\cdot(\Sigma x)_i}{\sqrt{x^{\top}\Sigma x}} \\
&= -x^{\top}\mu+\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\frac{x^{\top}(\Sigma x)}{\sqrt{x^{\top}\Sigma x}} \\
&= \mathrm{ES}_{\alpha}(x)
\end{aligned}$$

The expected shortfall then verifies the Euler allocation principle (TR-RPB, page 78).

2. (a) We have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	30.00%	18.40%	5.52%	45.57%
2	30.00%	22.95%	6.89%	56.84%
3	40.00%	−0.73%	−0.29%	−2.41%
$\mathrm{ES}_\alpha(x)$			12.11%	

(b) The ERC portfolio is:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	18.53%	14.83%	2.75%	33.33%
2	18.45%	14.89%	2.75%	33.33%
3	63.02%	4.36%	2.75%	33.33%
$\mathrm{ES}_\alpha(x)$			8.24%	

(c) If the risk budgets are equal to $b = (70\%, 20\%, 10\%)$, we obtain:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.16%	21.57%	7.15%	70.00%
2	15.91%	12.85%	2.04%	20.00%
3	50.93%	2.01%	1.02%	10.00%
$\mathrm{ES}_\alpha(x)$			10.22%	

(d) We have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	80.00%	21.54%	17.24%	57.93%
2	50.00%	21.49%	10.75%	36.12%
3	−30.00%	−5.89%	1.77%	5.94%
$\mathrm{ES}_\alpha(x)$			29.75%	

We notice that the risk contributions are all positive even if we consider a long-short portfolio. We can therefore think that there may be several solutions to the risk budgeting problem if we consider long-short portfolios.

(e) Here is a long-short ERC portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	38.91%	13.22%	5.14%	33.33%
2	−21.02%	−24.47%	5.14%	33.33%
3	82.11%	6.26%	5.14%	33.33%
$\mathrm{ES}_\alpha(x)$			15.43%	

Nevertheless, this solution is not unique. For instance, here is another long-short ERC portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	−58.65%	−23.04%	13.51%	33.33%
2	−40.03%	−33.75%	13.51%	33.33%
3	198.68%	6.80%	13.51%	33.33%
$\mathrm{ES}_\alpha(x)$			40.53%	

(f) Here are three long-short portfolios that satisfy the risk budgets $b = (70\%, 20\%, 10\%)$:

Solution	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
$\mathcal{S}_1$	1	115.31%	23.56%	27.17%	70.00%
	2	45.56%	17.04%	7.76%	20.00%
	3	−60.87%	−6.38%	3.88%	10.00%
	$\mathrm{ES}_\alpha(x)$			38.81%	
$\mathcal{S}_2$	1	60.76%	20.97%	12.74%	70.00%
	2	−20.82%	−17.48%	3.64%	20.00%
	3	60.07%	3.03%	1.82%	10.00%
	$\mathrm{ES}_\alpha(x)$			18.20%	
$\mathcal{S}_3$	1	−72.96%	−24.32%	17.74%	70.00%
	2	54.53%	9.30%	5.07%	20.00%
	3	118.43%	2.14%	2.53%	10.00%
	$\mathrm{ES}_\alpha(x)$			25.34%	

(g) Contrary to the long-only case, the RB portfolio may not be unique when the portfolio is long-short.

3. (a) We have:

$$\begin{aligned} L(x) &= -\sum_{i=1}^n x_i R_i \\ &= \sum_{i=1}^n L_i \end{aligned}$$

with $L_i = -x_i R_i$. We know that (TR-RPB, page 85):

$$\begin{aligned} \mathcal{RC}_i &= \mathbb{E}\left[L_i \mid L \geq \mathrm{VaR}_\alpha(L)\right] \\ &= \frac{\mathbb{E}\left[L_i \cdot \mathbb{1}\left\{L \geq \mathrm{VaR}_\alpha(L)\right\}\right]}{\mathbb{E}\left[\mathbb{1}\left\{L \geq \mathrm{VaR}_\alpha(L)\right\}\right]} \\ &= \frac{\mathbb{E}\left[L_i \cdot \mathbb{1}\left\{L \geq \mathrm{VaR}_\alpha(L)\right\}\right]}{1-\alpha} \end{aligned}$$

We deduce that:

$$\mathcal{RC}_i = -\frac{x_i}{1-\alpha}\mathbb{E}\left[R_i \cdot \mathbf{1}\left\{R(x) \leq -\mathrm{VaR}_\alpha(L)\right\}\right]$$

(b) We know that the random vector $(R, R(x))$ has a multivariate normal distribution:

$$\left(\begin{array}{c} R \\ R(x) \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu \\ x^\top \mu \end{array}\right), \left(\begin{array}{cc} \Sigma & \Sigma x \\ x^\top \Sigma & x^\top \Sigma x \end{array}\right)\right)$$

We deduce that:

$$\left(\begin{array}{c} R_i \\ R(x) \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_i \\ x^\top \mu \end{array}\right), \left(\begin{array}{cc} \Sigma_{i,i} & (\Sigma x)_i \\ (\Sigma x)_i & x^\top \Sigma x \end{array}\right)\right)$$

Let $I = \mathbb{E}[R_i \cdot \mathbb{1}\{R(x) \leq -\operatorname{VaR}_\alpha(L)\}]$. We note f the density function of the random vector $(R_i, R(x))$ and $\rho = \Sigma_{i,i}^{-1/2}\left(x^\top \Sigma x\right)^{-1/2}(\Sigma x)_i$ the correlation between R_i and $R(x)$. It follows that:

$$\begin{aligned} I &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r \cdot \mathbf{1}\{s \leq -\operatorname{VaR}_\alpha(L)\} f(r,s)\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{-\operatorname{VaR}_\alpha(L)} r f(r,s)\,\mathrm{d}r\,\mathrm{d}s \end{aligned}$$

Let $t = (r - \mu_i)/\sqrt{\Sigma_{i,i}}$ and $u = \left(s - x^\top \mu\right)/\sqrt{x^\top \Sigma x}$. We deduce that[6]:

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{\Phi^{-1}(1-\alpha)} \frac{\mu_i + \sqrt{\Sigma_{i,i}}t}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{t^2 + u^2 - 2\rho t u}{2(1-\rho^2)}\right)\mathrm{d}t\,\mathrm{d}u$$

By considering the change of variables $(t, u) = \varphi(t, v)$ such that $u = \rho t + \sqrt{1-\rho^2}v$, we obtain[7]:

$$\begin{aligned} I &= \int_{-\infty}^{+\infty}\int_{-\infty}^{g(t)} \frac{\mu_i + \sqrt{\Sigma_{i,i}}t}{2\pi} \exp\left(-\frac{t^2+v^2}{2}\right)\mathrm{d}t\,\mathrm{d}v \\ &= \mu_i \int_{-\infty}^{+\infty}\int_{-\infty}^{g(t)} \frac{1}{2\pi}\exp\left(-\frac{t^2+v^2}{2}\right)\mathrm{d}t\,\mathrm{d}v + \\ &\quad \sqrt{\Sigma_{i,i}}\int_{-\infty}^{+\infty}\int_{-\infty}^{g(t)} \frac{t}{2\pi}\exp\left(-\frac{t^2+v^2}{2}\right)\mathrm{d}t\,\mathrm{d}v + \\ &= \mu_i I_1 + \sqrt{\Sigma_{i,i}} I_2 \end{aligned}$$

where the bound $g(t)$ is defined as follows:

$$g(t) = \frac{\Phi^{-1}(1-\alpha) - \rho t}{\sqrt{1-\rho^2}}$$

[6]Because we have $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$.

[7]We use the fact that $\mathrm{d}t\,\mathrm{d}v = \sqrt{1-\rho^2}\,\mathrm{d}t\,\mathrm{d}u$ because the determinant of the Jacobian matrix containing the partial derivatives $D\varphi$ is $\sqrt{1-\rho^2}$.

For the first integral, we have[8]:

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \left(\int_{-\infty}^{g(t)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) \mathrm{d}v\right) \mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(1-\alpha) - \rho t}{\sqrt{1-\rho^2}}\right) \phi(t) \, \mathrm{d}t \\ &= 1 - \alpha \end{aligned}$$

The computation of the second integral I_2 is a little bit more tedious. Integration by parts with the derivative function $t\phi(t)$ gives:

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(1-\alpha) - \rho t}{\sqrt{1-\rho^2}}\right) t\phi(t) \, \mathrm{d}t \\ &= -\frac{\rho}{\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \phi\left(\frac{\Phi^{-1}(1-\alpha) - \rho t}{\sqrt{1-\rho^2}}\right) \phi(t) \, \mathrm{d}t \\ &= -\frac{\rho}{\sqrt{1-\rho^2}} \phi\left(\Phi^{-1}(1-\alpha)\right) \int_{-\infty}^{+\infty} \phi\left(\frac{t - \rho\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho^2}}\right) \mathrm{d}t \\ &= -\rho\phi\left(\Phi^{-1}(1-\alpha)\right) \end{aligned}$$

We could then deduce the value of I:

$$\begin{aligned} I &= \mu_i(1-\alpha) - \rho\sqrt{\Sigma_{i,i}}\phi\left(\Phi^{-1}(1-\alpha)\right) \\ &= \mu_i(1-\alpha) - \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \phi\left(\Phi^{-1}(\alpha)\right) \end{aligned}$$

We finally obtain that:

$$\mathcal{RC}_i = -x_i\mu_i + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)} \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

We obtain the same expression as found in Question 1(c).

(c) We have

$$\mathcal{RC}_i = -\frac{x_i}{1-\alpha} \mathbb{E}\left[R_i \cdot \mathbf{1}\left\{R(x) \leq -\operatorname{VaR}_\alpha(L)\right\}\right]$$

Let $R_{i,t}$ be the asset return for the observation t. The portfolio return is then $R_t(x) = \sum_{i=1}^n x_i R_{i,t}$ at time t. We note $R_{j:T}(x)$ the j^{th} order statistic. The estimated value-at-risk is then:

$$\operatorname{VaR}_\alpha = -R_{(1-\alpha)T:T}(x)$$

[8] We use the fact that:

$$\mathbb{E}\left[\Phi\left(\frac{\Phi^{-1}(p) - \rho T}{\sqrt{1-\rho^2}}\right)\right] = p$$

where $T \sim \mathcal{N}(0,1)$.

We deduce that the estimated risk contribution is:

$$\mathcal{RC}_i = -\frac{x_i}{(1-\alpha)T}\sum_{t=1}^{T} R_{i,t} \cdot \mathbb{1}\left\{R_t(x) \leq R_{(1-\alpha)T:T}(x)\right\}$$

(d) We note $\mathcal{RC}_i^{(j)}$ the estimated risk contribution of the asset i for the simulation j:

$$\mathcal{RC}_i^{(j)} = -\frac{x_i}{(1-\alpha)T}\sum_{t=1}^{T} R_{i,t}^{(j)} \cdot \mathbb{1}\left\{R_t^{(j)}(x) \leq R_{(1-\alpha)T:T}^{(j)}(x)\right\}$$

where $R_{i,t}^{(j)}$ is the simulated value of $R_{i,t}$ for the simulation j. We consider one million of simulated observations[9] and 100 Monte Carlo replications. We estimate the risk contribution as the average of $\mathcal{RC}_i^{(j)}$:

$$\mathcal{RC}_i = \frac{1}{100}\sum_{j=1}^{100} \mathcal{RC}_i^{(j)}$$

Using the numerical values of the parameters, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	30.00%	29.12%	8.74%	25.91%
2	30.00%	36.48%	10.94%	32.46%
3	40.00%	35.11%	14.04%	41.63%
$\mathrm{ES}_\alpha(x)$			33.73%	

If $\nu_i \to \infty$, we have:

$$\frac{R_i - \mu_i}{\sigma_i} \sim \mathcal{N}(0,1)$$

One would think that this numerical application is very close to the one given in Question 2(a). However, we obtain very different risk contributions. When we assume that asset returns are Gaussian, the risk contribution of the third asset is negative ($\mathcal{RC}_i^{\star} = -2.41\%$) whereas the third asset is the main contributor here ($\mathcal{RC}_i^{\star} = 41.63\%$). It is due to the fat tail effect of the Student t distribution. This effect also explains why the expected shortfall has been multiplied by a factor greater than 2.

[9] It means that $T = 1\,000\,000$.

2.8 ERC optimization problem

1. (a) The weights of the three portfolios are:

Asset	MV	ERC	EW
1	87.51%	37.01%	25.00%
2	4.05%	24.68%	25.00%
3	4.81%	20.65%	25.00%
4	3.64%	17.66%	25.00%

(b) The Lagrange function is:

$$\begin{aligned}\mathcal{L}(x;\lambda,\lambda_0,\lambda_c) &= \sqrt{x^\top \Sigma x} - \lambda^\top x - \lambda_0 \left(\mathbf{1}^\top x - 1\right) - \\ &\quad \lambda_c \left(\sum_{i=1}^n \ln x_i - c\right) \\ &= \left(\sqrt{x^\top \Sigma x} - \lambda_c \sum_{i=1}^n \ln x_i\right) - \lambda^\top x - \\ &\quad \lambda_0 \left(\mathbf{1}^\top x - 1\right) + \lambda_c c\end{aligned}$$

We deduce that an equivalent optimization problem is:

$$\begin{aligned}\tilde{x}^\star(\lambda_c) &= \arg\min \sqrt{\tilde{x}^\top \Sigma \tilde{x}} - \lambda_c \sum_{i=1}^n \ln \tilde{x}_i \\ &\text{u.c.} \quad \left\{\begin{array}{l}\mathbf{1}^\top \tilde{x} = 1 \\ \tilde{x} \geq \mathbf{0}\end{array}\right.\end{aligned}$$

We notice a strong difference between the two problems because they don't use the same control variable. However, the control variable c of the first problem may be deduced from the solution of the second problem:

$$c = \sum_{i=1}^n \ln \tilde{x}_i^\star(\lambda_c)$$

We also know that (TR-RPB, page 131):

$$c_- \leq \sum_{i=1}^n \ln x_i \leq c_+$$

where $c_- = \sum_{i=1}^n \ln (x_{\mathrm{mv}})_i$ and $c_+ = -n \ln n$. It follows that:

$$\left\{\begin{array}{lll}x^\star(c) = \tilde{x}^\star(0) & \text{if} & c \leq c_- \\ x^\star(c) = \tilde{x}^\star(\infty) & \text{if} & c \geq c_+\end{array}\right.$$

If $c \in \left]c_-, c_+\right[$, there exists a scalar $\lambda_c > 0$ such that:

$$x^\star(c) = \tilde{x}^\star(\lambda_c)$$

(c) For a given value $\lambda_c \in [0, +\infty[$, we solve numerically the second problem and find the optimized portfolio $\tilde{x}^\star(\lambda_c)$. Then, we calculate $c = \sum_{i=1}^n \ln \tilde{x}_i^\star(\lambda_c)$ and deduce that $x^\star(c) = \tilde{x}^\star(\lambda_c)$. We finally obtain $\sigma(x^\star(c)) = \sigma(\tilde{x}^\star(\lambda_c))$ and $\mathcal{I}^\star(x^\star(c)) = \mathcal{I}^\star(\tilde{x}^\star(\lambda_c))$. The relationships between λ_c, c, $\mathcal{I}^\star(x^\star(c))$ and $\sigma(x^\star(c))$ are reported in Figure 2.4.

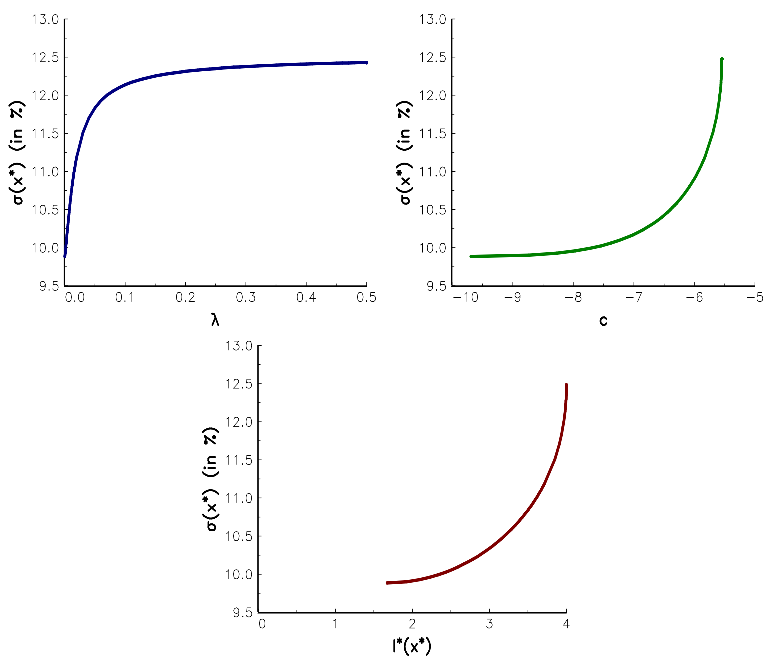

FIGURE 2.4: Relationship between λ_c, c, $\mathcal{I}^\star(x^\star(c))$ and $\sigma(x^\star(c))$

(d) If we consider $\mathcal{I}^\star(\mathcal{RC})$ in place of $\sigma(x^\star(c))$, we obtain Figure 2.5.

(e) In Figure 2.6, we have reported the relationship between $\sigma(x^\star(c))$ and $\mathcal{I}^\star(\mathcal{RC})$. The ERC portfolio satisfies the equation $\mathcal{I}^\star(\mathcal{RC}) = n$.

2. (a) Let us consider the optimization problem when we impose the constraint $\mathbf{1}^\top x = 1$. The first-order condition is:

$$\frac{\partial\,\sigma(x)}{\partial\,x_i} - \lambda_i - \lambda_0 - \frac{\lambda_c}{x_i} = 0$$

Because $x_i > 0$, we deduce that $\lambda_i = 0$ and:

$$x_i \frac{\partial\,\sigma(x)}{\partial\,x_i} = \lambda_0 x_i + \lambda_c$$

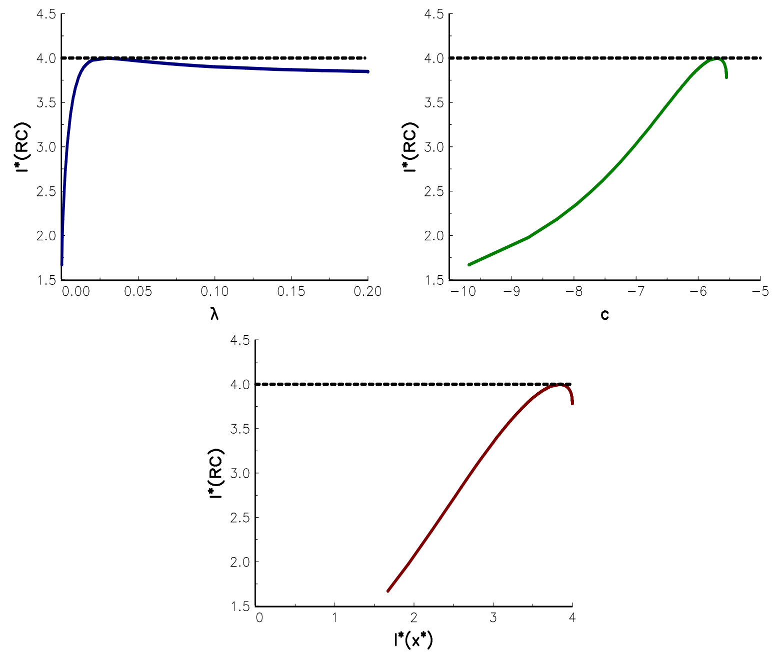

FIGURE 2.5: Relationship between λ_c, c, $\mathcal{I}^\star\left(x^\star\left(c\right)\right)$ and $\mathcal{I}^\star\left(\mathcal{RC}\right)$

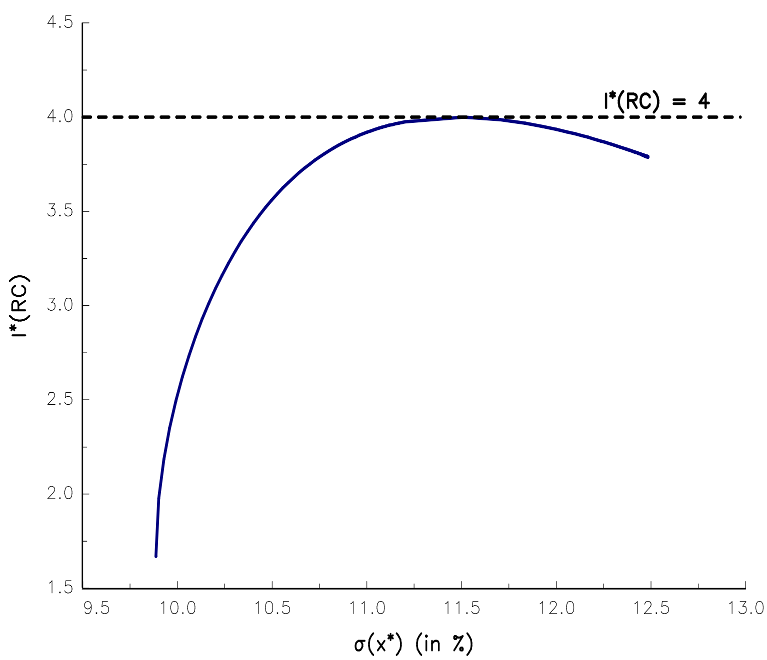

FIGURE 2.6: Relationship between $\sigma\left(x^\star\left(c\right)\right)$ and $\mathcal{I}^\star\left(\mathcal{RC}\right)$

If this solution corresponds to the ERC portfolio, we obtain:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \lambda_0 x_i + \lambda_c = \lambda_0 x_j + \lambda_c$$

If $\lambda_0 \neq 0$, we deduce that:

$$x_i = x_j$$

It corresponds to the EW portfolio meaning that the assumption $\mathcal{RC}_i = \mathcal{RC}_j$ is false.

(b) If c is equal to -10, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	12.65%	7.75%	0.98%	25.00%
2	8.43%	11.63%	0.98%	25.00%
3	7.06%	13.89%	0.98%	25.00%
4	6.03%	16.25%	0.98%	25.00%
$\sigma(x)$			3.92%	

(c) If c is equal to 0, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	154.07%	7.75%	11.94%	25.00%
2	102.72%	11.63%	11.94%	25.00%
3	85.97%	13.89%	11.94%	25.00%
4	73.50%	16.25%	11.94%	25.00%
$\sigma(x)$			47.78%	

(d) In this case, the first-order condition is:

$$\frac{\partial\, \sigma(x)}{\partial\, x_i} - \lambda_i - \frac{\lambda_c}{x_i} = 0$$

As previously, $\lambda_i = 0$ because $x_i > 0$ and we obtain:

$$x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = \lambda_c$$

The solution of the second optimization problem is then a non-normalized ERC portfolio because $\sum_{i=1}^n x_i$ is not necessarily equal to 1. If we note $c_{\text{erc}} = \sum_{i=1}^n \ln(x_{\text{erc}})_i$, we deduce that:

$$\begin{aligned} x_{\text{erc}} &= \arg\min \sqrt{x^\top \Sigma x} \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n \ln x_i \geq c_{\text{erc}} \\ x \geq \mathbf{0} \end{array} \right. \end{aligned}$$

Let $x^\star(c)$ be the portfolio defined by:

$$x^\star(c) = \exp\left(\frac{c - c_{\text{erc}}}{n}\right) x_{\text{erc}}$$

We have $x^\star(c) > \mathbf{0}$,

$$\sqrt{x^\star(c)^\top \Sigma x^\star(c)} = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right)\sqrt{x_{\mathrm{erc}}^\top \Sigma x_{\mathrm{erc}}}$$

and:

$$\begin{aligned}\sum_{i=1}^n \ln x_i^\star(c) &= \sum_{i=1}^n \ln\left(\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}\right)_i \\ &= c - c_{\mathrm{erc}} + \sum_{i=1}^n \ln(x_{\mathrm{erc}})_i \\ &= c\end{aligned}$$

We conclude that $x^\star(c)$ is the solution of the optimization problem. $x^\star(c)$ is then a leveraged ERC portfolio if $c > c_{\mathrm{erc}}$ and a deleveraged ERC portfolio if $c < c_{\mathrm{erc}}$. In our example, c_{erc} is equal to -5.7046. If $c = -10$, we have:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 34.17\%$$

We verify that the solution of Question 2(b) is such that $\sum_{i=1}^n x_i = 34.17\%$ and $\mathcal{RC}_i^\star = \mathcal{RC}_j^\star$. If $c = 0$, we obtain:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 416.26\%$$

In this case, the solution is a leveraged ERC portfolio.

(e) From the previous question, we know that the ERC optimization portfolio is the solution of the second optimization problem if we use c_{erc} for the control variable. In this case, we have $\sum_{i=1}^n x_i^\star(c_{\mathrm{erc}}) = 1$ meaning that x_{erc} is also the solution of the first optimization problem. We deduce that $\lambda_0 = 0$ if $c = c_{\mathrm{erc}}$. The first optimization problem is a convex problem with a convex inequality constraint. The objective function is then an increasing function of the control variable c:

$$c_1 \leq c_2 \Rightarrow \sigma(x^\star(c_1)) \geq \sigma(x^\star(c_2))$$

We have seen that the minimum variance portfolio corresponds to $c = -\infty$, that the EW portfolio is obtained with $c = -n\ln n$ and that the ERC portfolio is the solution of the optimization problem when c is equal to c_{erc}. Moreover, we have $-\infty \leq c_{\mathrm{erc}} \leq -n\ln n$. We deduce that the volatility of the ERC portfolio is between the volatility of the long-only minimum variance portfolio and the volatility of the equally weighted portfolio:

$$\sigma(x_{\mathrm{mv}}) \leq \sigma(x_{\mathrm{erc}}) \leq \sigma(x_{\mathrm{ew}})$$

2.9 Risk parity portfolios with skewness and kurtosis

1. (a) We use the formulas given in TR-RPB on page 94. The mean μ corresponds to M_1:

$$\mu = \begin{pmatrix} 0.225\% \\ 0.099\% \\ 0.087\% \end{pmatrix}$$

Let μ_r be the centered r-order moment. We have $\sigma = \sqrt{\mu_2}$, $\gamma_1 = \mu_3/\mu_2^{3/2}$ and $\gamma_2 = \mu_4/\mu_2^2 - 3$. The difficulty is to read the good value of $\mu_r^{(i)}$ for the asset i from the matrices M_2, M_3 and M_4. We have $\mu_r^{(i)} = (M_r)_{i,j}$ where the index j is given by the following table:

	r		
i	2	3	4
1	1	1	1
2	2	5	14
3	3	9	27

The volatility is then:

$$\sigma = \begin{pmatrix} 2.652\% \\ 0.874\% \\ 2.498\% \end{pmatrix}$$

For the skewness, we obtain:

$$\gamma_1 = \begin{pmatrix} -0.351 \\ -0.027 \\ -0.248 \end{pmatrix}$$

whereas the excess kurtosis is:

$$\gamma_2 = \begin{pmatrix} 1.242 \\ -0.088 \\ 0.930 \end{pmatrix}$$

(b) Let x be the portfolio. We know that (TR-RPB, page 94):

$$\mu_r(\Pi) = x M_r \left(\bigotimes_{j=1}^{r} x \right)$$

We obtain $\mu_1(\Pi) = 13.732 \times 10^{-4}$, $\mu_2(\Pi) = 2.706 \times 10^{-4}$, $\mu_3(\Pi) = -1.117 \times 10^{-6}$ and $\mu_4(\Pi) = 0.252 \times 10^{-6}$.

(c) We have:

$$\begin{aligned}\mu(L) &= -\mu_1(\Pi) = -0.137\% \\ \sigma(L) &= \sqrt{\mu_2(\Pi)} = 1.645\% \\ \gamma_1(L) &= -\frac{\mu_3(\Pi)}{\sigma^3(L)} = 0.251 \\ \gamma_2(L) &= \frac{\mu_4(\Pi)}{\sigma^4(L)} = 0.438\end{aligned}$$

(d) The risk allocation of the EW portfolio is:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.333%	3.719%	1.240%	48.272%
2	33.333%	0.372%	0.124%	4.825%
3	33.333%	3.614%	1.205%	46.903%
$\mathrm{VaR}_\alpha(x)$			2.568%	

(e) If we consider the Cornish-Fisher value-at-risk, we get:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.333%	3.919%	1.306%	48.938%
2	33.333%	0.319%	0.106%	3.977%
3	33.333%	3.770%	1.257%	47.085%
$\mathrm{VaR}_\alpha(x)$			2.669%	

2. (a) We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	17.371%	3.151%	0.547%	33.333%
2	65.208%	0.840%	0.547%	33.333%
3	17.421%	3.142%	0.547%	33.333%
$\mathrm{VaR}_\alpha(x)$			1.642%	

(b) If we consider the Cornish-Fisher value-at-risk, the results become:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	17.139%	3.253%	0.558%	33.333%
2	65.659%	0.849%	0.558%	33.333%
3	17.202%	3.241%	0.558%	33.333%
$\mathrm{VaR}_\alpha(x)$			1.673%	

We notice that the weights of the portfolio are very close to the weights obtained with the Gaussian value-at-risk. The impact of the skewness and kurtosis is thus limited.

(c) If α is equal to 99%, the ERC portfolio with the Gaussian value-at-risk is:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	17.403%	4.559%	0.793%	33.333%
2	64.950%	1.222%	0.793%	33.333%
3	17.647%	4.497%	0.793%	33.333%
$\mathrm{VaR}_\alpha(x)$			2.380%	

whereas the ERC portfolio with the Cornish-Fisher value-at-risk is:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	16.467%	4.770%	0.785%	33.333%
2	67.860%	1.157%	0.785%	33.333%
3	15.672%	5.012%	0.785%	33.333%
$\mathrm{VaR}_\alpha(x)$			2.356%	

The impact is higher. In particular, we see that the weight of bonds increases if we take into account skewness and kurtosis.

Chapter 3

Exercises related to risk parity applications

3.1 Computation of heuristic portfolios

1. All the results are expressed in %.

   (a) To compute the unconstrained tangency portfolio, we use the analytical formula (TR-RPB, page 14):

   $$x^{\star} = \frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^{\top}\Sigma^{-1}(\mu - r\mathbf{1})}$$

   We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	11.11%	6.56%	0.73%	5.96%
2	17.98%	13.12%	2.36%	19.27%
3	2.55%	6.56%	0.17%	1.37%
4	33.96%	9.84%	3.34%	27.31%
5	34.40%	16.40%	5.64%	46.09%

   (b) We obtain the following results for the equally weighted portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	20.00%	7.47%	1.49%	13.43%
2	20.00%	15.83%	3.17%	28.48%
3	20.00%	9.98%	2.00%	17.96%
4	20.00%	9.89%	1.98%	17.80%
5	20.00%	12.41%	2.48%	22.33%

   (c) For the minimum variance portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	74.80%	9.08%	6.79%	74.80%
2	−15.04%	9.08%	−1.37%	−15.04%
3	21.63%	9.08%	1.96%	21.63%
4	10.24%	9.08%	0.93%	10.24%
5	8.36%	9.08%	0.76%	8.36%

(d) For the most diversified portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	−14.47%	4.88%	−0.71%	−5.34%
2	4.83%	9.75%	0.47%	3.56%
3	18.94%	7.31%	1.38%	10.47%
4	49.07%	12.19%	5.98%	45.24%
5	41.63%	14.63%	6.09%	46.06%

(e) For the ERC portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	27.20%	7.78%	2.12%	20.00
2	13.95%	15.16%	2.12%	20.00
3	20.86%	10.14%	2.12%	20.00
4	19.83%	10.67%	2.12%	20.00
5	18.16%	11.65%	2.12%	20.00

(f) We recall the definition of the statistics:

$$\begin{aligned}
\mu(x) &= \mu^\top x \\
\sigma(x) &= \sqrt{x^\top \Sigma x} \\
\mathrm{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\
\sigma(x \mid b) &= \sqrt{(x-b)^\top \Sigma (x-b)} \\
\beta(x \mid b) &= \frac{x^\top \Sigma b}{b^\top \Sigma b} \\
\rho(x \mid b) &= \frac{x^\top \Sigma b}{\sqrt{x^\top \Sigma x}\sqrt{b^\top \Sigma b}}
\end{aligned}$$

We obtain the following results:

Statistic	$x^\star$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.11%	9.67%	8.04%
$\sigma(x)$	12.24%	11.12%	9.08%	13.22%	10.58%
$\mathrm{SR}(x \mid r)$	60.96%	57.57%	45.21%	58.03%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	8.21%	4.06%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	55.01%	102.82%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	74.17%	95.19%	93.76%

We notice that all the portfolios present similar performance in terms of Sharpe Ratio. The minimum variance portfolio shows the smallest Sharpe ratio, but it also shows the lowest correlation with the tangency portfolio.

2. The tangency portfolio, the equally weighted portfolio and the ERC portfolio are already long-only. For the minimum variance portfolio, we obtain:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	65.85%	9.37%	6.17%	65.85%
2	0.00%	13.11%	0.00%	0.00%
3	16.72%	9.37%	1.57%	16.72%
4	9.12%	9.37%	0.85%	9.12%
5	8.32%	9.37%	0.78%	8.32%

whereas we have for the most diversified portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	0.00%	5.50%	0.00%	0.00%
2	1.58%	9.78%	0.15%	1.26%
3	16.81%	7.34%	1.23%	10.04%
4	44.13%	12.23%	5.40%	43.93%
5	37.48%	14.68%	5.50%	44.77%

The results become:

Statistic	$x^\star$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.68%	9.19%	8.04%
$\sigma(x)$	12.24%	11.12%	9.37%	12.29%	10.58%
$\mathrm{SR}(x \mid r)$	60.96%	57.57%	49.99%	58.56%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	7.04%	3.44%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	62.74%	96.41%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	82.00%	96.06%	93.76%

3.2 Equally weighted portfolio

1. (a) The elements of the covariance matrix are $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$. If we consider a portfolio $x = (x_1, \ldots, x_n)$, its volatility is:

$$\begin{aligned}\sigma(x) &= \sqrt{x^\top \Sigma x} \\ &= \sqrt{\sum_{i=1}^{n} x_i^2\sigma_i^2 + 2\sum_{i>j} x_i x_j \rho_{i,j}\sigma_i\sigma_j}\end{aligned}$$

For the equally weighted portfolio, we have $x_i = n^{-1}$ and:

$$\sigma(x) = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2 + 2\sum_{i>j}\rho_{i,j}\sigma_i\sigma_j}$$

(b) We have:

$$\sigma_0(x) = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2}$$

and:

$$\begin{aligned}
\sigma_1(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j} \\
&= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i\sum_{j=1}^{n}\sigma_j} \\
&= \frac{1}{n}\sqrt{\left(\sum_{i=1}^{n}\sigma_i\right)^2} \\
&= \frac{\sum_{i=1}^{n}\sigma_i}{n} \\
&= \bar{\sigma}
\end{aligned}$$

(c) If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\sigma(x) = \frac{\sigma}{n}\sqrt{n + 2\sum_{i>j}\rho_{i,j}}$$

Let $\bar{\rho}$ be the mean correlation. We have:

$$\bar{\rho} = \frac{2}{n^2 - n}\sum_{i>j}\rho_{i,j}$$

We deduce that:

$$\sum_{i>j}\rho_{i,j} = \frac{n(n-1)}{2}\bar{\rho}$$

We finally obtain:

$$\begin{aligned}
\sigma(x) &= \frac{\sigma}{n}\sqrt{n + n(n-1)\bar{\rho}} \\
&= \sigma\sqrt{\frac{1 + (n-1)\bar{\rho}}{n}}
\end{aligned}$$

When $\bar{\rho}$ is equal to zero, the volatility $\sigma(x)$ is equal to $\sigma/\sqrt{n}$. When the number of assets tends to $+\infty$, it follows that:

$$\lim_{n\to\infty}\sigma(x) = \sigma\sqrt{\bar{\rho}}$$

(d) If $\rho_{i,j} = \rho$, we obtain:

$$\begin{aligned}
\sigma(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j} \\
&= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j - \rho\sum_{i=1}^{n}\sigma_i^2} \\
&= \frac{1}{n}\sqrt{(1-\rho)\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j}
\end{aligned}$$

We have:

$$\sum_{i=1}^{n}\sigma_i^2 = n^2\sigma_0^2(x)$$

and:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j = n^2\sigma_1^2(x)$$

It follows that:

$$\sigma(x) = \sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}$$

When the correlation is uniform, the variance $\sigma^2(x)$ is the weighted average between $\sigma_0^2(x)$ and $\sigma_1^2(x)$.

2. (a) The risk contributions are equal to:

$$\mathcal{RC}_i^\star = \frac{x_i \cdot (\Sigma x)_i}{\sigma^2(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}
\mathcal{RC}_i^\star &= \frac{\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j}{n^2\sigma^2(x)} \\
&= \frac{\sigma_i^2 + \sigma_i\sum_{j\neq i}\rho_{i,j}\sigma_j}{n^2\sigma^2(x)}
\end{aligned}$$

(b) If asset returns are independent, we have:

$$\mathcal{RC}_i^\star = \frac{\sigma_i^2}{\sum_{i=1}^{n}\sigma_i^2}$$

In the case of perfect correlation, we obtain:

$$\begin{aligned}\mathcal{RC}_i^{\star} &= \frac{\sigma_i^2 + \sigma_i \sum_{j \neq i} \sigma_j}{n^2 \bar{\sigma}^2} \\ &= \frac{\sigma_i \sum_j \sigma_j}{n^2 \bar{\sigma}^2} \\ &= \frac{\sigma_i}{n \bar{\sigma}}\end{aligned}$$

(c) If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\begin{aligned}\mathcal{RC}_i^{\star} &= \frac{\sigma^2 + \sigma^2 \sum_{j \neq i} \rho_{i,j}}{n^2 \sigma^2 (x)} \\ &= \frac{\sigma^2 + (n-1)\,\sigma^2 \bar{\rho}_i}{n^2 \sigma^2 (x)} \\ &= \frac{1 + (n-1)\,\bar{\rho}_i}{n\,(1 + (n-1)\,\bar{\rho})}\end{aligned}$$

It follows that:

$$\lim_{n \to \infty} \frac{1 + (n-1)\,\bar{\rho}_i}{1 + (n-1)\,\bar{\rho}} = \frac{\bar{\rho}_i}{\bar{\rho}}$$

We deduce that the risk contributions are proportional to the ratio between the mean correlation of asset i and the mean correlation of the asset universe.

(d) We recall that we have:

$$\sigma (x) = \sqrt{(1-\rho)\,\sigma_0^2 (x) + \rho \sigma_1^2 (x)}$$

It follows that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \frac{(1-\rho)\,\sigma_0 (x)\,\partial_{x_i} \sigma_0 (x) + \rho \sigma_1 (x)\,\partial_{x_i} \sigma_1 (x)}{\sqrt{(1-\rho)\,\sigma_0^2 (x) + \rho \sigma_1^2 (x)}} \\ &= \frac{(1-\rho)\,\sigma_0 (x)\,\mathcal{RC}_{0,i} + \rho \sigma_1 (x)\,\mathcal{RC}_{1,i}}{\sqrt{(1-\rho)\,\sigma_0^2 (x) + \rho \sigma_1^2 (x)}}\end{aligned}$$

We then obtain:

$$\mathcal{RC}_i^{\star} = \frac{(1-\rho)\,\sigma_0^2 (x)}{\sigma^2 (x)} \mathcal{RC}_{0,i}^{\star} + \frac{\rho \sigma_1 (x)}{\sigma^2 (x)} \mathcal{RC}_{1,i}^{\star}$$

We verify that the risk contribution $\mathcal{RC}_i$ is a weighted average of $\mathcal{RC}_{0,i}^{\star}$ and $\mathcal{RC}_{1,i}^{\star}$.

3. (a) We have:

$$\Sigma = \beta \beta^{\top} \sigma_m^2 + D$$

We deduce that:

$$\sigma(x) = \frac{1}{n}\sqrt{\sigma_m^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_i \beta_j + \sum_{i=1}^{n} \tilde{\sigma}_i^2}$$

(b) The risk contributions are equal to:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sigma(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}
\mathcal{RC}_i &= \frac{x_i \cdot \left(\sigma_m^2 \beta_i \sum_{j=1}^{n} x_j \beta_j + x_i \tilde{\sigma}_i^2\right)}{n^2 \sigma(x)} \\
&= \frac{\sigma_m^2 \beta_i \sum_{j=1}^{n} \beta_j + \tilde{\sigma}_i^2}{n^2 \sigma(x)} \\
&= \frac{n \sigma_m^2 \beta_i \bar{\beta} + \tilde{\sigma}_i^2}{n^2 \sigma(x)}
\end{aligned}$$

(c) When the number of assets is large and $\beta_i > 0$, we obtain:

$$\mathcal{RC}_i \simeq \frac{\sigma_m^2 \beta_i \bar{\beta}}{n \sigma(x)}$$

because $\bar{\beta} > 0$. We deduce that the risk contributions are approximately proportional to the beta coefficients:

$$\mathcal{RC}_i^\star \simeq \frac{\beta_i}{\sum_{j=1}^{n} \beta_j}$$

In Figure 3.1, we compare the exact and approximated values of $\mathcal{RC}_i^\star$. For that, we simulate β_i and $\tilde{\sigma}_i$ with $\beta_i \sim \mathcal{U}_{[0.5,1.5]}$ and $\tilde{\sigma}_i \sim \mathcal{U}_{[0,20\%]}$ whereas σ_m is set to 25%. We notice that the approximated value is very close to the exact value when n increases.

3.3 Minimum variance portfolio

1. (a) The optimization program is:

$$\begin{aligned}
x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x \\
\text{u.c.} &\quad \mathbf{1}^\top x = 1
\end{aligned}$$

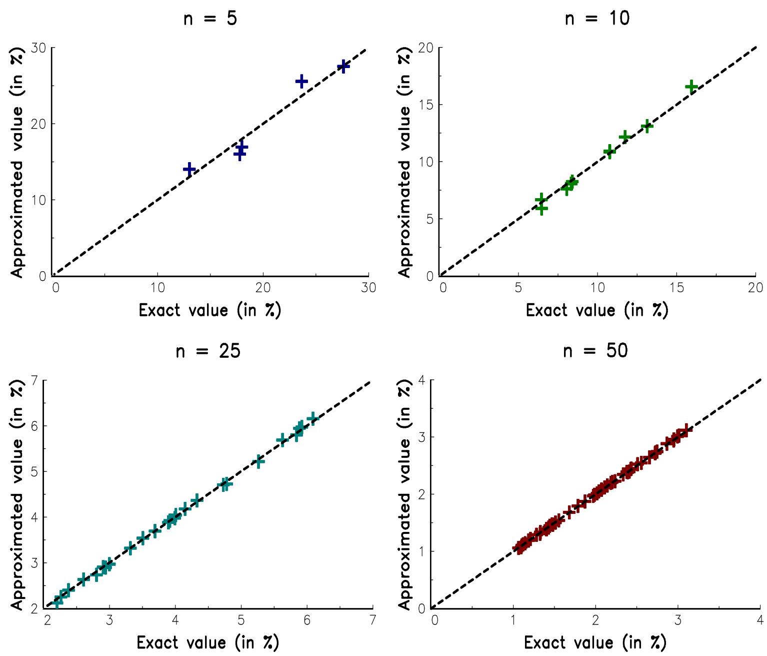

FIGURE 3.1: Comparing the exact and approximated values of $\mathcal{RC}_i^\star$

We can show that the optimal portfolio is then equal to (TR-RPB, page 11):

$$x^\star = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}$$

Let $C = C_n(\rho)$ be the constant correlation matrix. We have $\Sigma = \sigma\sigma^\top \circ C$ and $\Sigma^{-1} = \Gamma \circ C^{-1}$ with $\Gamma_{i,j} = (\sigma_i\sigma_j)^{-1}$. The computation of C^{-1} gives[1]:

$$C^{-1} = \frac{\rho\mathbf{1}\mathbf{1}^\top - ((n-1)\rho + 1) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

Because $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we obtain:

$$\left(\Sigma^{-1}\right)_{i,j} = \frac{\rho}{(n-1)\rho^2 - (n-2)\rho - 1}(\sigma_i\sigma_j)^{-1}$$

if $i \neq j$ and:

$$\left(\Sigma^{-1}\right)_{i,j} = -\frac{(n-2)\rho + 1}{(n-1)\rho^2 - (n-2)\rho - 1}\sigma_i^{-2}$$

[1] We use the relationship:

$$C^{-1} = \frac{1}{\det C}\tilde{C}^\top$$

with $\tilde{C}$ the cofactor matrix of C.

It follows that:

$$\begin{aligned}\left(\Sigma^{-1}\mathbf{1}\right)_i &= -\frac{(n-2)\,\rho+1}{(n-1)\,\rho^2-(n-2)\,\rho-1}\sigma_i^{-2}+ \\ &\quad \sum_{j\neq i}\frac{\rho}{(n-1)\,\rho^2-(n-2)\,\rho-1}\left(\sigma_i\sigma_j\right)^{-1} \\ &= \frac{-\left((n-1)\,\rho+1\right)\sigma_i^{-2}+\rho\sum_{j=1}^n\left(\sigma_i\sigma_j\right)^{-1}}{(n-1)\,\rho^2-(n-2)\,\rho-1}\end{aligned}$$

We deduce that:

$$\begin{aligned}x_i^\star &= \frac{-\left((n-1)\,\rho+1\right)\sigma_i^{-2}+\rho\sum_{j=1}^n\left(\sigma_i\sigma_j\right)^{-1}}{\sum_{k=1}^n\left(-\left((n-1)\,\rho+1\right)\sigma_k^{-2}+\rho\sum_{j=1}^n\left(\sigma_k\sigma_j\right)^{-1}\right)} \\ &\propto -\left((n-1)\,\rho+1\right)\sigma_i^{-2}+\rho\sum_{j=1}^n\left(\sigma_i\sigma_j\right)^{-1}\end{aligned}$$

(b) When $\rho = 1$, we obtain:

$$\begin{aligned}x_i^\star &\propto -n\sigma_i^{-2}+\sum_{j=1}^n\left(\sigma_i\sigma_j\right)^{-1} \\ &= \sigma_i^{-1}\left(\sum_{j=1}^n\sigma_j^{-1}-n\sigma_i^{-1}\right) \\ &= n\sigma_i^{-1}\left(H^{-1}-\sigma_i^{-1}\right) \\ &= \frac{n\sigma_i^{-1}}{H\sigma_i}\left(\sigma_i-H\right)\end{aligned}$$

where H is the harmonic mean:

$$H = \left(\frac{1}{n}\sum_{j=1}^n\sigma_j^{-1}\right)^{-1}$$

The weights are all positive if the sign of $\sigma_i - H$ is the same for all the assets. By definition of the harmonic mean, the only solution is when the volatilities are the same for all the assets.

(c) When $\rho = 0$, we obtain $x_i^\star \propto -\sigma_i^{-2}$. Because $\mathbf{1}^\top x^\star = 1$, the solution is:

$$x_i^\star = \frac{\sigma_i^{-2}}{\sum_{j=1}^n\sigma_i^{-2}}$$

The weight of asset i is inversely proportional to its variance.

(d) We have:

$$\begin{aligned} C &= \rho \mathbf{1}\mathbf{1}^\top - (\rho - 1) I_n \\ &= \rho \left(\mathbf{1}\mathbf{1}^\top - \frac{(\rho - 1)}{\rho} I_n \right) \end{aligned}$$

Let $\mathbb{P}_A(\lambda)$ be the characteristic polynomial of A. It follows that:

$$\begin{aligned} \det C &= \rho^n \det \left(\mathbf{1}\mathbf{1}^\top - \frac{(\rho - 1)}{\rho} I_n \right) \\ &= \rho^n \mathbb{P}_{\mathbf{1}\mathbf{1}^\top} \left(\frac{\rho - 1}{\rho} \right) \\ &= \rho^n (-1)^n \left(\frac{\rho - 1}{\rho} \right)^{n-1} \left(\frac{\rho - 1}{\rho} - n \right) \\ &= (1 - \rho)^{n-1} ((n - 1)\rho + 1) \end{aligned}$$

A necessary condition for C to be definite positive is that the determinant is positive:

$$\det C > 0 \Leftrightarrow \rho > -\frac{1}{n-1}$$

The lower bound is then $\rho^- = -(n-1)^{-1}$. In this case, we obtain:

$$x_i^\star \propto -\frac{1}{n-1} \sum_{j=1}^n (\sigma_i \sigma_j)^{-1}$$

We deduce that:

$$\begin{aligned} x_i^\star &= \frac{\sum_{j=1}^n (\sigma_i \sigma_j)^{-1}}{\sum_{k=1}^n \sum_{j=1}^n (\sigma_i \sigma_j)^{-1}} \\ &= \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_i^{-1}} \end{aligned}$$

When $\rho = \rho^-$, the MV portfolio coincides with the ERC portfolio.

(e) We have

$$x_i^\star = \frac{a_i + b_i}{\sum_{j=1}^n a_j + b_j}$$

with:

$$\begin{aligned} a_i &= (n - 1) \rho \sigma_i^{-2} \\ b_i &= \sigma_i^{-2} - \rho \sum_{j=1}^n (\sigma_i \sigma_j)^{-1} \end{aligned}$$

If $\rho^{-} \leq \rho \leq 0$, we have $a_i \leq 0$ and $b_i \geq 0$. We would like to show that $x_i^{\star}$ or equivalently $a_i + b_i$ is positive. We then have to show that:

$$\rho n \left(\sigma_i^{-1} - H^{-1}\right) + (1-\rho)\,\sigma_i^{-1} \geq 0$$

for every asset. If $\sigma_i \geq H$, this inequality is satisfied. If $\sigma_i < H$, it means that:

$$\sigma_i < \left(\frac{1}{n}\sum_{j=1}^{n}\sigma_j^{-1}\right)^{-1}$$

If $\rho^{-} \leq \rho \leq 0$, we conclude that $x_i^{\star} \geq 0$. Moreover, we know that if $\rho = 1$, at least one asset has a negative weight. It implies that there exists a correlation $\rho^{\star} > 0$ such that the weights are all positive if $\rho \leq \rho^{\star}$. It is obvious that $\rho^{\star}$ must satisfy this equation:

$$-\left((n-1)\,\rho + 1\right)\sigma_i^{-1} + \rho\sum_{j=1}^{n}\sigma_j^{-1} = 0$$

for one asset. We finally obtain:

$$\begin{aligned}\rho^{\star} &= \inf_{[0,1]} \frac{\sigma_i^{-1}}{\sum_{j=1}^{n}\sigma_j^{-1} - (n-1)\,\sigma_i^{-1}} \\ &= \frac{\sigma_+^{-1}}{\sum_{j=1}^{n}\sigma_j^{-1} - (n-1)\,\sigma_+^{-1}}\end{aligned}$$

with $\sigma_+ = \sup \sigma_i$.

(f) We have reported the relationship between $\inf x_i^{\star}$ and ρ in Figure 3.2. We notice that $\inf x_i^{\star}$ is close to one for the second set of parameters (i.e. when the dispersion across σ_i is high) and $\inf x_i^{\star}$ is close to zero for the third set of parameters (i.e. when the dispersion across σ_i is low). We may then postulate these two rules[2]:

$$\lim_{\operatorname{var}(\sigma_i)\to 0} \rho^{\star} = 0$$

and:

$$\lim_{\operatorname{var}(\sigma_i)\to \infty} \rho^{\star} = 1$$

[2]If you are interested to prove these two rules, you have to use the following inequality:

$$A(\sigma_i) - H(\sigma_i) \geq \frac{\operatorname{var}(\sigma_i)}{2\sup \sigma_i}$$

where $A(\sigma_i)$ and $H(\sigma_i)$ are the arithmetic and harmonic means of $\{\sigma_1, \ldots, \sigma_n\}$.

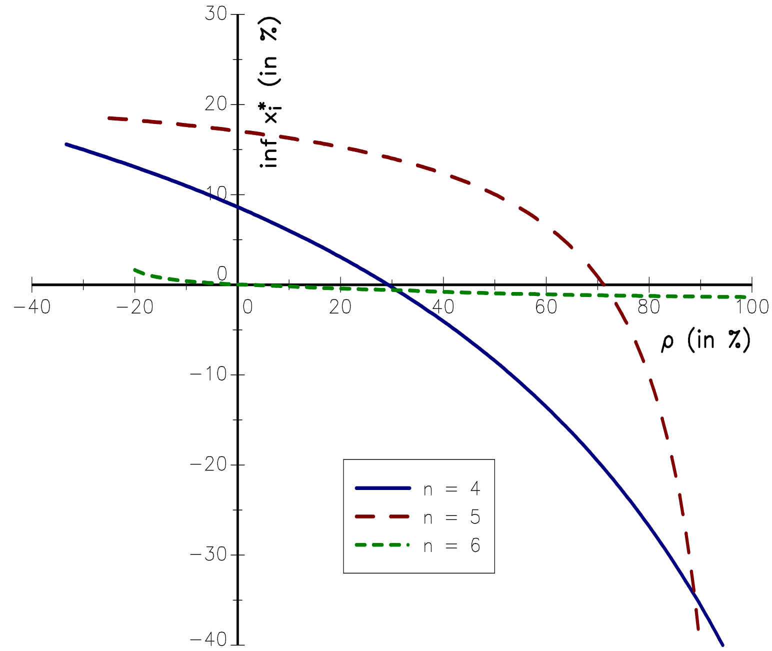

FIGURE 3.2: Relationship between ρ and $\inf x_i^\star$

(g) For the first set of parameters, $\rho^\star$ is equal to 29.27% and the optimal weights are:

ρ	ρ^-	0	$\rho^\star$	0.9	1
$x_1^\star$	38.96%	53.92%	72.48%	149.07%	172.69%
$x_2^\star$	25.97%	23.97%	21.48%	11.20%	8.03%
$x_3^\star$	19.48%	13.48%	6.04%	−24.67%	−34.14%
$x_4^\star$	15.58%	8.63%	0.00%	−35.61%	−46.59%

For the second set of parameters, $\rho^\star$ is equal to 71.15% and the optimal weights are:

ρ	ρ^-	0	$\rho^\star$	0.9	1
$x_1^\star$	21.42%	22.89%	40.38%	82.16%	621.36%
$x_2^\star$	20.35%	20.65%	24.30%	33.00%	145.26%
$x_3^\star$	19.38%	18.73%	11.02%	−7.40%	−245.13%
$x_4^\star$	18.50%	17.07%	0.00%	−40.75%	−566.75%
$x_5^\star$	20.35%	20.65%	24.30%	33.00%	145.26%

For the third set of parameters, $\rho^\star$ is equal to 1.77% and the optimal weights are:

ρ	ρ^-	0	$\rho^\star$	0.9	1
$x_1^\star$	81.41%	98.56%	98.98%	104.07%	104.21%
$x_2^\star$	8.14%	0.99%	0.81%	−1.32%	−1.37%
$x_3^\star$	4.07%	0.25%	0.15%	−0.98%	−1.01%
$x_4^\star$	2.71%	0.11%	0.04%	−0.73%	−0.75%
$x_5^\star$	2.04%	0.06%	0.01%	−0.57%	−0.59%
$x_6^\star$	1.63%	0.04%	0.00%	−0.47%	−0.48%

2. (a) We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

The Sherman-Morrison-Woodbury formula is (TR-RPB, page 167):

$$\left(A + uv^\top\right)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1} u v^\top A^{-1}$$

where u and v are two vectors and A is an invertible square matrix. By setting $A = D$ and $u = v = \sigma_m \beta$, we obtain:

$$\Sigma^{-1} = D^{-1} - \frac{\sigma_m^2}{1 + \sigma_m^2 \beta^\top D^{-1} \beta} D^{-1} \beta\beta^\top D^{-1}$$

We note $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$ and $\kappa = \tilde{\beta}^\top \beta$. We have $\tilde{\beta} = D^{-1}\beta$ and:

$$\Sigma^{-1} = D^{-1} - \frac{\sigma_m^2}{1 + \sigma_m^2 \kappa} \tilde{\beta}\tilde{\beta}^\top$$

(b) The analytical expression of the minimum variance portfolio is:

$$x^\star = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}$$

We have:

$$\begin{aligned} \sigma^2(x^\star) &= {x^\star}^\top \Sigma x^\star \\ &= \frac{\mathbf{1}^\top \Sigma^{-1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \Sigma \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \\ &= \frac{1}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \end{aligned}$$

We deduce that:

$$x^\star = \sigma^2(x^\star) \left(D^{-1}\mathbf{1} - \frac{\sigma_m^2}{1 + \sigma_m^2 \kappa} \tilde{\beta}\tilde{\beta}^\top \mathbf{1} \right)$$

(c) We have:

$$\begin{aligned} x_i^\star &= \sigma^2\left(x^\star\right)\left(\frac{1}{\tilde{\sigma}_i^2} - \frac{\sigma_m^2 \tilde{\beta}^\top \mathbf{1}}{1+\sigma_m^2 \kappa}\tilde{\beta}_i\right) \\ &= \sigma^2\left(x^\star\right)\left(\frac{1}{\tilde{\sigma}_i^2} - \frac{\sigma_m^2 \tilde{\beta}^\top \mathbf{1}}{1+\sigma_m^2 \kappa}\frac{\beta_i}{\tilde{\sigma}_i^2}\right) \\ &= \frac{\sigma^2\left(x^\star\right)}{\tilde{\sigma}_i^2}\left(1-\frac{\beta_i}{\beta^\star}\right) \end{aligned}$$

with:

$$\begin{aligned} \beta^\star &= \frac{1+\sigma_m^2\kappa}{\sigma_m^2\tilde{\beta}^\top\mathbf{1}} \\ &= \frac{1+\sigma_m^2\sum_{j=1}^n \beta_j^2/\tilde{\sigma}_j^2}{\sigma_m^2\sum_{j=1}^n \beta_j/\tilde{\sigma}_j^2} \end{aligned}$$

(d) The optimal weight $x_i^\star$ is positive if:

$$1-\frac{\beta_i}{\beta^\star}\geq 0$$

or equivalently:

$$\beta^\star \geq \beta_i$$

If $\beta_i = \beta_j = \beta$, we obtain:

$$\begin{aligned} \beta^\star &= \frac{1+\sigma_m^2\beta^2\sum_{j=1}^n 1/\tilde{\sigma}_j^2}{\sigma_m^2\beta\sum_{j=1}^n 1/\tilde{\sigma}_j^2} \\ &= \frac{1}{\sigma_m^2\beta\sum_{j=1}^n 1/\tilde{\sigma}_j^2}+\beta \\ &\geq \beta \end{aligned}$$

We deduce that the weights are positive for all the assets if the betas are the same. If $\tilde{\sigma}_i = \tilde{\sigma}_j = \tilde{\sigma}$, we have:

$$\begin{aligned} \beta^\star - \beta_i &= \frac{1+\sigma_m^2/\tilde{\sigma}^2\sum_{j=1}^n \beta_j^2}{\sigma_m^2/\tilde{\sigma}^2\sum_{j=1}^n \beta_j} - \beta_i \\ &= \frac{1}{\sum_{j=1}^n \beta_j}\left(\frac{\tilde{\sigma}^2}{\sigma_m^2}+\sum_{j=1}^n\left(\beta_j-\beta_i\right)\beta_j\right) \end{aligned}$$

The weights are all positive if and only if:

$$\frac{\tilde{\sigma}^2}{\sigma_m^2}\geq\sum_{j=1}^n\left(\sup\beta_j-\beta_j\right)\beta_j$$

If $\tilde{\sigma} \gg \sigma_m$, the previous inequality holds. Except in this case, the inequality cannot be verified, i.e. the weights cannot be all positive.

(e) We obtain the following results:

σ_m	5.00%	10.00%	15.00%	20.00%	25.00%
$x_1^\star$	49.41%	90.71%	125.69%	149.41%	164.78%
$x_2^\star$	10.26%	16.52%	21.83%	25.43%	27.76%
$x_3^\star$	8.16%	10.37%	12.24%	13.50%	14.32%
$x_4^\star$	24.24%	16.84%	10.57%	6.31%	3.56%
$x_5^\star$	7.46%	−32.41%	−66.18%	−89.08%	−103.93%
$x_6^\star$	0.47%	−2.03%	−4.14%	−5.57%	−6.50%
$\beta^\star$	1.29	1.07	1.03	1.01	1.01

3.4 Most diversified portfolio

1. (a) Let $\mathcal{R}(x)$ be the risk measure of the portfolio x. We note $\mathcal{R}_i = \mathcal{R}(\mathbf{e}_i)$ the risk associated to the i^{th} asset. The diversification ratio is the ratio between the weighted mean of the individual risks and the portfolio risk (TR-RPB, page 168):

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^n x_i \mathcal{R}_i}{\mathcal{R}(x)}$$

If we assume that the risk measure satisfies the Euler allocation principle, we have:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^n x_i \mathcal{R}_i}{\sum_{i=1}^n \mathcal{RC}_i}$$

(b) If $\mathcal{R}(x)$ satisfies the Euler allocation principle, we know that $\mathcal{R}_i \geq \mathcal{MR}_i$ (TR-RPB, page 78). We deduce that:

$$\begin{aligned} \mathcal{DR}(x) &\geq \frac{\sum_{i=1}^n x_i \mathcal{R}_i}{\sum_{i=1}^n x_i \mathcal{R}_i} \\ &\geq 1 \end{aligned}$$

Let x_{mr} be the portfolio that minimizes the risk measure. We have:

$$\mathcal{DR}(x) \leq \frac{\sup \mathcal{R}_i}{\mathcal{R}(x_{\text{mr}})}$$

(c) If we consider the volatility risk measure, the minimum risk portfolio is the minimum variance portfolio. We have (TR-RPB, page 164):

$$\sigma(x_{\text{mv}}) = \frac{1}{\sqrt{\mathbf{1}^\top \Sigma \mathbf{1}}}$$

We deduce that:

$$\mathcal{DR}(x) \leq \sqrt{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \cdot \sup \sigma_i$$

(d) The MDP is the portfolio which maximizes the diversification ratio when the risk measure is the volatility (TR-RPB, page 168). We have:

$$\begin{aligned} x^\star &= \arg\max \mathcal{DR}(x) \\ &\text{u.c.} \quad \mathbf{1}^\top x = 1 \end{aligned}$$

If we consider that the risk premium $\pi_i = \mu_i - r$ of the asset i is proportional to its volatility σ_i, we obtain:

$$\begin{aligned} \text{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\ &= \frac{\sum_{i=1}^n x_i (\mu_i - r)}{\sigma(x)} \\ &= s \frac{\sum_{i=1}^n x_i \sigma_i}{\sigma(x)} \\ &= s \cdot \mathcal{DR}(x) \end{aligned}$$

where s is the coefficient of proportionality. Maximizing the diversification ratio is equivalent to maximizing the Sharpe ratio. We recall that the expression of the tangency portfolio is:

$$x^\star = \frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r\mathbf{1})}$$

We deduce that the weights of the MDP are:

$$x^\star = \frac{\Sigma^{-1}\sigma}{\mathbf{1}^\top \Sigma^{-1}\sigma}$$

The volatility of the MDP is then:

$$\begin{aligned} \sigma(x^\star) &= \sqrt{\frac{\sigma^\top \Sigma^{-1}}{\mathbf{1}^\top \Sigma^{-1}\sigma} \Sigma \frac{\Sigma^{-1}\sigma}{\mathbf{1}^\top \Sigma^{-1}\sigma}} \\ &= \frac{\sqrt{\sigma^\top \Sigma^{-1}\sigma}}{\mathbf{1}^\top \Sigma^{-1}\sigma} \end{aligned}$$

(e) We have seen in Exercise 1.11 that another expression of the unconstrained tangency portfolio is:

$$x^\star = \frac{\sigma^2(x^\star)}{(\mu(x^\star) - r)} \Sigma^{-1}(\mu - r\mathbf{1})$$

We deduce that the MDP is also:

$$x^\star = \frac{\sigma^2(x^\star)}{\bar{\sigma}(x^\star)} \Sigma^{-1} \sigma$$

where $\bar{\sigma}(x^\star) = x^{\star\top} \sigma$. Nevertheless, this solution is endogenous.

2. (a) We have:

$$\operatorname{cov}(R_i, R_m) = \beta_i \sigma_m^2$$

We deduce that:

$$\begin{aligned} \rho_{i,m} &= \frac{\operatorname{cov}(R_i, R_m)}{\sigma_i \sigma_m} \\ &= \beta_i \frac{\sigma_m}{\sigma_i} \end{aligned} \tag{3.1}$$

and:

$$\begin{aligned} \tilde{\sigma}_i &= \sqrt{\sigma_i^2 - \beta_i^2 \sigma_m^2} \\ &= \sigma_i \sqrt{1 - \rho_{i,m}^2} \end{aligned} \tag{3.2}$$

(b) We know that (TR-RPB, page 167):

$$\Sigma^{-1} = D^{-1} - \frac{1}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta} \tilde{\beta}^\top$$

where $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$. We deduce that:

$$x^\star = \frac{\sigma^2(x^\star)}{\bar{\sigma}(x^\star)} \left(D^{-1} \sigma - \frac{1}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta} \tilde{\beta}^\top \sigma \right)$$

and:

$$\begin{aligned} x_i^\star &= \frac{\sigma^2(x^\star)}{\bar{\sigma}(x^\star)} \left(\frac{\sigma_i}{\tilde{\sigma}_i^2} - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta}_i \right) \\ &= \frac{\sigma_i \sigma^2(x^\star)}{\bar{\sigma}(x^\star) \tilde{\sigma}_i^2} \left(1 - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta} \frac{\sigma_m \tilde{\sigma}_i^2 \tilde{\beta}_i}{\sigma_i} \right) \\ &= \frac{\sigma_i \sigma^2(x^\star)}{\bar{\sigma}(x^\star) \tilde{\sigma}_i^2} \left(1 - \frac{\tilde{\beta}^\top \sigma}{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta} \rho_{i,m} \right) \\ &= \mathcal{DR}(x^\star) \frac{\sigma_i \sigma(x^\star)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star} \right) \end{aligned}$$

Using Equations (3.1) and (3.2), $\rho^\star$ is defined as follows:

$$\begin{aligned}\rho^\star &= \frac{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta}{\tilde{\beta}^\top \sigma} \\ &= \left(1 + \sum_{j=1}^{n} \frac{\sigma_m^2 \beta_j^2}{\tilde{\sigma}_j^2}\right) \Big/ \left(\sum_{j=1}^{n} \frac{\sigma_m \beta_j \sigma_j}{\tilde{\sigma}_j^2}\right) \\ &= \left(1 + \sum_{j=1}^{n} \frac{\rho_{j,m}^2}{1 - \rho_{j,m}^2}\right) \Big/ \left(\sum_{j=1}^{n} \frac{\rho_{j,m}}{1 - \rho_{j,m}^2}\right)\end{aligned}$$

(c) The optimal weight $x_i^\star$ is positive if:

$$1 - \frac{\rho_{i,m}}{\rho^\star} \geq 0$$

or equivalently:

$$\rho_{i,m} \leq \rho^\star$$

(d) We recall that:

$$\begin{aligned}\rho_{i,m} &= \beta_i \frac{\sigma_m}{\sigma_i} \\ &= \frac{\beta_i \sigma_m}{\sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2}}\end{aligned}$$

If $\beta_i < 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ increases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We deduce that the weight is a decreasing function of the specific volatility $\tilde{\sigma}_i$. If $\beta_i > 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ decreases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We cannot conclude in this case.

3. (a) The MDP coincide with the MV portfolio when the volatility is the same for all the assets.

(b) The formula cannot be used directly, because it depends on $\sigma(x^\star)$ and $\mathcal{DR}(x^\star)$. However, we notice that:

$$x_i^\star \propto \frac{\sigma_i}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star}\right)$$

It suffices then to rescale these weights to obtain the solution. Using the numerical values of the parameters, $\rho^\star = 98.92\%$ and we obtain the following results:

	β_i	$\rho_{i,m}$	$x_i \in \mathbb{R}$ MDP	$x_i \in \mathbb{R}$ MV	$x_i \geq 0$ MDP	$x_i \geq 0$ MV
$x_1^\star$	0.80	99.23%	−27.94%	211.18%	0.00%	100.00%
$x_2^\star$	0.90	96.35%	43.69%	−51.98%	25.00%	0.00%
$x_3^\star$	1.10	82.62%	43.86%	−24.84%	39.24%	0.00%
$x_4^\star$	1.20	84.80%	40.39%	−34.37%	35.76%	0.00%
$\sigma(x^\star)$			24.54%	13.42%	23.16%	16.12%

(c) The results are:

	$x_i \in \mathbb{R}$		$x_i \geq 0$	
	MDP	MV	MDP	MV
$x_1^\star$	-36.98%	60.76%	0.00%	48.17%
$x_2^\star$	-36.98%	60.76%	0.00%	48.17%
$x_3^\star$	91.72%	-18.54%	50.00%	0.00%
$x_4^\star$	82.25%	-2.98%	50.00%	3.66%
$\sigma(x^\star)$	48.59%	6.43%	30.62%	9.57%

(d) These two examples show that the MDP may have a different behavior than the minimum variance portfolio. Contrary to the latter, the most diversified portfolio is not necessarily a low-beta or a low-volatility portfolio.

3.5 Risk allocation with yield curve factors

1. (a) Let v_i be the i^{th} eigenvector. We have:

$$Av_i = \lambda_i v_i$$

Generally, we assume that the eigenvector is normalized, that is $v_i^\top v_i = 1$. In a matrix form, the previous relationship becomes:

$$AV = V\Lambda$$

with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $V = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$. We deduce that:

$$A = V\Lambda V^{-1}$$

We have:

$$\begin{aligned}
\operatorname{tr}(A) &= \operatorname{tr}\left(V\Lambda V^{-1}\right) \\
&= \operatorname{tr}\left(\Lambda V^{-1} V\right) \\
&= \operatorname{tr}(\Lambda) \\
&= \sum_{i=1}^{n} \lambda_i
\end{aligned}$$

and[3]:

$$\begin{aligned}\det(A) &= \det\left(V\Lambda V^{-1}\right)\\ &= \det(V)\cdot\det(\Lambda)\cdot\det\left(V^{-1}\right)\\ &= \det(\Lambda)\\ &= \prod_{i=1}^{n}\lambda_i\end{aligned}$$

(b) We have $\Sigma v_i = \lambda_i v_i$ or $v_i^\top \Sigma = \lambda_i v_i^\top$. We deduce that:

$$v_i^\top \Sigma v_j = \lambda_i v_i^\top v_j$$

and:

$$v_j^\top \Sigma v_i = \lambda_j v_j^\top v_i = \lambda_j v_i^\top v_j$$

Moreover, we have:

$$v_i^\top \Sigma v_j = \left(v_i^\top \Sigma v_j\right)^\top = v_j^\top \Sigma v_i$$

We finally obtain $\lambda_i v_i^\top v_j = \lambda_j v_i^\top v_j$ or:

$$(\lambda_i - \lambda_j)\, v_i^\top v_j = 0$$

Because $\lambda_i \neq \lambda_j$, we conclude that $v_i^\top v_j$ meaning that eigenvectors are orthogonal. We then have $V^\top V = I$ and:

$$\begin{aligned}\Sigma &= V\Lambda V^{-1}\\ &= V\Lambda V^\top\end{aligned}$$

(c) If $C = C_n(\rho)$, we know that $\lambda_2 = \ldots = \lambda_n = \lambda$. It follows that:

$$\sum_{i=1}^{n}\lambda_i = \lambda_1 + (n-1)\lambda = n$$

and[4]:

$$\prod_{i=1}^{n}\lambda_i = \lambda_1\lambda^{n-1} = (1-\rho)^{n-1}\left((n-1)\rho + 1\right)$$

We deduce that:

$$\lambda_1 = 1 + (n-1)\rho$$

and:

$$\lambda_i = 1 - \rho \quad \text{if} \quad i > 1$$

[3]Because $\det(V)\cdot\det\left(V^{-1}\right) = \det(I) = 1$.

[4]See Exercise 3.2.

It proves the result because $\bar{\rho} = \rho$. We note π_1 the percentage of variance explained by the first eigenvalue. We have:

$$\begin{aligned} \pi_1 &= \frac{\lambda_1}{\operatorname{tr}(C)} \\ &= \frac{\lambda_1}{n} \end{aligned}$$

We get $\pi_1 \geq \pi_1^-$ with:

$$\begin{aligned} \pi_1^- &= \frac{1 + (n-1)\bar{\rho}}{n} \\ &= \bar{\rho} + \frac{(1-\bar{\rho})}{n} \end{aligned}$$

π_1^- takes the following values:

$n/\bar{\rho}$	10%	20%	50%	70%	90%
2	55.0%	60.0%	75.0%	85.0%	95.0%
3	40.0%	46.7%	66.7%	80.0%	93.3%
5	28.0%	36.0%	60.0%	76.0%	92.0%
10	19.0%	28.0%	55.0%	73.0%	91.0%

We notice that $\pi_1^- \simeq \bar{\rho}$ when n is large.

(d) We obtain the following results:

Asset	v_1	v_2	v_3	v_4
1	0.2704	0.5900	0.3351	0.6830
2	0.4913	0.3556	0.3899	−0.6929
3	0.4736	0.2629	−0.8406	−0.0022
4	0.6791	−0.6756	0.1707	0.2309
λ_i	0.0903	0.0416	0.0358	0.0156

Let us consider the following optimization problem:

$$\begin{aligned} x^\star &= \arg\max \frac{1}{2} x^\top \Sigma x \\ &\text{u.c.} \quad x^\top x = 1 \end{aligned}$$

The first-order condition is $\Sigma x - \lambda x = 0$ where λ is the Lagrange coefficient associated to the normalization constraint $x^\top x = 1$. It corresponds precisely to the definition of the eigendecomposition. We deduce that $x^\star$ is the first eigenvector:

$$x^\star = v_1$$

It means that the first eigenvector is the maximum variance portfolio under the normalization constraint. In the same way, we can show that the last eigenvector is the minimum variance portfolio under the normalization constraint:

$$\begin{aligned} v_n &= \arg\min \frac{1}{2} x^\top \Sigma x \\ \text{u.c.} & \quad x^\top x = 1 \end{aligned}$$

(e) In the case of the correlation matrix, the eigendecomposition is:

Asset	v_1	v_2	v_3	v_4
1	0.4742	0.6814	0.0839	−0.5512
2	0.6026	0.2007	−0.2617	0.7267
3	0.4486	−0.3906	0.8035	0.0253
4	0.4591	−0.5855	−0.5281	−0.4092
λ_i	1.9215	0.9467	0.7260	0.4059

We obtain $\pi_1 = 48.04\%$ and $\pi_1^- = 47.50\%$ because $\bar{\rho} = 30\%$. We notice that the lower bound is close to the true value.

(f) Let us specify the risk model as follows (TR-RPB, page 38):

$$R_t = A\mathcal{F}_t + \varepsilon_t$$

where R_t is the vector of asset returns, $\mathcal{F}_t$ is the vector of risk factors and ε_t is the vector of idiosyncratic factors. We assume that $\operatorname{cov}(R_t) = \Sigma$, $\operatorname{cov}(\mathcal{F}_t) = \Omega$ and $\operatorname{cov}(\varepsilon_t) = D$. Moreover, we suppose that the risk factors and the idiosyncratic factors are independent. We know that (TR-RPB, page 38):

$$\Sigma = A\Omega A^\top + D$$

If we consider a principal component analysis, we have (TR-RPB, page 216):

$$\Sigma = V\Lambda V^\top$$

We could then consider that the risk factors correspond to the eigenfactors and we have $A = V$, $\Omega = \Lambda$ and $D = \mathbf{0}$. In the case of two factors, the risk factors are the first two eigenfactors and we have $A = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, $\Omega = \operatorname{diag}(\lambda_1, \lambda_2)$ and $D = \Sigma - A\Omega A^\top$. We notice that:

$$\begin{aligned} R_t &= \sum_{j=1}^{4} v_j \mathcal{F}_{j,t} \\ &= \sum_{j=1}^{2} v_j \mathcal{F}_{j,t} + \sum_{j=3}^{4} v_j \mathcal{F}_{j,t} \\ &= \sum_{j=1}^{2} v_j \mathcal{F}_{j,t} + \varepsilon_t \end{aligned}$$

Another expression of D is then $D = B\Phi B^\top$ with $B = \begin{pmatrix} v_3 & v_4 \end{pmatrix}$ and $\Phi = \operatorname{diag}(\lambda_3, \lambda_4)$. We also verify that $\operatorname{cov}(\mathcal{F}_t, \varepsilon_t) = \mathbf{0}$ by definition of the eigendecomposition. If we would like to impose that D is diagonal, a simple way is to consider only the diagonal elements of the matrix $\Sigma - A\Omega A^\top$. We have:

$$\begin{aligned} \mathcal{F}_{1,t} &= 0.47 \cdot R_{1,t} + 0.60 \cdot R_{2,t} + 0.45 \cdot R_{3,t} + 0.46 \cdot R_{4,t} \\ \mathcal{F}_{1,t} &= 0.68 \cdot R_{1,t} + 0.20 \cdot R_{2,t} - 0.39 \cdot R_{3,t} - 0.59 \cdot R_{4,t} \end{aligned}$$

The first factor is a long-only portfolio. It represents the market factor. The second factor is a long-short portfolio. It represents an arbitrage factor between the first two assets and the last two assets.

2. (a) We consider that the risk factors are the zero-coupon rates. For each maturity, we compute the sensitivity (TR-RPB, page 204):

$$\delta(T_i) = -D_t(T_i) \cdot B_t(T_i) \cdot C(T_i)$$

where $D_t(T_i)$, $B_t(T_i)$ and $C(T_i)$ are the duration, the price and the coupon of the zero-coupon bond with maturity T_i. Using Equation (4.2) of TR-RPB on page 204, we obtain the following results:

T_i	$\delta(T_i)$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1Y	−1.00	−0.21%	0.21%	1.98%
3Y	−2.94	−0.41%	1.20%	11.20%
5Y	−4.77	−0.47%	2.25%	21.07%
7Y	−6.37	−0.48%	3.08%	28.75%
10Y	−8.36	−0.47%	3.96%	37.01%
$\text{VaR}_\alpha(x)$			10.70%	

If we prefer to consider the number n_i of the zero-coupon bond with maturity T_i, we have:

T_i	n_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1Y	1.00	0.21%	0.21%	1.98%
3Y	1.00	1.20%	1.20%	11.20%
5Y	1.00	2.25%	2.25%	21.07%
7Y	1.00	3.08%	3.08%	28.75%
10Y	1.00	3.96%	3.96%	37.01%
$\text{VaR}_\alpha(x)$			10.70%	

Finally, if we measure the exposures in terms of relative weights, we obtain (TR-RPB, Equation (4.3), page 205):

T_i	ϖ_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1Y	21.28%	1.00%	0.21%	1.98%
3Y	20.99%	5.71%	1.20%	11.20%
5Y	20.39%	11.06%	2.25%	21.07%
7Y	19.46%	15.81%	3.08%	28.75%
10Y	17.88%	22.16%	3.96%	37.01%
$\text{VaR}_\alpha(x)$			10.70%	

(b) The eigendecomposition of Σ is:

T_i	v_1	v_2	v_3	v_4	v_5
1Y	0.2475	−0.7909	−0.5471	0.1181	0.0016
3Y	0.4427	−0.3469	0.5747	−0.5898	0.0739
5Y	0.5012	0.0209	0.3292	0.6215	−0.5038
7Y	0.5040	0.2487	−0.0736	0.2583	0.7823
10Y	0.4874	0.4380	−0.5066	−0.4304	−0.3588
λ_i ($\times 10^{-6}$)	16.6446	1.5685	0.2421	0.0273	0.0045
π_i (in %)	90.0340	8.4846	1.3095	0.1478	0.0241
$\pi_i^\star$ (in %)	90.0340	98.5186	99.8281	99.9759	100.0000

where $\pi_i^\star = \sum_{j=1}^{i} \pi_i$ is the percentage of variance explained by the top i eigenvectors. The first three factors are the level, slope and convexity factors. We have represented them in Figure 3.3. They differ slightly from those reported in TR-RPB on page 197, because the set of maturities is different. Note also that the convexity factor is the opposite of the one obtained in TR-RPB because eigenvectors are not signed.

(c) Let $V\Lambda V^\top$ be the eigendecomposition of Σ. We have $A = V$, $\Omega = \Lambda$ and $D = \mathbf{0}$. The risk contribution of the j^{th} factor is then (TR-RPB, page 142):

$$\mathcal{RC}(\mathcal{F}_j) = \Phi^{-1}(\alpha) \cdot \left(A^\top \delta\right)_j \cdot \left(\frac{A^{+}\Sigma\delta}{\sqrt{\delta^\top \Sigma \delta}}\right)_j$$

We obtain the following risk decomposition:

PCA factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	−11.22	−0.94%	10.60%	99.03%
$\mathcal{F}_2$	−3.54	−0.03%	0.10%	0.93%
$\mathcal{F}_3$	1.99	0.00%	0.00%	0.05%
$\mathcal{F}_4$	0.61	0.00%	0.00%	0.00%
$\mathcal{F}_5$	0.20	0.00%	0.00%	0.00%
$\text{VaR}_\alpha(x)$			10.70%	

We notice that most of the risk is explained by the level factor (about 99%).

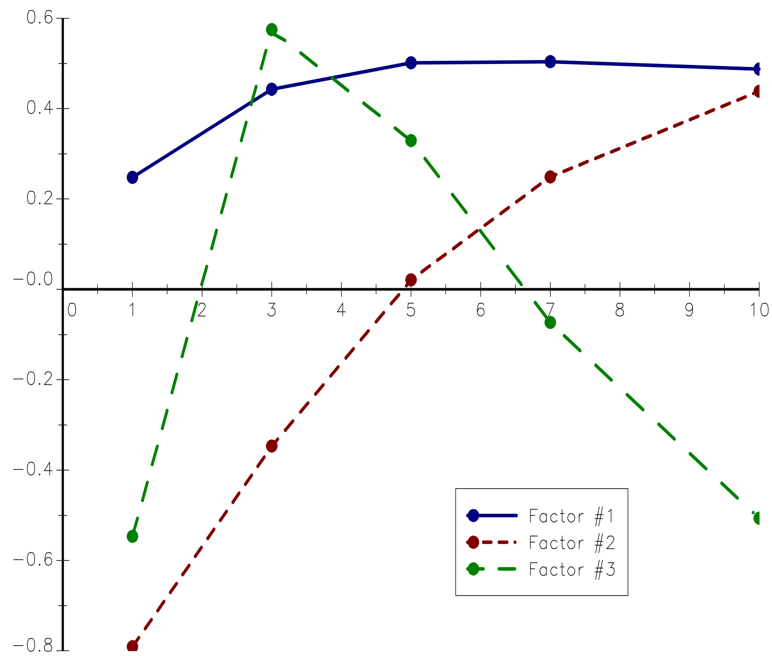

FIGURE 3.3: Representation of the first three PCA factors

(d) The ERC portfolio is the following:

T_i	n_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1Y	14.08	0.26%	3.71%	20.00%
3Y	2.95	1.26%	3.71%	20.00%
5Y	1.66	2.24%	3.71%	20.00%
7Y	1.25	2.96%	3.71%	20.00%
10Y	1.00	3.71%	3.71%	20.00%
$\mathrm{VaR}_\alpha(x)$			18.54%	

whereas its risk decomposition with respect to PCA factors is:

PCA factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	−19.36	−0.94%	18.21%	98.22%
$\mathcal{F}_2$	8.28	0.04%	0.31%	1.69%
$\mathcal{F}_3$	4.90	0.00%	0.02%	0.09%
$\mathcal{F}_4$	0.09	0.00%	0.00%	0.00%
$\mathcal{F}_5$	0.07	0.00%	0.00%	0.00%
$\mathrm{VaR}_\alpha(x)$			18.54%	

We notice that the ERC portfolio is more exposed to the slope factor than the previous EW portfolio.

(e) We suppose now that portfolio x is equal to the eigenvector v_i. In the following table, we have reported the normalized risk contributions $\mathcal{RC}_j^\star$:

PCA factor	v_1	v_2	v_3	v_4	v_5
$\mathcal{F}_1$	98.72%	81.43%	55.57%	22.62%	14.62%
$\mathcal{F}_2$	1.24%	17.24%	28.41%	23.43%	17.07%
$\mathcal{F}_3$	0.04%	1.32%	15.72%	26.63%	15.44%
$\mathcal{F}_4$	0.00%	0.01%	0.29%	27.16%	6.75%
$\mathcal{F}_5$	0.00%	0.00%	0.00%	0.17%	46.13%

We notice that the risk contribution of the level factor decreases when we consider higher orders of eigenvectors. For instance, the risk contribution is equal to 14.62% if the portfolio is the fifth eigenvector whereas it is equal to 98.72% if the portfolio is the first eigenvector. We also observe that the risk contribution $\mathcal{RC}_j^\star$ of the j^{th} factor is generally high when the portfolio corresponds to the j^{th} eigenvector.

3.6 Credit risk analysis of sovereign bond portfolios

1. We recall that the credit risk measure of a bond portfolio is (TR-RPB, page 227):

$$\mathcal{R}(x) = \sqrt{x^\top \Sigma x}$$

where $\Sigma = (\Sigma_{i,j})$ and $\Sigma_{i,j}$ is the credit covariance between the bond i and the bond j. We have:

$$\Sigma_{i,j} = \rho_{i,j} \sigma_i^{\mathfrak{c}} \sigma_i^{\mathfrak{c}}$$

and:

$$\sigma_i^{\mathfrak{c}} = D_i \sigma_i^{\mathfrak{s}} \mathfrak{s}_i(t)$$

where D_i is the duration of bond i.

(a) We obtain the following results:

Bond	$x_i^{(1)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	10	0.022	0.221	14.8%
2	12	0.012	0.147	9.8%
3	8	0.066	0.526	35.2%
4	7	0.086	0.602	40.2%
$\mathcal{R}(x^{(1)})$			1.495	

(b) We obtain the following results:

Bond	$x_i^{(2)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	8	0.023	0.186	6.6%
2	8	0.010	0.084	3.0%
3	12	0.065	0.785	27.6%
4	14	0.128	1.788	62.9%
$\mathcal{R}\left(x^{(2)}\right)$			2.843	

(c) The portfolio is now composed by two bonds per country, i.e. eight bonds. The correlation matrix becomes:

$$\rho = \left(\begin{array}{cccc|cccc}
1.00 & 0.65 & 0.67 & 0.64 & 1.00 & 0.65 & 0.67 & 0.64 \\
0.65 & 1.00 & 0.70 & 0.67 & 0.65 & 1.00 & 0.70 & 0.67 \\
0.67 & 0.70 & 1.00 & 0.83 & 0.67 & 0.70 & 1.00 & 0.83 \\
0.64 & 0.67 & 0.83 & 1.00 & 0.64 & 0.67 & 0.83 & 1.00 \\
\hline
1.00 & 0.65 & 0.67 & 0.64 & 1.00 & 0.65 & 0.67 & 0.64 \\
0.65 & 1.00 & 0.70 & 0.67 & 0.65 & 1.00 & 0.70 & 0.67 \\
0.67 & 0.70 & 1.00 & 0.83 & 0.67 & 0.70 & 1.00 & 0.83 \\
0.64 & 0.67 & 0.83 & 1.00 & 0.64 & 0.67 & 0.83 & 1.00
\end{array}\right)$$

We obtain the following results:

Bond	$x_i^{(1+2)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1 (#1)	10	0.021	0.210	4.8%
2 (#1)	12	0.012	0.141	3.3%
3 (#1)	8	0.065	0.520	12.0%
4 (#1)	7	0.088	0.618	14.3%
1 (#2)	8	0.024	0.192	4.4%
2 (#2)	8	0.011	0.086	2.0%
3 (#2)	12	0.066	0.793	18.3%
4 (#2)	14	0.126	1.769	40.9%
$\mathcal{R}\left(x^{(1+2)}\right)$			4.329	

We notice that $\mathcal{R}\left(x^{(1+2)}\right) \simeq \mathcal{R}\left(x^{(1)}\right) + \mathcal{R}\left(x^{(2)}\right)$. The diversification effect is limited because the two portfolios $x^{(1)}$ and $x^{(2)}$ are highly correlated:

$$\rho\left(x^{(1)}, x^{(2)}\right) = 99.01\%$$

(d) The notional of the meta-bond for the country i is the sum of notional of the two bonds, which belong to this country:

$$x_i^{(3)} = x_i^{(1)} + x_i^{(2)}$$

Its duration is the weighted average:

$$D_i^{(3)} = \frac{x_i^{(1)}}{x_i^{(1)} + x_i^{(2)}} D_i^{(1)} + \frac{x_i^{(2)}}{x_i^{(1)} + x_i^{(2)}} D_i^{(2)}$$

We obtain the following results:

Bond	$x_i^{(3)}$	$D_i^{(3)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	18	6.600	0.022	0.402	9.3%
2	20	7.260	0.011	0.227	5.2%
3	20	6.460	0.066	1.313	30.3%
4	21	7.467	0.114	2.387	55.1%
$\mathcal{R}\left(x^{(3)}\right)$			4.329		

Let us consider the results of the previous question. If we aggregate the risk contributions by countries, we obtain:

Country	$x_j^{(1+2)}$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
France	18	0.402	9.3%
Germany	20	0.227	5.2%
Italy	20	1.313	30.3%
Spain	21	2.387	55.1%

We notice that we have exactly the same results. We can postulate the hypothesis that computing the risk contributions based on the portfolio in its entirety is equivalent to compute the risk contributions by considering one meta-bond by country.

(e) The notional invested in each country is very close to 20 billions of dollars. However, most of the credit risk in concentrated in Italian and Spanish bonds. Indeed, these two countries represent about 85% of the credit risk of the portfolio.

2. (a) The last two assets are perfectly correlated. It means that:

$$\rho_{i,n} = \rho_{i,n+1}$$

If $i < n$, we have:

$$\begin{aligned} \mathcal{RC}_i(x) &= \frac{x_i \sigma_i}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j \rho_{i,j} \sigma_j + x_n \rho_{i,n} \sigma_n + x_{n+1} \rho_{i,n+1} \sigma_{n+1} \right) \\ &= \frac{x_i \sigma_i}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j \rho_{i,j} \sigma_j + \rho_{i,n} \left(x_n \sigma_n + x_{n+1} \sigma_{n+1} \right) \right) \end{aligned}$$

For the last two assets, we have:

$$\mathcal{RC}_n(x) = \frac{x_n \sigma_n}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j \rho_{n,j} \sigma_j + \left(x_n \sigma_n + x_{n+1} \sigma_{n+1} \right) \right)$$

and:

$$\mathcal{RC}_{n+1}(x) = \frac{x_{n+1}\sigma_{n+1}}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j \rho_{n,j} \sigma_j + (x_n\sigma_n + x_{n+1}\sigma_{n+1}) \right)$$

because $\rho_{n,j} = \rho_{n+1,j}$.

(b) We have:

$$\mathcal{RC}_i(y) = \frac{y_i\sigma'_i}{\sigma(y)} \left(\sum_{j=1}^{n-1} y_j \rho'_{i,j} \sigma'_j + y_n \rho'_{i,n} \sigma'_n \right)$$

(c) By construction, we have:

$$\begin{array}{rcl} \sigma(x) & = & \sum_{i=1}^{n-1} \mathcal{RC}_i(x) + \mathcal{RC}_n(x) + \mathcal{RC}_{n+1}(x) \\ & = & \sum_{i=1}^{n-1} \mathcal{RC}_i(y) + \mathcal{RC}_n(y) \\ & = & \sigma(y) \end{array}$$

For $i < n$, $\mathcal{RC}_i(y) = \mathcal{RC}_i(x)$ implies then:

$$\left\{ \begin{array}{l} y_i\sigma'_i = x_i\sigma_i \\ y_j\rho'_{i,j}\sigma'_j = x_j\rho_{i,j}\sigma_j \\ y_n\rho'_{i,n}\sigma'_n = \rho_{i,n}(x_n\sigma_n + x_{n+1}\sigma_{n+1}) \end{array} \right. \tag{3.3}$$

If the previous constraints are verified and if we assume that $\rho'_{i,n} = \rho_{i,n}$, we deduce that the restriction $\mathcal{RC}_n(y) = \mathcal{RC}_n(x) + \mathcal{RC}_{n+1}(x)$ is satisfied too:

$$\begin{array}{rcl} S & = & \mathcal{RC}_n(x) + \mathcal{RC}_{n+1}(x) \\ & = & \frac{x_n\sigma_n}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j\rho_{n,j}\sigma_j + (x_n\sigma_n + x_{n+1}\sigma_{n+1}) \right) + \\ & & \frac{x_{n+1}\sigma_{n+1}}{\sigma(x)} \left(\sum_{j=1}^{n-1} x_j\rho_{n,j}\sigma_j + (x_n\sigma_n + x_{n+1}\sigma_{n+1}) \right) \\ & = & \left(\frac{x_n\sigma_n}{\sigma(x)} + \frac{x_{n+1}\sigma_{n+1}}{\sigma(x)} \right) \left(\sum_{j=1}^{n-1} y_j\rho'_{n,j}\sigma'_j + \frac{\rho'_{i,n}}{\rho_{i,n}} y_n\sigma'_n \right) \\ & = & \frac{y_n\sigma'_n}{\sigma(y)} \left(\sum_{j=1}^{n-1} y_j\rho'_{n,j}\sigma'_j + y_n\sigma'_n \right) \\ & = & \mathcal{RC}_n(y) \end{array}$$

A solution of the system (3.3) is:

$$\begin{cases} y_i = x_i \\ \sigma'_i = \sigma_i \\ \rho'_{i,j} = \rho_{i,j} \\ y_n \sigma'_n = (x_n \sigma_n + x_{n+1} \sigma_{n+1}) \end{cases}$$

In fact, the asset universe and the portfolio are the same for $i < n$. The only change concerns the n-th asset.

(d) It suffices to choose an arbitrary value for σ'_n and we have:

$$y_n = \frac{x_n \sigma_n + x_{n+1} \sigma_{n+1}}{\sigma'_n}$$

It implies that there are infinite solutions. If we set $y_n = x_n + x_{n+1}$, we obtain:

$$\sigma'_n = \frac{x_n}{x_n + x_{n+1}} \sigma_n + \frac{x_{n+1}}{x_n + x_{n+1}} \sigma_{n+1}$$

In this case, the volatility σ'_n of the n-th asset is a weighted average of the volatilities of the last two assets. If $\sigma'_n = \sigma_n + \sigma_{n+1}$, we obtain:

$$y_n = x_n \frac{\sigma_n}{\sigma_n + \sigma_{n+1}} + x_{n+1} \frac{\sigma_{n+1}}{\sigma_n + \sigma_{n+1}}$$

The weight y_n of the n-th asset is a weighted average of the weights of the last two assets. From a financial point of view, we prefer the first solution $y_n = x_n + x_{n+1}$, because the exposures of the portfolio y are coherent with the exposures of the portfolio x.

(e) We obtain the following results (expressed in %) for portfolio x:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	8.82	1.76	11.60
2	30.00	14.85	4.46	29.28
3	10.00	18.40	1.84	12.09
4	10.00	23.86	2.39	15.68
5	30.00	15.90	4.77	31.35
$\sigma(x)$			15.22	

If we consider the first solution, we have:

Asset	y_i	σ'_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	15.00	8.82	1.76	11.60
2	30.00	20.00	14.85	4.46	29.28
3	10.00	25.00	18.40	1.84	12.09
4′	40.00	22.50	17.89	7.16	47.03
$\sigma(y)$			15.22		

For the second solution, we obtain:

Asset	y_i	σ'_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}^\star_i$
1	20.00	15.00	8.82	1.76	11.60
2	30.00	20.00	14.85	4.46	29.28
3	10.00	25.00	18.40	1.84	12.09
4′	18.00	50.00	39.76	7.16	47.03
$\sigma(y)$			15.22		

Finally, if we impose $y_4 = 80\%$, we have:

Asset	y_i	σ'_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}^\star_i$
1	20.00	15.00	8.82	1.76	11.60
2	30.00	20.00	14.85	4.46	29.28
3	10.00	25.00	18.40	1.84	12.09
4′	80.00	11.25	8.95	7.16	47.03
$\sigma(y)$			15.22		

(f) The previous analysis shows that considering a meta-bond by country is equivalent to consider the complete universe of individual bonds.

3. To find the RB portfolios y, we proceed in two steps. First, we calculate the normalized portfolio $\tilde{y}$ such that the weights are equal to 1. For that, we optimize the objective function (TR-RPB, page 102):

$$\begin{aligned} \tilde{y} &= \arg\min \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\tilde{y}_i\cdot(\Sigma\tilde{y})_i}{b_i}-\frac{\tilde{y}_j\cdot(\Sigma\tilde{y})_j}{b_j}\right)^2 \\ &\text{u.c.}\quad \mathbf{1}^\top\tilde{y}=1 \text{ and } \tilde{y}\geq\mathbf{0} \end{aligned}$$

Then, we deduce the portfolio y in the following way:

$$y = \left(\sum x_i\right)\cdot\tilde{y}$$

(a) We obtain the following results:

Bond	$y_i^{(1)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}^\star_i$
1	9.95	0.023	0.229	20.0%
2	17.62	0.013	0.229	20.0%
3	5.31	0.065	0.344	30.0%
4	4.12	0.083	0.344	30.0%
$\mathcal{R}\left(y^{(1)}\right)$			1.145	

(b) We obtain the following results:

Bond	$y_i^{(2)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	10.15	0.026	0.268	20.0%
2	22.37	0.012	0.268	20.0%
3	6.11	0.066	0.401	30.0%
4	3.37	0.119	0.401	30.0%
$\mathcal{R}\left(y^{(2)}\right)$			1.338	

(c) We obtain the following results:

Bond	$y_i^{(1+2)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1 (#1)	9.95	0.023	0.229	9.2%
2 (#1)	17.62	0.013	0.229	9.2%
3 (#1)	5.31	0.065	0.344	13.8%
4 (#1)	4.12	0.083	0.344	13.8%
1 (#2)	10.15	0.026	0.268	10.8%
2 (#2)	22.37	0.012	0.268	10.8%
3 (#2)	6.11	0.066	0.401	16.2%
4 (#2)	3.37	0.119	0.401	16.2%
$\mathcal{R}\left(y^{(1+2)}\right)$			2.483	

If we aggregate the exposures by country, we have:

Country	$y_j^{(1+2)}$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
France	20.10	0.497	20.0%
Germany	39.99	0.497	20.0%
Italy	11.42	0.745	30.0%
Spain	7.49	0.745	30.0%

(d) The optimization program becomes:

$$\begin{aligned}
\tilde{y} &= \arg\min \sum_{i=1}^{4}\sum_{j=1}^{4}\left(\frac{\tilde{b}_i}{b_i}-\frac{\tilde{b}_j}{b_j}\right)^2 \\
&\text{u.c.} \quad \left\{\begin{array}{l}
\tilde{b}_i = \tilde{y}_i \cdot (\Sigma\tilde{y})_i + \tilde{y}_{i+4}\cdot(\Sigma\tilde{y})_{i+4} \\
\mathbf{1}^\top \tilde{y} = 1 \\
\tilde{y} \geq \mathbf{0}
\end{array}\right.
\end{aligned}$$

In fact, we have specified four constraints on the risk contributions (one by country), but we have eight unknown variables. It is obvious that there are several solutions. The problem with the meta-bonds is that their characteristics depend on the weights of the portfolio. For instance, in Question 1(c), we have specified the duration of the meta-bond i as follows:

$$D_i^{(3)} = \frac{x_i^{(1)}D_i^{(1)} + x_i^{(2)}D_i^{(2)}}{x_i^{(1)} + x_i^{(2)}} \tag{3.4}$$

If we assume that:

$$D_i^{(3)} = \frac{\tilde{y}_i^{(1)} D_i^{(1)} + \tilde{y}_i^{(2)} D_i^{(2)}}{\tilde{y}_i^{(1)} + \tilde{y}_i^{(2)}} \tag{3.5}$$

we see that we face an endogeneity problem because the duration of the meta-bond depends on the solution of the RB optimization problem. Nevertheless, the analysis conducted in Question 2(d) helps us to propose a practical solution. The idea is to specify the meta-bonds using Equation (3.4) and not Equation (3.5). It means that we maintain the same proportion of individual bonds in the portfolio $\tilde{y}$ than previously. In this case, we obtain the following results:

Country	$y_j^{(4)}$	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
France	20.51	0.025	0.502	20.0%
Germany	39.93	0.013	0.502	20.0%
Italy	11.54	0.065	0.754	30.0%
Spain	7.02	0.107	0.754	30.0%
$\mathcal{R}\left(y^{(4)}\right)$			2.512	

To obtain the allocation in terms of bonds, we define the portfolio $y^{(5)}$ as follows:

$$y^{(5)} = \begin{bmatrix} w \circ y^{(4)} \\ (1 - w) \circ y^{(4)} \end{bmatrix}$$

where w is the vector such that:

$$w_i = \frac{x_i^{(1)}}{x_i^{(1)} + x_i^{(2)}}$$

This allocation principle is derived from the specification of the meta-bonds. Finally, we obtain:

Bond	$y_i^{(5)}$	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1 (#1)	11.39	0.023	0.262	10.4%
2 (#1)	23.96	0.013	0.311	12.4%
3 (#1)	4.62	0.065	0.299	11.9%
4 (#1)	2.34	0.083	0.195	7.8%
1 (#2)	9.11	0.026	0.240	9.6%
2 (#2)	15.97	0.012	0.191	7.6%
3 (#2)	6.93	0.066	0.455	18.1%
4 (#2)	4.68	0.119	0.559	22.2%
$\mathcal{R}\left(y^{(5)}\right)$			2.512	

This solution is different from the previous merged portfolio. We have then found at least two solutions. By the property of the Euler allocation, it implies that every combination of these two solutions is also another solution. It means that there are infinite solutions to this problem.

3.7 Risk contributions of long-short portfolios

1. (a) We have:

$$\sigma(x) = \sqrt{x_1^2\sigma_1^2 + 2x_1x_2\rho\sigma_1\sigma_2 + x_2^2\sigma_2^2}$$

It follows that:

$$\frac{\partial\,\sigma(x)}{\partial\,x_1} = \frac{x_1\sigma_1^2 + x_2\rho\sigma_1\sigma_2}{\sigma(x)}$$

Finally, we get:

$$\mathcal{RC}_1 = \frac{x_1^2\sigma_1^2 + x_1x_2\rho\sigma_1\sigma_2}{\sigma(x)}$$

and:

$$\mathcal{RC}_2 = \frac{x_2^2\sigma_2^2 + x_1x_2\rho\sigma_1\sigma_2}{\sigma(x)}$$

(b) The inequality $\mathcal{RC}_2 \leq 0$ implies that:

$$\begin{aligned} & \frac{x_2^2\sigma_2^2 + x_1x_2\rho\sigma_1\sigma_2}{\sigma(x)} \leq 0 \\ \Leftrightarrow\quad & x_2^2\sigma_2^2 + x_1x_2\rho\sigma_1\sigma_2 \leq 0 \\ \Leftrightarrow\quad & x_2\left(x_2\sigma_2^2 + x_1\rho\sigma_1\sigma_2\right) \leq 0 \\ \Leftrightarrow\quad & \left\{\begin{array}{lll} x_2 \leq 0 & \text{and} & x_2 \geq -x_1\rho\sigma_1/\sigma_2 \\ x_2 \geq 0 & \text{and} & x_2 \leq -x_1\rho\sigma_1/\sigma_2 \end{array}\right. \end{aligned}$$

We distinguish two cases. If $x_1\rho < 0$, we get $x_2 \in [0, -x_1\rho\sigma_1/\sigma_2]$. If $x_1\rho > 0$, the solution set becomes $[-x_1\rho\sigma_1/\sigma_2, 0]$. We notice that if the risk contribution is negative for a negative (resp. positive) value of x_2, it cannot be negative for a positive (resp. negative) value of x_2.

(c) The relationship between x_2 and $\mathcal{RC}_2$ is reported in Figure 3.4.

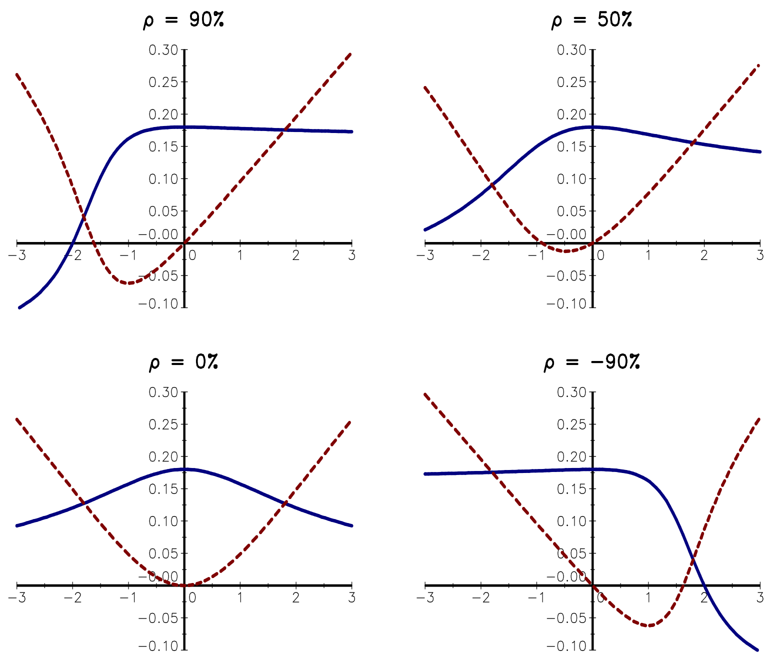

FIGURE 3.4: Risk contribution $\mathcal{RC}_2$

2. (a) We have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	100.00%	4.73%	4.73%	10.66%
2	−100.00%	−3.94%	3.94%	8.88%
3	100.00%	12.50%	12.50%	28.17%
4	−100.00%	−20.50%	20.50%	46.19%
5	100.00%	1.01%	1.01%	2.28%
6	−100.00%	−1.69%	1.69%	3.81%
$\sigma(x)$			44.38%	

(b) The covariance matrix of the long-short assets is $A\Sigma A^\top$ with:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

We deduce that the volatilities of the long-short assets are 20.62%, 38.67% and 10.95% whereas the correlation matrix is:

$$C = \begin{pmatrix} 100.00\% & & \\ -4.39\% & 100.00\% & \\ -2.21\% & 1.18\% & 100\% \end{pmatrix}$$

Finally, the risk decomposition is:

L/S asset	y_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	100.00%	8.67%	8.67%	19.54%
2	100.00%	33.01%	33.01%	74.37%
3	100.00%	2.70%	2.70%	6.09%
$\sigma(y)$			44.38%	

(c) The ERC portfolio is:

L/S asset	y_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	29.81%	11.51%	3.43%	33.33%
2	15.63%	21.95%	3.43%	33.33%
3	54.56%	6.29%	3.43%	33.33%
$\sigma(y)$			10.29%	

If we don't take into account the correlations, we obtain a similar portfolio:

L/S asset	y_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	29.28%	11.90%	3.49%	33.33%
2	15.61%	22.32%	3.49%	33.33%
3	55.11%	6.32%	3.49%	33.33%
$\sigma(y)$			10.46%	

(d) The risk allocation with respect to the six assets is:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	29.81%	5.32%	1.59%	15.41%
2	−29.81%	−6.19%	1.84%	17.92%
3	15.63%	7.29%	1.14%	11.07%
4	−15.63%	−14.66%	2.29%	22.26%
5	54.56%	2.88%	1.57%	15.28%
6	−54.56%	−3.41%	1.86%	18.06%
$\sigma(x)$			10.29%	

We notice that we do not have $\mathcal{RC}_1 = \mathcal{RC}_2$, $\mathcal{RC}_3 = \mathcal{RC}_4$ and $\mathcal{RC}_5 = \mathcal{RC}_6$.

(e) A first route is to perform an optimization by tacking into account the weight constraints $x_1 x_2 < 0$, $x_3 x_4 < 0$, $x_5 x_6 < 0$. Nevertheless, if there is a solution x to this problem, the portfolio $y = \alpha x$ is also a solution (TR-RPB, page 256). This is why we need to impose that:

$$\mathcal{RC}_1 = c$$

with c a positive scalar. For example, if $c = 4\%$, we obtain the following portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	64.77%	5.91%	3.82%	17.09%
2	−64.77%	−5.66%	3.67%	16.39%
3	38.90%	9.48%	3.69%	16.47%
4	−30.42%	−12.36%	3.76%	16.80%
5	115.20%	2.88%	3.32%	14.84%
6	−123.68%	−3.33%	4.12%	18.41%
$\sigma(x)$			22.38%	

We notice that this portfolio does not match the constraints. In fact, this optimization problem is tricky from a numerical point of view. A second route consists in using the following parametrization $y = Ax$ with:

$$A = \operatorname{diag}(1, -1, 1, -1, 1, -1)$$

We have then transformed the assets and the new covariance matrix is $A\Sigma A^\top$. We can then compute the ERC portfolio y and deduce the long-short portfolio $x = A^{-1}y$. In this case, we obtain a long-short portfolio that matches all the constraints:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	14.29%	5.95%	0.85%	16.67%
2	−15.03%	−5.66%	0.85%	16.67%
3	8.94%	9.51%	0.85%	16.67%
4	−6.96%	−12.23%	0.85%	16.67%
5	27.68%	3.07%	0.85%	16.67%
6	−27.10%	−3.14%	0.85%	16.67%
$\sigma(x)$			5.11%	

Note that solution x is not unique and every portfolio of the form $y = \alpha x$ is also a solution.

3.8 Risk parity funds

1. (a) The RP portfolio is defined as follows:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.89%	18.75%	38.35%	23.57%	18.07%	22.63%
B	52.81%	52.71%	43.60%	55.45%	61.35%	55.02%
C	23.29%	28.54%	18.05%	20.98%	20.58%	22.36%
$\sigma(x)$	4.83%	6.08%	6.26%	5.51%	11.64%	8.38%

(b) In the ERC portfolio, the risk contributions are equal for all the assets:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

with:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \tag{3.6}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.66%	18.18%	37.85%	23.28%	17.06%	20.33%
B	53.12%	58.64%	43.18%	59.93%	66.39%	59.61%
C	23.22%	23.18%	18.97%	16.79%	16.54%	20.07%
$\sigma(x)$	4.82%	5.70%	6.32%	5.16%	10.77%	7.96%

(c) We notice that $\sigma(x_{\mathrm{erc}}) \leq \sigma(x_{\mathrm{rp}})$ except for the year 2005. This date corresponds to positive correlations between assets. Moreover, the correlation between stocks and bonds is the highest. Starting from the RP portfolio, it is then possible to approach the ERC portfolio by reducing the weights of stocks and bonds and increasing the weight of commodities. At the end, we find an ERC portfolio that has a slightly higher volatility.

(d) The volatility of the RP portfolio is:

$$\begin{aligned}
\sigma(x) &= \frac{1}{\sum_{j=1}^n \sigma_j^{-1}} \sqrt{\left(\sigma^{-1}\right)^\top \Sigma \sigma^{-1}} \\
&= \frac{1}{\sum_{j=1}^n \sigma_j^{-1}} \sqrt{\sum_{i=1}^n \sum_{j=1}^n \frac{1}{\sigma_i \sigma_j} \rho_{i,j} \sigma_i \sigma_j} \\
&= \frac{1}{\sum_{j=1}^n \sigma_j^{-1}} \sqrt{n + 2 \sum_{i>j} \rho_{i,j}} \\
&= \frac{1}{\sum_{j=1}^n \sigma_j^{-1}} \sqrt{n\left(1 + (n-1)\bar{\rho}\right)}
\end{aligned}$$

where $\bar{\rho}$ is the average correlation between asset returns. For the marginal risk, we obtain:

$$\begin{aligned}
\mathcal{MR}_i &= \frac{\left(\Sigma\sigma^{-1}\right)_i}{\sigma(x)\sum_{j=1}^n \sigma_j^{-1}} \\
&= \frac{1}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j\frac{1}{\sigma_j} \\
&= \frac{\sigma_i}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j} \\
&= \frac{\sigma_i\bar{\rho}_i\sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}}
\end{aligned}$$

where $\bar{\rho}_i$ is the average correlation of asset i with the other assets (including itself). The expression of the risk contribution is then:

$$\begin{aligned}
\mathcal{RC}_i &= \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}\frac{\sigma_i\bar{\rho}_i\sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}} \\
&= \frac{\bar{\rho}_i\sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}\sum_{j=1}^n \sigma_j^{-1}}
\end{aligned}$$

We deduce that the normalized risk contribution is:

$$\begin{aligned}
\mathcal{RC}_i^\star &= \frac{\bar{\rho}_i\sqrt{n}}{\sigma(x)\sqrt{1+(n-1)\bar{\rho}}\sum_{j=1}^n \sigma_j^{-1}} \\
&= \frac{\bar{\rho}_i}{1+(n-1)\bar{\rho}}
\end{aligned}$$

(e) We obtain the following normalized risk contributions:

Date	1999	2002	2005	2007	2008	2010
S	33.87%	34.96%	34.52%	32.56%	34.45%	36.64%
B	32.73%	20.34%	34.35%	24.88%	24.42%	26.70%
C	33.40%	44.69%	31.14%	42.57%	41.13%	36.67%

We notice that the risk contributions are not exactly equal for all the assets. Generally, the risk contribution of bonds is lower than the risk contribution of equities, which is itself lower than the risk contribution of commodities.

2. (a) We obtain the following RB portfolios:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	26.83%	22.14%	42.83%	27.20%	20.63%	25.92%
B	45%	59.78%	66.10%	48.77%	66.15%	73.35%	67.03%
C	10%	13.39%	11.76%	8.40%	6.65%	6.02%	7.05%
S	70%	40.39%	29.32%	65.53%	39.37%	33.47%	46.26%
B	10%	37.63%	51.48%	19.55%	47.18%	52.89%	37.76%
C	20%	21.98%	19.20%	14.93%	13.45%	13.64%	15.98%
S	20%	17.55%	16.02%	25.20%	18.78%	12.94%	13.87%
B	70%	69.67%	71.70%	66.18%	74.33%	80.81%	78.58%
C	10%	12.78%	12.28%	8.62%	6.89%	6.24%	7.55%
S	25%	21.69%	15.76%	34.47%	20.55%	14.59%	16.65%
B	25%	48.99%	54.03%	39.38%	55.44%	61.18%	53.98%
C	50%	29.33%	30.21%	26.15%	24.01%	24.22%	29.37%

(b) To compute the implied risk premium $\tilde{\pi}_i$, we use the following formula (TR-RPB, page 274):

$$\begin{aligned}\tilde{\pi}_i &= \operatorname{SR}(x \mid r) \cdot \mathcal{MR}_i \\ &= \operatorname{SR}(x \mid r) \cdot \frac{(\Sigma x)_i}{\sigma(x)}\end{aligned}$$

where $\operatorname{SR}(x \mid r)$ is the Sharpe ratio of the portfolio. We obtain the following results:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	3.19%	4.60%	2.49%	3.15%	8.64%	5.20%
B	45%	1.43%	1.54%	2.19%	1.29%	2.43%	2.01%
C	10%	1.42%	1.92%	2.82%	2.86%	6.58%	4.24%
S	70%	4.05%	6.45%	2.86%	4.31%	11.56%	6.32%
B	10%	0.62%	0.52%	1.37%	0.51%	1.04%	1.11%
C	20%	2.13%	2.81%	3.59%	3.61%	8.11%	5.23%
S	20%	2.06%	2.68%	1.91%	1.93%	5.61%	3.91%
B	70%	1.82%	2.10%	2.54%	1.71%	3.14%	2.42%
C	10%	1.42%	1.75%	2.79%	2.64%	5.82%	3.60%
S	25%	2.33%	3.78%	1.98%	2.74%	8.06%	5.13%
B	25%	1.03%	1.10%	1.74%	1.02%	1.92%	1.58%
C	50%	3.45%	3.95%	5.23%	4.69%	9.71%	5.82%

(c) We have:

$$x_i \tilde{\pi}_i = \operatorname{SR}(x \mid r) \cdot \mathcal{RC}_i$$

We deduce that:

$$\tilde{\pi}_i \propto \frac{b_i}{x_i}$$

x_i is generally an increasing function of b_i. As a consequence, the relationship between the risk budgets b_i and the risk premiums $\tilde{\pi}_i$ is not necessarily increasing. However, we notice that the bigger the risk budget, the higher the risk premium. This is easily explained. If an investor allocates more risk budget to one asset class than another investor, he thinks that the risk premium of this asset class is higher than the other investor. However, we must be careful. This interpretation is valid if we compare two sets of risk budgets. It is false if we compare the risk budgets among themselves. For instance, if we consider the third parameter set, the risk budget of bonds is 70% whereas the risk budget of stocks is 20%. It does not mean that the risk premium of bonds is higher than the risk premium of equities. In fact, we observe the contrary. If we would like to compare risk budgets among themselves, the right measure is the implied Sharpe ratio, which is equal to:

$$\begin{aligned} \mathrm{SR}_i &= \frac{\tilde{\pi}_i}{\sigma_i} \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \frac{\mathcal{MR}_i}{\sigma_i} \end{aligned}$$

For instance, if we consider the most diversified portfolio, the marginal risk is proportional to the volatility and we retrieve the result that Sharpe ratios are equal if the MDP is optimal.

3.9 Frazzini-Pedersen model

1. (a) The Lagrange function of the optimization problem is:

$$\begin{aligned} \mathcal{L}\left(x;\lambda\right) &= x_j^\top \mathbb{E}_t\left[P_{t+1} + D_{t+1} - \left(1+r\right)P_t\right] - \frac{\phi_j}{2} x_j^\top \Sigma x_j - \\ &\quad \lambda_j \left(m_j \left(x_j^\top P_t\right) - W_j\right) \end{aligned}$$

where λ_j is the Lagrange multiplier associated with the constraint $m_j\left(x_j^\top P_t\right) \leq W_j$. The solution x_j then verifies the first-order condition:

$$\partial_x \mathcal{L}\left(x;\lambda\right) = \mathbb{E}_t\left[P_{t+1} + D_{t+1} - \left(1+r\right)P_t\right] - \phi_j \Sigma x_j - \lambda_j m_j P_t = 0$$

We deduce that:

$$x_j = \frac{1}{\phi_j}\Sigma^{-1}\left(\mathbb{E}_t\left[P_{t+1} + D_{t+1}\right] - \left(1 + r + \lambda_j m_j\right)P_t\right)$$

(b) At the equilibrium, we have:

$$\begin{aligned}
\bar{x} &= \sum_{j=1}^{m} \frac{1}{\phi_j} \Sigma^{-1} \left(\mathbb{E}_t \left[P_{t+1} + D_{t+1}\right] - \left(1 + r + \lambda_j m_j\right) P_t\right) \\
&= \sum_{j=1}^{m} \frac{1}{\phi_j} \Sigma^{-1} \mathbb{E}_t \left[P_{t+1} + D_{t+1}\right] - \sum_{j=1}^{m} \frac{1}{\phi_j} \left(1 + r + \lambda_j m_j\right) \Sigma^{-1} P_t \\
&= \left(\sum_{j=1}^{m} \phi_j^{-1}\right) \Sigma^{-1} \mathbb{E}_t \left[P_{t+1} + D_{t+1}\right] - \\
&\quad \left(\sum_{j=1}^{m} \phi_j^{-1}\right) \sum_{j=1}^{m} \frac{1}{\left(\sum_{k=1}^{m} \phi_k^{-1}\right) \phi_j} \left(1 + r + \lambda_j m_j\right) P_t
\end{aligned}$$

Frazzini and Pedersen (2010) introduce the notations:

$$\phi = \frac{1}{\left(\sum_{j=1}^{m} \phi_j^{-1}\right)}$$

and:

$$\begin{aligned}
\psi &= \sum_{j=1}^{m} \frac{1}{\left(\sum_{k=1}^{m} \phi_k^{-1}\right) \phi_j} \lambda_j m_j \\
&= \sum_{j=1}^{m} \phi \phi_j^{-1} \lambda_j m_j
\end{aligned}$$

We finally obtain:

$$\bar{x} = \frac{1}{\phi} \Sigma^{-1} \left(\mathbb{E}_t \left[P_{t+1} + D_{t+1}\right] - \left(1 + r + \psi\right) P_t\right)$$

because:

$$\begin{aligned}
\sum_{j=1}^{m} \frac{1}{\left(\sum_{k=1}^{m} \phi_k^{-1}\right) \phi_j} &= \sum_{j=1}^{m} \frac{\phi}{\phi_j} \\
&= \phi \left(\sum_{j=1}^{m} \phi_j^{-1}\right) \\
&= 1
\end{aligned}$$

(c) The equilibrium prices are then given by:

$$P_t = \frac{\mathbb{E}_t \left[P_{t+1} + D_{t+1}\right] - \phi \Sigma \bar{x}}{1 + r + \psi}$$

It follows that:

$$P_{i,t} = \frac{\mathbb{E}_t\left[P_{i,t+1} + D_{i,t+1}\right] - \phi\left(\Sigma\bar{x}\right)_i}{1 + r + \psi}$$

(d) Following Frazzini and Pedersen (2010), the asset return $R_{i,t+1}$ satisfies the following equation:

$$\begin{aligned}
\mathbb{E}_t\left[R_{i,t+1}\right] &= \frac{\mathbb{E}_t\left[P_{i,t+1} + D_{i,t+1}\right]}{P_{i,t}} - 1 \\
&= r + \psi + \frac{\phi}{P_{i,t}}\left(\Sigma\bar{x}\right)_i
\end{aligned}$$

We know that:

$$\begin{aligned}
\left(\Sigma\bar{x}\right)_i &= \operatorname{cov}\left(P_{i,t+1} + D_{i,t+1}, \bar{x}^\intercal\left(P_{t+1} + D_{t+1}\right)\right) \\
&= P_{i,t} \cdot \operatorname{cov}\left(R_{i,t+1}, R_{t+1}\left(\bar{x}\right)\right) \cdot \left(\bar{x}^\intercal P_t\right)
\end{aligned}$$

Let $\bar{w}_i = \left(\bar{x}_i P_{i,t}\right) / \left(\bar{x}^\intercal P_t\right)$ be the weight of asset i in the market portfolio. It follows that:

$$\begin{aligned}
\mathbb{E}_t\left[R_{t+1}\left(\bar{x}\right)\right] &= \sum_{j=1}^{m} \bar{w}_j \mathbb{E}_t\left[R_{i,t+1}\right] \\
&= r + \psi + \phi\left(\bar{x}^\intercal P_t\right)\sigma^2\left(\bar{x}\right)
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathbb{E}_t\left[R_{i,t+1}\right] &= r + \psi + \phi\beta_i\sigma^2\left(\bar{x}\right)\left(\bar{x}^\intercal P_t\right) \\
&= r + \psi + \beta_i\left(\mathbb{E}_t\left[R_{t+1}\left(\bar{x}\right)\right] - r - \psi\right) \\
&= r + \psi\left(1 - \beta_i\right) + \beta_i\left(\mathbb{E}_t\left[R_{t+1}\left(\bar{x}\right)\right] - r\right)
\end{aligned}$$

We finally obtain that:

$$\mathbb{E}_t\left[R_{i,t+1}\right] - r = \alpha_i + \beta_i\left(\mathbb{E}_t\left[R_{t+1}\left(\bar{x}\right)\right] - r\right)$$

where $\alpha_i = \psi\left(1 - \beta_i\right)$.

(e) In the CAPM, the traditional relationship between the risk premium and the beta of asset i is:

$$\mathbb{E}_t\left[R_{i,t+1}\right] - r = \beta_i\left(\mathbb{E}_t\left[R_{t+1}\left(\bar{x}\right)\right] - r\right)$$

If we compare this equation with the expression obtained by Frazzini and Pedersen (2010), we notice the presence of a new term α_i, which is Jensen's alpha. Moreover, α_i is a decreasing function of β_i.

(a) The optimal value of ϕ is (TR-RPB, page 14):

$$\phi = \mathbf{1}^\top \Sigma^{-1} (\mu - r\mathbf{1})$$

The tangency portfolio is then:

$$x^\star = \frac{1}{\phi} \Sigma^{-1} (\mu - r\mathbf{1})$$

The beta of asset i is defined as follows (TR-RPB, page 17):

$$\beta_i = \frac{\mathbf{e}_i \Sigma x^\star}{x^{\star\top} \Sigma x^\star}$$

We can also compute the beta component of the risk premium:

$$\pi (\mathbf{e}_i \mid x^\star) = \beta_i (\mu (x^\star) - r)$$

In our case, we obtain $\phi = 2.1473$. The composition of the portfolio is $(47.50\%, 19.83\%, 27.37\%, 5.30\%)$. We deduce that the expected return of the portfolio is $\mu (x^\star) = 6.07\%$. Finally, we obtain the following results:

Asset	$x_i^\star$	π_i	β_i	$\pi (\mathbf{e}_i \mid x^\star)$
1	47.50%	3.00%	0.737	3.00%
2	19.83%	4.00%	0.982	4.00%
3	27.37%	6.00%	1.473	6.00%
4	5.30%	4.00%	0.982	4.00%

We verify that:

$$\pi_i = \mu_i - r = \pi (\mathbf{e}_i \mid x^\star) = \beta_i (\mu (x^\star) - r)$$

(b) We obtain the following portfolio weights[5]:

Asset	$x_{i,1}$	$x_{i,2}$	$\bar{x}_i$
1	47.50%	15.82%	42.21%
2	19.83%	3.72%	15.70%
3	27.37%	27.09%	36.31%
4	5.30%	3.37%	5.78%

The corresponding Lagrange coefficients are $\lambda_1 = 0.9314\%$ and $\lambda_1 = 1.7178\%$. The expected return $\mu (\bar{x})$ of the market portfolio is 6.30%. Finally, we obtain the following results:

Asset	π_i	α_i	β_i	$\pi (\mathbf{e}_i \mid \bar{x})$	$\alpha_i + \beta_i (\mu (\bar{x}) - r)$
1	3.00%	0.32%	0.62	2.68%	3.00%
2	4.00%	0.07%	0.91	3.93%	4.00%
3	6.00%	−0.41%	1.49	6.41%	6.00%
4	4.00%	0.07%	0.91	3.93%	4.00%

[5]For the second investor, the risky assets only represent 50% of his wealth.

(c) The second investor has a cash constraint and invests only 50% of his wealth in risky assets. His portfolio is then highly exposed to the third asset. This implies that the market portfolio is overweighted in the third asset with respect to the tangency portfolio. This asset has then a negative alpha. At the opposite, the first asset has a positive alpha, because its beta is low and it is underweighted in the market portfolio.

3.10 Dynamic risk budgeting portfolios

1. (a) The optimization problem is:

$$\begin{aligned} x^\star &= \arg\max x^\top (\mu - r\mathbf{1}) - \frac{\phi}{2} x^\top \Sigma x \\ &\text{u.c.} \quad \mathbf{1}^\top x = 1 \end{aligned}$$

The first-order condition is:

$$\mu - r\mathbf{1} - \phi \Sigma x = 0$$

We have:

$$x = \frac{1}{\phi} \Sigma^{-1} (\mu - r\mathbf{1})$$

Finally, we obtain:

$$x^\star = c \cdot \Sigma^{-1} (\mu - r\mathbf{1})$$

with:

$$c = \frac{1}{\mathbf{1}^\top \Sigma^{-1} (\mu - r\mathbf{1})}$$

When the correlations are equal to zero, the optimal weights are proportional to the risk premium $\pi_i = \mu_i - r$ and inversely proportional to the variance σ_i^2 of asset returns:

$$x_i^\star = c \cdot \frac{(\mu_i - r)}{\sigma_i^2}$$

(b) Let $\sigma(x) = \sqrt{x^\top \Sigma x}$ be the volatility of the portfolio. We have:

$$\begin{aligned}
\mathcal{RC}_i &= x_i \cdot \frac{\partial\, \sigma(x)}{\partial\, x_i} \\
&= \frac{x_i \cdot (\Sigma x)_i}{\sigma(x)} \\
&= \frac{x_i \left(\sum_{j=1}^n x_j \rho_{i,j} \sigma_i \sigma_j\right)}{\sigma(x)} \\
&= \frac{x_i^2 \sigma_i^2 + x_i \sigma_i \left(\sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right)}{\sigma(x)}
\end{aligned}$$

(c) When the correlations are equal to zero, we obtain:

$$\mathcal{RC}_i = \frac{x_i^2 \sigma_i^2}{\sigma(x)}$$

The RB portfolio satisfies:

$$\frac{\mathcal{RC}_i}{b_i} = \frac{\mathcal{RC}_j}{b_j}$$

or:

$$\frac{x_i^2 \sigma_i^2}{b_i} = \frac{x_j^2 \sigma_j^2}{b_j}$$

We deduce that:

$$x_i \propto \frac{\sqrt{b_i}}{\sigma_i}$$

The RB portfolio is the tangency portfolio when the risk budgets are proportional to the square of the Sharpe ratios:

$$\begin{aligned}
b_i &\propto \left(\frac{\mu_i - r}{\sigma_i}\right)^2 \\
&= \frac{\pi_i^2}{\sigma_i^2}
\end{aligned}$$

2. (a) If $\alpha = \beta = \gamma = \delta = 0$, we get:

$$b_i(t) = b_i(\infty) = \frac{\pi_i^2(\infty)}{\sigma_i^2(\infty)}$$

The risk budgets at time t are equal to the long-run risk budgets. However, it does not mean that the allocation is static, because it will depend on the covariance matrix. If $\alpha = \beta = \gamma = \delta = 2$, we get:

$$b_i(t) = \frac{\pi_i^2(t)}{\sigma_i^2(t)}$$

These risk budgets correspond to those given by the tangency portfolio when the correlations are equal to zero.

(b) If the correlations are equal to zero, we have:

$$x_i(t) = \frac{\pi_i(\infty)}{\sigma_i^2(\infty)} \Leftrightarrow b_i(t) = \frac{\pi_i^2(\infty)\,\sigma_i^2(t)}{\sigma_i^4(\infty)}$$
$$\Leftrightarrow b_i(t) = b_i(\infty)\,\frac{\sigma_i^2(t)}{\sigma_i^2(\infty)}$$

It implies that $\alpha = \beta = 0$ and $\gamma = \delta = -2$.

(c) The long-run tangency portfolio is $x_1^\star(\infty) = 38.96\%$, $x_2^\star(\infty) = 25.97\%$, $x_3^\star(\infty) = 19.48\%$ and $x_4^\star(\infty) = 15.58\%$. The risk budgets $b_i(\infty)$ are all equal to 25% meaning that the long-run tangency portfolio is the ERC portfolio. The relationship between the parameters θ, the risk budgets $b_i(t)$ and the RB weights $x_i(t)$ is reported in Figure 3.5. The allocation at time t differs from the long-run allocation when the parameter θ increases. θ may then be viewed as a parameter that controls the relative weight of the tactical asset allocation with respect to the strategic asset (or long-run) allocation.

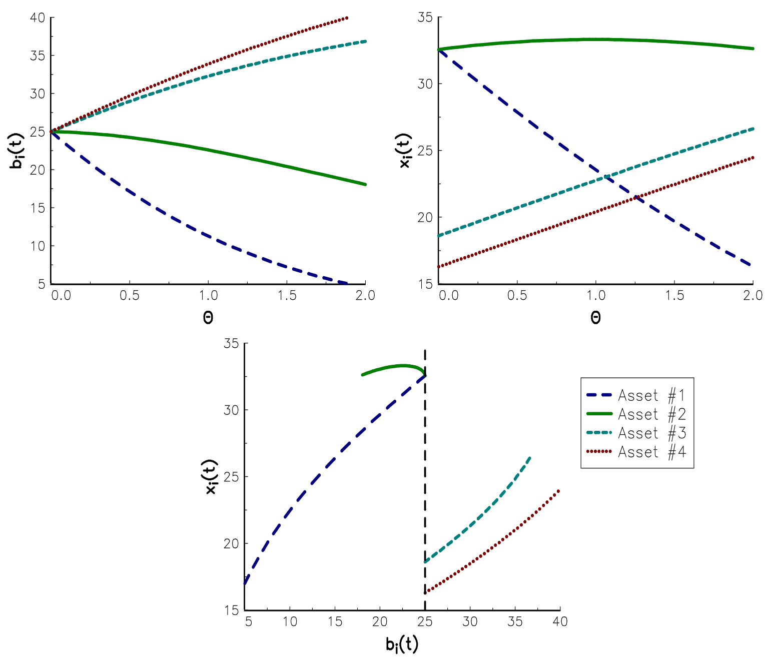

FIGURE 3.5: Relationship between θ, $b_i(t)$ and $x_i(t)$

3. (a) The backtests are reported in Figure 3.6. We notice that the risk budgets change from one period to another period in the case of the dynamic risk parity strategy. However, the weights are not so far from those obtained with the ERC strategy. We also observe that the performance is better for the dynamic risk parity strategy whereas it has the same volatility than the ERC strategy.

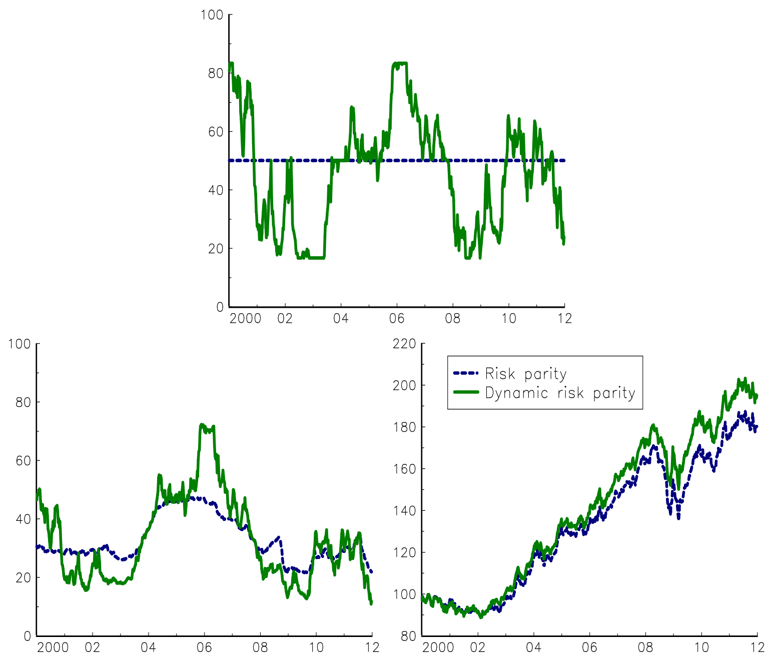

FIGURE 3.6: Simulation of the dynamic risk parity strategy

   (b) We have reported the evolution of equity weights in Figure 3.7. We notice that the tangency portfolio produces a higher turnover. Moreover, it is generally invested in only one asset class, either stocks or bonds.

   (c) Results are given in Figure 3.8.

   (d) We notice that the first simulation is based on the Sharpe ratio whereas the second simulation considers the risk premium. Nevertheless, the risk premium of stocks is not homogeneous to the risk premium of bonds. This is why it is better to use the Sharpe ratio.

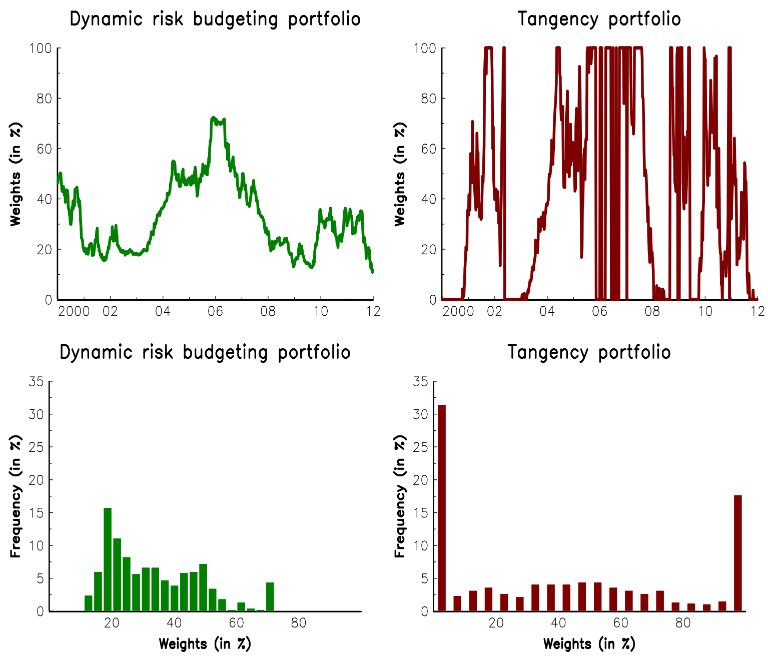

FIGURE 3.7: Comparison of the weights

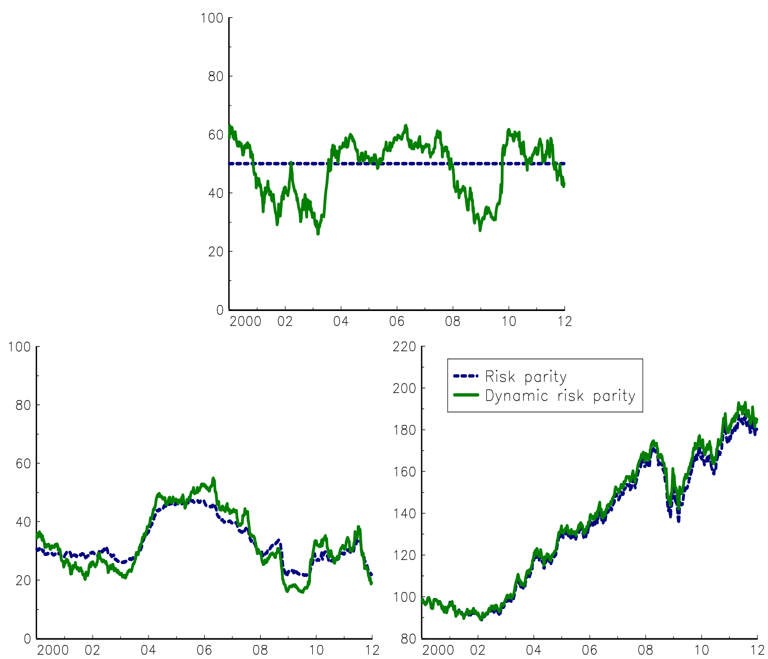

FIGURE 3.8: Simulation of the second dynamic risk parity strategy