INTERPRETATION AND ESTIMATION OF DEFAULT CORRELATIONS

Petit Déjeuner de la Finance

Maison des Polytechniciens - Frontières en Finance,
29 Septembre 2004

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Agenda

- Motivations
- 1st Case : Default Correlations and Loss distribution of a Credit Book
  1. Raroc and credit pricing
  2. Credit portfolio management
  3. Definition of default correlations
  4. MLE of default correlations
- 2nd Case : Default Correlations and Credit Basket Pricing/Hedging
  1. Duality between factor models and copula models
  2. Default correlations and spread jumps
  3. Trac-X implied correlation
  4. Implications for CDO pricing

1 Motivations

Correlations = Parameter of the multivariate Normal distribution / Linear dependence between gaussian random variables

$\Rightarrow$ The second point of view is the reference in finance (regression, factor analysis, etc.)

In asset management, correlations are used to represent the dependence between returns.
Objective: computing risk/return of portfolios.

$\Rightarrow$ Correlation = a good tool for credit risk modelling ?

Our point of view = Correlation is a mathematical tool to define the loss of credit portfolios (CreditRisk+).

Default correlation $\neq$ dependence between default times (KMV, CreditMetrics).

$\Rightarrow$ Credit Portfolio Management = Loss distribution of a portfolio (Credit VaR, Risk Contribution, Stress Testing, etc.)

$\Rightarrow$ Credit Derivatives Pricing/Hedging = Loss distribution of a tranche & Spread dynamics

2 Default Correlations and Loss distribution of a Credit Book

2.1 Raroc and credit pricing

Some notations:

L	Loss of a loan or a portfolio
$\mathrm{EL} = \mathbb{E}\left[L\right]$	Expected Loss or risk cost
UL	Unexpected Loss
	$= \mathrm{VaR}\left[L;\alpha\right] - \mathrm{EL}$

Definition of Raroc:

$$\begin{aligned} \text{Raroc} &= \frac{\text{Expected Return}}{\text{Economic Capital}} \\ &= \frac{\text{PNB} - \text{Cost} - \text{EL}}{\text{UL}} \end{aligned}$$

Objective:

$$\text{Target on Return on Equity} \Leftarrow \text{Target on Raroc}$$

2.1.1 Ex-Ante Raroc of a loan

Economic Capital = Risk contribution of the loan to the total risk of the portfolio

$$\text{Raroc} = \frac{\text{Expected Return}}{\text{Risk Contribution of the loan}}$$

Problems: What is the target portfolio of the bank ? Given this portfolio, how to calibrate the parameters of the Raroc model ? How to approximate the risk contribution when the credit is well modelled (ex-ante raroc) ?

2.1.2 An example with an infinitely fine-grained portfolio and a one factor model

Let $\text{UL} = \text{VaR}\left[L;\alpha\right] - \text{EL}$. If the portfolio is infinitely fine-grained, we have $\text{RC}_i = \mathbb{E}\left[L_i \mid L = \text{EL} + \text{UL}\right] - \mathbb{E}\left[L_i\right]$. We consider the following proxy $\text{UL}^\star = k \times \sigma\left(L\right)$. Because we have:

$$\sigma\left(L\right) = \sum_i \sigma\left(L_i\right) \frac{\text{cov}\left(L, L_i\right)}{\sigma\left(L\right)\sigma\left(L_i\right)} = \sum_i f_i \times \sigma\left(L_i\right)$$

we deduce that a proxy of the risk contribution is $\text{RC}_i^\star = k \times f_i \times \sigma\left(L_i\right)$. f_i is called the diversification factor, because it depends on the dependence structure of the portfolio. In the case of one-factor model and an homogeneous portfolio, we obtain:

$$f = \sqrt{\frac{\mathbf{C}\left(\text{PD}, \text{PD}; \rho\right) - \text{PD}^2}{\frac{\sigma^2[\text{LGD}]}{\mathbb{E}^2[\text{LGD}]}\text{PD} + \left(\text{PD} - \text{PD}^2\right)}}$$

$\Rightarrow f$ depends on the default correlation ρ.

2.2 Credit Portfolio Management

$\Rightarrow$ Moving the original portfolio to obtain the target portfolio

- the original portfolio without management is generally concentrated (either at a name, industry or geography level, etc.)
- the target portfolio is generally an infinitely fine-grained portfolio which has some other good properties (= optimise the capital)

Dis-investment / Re-investment

- Single-name hedges (CDS) / Multi-name hedges (F2D, CDO)
- Securitisations (CBO)
- Investment opportunities (CDS / CDO)

$\Rightarrow$ CPM needs default correlations.

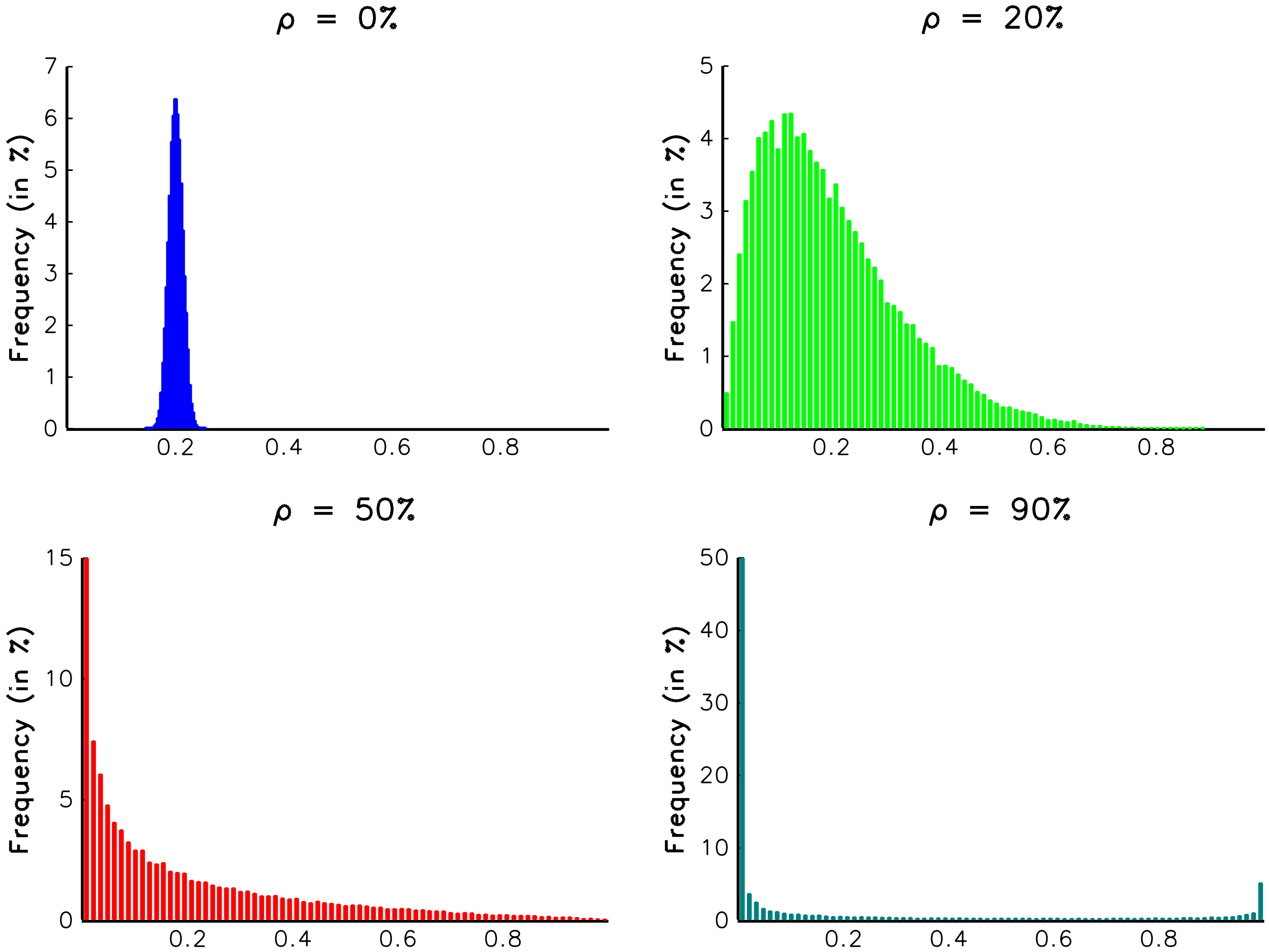
2.3 Definition of default correlations

- Default time correlation $\rho(\tau_1, \tau_2)$
- Default event correlations $\rho(\mathbf{1}\{\tau_1 \leq t_1\}, \mathbf{1}\{\tau_2 \leq t_2\})$
- Spread jumps $s_1(t_1 \mid \tau_2 = t_2, \tau_1 \geq t_2)$
- Asset / Equity correlations

$\Rightarrow$ How to calibrate correlations needed by CPM ?

In the target portfolio, the credit risk is principally a risk on default rates.

$$\begin{aligned} \text{probability of default} \quad &\Leftrightarrow \quad \text{mean of default rates} \\ \text{default correlations} \quad &\Leftrightarrow \quad \text{volatility of default rates} \end{aligned}$$



ρ = 0%
Frequency (in %)
0
1
2
3
4
5
6
7
0.2
0.4
0.6
0.8
ρ = 20%
Frequency (in %)
0
1
2
3
4
5
0.2
0.4
0.6
0.8
ρ = 50%
Frequency (in %)
0
5
10
15
0.2
0.4
0.6
0.8
ρ = 90%
Frequency (in %)
0
10
20
30
40
50
0.2
0.4
0.6
0.8

2.4 Data

- History of annual default rates by risk class
- Risk classes are typically industrial sectors, rating grades, geographical zones, ...

For example : S&P provides this data, between 1980 and 2002 by industrial sector and by rating.

2.5 The model

- Merton model : obligor n defaults if and only if $Z_n \leq B_n$.
- The latent variable Z_n is gaussian
- Homogeneity of risk classes : $B^n = B^c$
- Within a given class of risk the correlation between two firms is constant, that is:

$$\rho_{m,n} = \rho_c, \quad \forall m, n \in c$$

- Given any pair of risk classes (c, d) there is a unique correlation between any couple of firms (m, n) belonging to each class, that is:

$$\rho_{m,n} = \rho_{c,d}, \quad \forall m \in c, n \in d$$

Let's define

$$\Sigma = \begin{pmatrix} \rho_1 & \rho_{1,2} & \cdots & \rho_{1,C} \\ \rho_{2,1} & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_{C-1,C} \\ \rho_{C,1} & \cdots & \rho_{C,C-1} & \rho_C \end{pmatrix}$$

then we can rewrite Z_n as a linear function of F factors X_f (with $A^\top A = \Sigma$)

$$Z_n = \sum_{f=1}^{F} A_{f,c} X_f + \sqrt{1-\rho_c}\,\varepsilon_n, \quad n \in c$$

2.6 MLE of default correlations

The number of default in risk class $D^c \mid X = \mathbf{x} \sim \mathcal{B}\left(n_t^c; P_c(\mathbf{x})\right)$.

The default probability conditionally to the factors $\mathbf{X}$ is:

$$P_c(\mathbf{x}) = \mathbf{\Phi}\left(\frac{B^c - \sum_{f=1}^{F} A_{f,c} x_f}{\sqrt{1-\rho_c}}\right)$$

The unconditional log-likelihood is then:

$$\ell_t(\theta) = \ln \int \cdots \int_{\mathbb{R}^F} \prod_{c=1}^{C} \mathbf{Bin}_{c,t}(\mathbf{x})\, \mathrm{d}\mathbf{\Phi}(\mathbf{x})$$

with:

$$\mathbf{Bin}_{c,t}(\mathbf{x}) = \binom{n_t^c}{d_t^c} P_c(\mathbf{x})^{d_t^c} \left(1 - P_c(\mathbf{x})\right)^{n_t^c - d_t^c}$$

$\Rightarrow$ The loglikelihood is not tractable (in particular when C increases), due to the multi-dimensional integration.

2.7 Constrained Model

$$\Sigma = \begin{pmatrix} \rho_1 & \rho & \cdots & \rho \\ \rho & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & \rho_C \end{pmatrix}$$

$$Z_n = \sqrt{\rho} X + \sqrt{\rho_c - \rho} X_c + \sqrt{1 - \rho_c} \varepsilon_n$$

Interpretation : Z_n is explained by a common factor X and by a specific factor X_c depending on the risk class.

Why : robustness of estimation; this assumption seems intuitive

$$P_c(x, x_c) = \Phi \left(\frac{B^c - \sqrt{\rho} x - \sqrt{\rho_c - \rho} x_c}{\sqrt{1 - \rho_c}} \right)$$

2.7.1 Binomial MLE

The conditional likelihood is first computed and then integrated successively on the distribution of each sectorial factor and on the distribution of the common factor:

$$\ell_t(\theta) = \ln \int_{\mathbb{R}} \left(\prod_{c=1}^{C} \int_{\mathbb{R}} \mathbf{Bin}_{c,t}(x, x_c)\, \mathsf{d}\Phi(x_c) \right) \mathsf{d}\Phi(x)$$

This is the 'binomial' MLE.

2.7.2 Asymptotic MLE

Let $\mu_t^c = \frac{d_t^c}{n_t^c}$ be the default rate at time t in class c.

$$\mu_t^c \mid X = \mathbf{x}, X_c = x_c \rightarrow P\left(x, x_c\right)$$

The loglikelihood function is then:

$$\ell_t\left(\theta\right) = \ln \int_0^1 \prod_{c=1}^{C} \phi\left(f\left(y\right)\right) \frac{\sqrt{1-\rho_c}}{\sqrt{\rho_c - \rho}} \frac{1}{\phi\left(\mathbf{\Phi}^{-1}\left(\mu_t^c\right)\right)} \, \mathrm{d}y$$

with:

$$f\left(y\right) = \frac{B^c - \sqrt{1-\rho_c}\mathbf{\Phi}^{-1}\left(\mu_t^c\right) - \sqrt{\rho}\mathbf{\Phi}^{-1}\left(y\right)}{\sqrt{\rho_c - \rho}}$$

2.8 Monte Carlo simulations

Single-factor $T = 20$ years, number of firms $n_t = N = 200$, homogeneous class (PD = 200 bp), $\rho = 25\%$

- MLE1: full information estimator ($B = \Phi^{-1}$(PD) is known)
- MLE2: limited information estimator (B is estimated)

	Asymptotic		Binomial	
Statistics (in %)	MLE1	MLE2	MLE1	MLE2
mean	23.7	22.5	25.2	23.6
std error	5.8	7.2	7.6	8.5

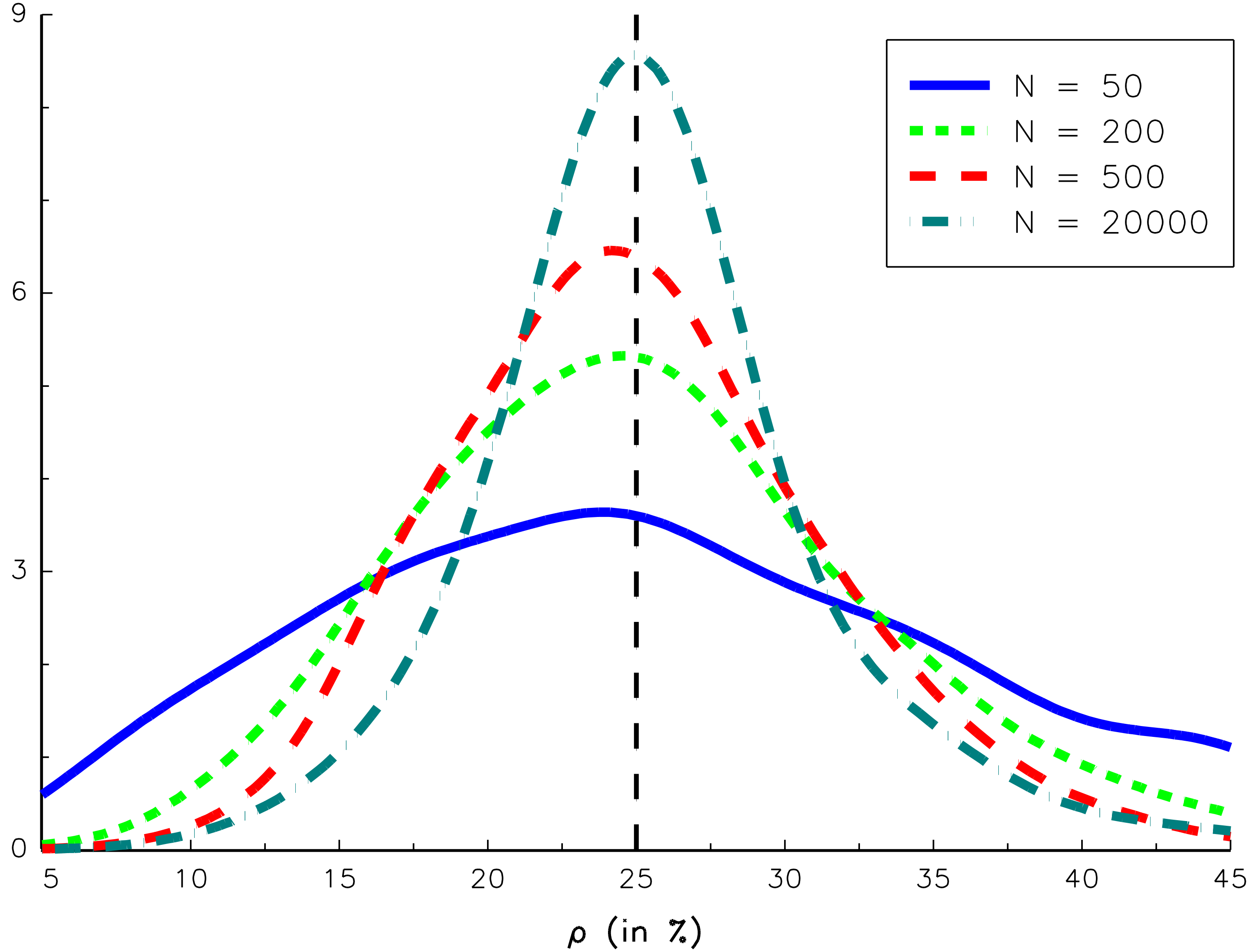
Statistics of the estimates (PD = 200 bp)

$\Rightarrow$ Bias for Asymptotic estimators

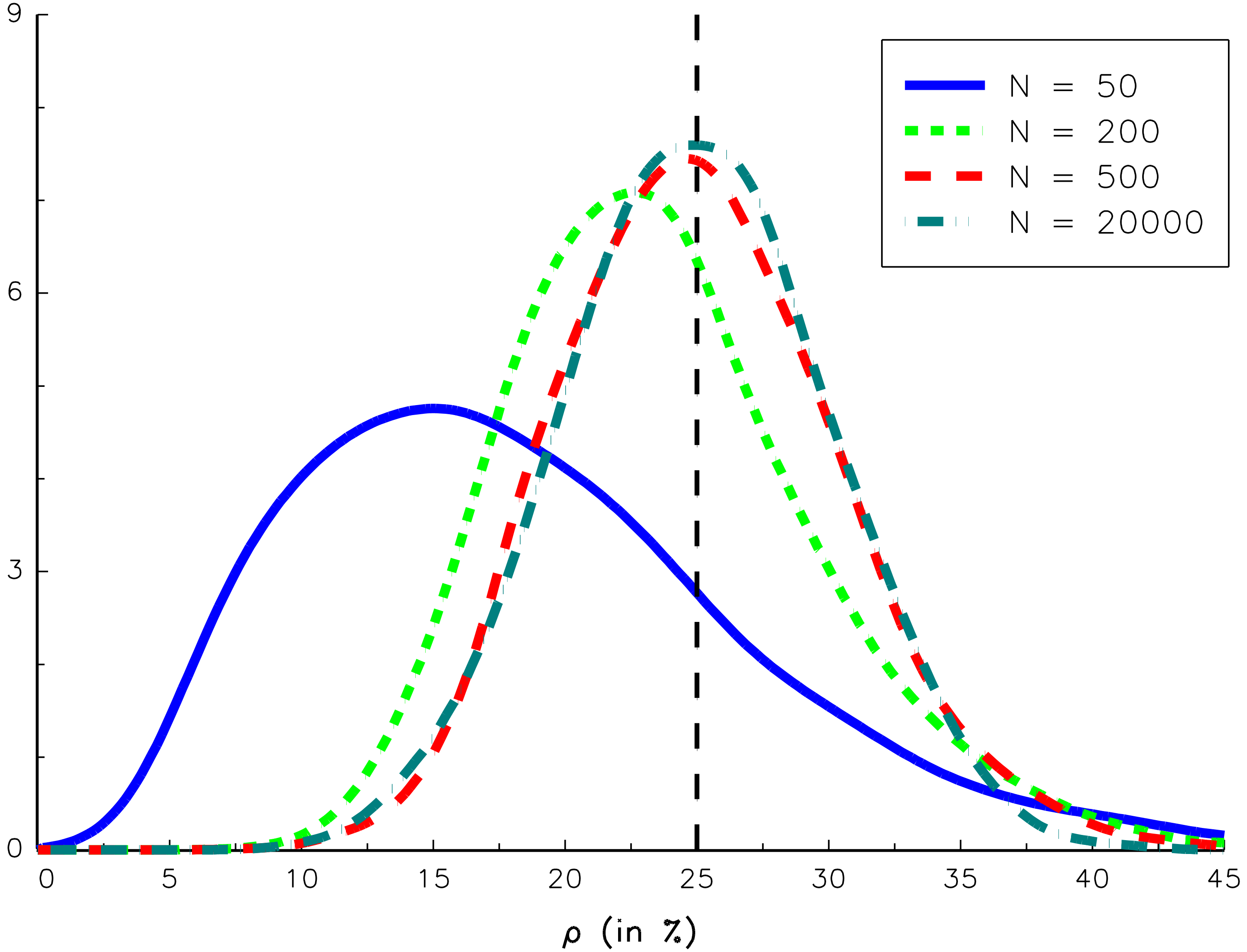
$\Rightarrow$ Downward bias for MLE2

$\Rightarrow$ Standard error is important

Impact of N on binomial MLE



Impact of N on asymptotic MLE



Two risk classes

$$\Sigma = \begin{pmatrix} \rho_1 & \rho \\ \rho & \rho_2 \end{pmatrix} = \begin{pmatrix} 20\% & 7\% \\ 7\% & 10\% \end{pmatrix}$$

Statistics (in %)	Asymptotic ρ_1	Asymptotic ρ_2	Asymptotic ρ	Binomial ρ_1	Binomial ρ_2	Binomial ρ
mean	19.9	12.9	6.5	19.9	10.7	7.5
std error	4.8	3.1	3.1	6.4	4.3	3.7

Statistics of the estimates (PD = 200 bp)

Remark 1 *The bias seems lower than in the one risk class experiment.*

2.9 Estimation using S&P data

			two-factor		Single-factor	
	$\bar{N}_c$	$\bar{\mu}_c$	Asymp.	Bin.	Asymp.	Bin.
Aerospace / Automobile	301	2.08%	13.3%	13.9%	13.7%	11.6%
Consumer / Service sector	355	2.37%	12.2%	10.6%	12.2%	8.9%
Energy / Natural ressources	177	2.10%	23.2%	25.5%	16.2%	14.5%
Financial institutions	424	0.57%	17.0%	16.4%	12.0%	9.5%
Forest / Building products	282	1.90%	18.1%	18.8%	28.6%	31.5%
Health	135	1.27%	12.9%	10.6%	13.1%	13.2%
High technology	131	1.66%	15.0%	16.4%	12.9%	10.6%
Insurance	166	0.61%	26.3%	34.3%	13.6%	17.8%
Leisure time / Media	232	3.01%	13.8%	9.4%	17.2%	12.0%
Real estate	133	1.01%	43.2%	52.4%	48.7%	53.0%
Telecoms	100	1.91%	22.9%	29.1%	27.0%	34.0%
Transportation	146	2.02%	17.7%	11.1%	12.8%	10.4%
Utilities	206	0.43%	14.4%	18.7%	10.4%	17.5%
Inter-sector			7.2%	9.4%	✓	✓

Conclusion

- we extend the study of Gordy and Heitfield (2002)
- we apply our methodology to S&P data
- there is a downward bias that one could try to correct

Application to Stress-Testing

$\Rightarrow$ Pillar II.

3 Default Correlations and Credit Basket Pricing/Hedging

3.1 Duality between factor models and copula models

Let $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$ be a latent variable with X the common factor and ε_i the specific factor. We have

$$D_i(t) = 1 \Leftrightarrow Z_i < B_i = \Phi^{-1}(\mathrm{PD}_i(t))$$

Let $\Sigma = C(\rho)$ be the constant correlation matrix. We have

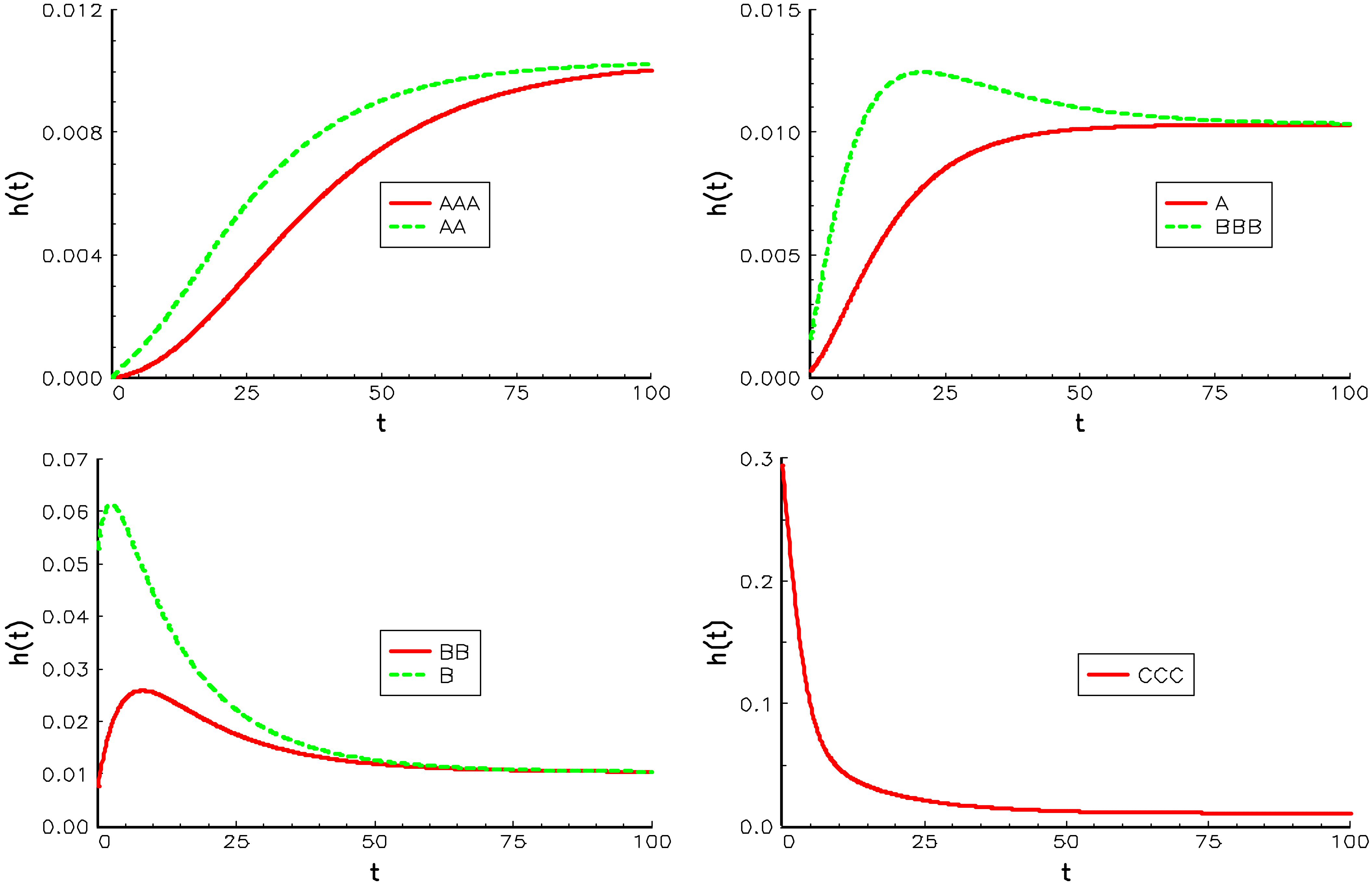
$$\begin{aligned}
\mathbf{S}(t_1,\ldots,t_I) &= \Pr\{\tau_1 > t_1,\ldots,\tau_I > t_I\} \\
&= \Pr\left\{Z_1 > \Phi^{-1}(\mathrm{PD}_1(t_1)),\ldots,Z_I > \Phi^{-1}(\mathrm{PD}_I(t_I))\right\} \\
&= \mathbf{C}(1-\mathrm{PD}_1(t_1),\ldots,1-\mathrm{PD}_I(t_I);\Sigma) \\
&= \mathbf{C}(\mathbf{S}_1(t_1),\ldots,\mathbf{S}_I(t_I);\Sigma)
\end{aligned}$$

where $\mathbf{C}$ is the Normal copula.

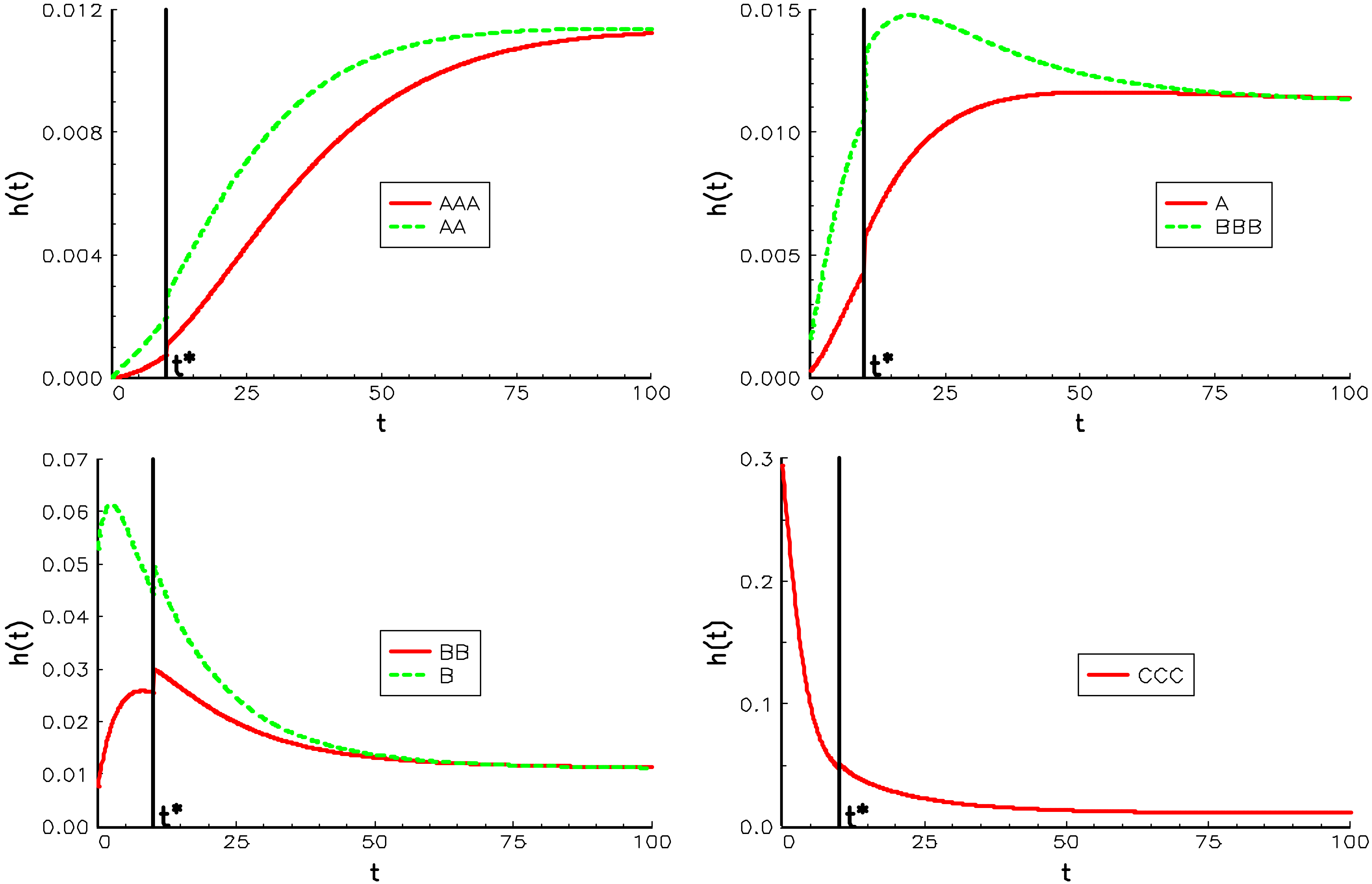
Remark 2 *Let τ_1 et τ_2 be two default times with the joint survival function $\mathbf{S}(t_1,t_2) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1),\mathbf{S}_2(t_2))$. We have $\mathbf{S}_1(t \mid \tau_2 = t^\star) = \partial_2\breve{\mathbf{C}}(\mathbf{S}_1(t),\mathbf{S}_2(t^\star))$. If $\mathbf{C} \neq \mathbf{C}^\perp$, default probability of one firm changes when the other has defaulted.*

Example 1 *The next figures show jumps of the hazard function* $\lambda(t) = f(t)/\mathbf{S}(t)$ *of the annual S&P transition matrix. With a Normal copula and* $\Sigma = C_I(\rho)$*, we have*

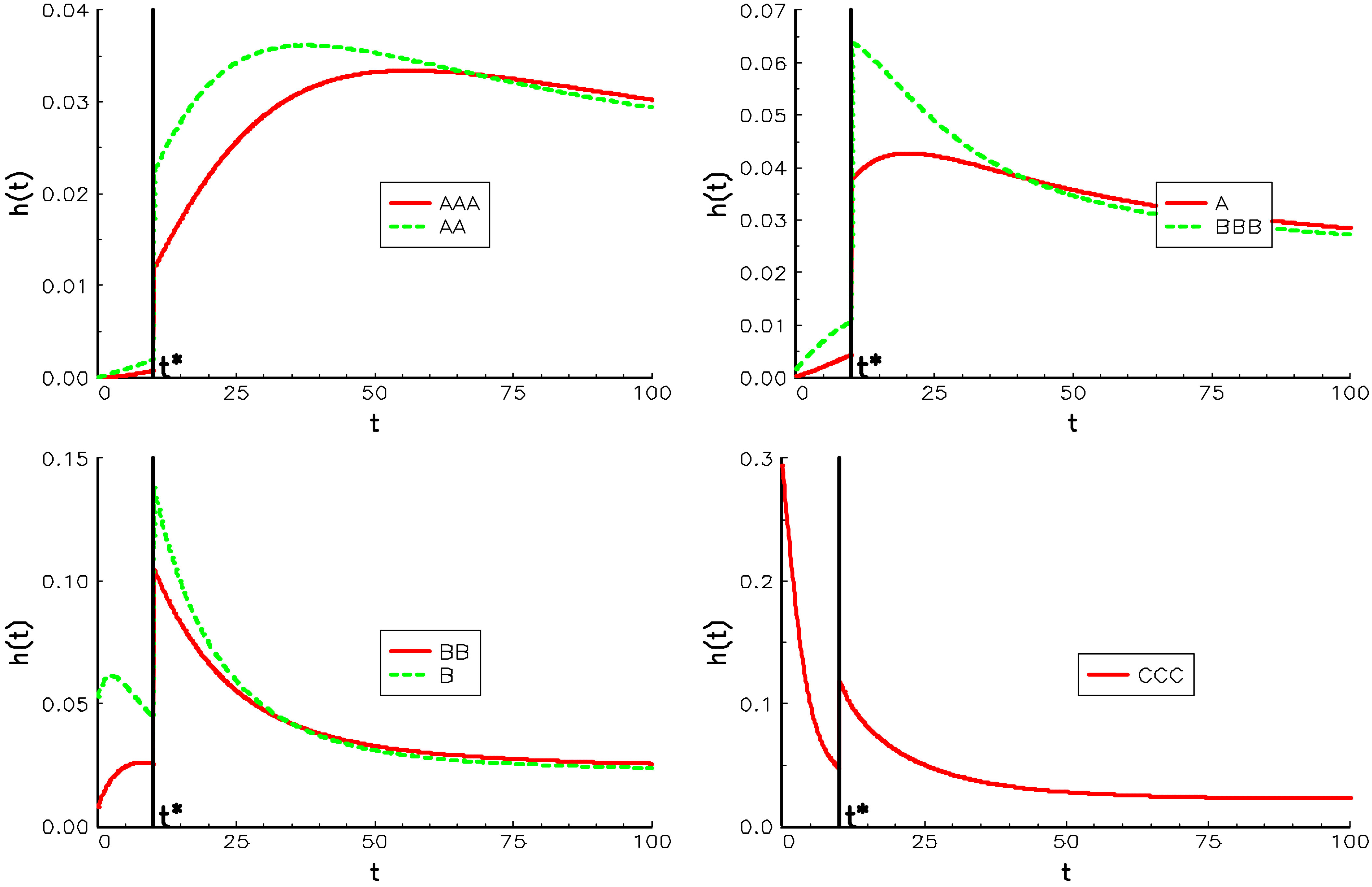
$$\mathbf{S}_1(t \mid \tau_2 = t^\star) = \Phi\left(\frac{\Phi^{-1}(\mathbf{S}_1(t)) - \rho\Phi^{-1}(\mathbf{S}_2(t^\star))}{\sqrt{1-\rho^2}}\right)$$



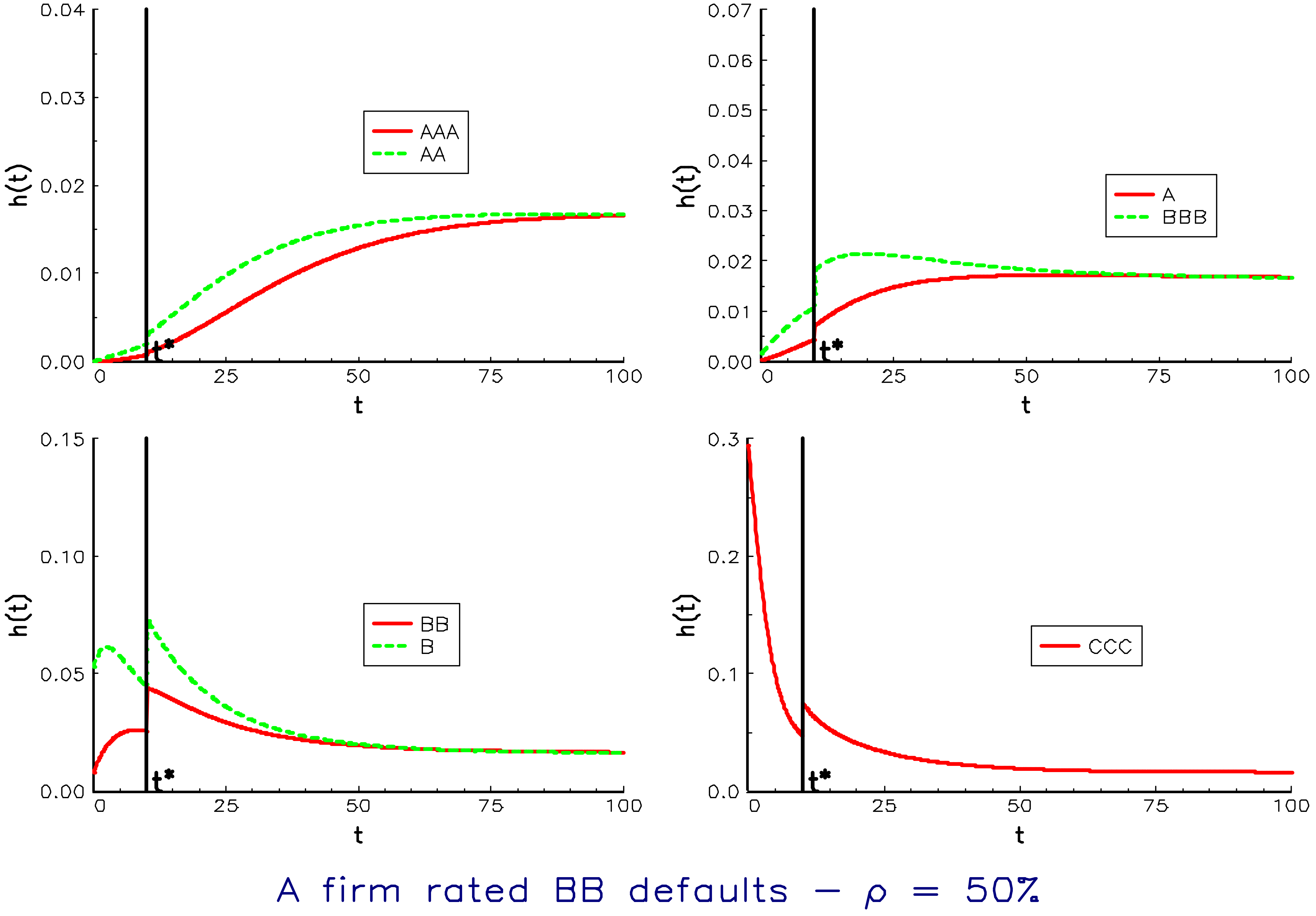
Hazard rate of the ratings

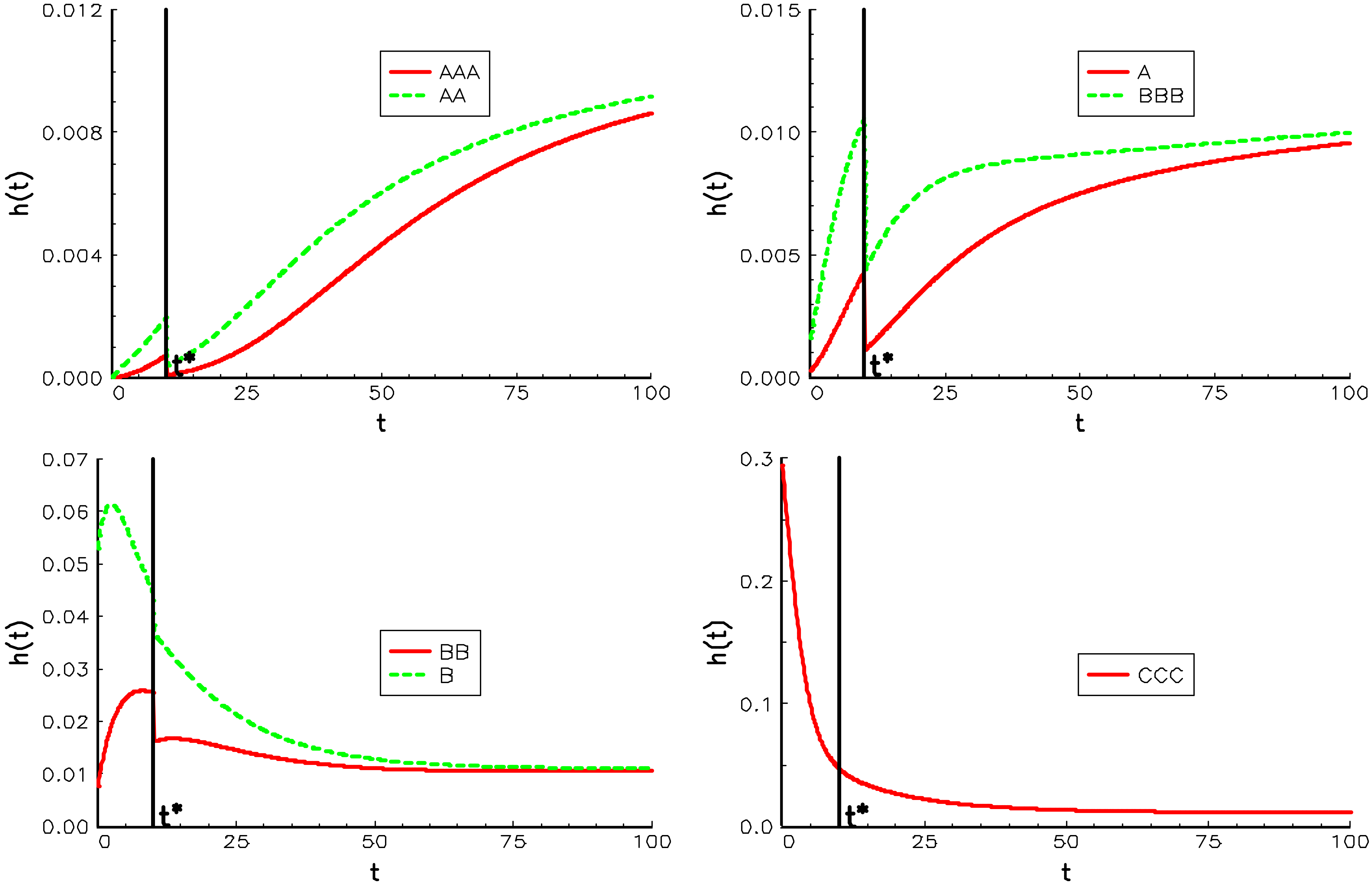


A firm rated AAA defaults – ρ = 5%



A firm rated AAA defaults – ρ = 50%


0.04
0.03
0.02
0.01
0.00
h(t)
t
t*
0
25
50
75
100
AAA
AA
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
A
BBB
0.15
0.10
0.05
0.00
BB
B
0.3
0.2
0.1
0.0
CCC
A firm rated BB defaults – ρ = 50%



A firm rated CCC defaults – ρ = 50%

3.2 Default correlations and spread jumps

We assume an exponential default model with intensity λ. Let s and R be the spread of the CDS and the recovery rate. We have

$$s = \lambda (1 - R)$$

It comes that the default probability is

$$\mathrm{PD}(t) = 1 - \exp\left(-\frac{s}{1-R} t\right)$$

The conditional probability of the first name given that the second name has defaulted at time $t^\star$ is then

$$\mathrm{PD}_1 (t \mid \tau_2 = t^\star) = \partial_2 \mathbf{C} (\mathrm{PD}_1 (t), \mathrm{PD}_2 (t^\star)) \qquad (t \geq t^\star)$$

we deduce that the spread of the first name after the default of the second name becomes

$$s_1 (t \mid \tau_2 = t^\star, \tau_1 \geq t^\star) = -\frac{(1 - R_1)}{(t - t^\star)} \ln (1 - \mathrm{PD}_1 (t \mid \tau_2 = t^\star, \tau_1 \geq t^\star))$$

Correlation implied to Ahold default

		Start	Wide	Jump	Correlation implied	
	Recovery	28/10/2002	28/02/2003		Normal	T4
AHOLD	**40%**	**235**	**1205**	**970**		
CASINO	40%	235	152	-83	-8	-48
SAINSBURY	40%	48	95	47	**12**	-31
CARREFOUR	40%	60	47	-13	-4	-52
KROGER	40%	127,5	108	-19,5	-3	-47
SAFEWAY	40%	66,5	145	78,5	**15**	-27

		Start	Wide	Jump	Correlation implied	
	Recovery	20/02/2003	28/02/2003		Normal	T4
AHOLD	**40%**	**195**	**1205**	**1010**		
CASINO	40%	135	160	25	-7	-79
SAINSBURY	40%	68	95	27	**3**	-76
CARREFOUR	40%	43	47	4	**1**	-80
KROGER	40%	90	95	5	**1**	-78
SAFEWAY	40%	195	145	-50	-3	-78

Correlation implied to Worldcom default

		Start	Wide	Jump	Correlation implied	
	Recovery	05/07/2001	01/05/2002		Normal	T4
WORLDCOM	**15%**	**165**	**1700**	**1535**		
TELECOMI	15%	165	130	-35	-5	-41
TELEFONI	15%	95	80	-15	-3	-43
BELLSOUT	15%	47	75	28	9	-31
BRITELEC	15%	105	105	0	0	-39
MOTOROLA	15%	285	300	15	**1**	-29
ATTCORP	15%	110	600	490	45	19
TELECOM	15%	185	345	160	15	-16

Correlation implied to TXU Corp. default

		Start	Wide	Jump	Correlation implied	
	Recovery	13/08/2002	10/10/2002		Normal	T4
TXU Corp.	**40%**	**450**	**1250**	**800**		
SEMPRA	40%	275	400	125	7	-33
DUKEENER	40%	170	225	55	5	-39
VIVENENV	40%	170	152,5	-17,5	-2	-48
SUEZ	40%	105	130	25	4	-43
AMELECPO	40%	380	925	545	20	-15
RWEAG	40%	67	98	31	6	-41
ENEL	40%	68	87	19	4	-44

3.3 Trac-X implied correlation

Model : $Z_i = \beta X + \sqrt{1-\beta^2}\varepsilon_i. \Rightarrow \beta = \sqrt{\rho}.$

Expectation of losses (5Y maturity)

Value of the floating leg (5Y maturity)

Implied correlation

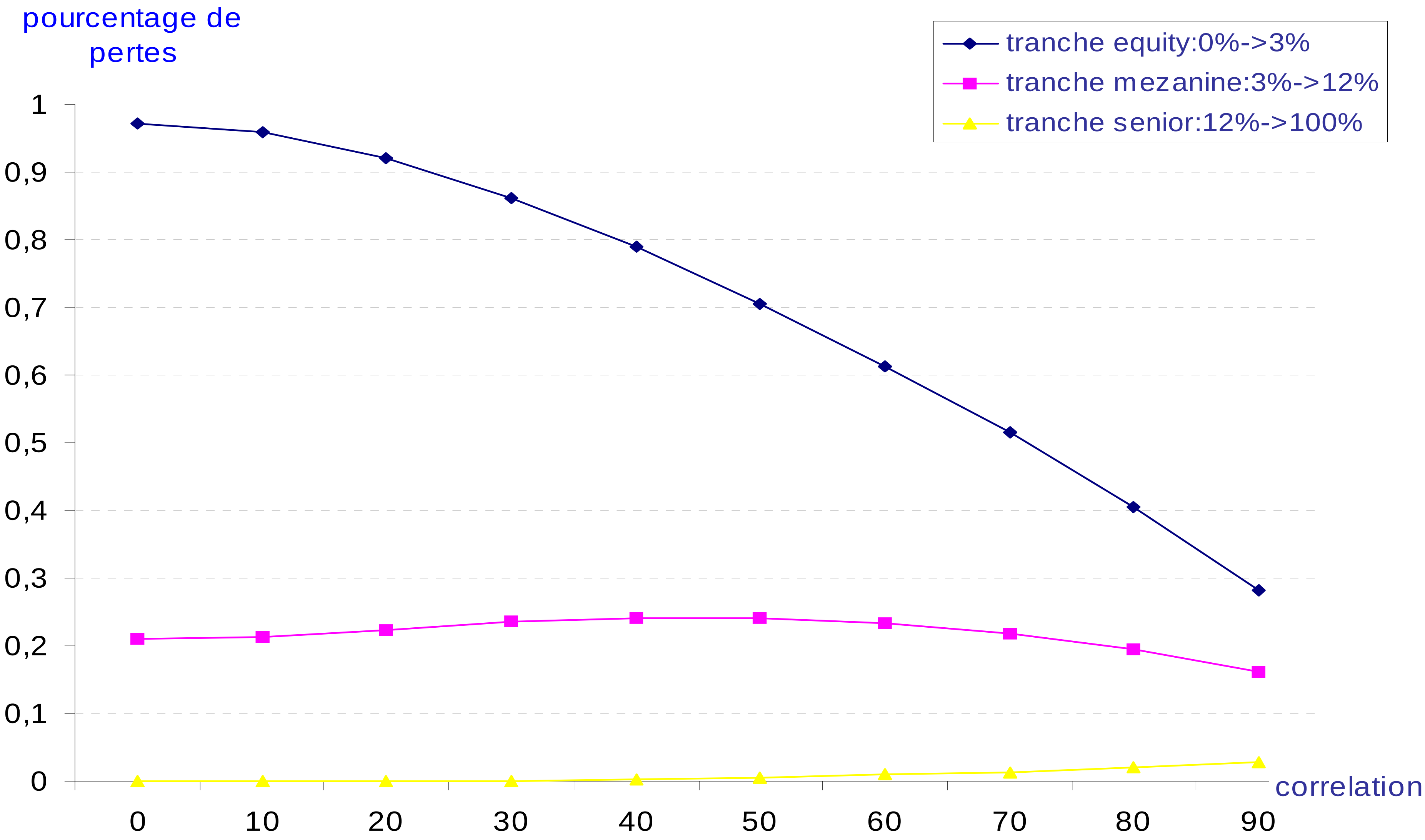
Attachment points: $0 = A_0 < A_1 < ... < A_M \leq 1$

Marked spread: $s(A_{i_{-1}}, A_i)^{obs}$ for the tranche $[A_{i-1}, A_i]$

The implied correlation for the tranche $[A_{i-1}, A_i]$ verify:

$$\forall i \neq 1,\ s(A_{i-1}, A_i, \rho(A_{i-1}, A_i)) = s(A_{i-1}, A_i)^{obs},$$

(correction for the equity tranche because of upfront payment)


pourcentage de pertes
tranche equity:0%->3%
tranche mezanine:3%->12%
tranche senior:12%->100%
1
0,9
0,8
0,7
0,6
0,5
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0,3
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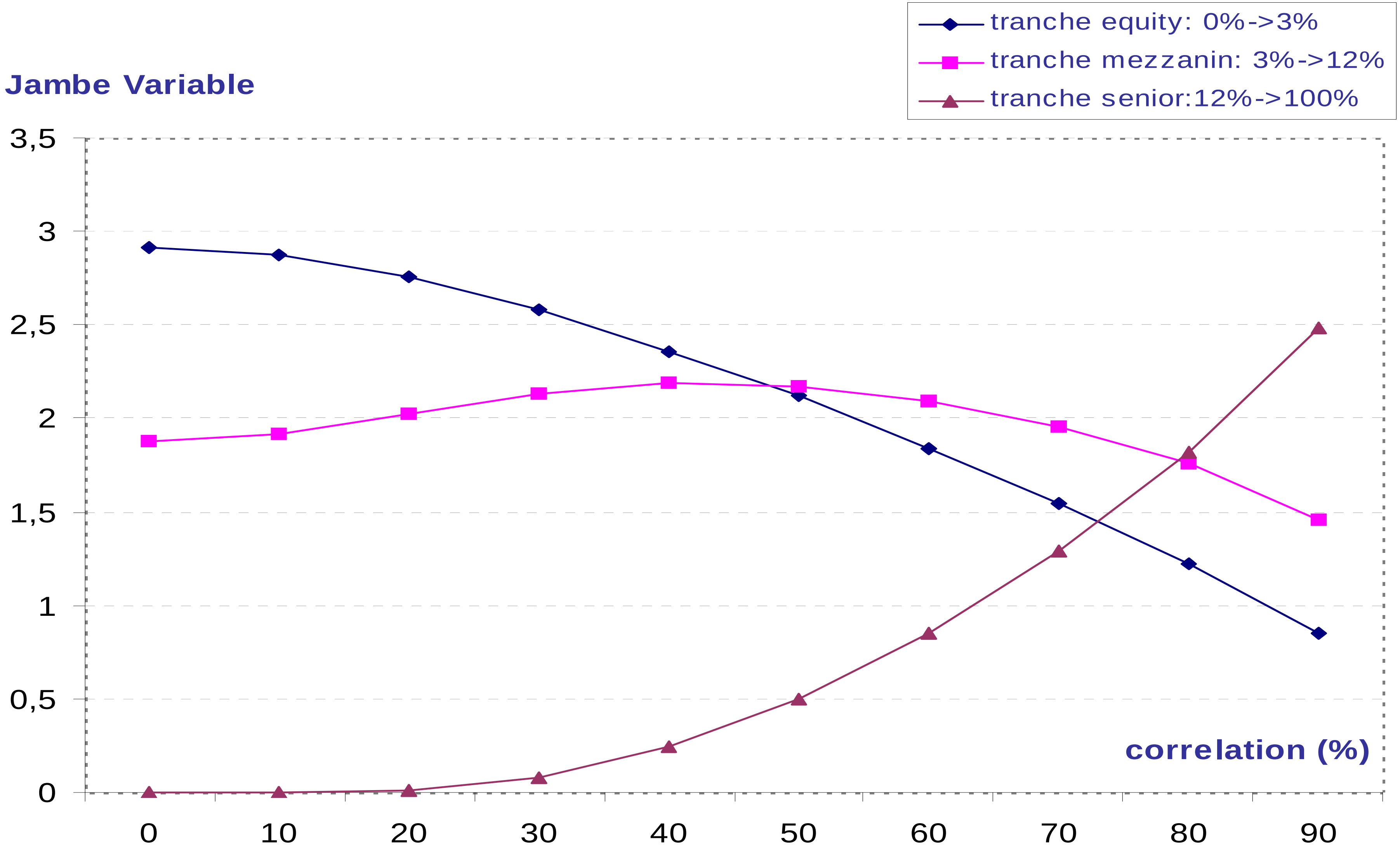
0
correlation
0
10
20
30
40
50
60
70
80
90


Jambe Variable
tranche equity: 0%->3%
tranche mezzanin: 3%->12%
tranche senior:12%->100%
correlation (%)
0
0,5
1
1,5
2
2,5
3
3,5
0
10
20
30
40
50
60
70
80
90

Example: Trac-X Euro 02/06/2004.

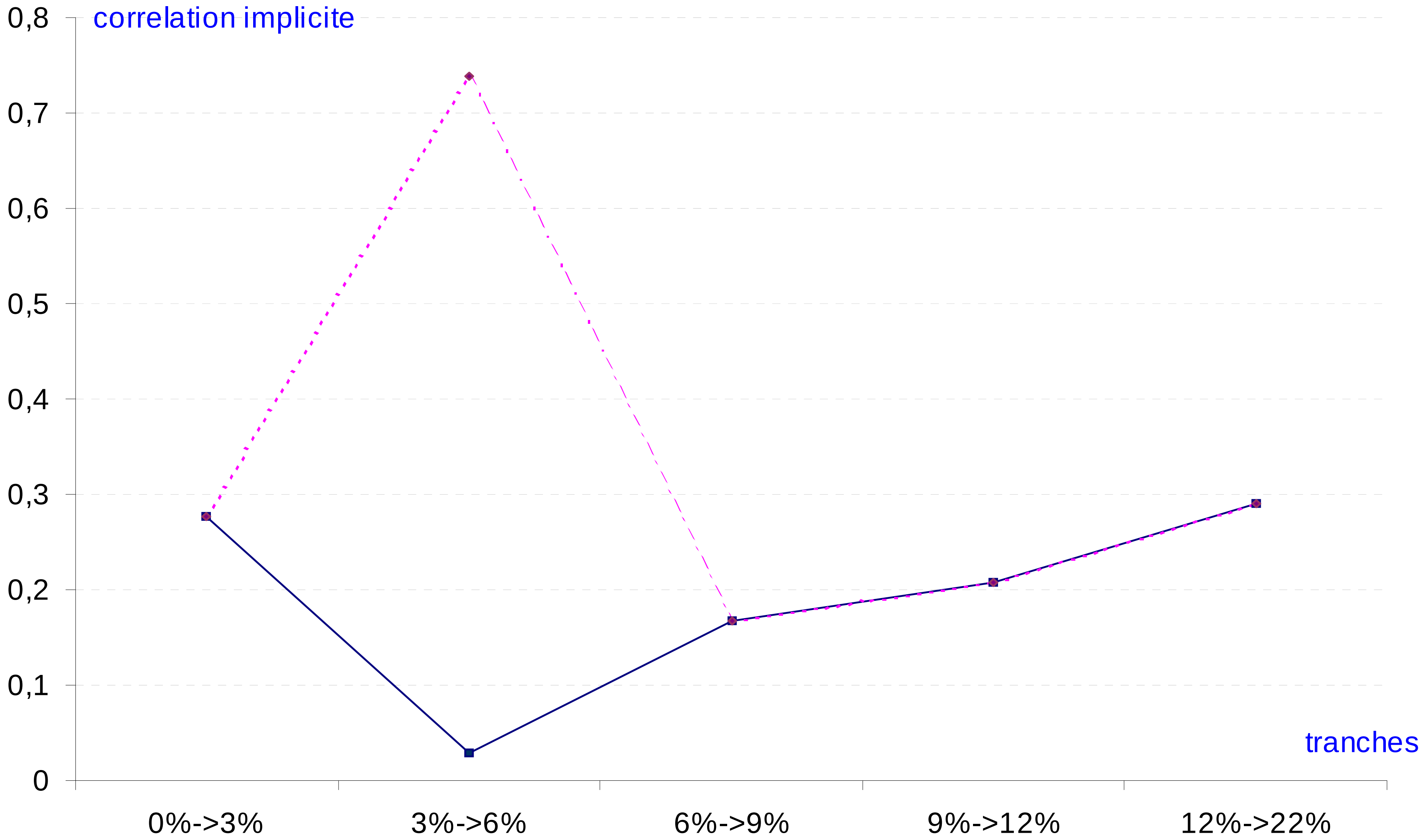
A	B	Upfront payment	Running spread (bp)
0%	3%	34%	500
3%	6%		279
6%	9%		114
9%	12%		58
12%	22%		23

Implied correlation of Trac-X

Loss distribution for the second tranche

Base correlation of Trac-X

Implied correlation of Trac-X (T9 copula)



0,8
0,7
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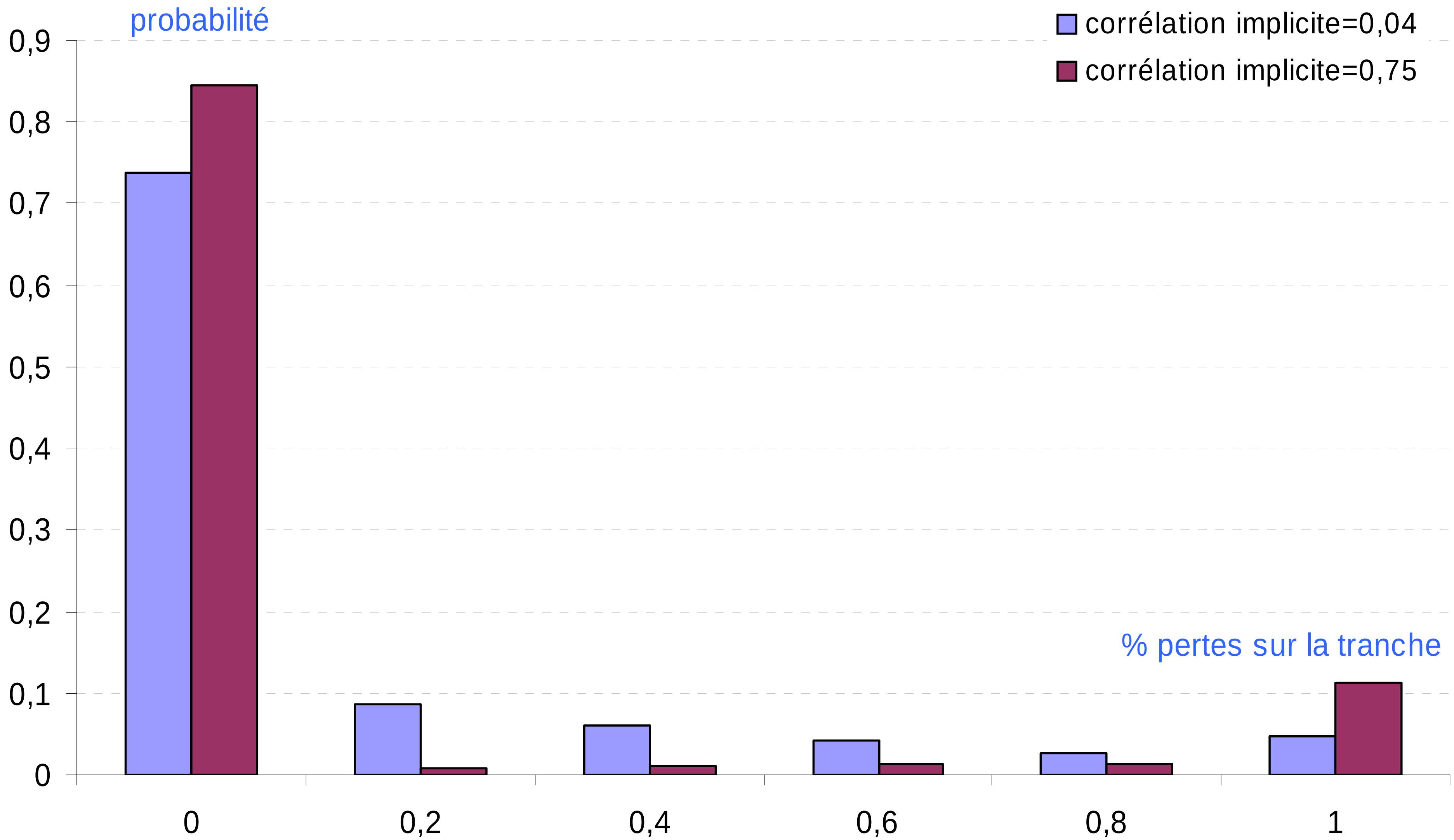
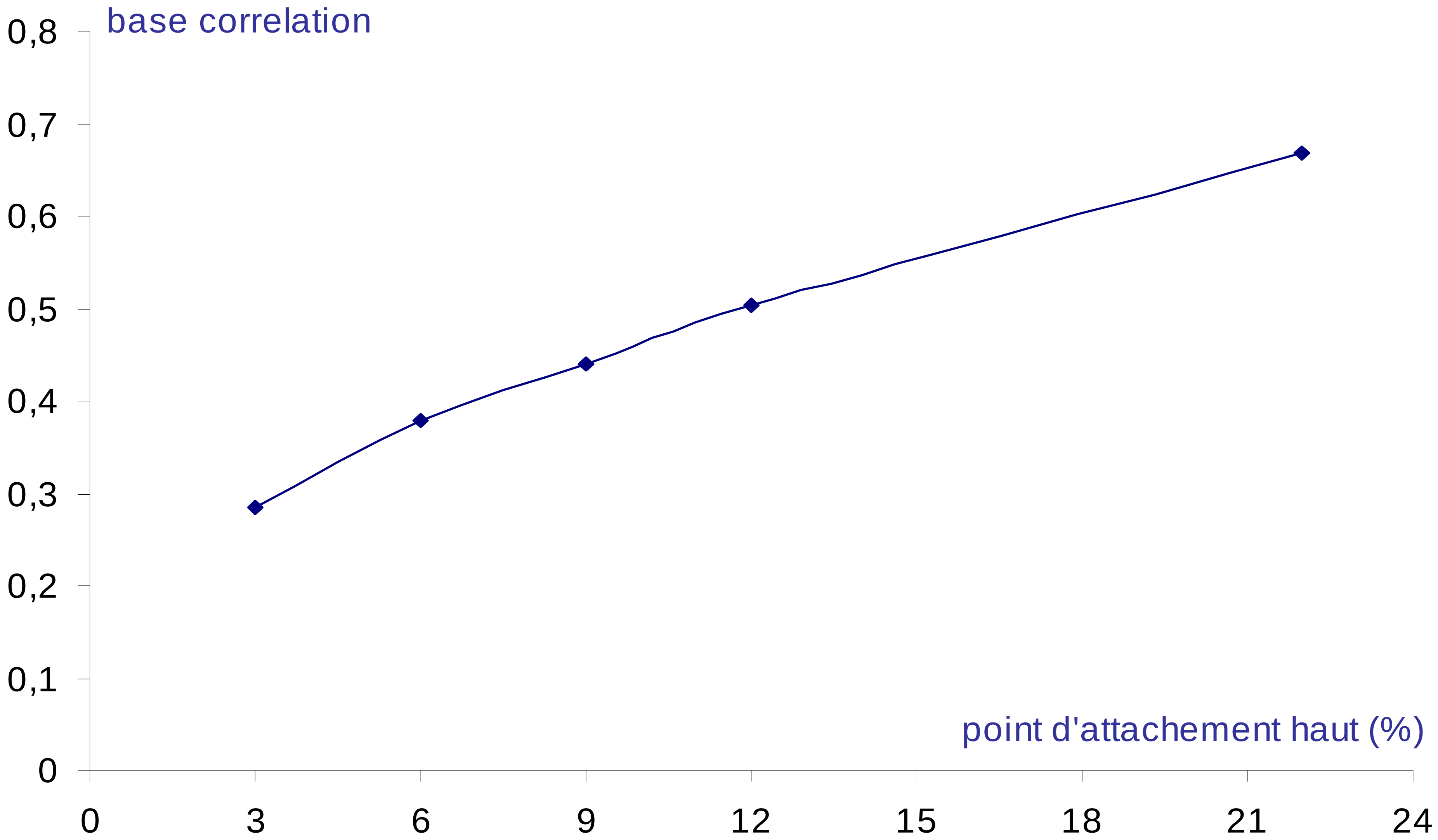
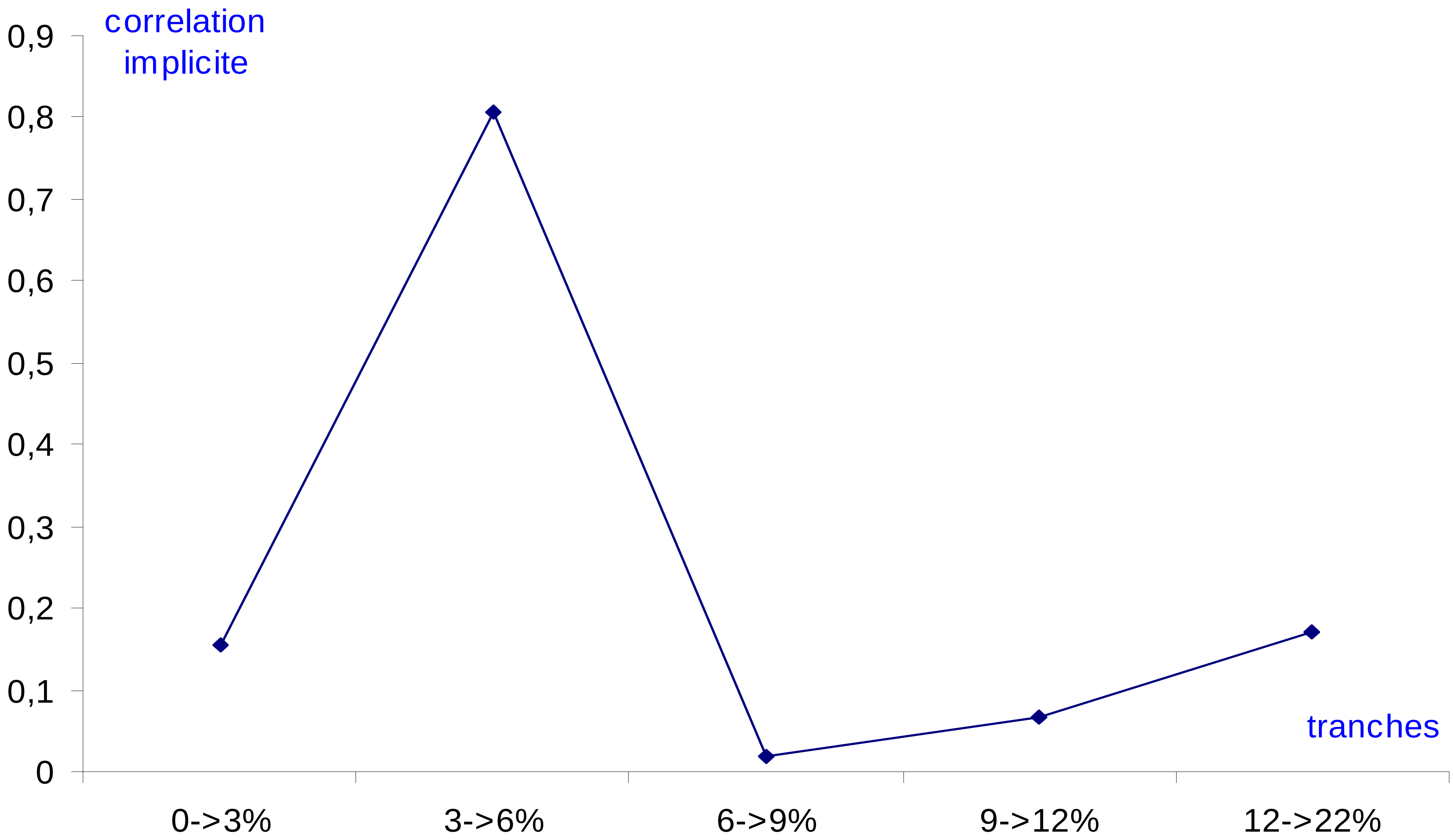
0,1
0
correlation implicite
tranches
0%->3%
3%->6%
6%->9%
9%->12%
12%->22%


probabilité
corrélation implicite=0,04
corrélation implicite=0,75
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
% pertes sur la tranche
0
0,2
0,4
0,6
0,8
1



base correlation
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
point d'attachement haut (%)
0
3
6
9
12
15
18
21
24

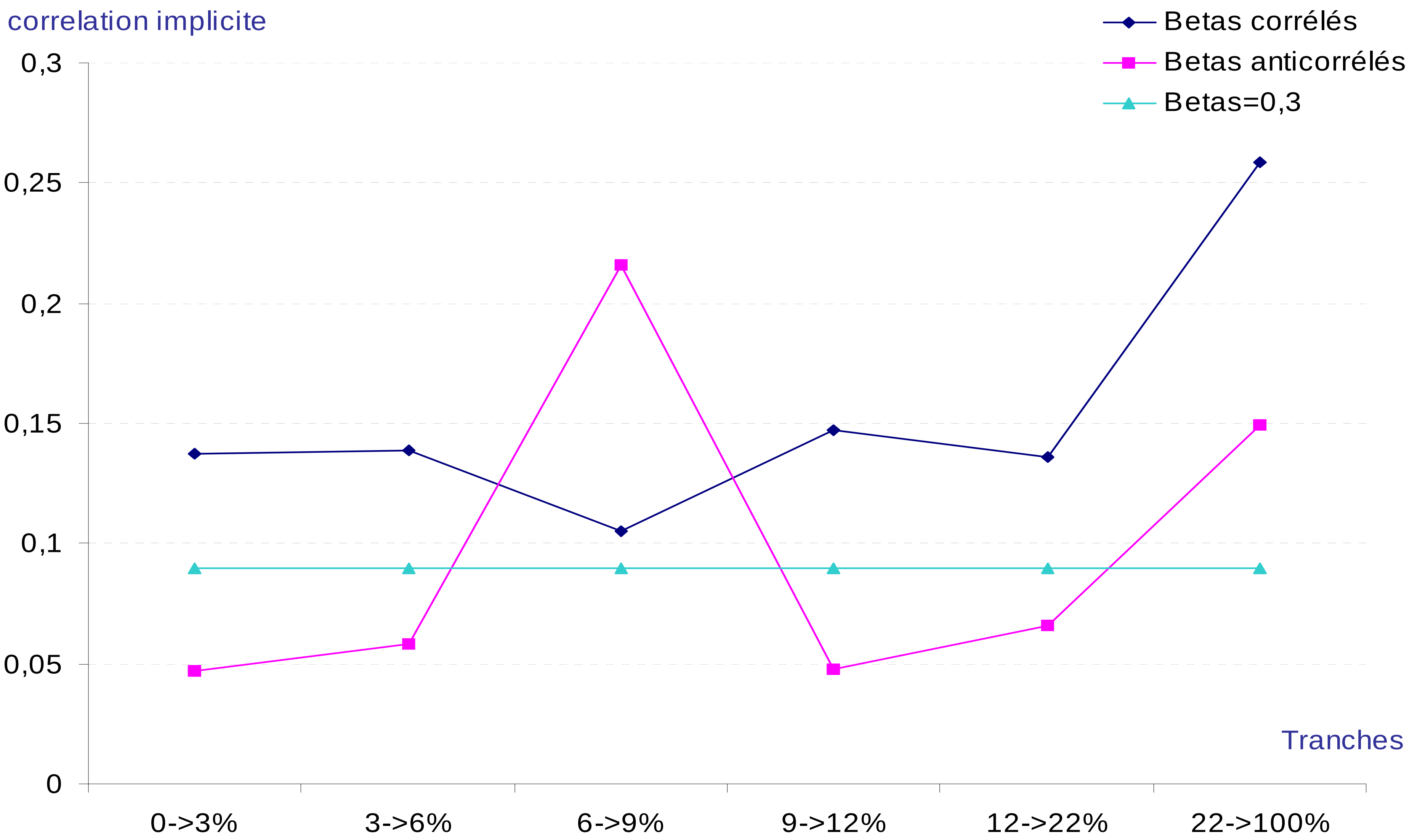

correlation
implicite
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
tranches
0->3%
3->6%
6->9%
9->12%
12->22%

Gaussian factors with three types of names (spread = 50 bp, 150 bp and 250 bp).

Three structures of correlation :

$$\Sigma_1 = \begin{pmatrix} 1 & 0.3^2 & 0.3^2 \\ 0.3^2 & 1 & 0.3^2 \\ 0.3^2 & 0.3^2 & 1 \end{pmatrix}, \; \Sigma_2 = \begin{pmatrix} 1 & 0.1 \times 0.3 & 0.1 \times 0.5 \\ 0.1 \times 0.3 & 1 & 0.3 \times 0.5 \\ 0.1 \times 0.5 & 0.3 \times 0.5 & 1 \end{pmatrix}$$

$$\Sigma_3 = \begin{pmatrix} 1 & 0.5 \times 0.3 & 0.1 \times 0.5 \\ 0.5 \times 0.3 & 1 & 0.3 \times 0.1 \\ 0.1 \times 0.5 & 0.3 \times 0.1 & 1 \end{pmatrix}$$


correlation implicite
Betas corrélés
Betas anticorrélés
Betas=0,3
0,3
0,25
0,2
0,15
0,1
0,05
0
Tranches
0->3%
3->6%
6->9%
9->12%
12->22%
22->100%

3.4 Implications for CDO pricing

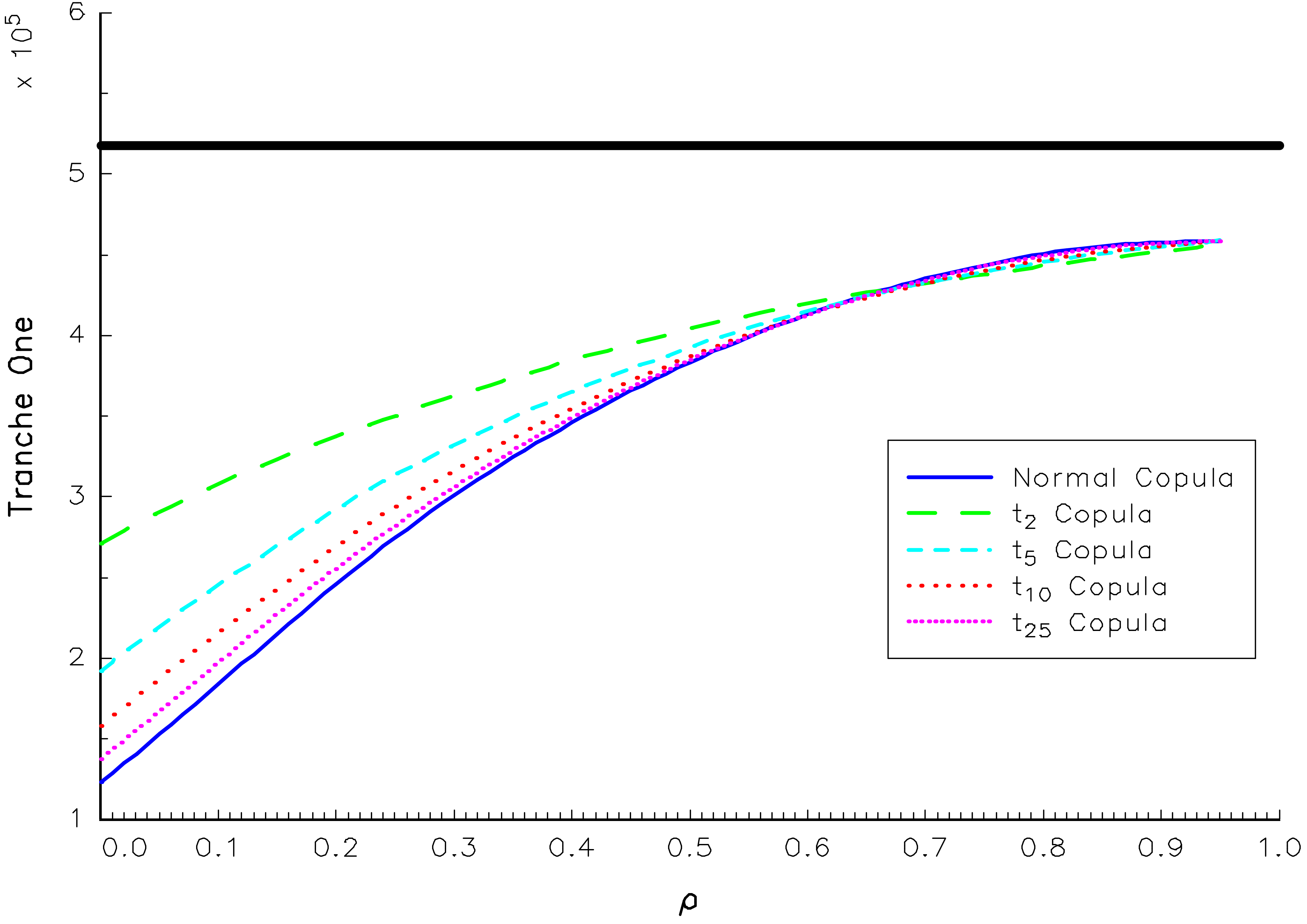
Implied correlation = not useful for CDO pricing.

Implied correlation of CDO $\neq$ Implied correlation of spread of two equity indices

A new dimension = TRAC-X PORTFOLIO.

What is the meaning of implied correlation ?

$\Rightarrow$ the mathematical root of an equation



x 10^5
6
5
4
3
2
1
Tranche One
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
ρ
Normal Copula
t_2 Copula
t_5 Copula
t_{10} Copula
t_{25} Copula