Financial Risk Management
Tutorial Class — Session 4

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Estimation of the loss severity distribution

Exercise

We consider a sample of n individual losses $\{x_1, \ldots, x_n\}$. We assume that they can be described by different probability distributions:

(i) X follows a log-normal distribution $\mathcal{LN}\left(\mu, \sigma^2\right)$.

(ii) X follows a Pareto distribution $\mathcal{P}\left(\alpha, x^-\right)$ defined by:

$$\Pr\left\{X \leq x\right\} = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

with $x \geq x_-$ and $\alpha > 0$.

(iii) X follows a gamma distribution $\Gamma\left(\alpha, \beta\right)$ defined by:

$$\Pr\left\{X \leq x\right\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma\left(\alpha\right)} \,\mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss X follows a gamma distribution: $\ln X \sim \Gamma\left(\alpha; \beta\right)$.

Estimation of the loss severity distribution

Question 1

We consider the case *(i)*.

(i) X follows a log-normal distribution $\mathcal{LN}\left(\mu, \sigma^2\right)$.

Estimation of the loss severity distribution

Question 1.a

Show that the probability density function is:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

Estimation of the loss severity distribution

The density of the Gaussian distribution $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is:

$$g\left(y\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$$

Let $X \sim \mathcal{LN}\left(\mu, \sigma^2\right)$. We have $X = \exp Y$. It follows that:

$$f\left(x\right) = g\left(y\right)\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|$$

with $y = \ln x$. We deduce that:

$$\begin{aligned} f\left(x\right) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right) \times \frac{1}{x} \\ &= \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right) \end{aligned}$$

Estimation of the loss severity distribution

Question 1.b

Calculate the two first moments of X. Deduce the orthogonal conditions of the generalized method of moments.

Estimation of the loss severity distribution

For $m \geq 1$, the non-centered moment is equal to:

$$\mathbb{E}\left[X^m\right] = \int_0^\infty x^m \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) \,\mathrm{d}x$$

Estimation of the loss severity distribution

By considering the change of variables $y = \sigma^{-1}(\ln x - \mu)$ and $z = y - m\sigma$, we obtain:

$$
\begin{aligned}
\mathbb{E}\left[X^m\right] &= \int_{-\infty}^{\infty} e^{m\mu + m\sigma y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \,\mathrm{d}y \\
&= e^{m\mu} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2 + m\sigma y} \,\mathrm{d}y \\
&= e^{m\mu} \times e^{\frac{1}{2}m^2\sigma^2} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - m\sigma)^2} \,\mathrm{d}y \\
&= e^{m\mu + \frac{1}{2}m^2\sigma^2} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \mathrm{d}z \\
&= e^{m\mu + \frac{1}{2}m^2\sigma^2}
\end{aligned}
$$

Estimation of the loss severity distribution

We deduce that:

$$\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$$

and:

$$\begin{aligned}
\mathrm{var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] \\
&= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\
&= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)
\end{aligned}$$

We can estimate the parameters μ and σ with the generalized method of moments by using the following empirical moments:

$$\left\{\begin{array}{l}
h_{i,1}(\mu, \sigma) = x_i - e^{\mu + \frac{1}{2}\sigma^2} \\
h_{i,2}(\mu, \sigma) = \left(x_i - e^{\mu + \frac{1}{2}\sigma^2}\right)^2 - e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)
\end{array}\right.$$

Estimation of the loss severity distribution

Question 1.c

Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$.

Estimation of the loss severity distribution

The log-likelihood function of the sample $\{x_1, \ldots, x_n\}$ is:

$$\begin{aligned} \ell(\mu, \sigma) &= \sum_{i=1}^{n} \ln f(x_i) \\ &= -\frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln 2\pi - \sum_{i=1}^{n} \ln x_i - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\ln x_i - \mu}{\sigma} \right)^2 \end{aligned}$$

To find the ML estimators $\hat{\mu}$ and $\hat{\sigma}$, we can proceed in two different way.

Estimation of the loss severity distribution

#1 $X \sim \mathcal{LN}\left(\mu, \sigma^2\right)$ implies that $Y = \ln X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. We know that the ML estimators $\hat{\mu}$ and $\hat{\sigma}$ associated to Y are:

$$\begin{aligned} \hat{\mu} &= \frac{1}{n} \sum_{i=1}^{n} y_i \\ \hat{\sigma} &= \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(y_i - \hat{\mu}\right)^2} \end{aligned}$$

We deduce that the ML estimators $\hat{\mu}$ and $\hat{\sigma}$ associated to the sample $\{x_1, \ldots, x_n\}$ are:

$$\begin{aligned} \hat{\mu} &= \frac{1}{n} \sum_{i=1}^{n} \ln x_i \\ \hat{\sigma} &= \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\ln x_i - \hat{\mu}\right)^2} \end{aligned}$$

Estimation of the loss severity distribution

#2 We maximize the log-likelihood function. The first-order conditions are $\partial_\mu \boldsymbol{\ell}(\mu,\sigma) = 0$ and $\partial_\sigma \boldsymbol{\ell}(\mu,\sigma) = 0$. We deduce that:

$$\partial_\mu \boldsymbol{\ell}(\mu,\sigma) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu) = 0$$

and:

$$\partial_\sigma \boldsymbol{\ell}(\mu,\sigma) = -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(\ln x_i - \mu)^2}{\sigma^3} = 0$$

We finally obtain:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i$$

and:

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\ln x_i - \hat{\mu})^2}$$

Estimation of the loss severity distribution

Question 2

We consider the case *(ii)*.

(ii) X follows a Pareto distribution $\mathcal{P}(\alpha, x^{-})$ defined by:

$$\Pr\{X \leq x\} = 1 - \left(\frac{x}{x_{-}}\right)^{-\alpha}$$

with $x \geq x_{-}$ and $\alpha > 0$.

Estimation of the loss severity distribution

Question 2.a

Calculate the two first moments of X. Deduce the GMM conditions for estimating the parameter α.

Estimation of the loss severity distribution

The probability density function is:

$$\begin{aligned} f(x) &= \frac{\partial \Pr\{X \leq x\}}{\partial x} \\ &= \alpha \frac{x^{-(\alpha+1)}}{x_-^{-\alpha}} \end{aligned}$$

For $m \geq 1$, we have:

$$\begin{aligned} \mathbb{E}[X^m] &= \int_{x_-}^{\infty} x^m \alpha \frac{x^{-(\alpha+1)}}{x_-^{-\alpha}} \,\mathrm{d}x \\ &= \frac{\alpha}{x_-^{-\alpha}} \int_{x_-}^{\infty} x^{m-\alpha-1} \,\mathrm{d}x \\ &= \frac{\alpha}{x_-^{-\alpha}} \left[\frac{x^{m-\alpha}}{m-\alpha} \right]_{x_-}^{\infty} \\ &= \frac{\alpha}{\alpha - m} x_-^m \end{aligned}$$

Estimation of the loss severity distribution

We deduce that:

$$\mathbb{E}[X] = \frac{\alpha}{\alpha - 1} x_-$$

and:

$$\begin{aligned}
\operatorname{var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] \\
&= \frac{\alpha}{\alpha - 2} x_-^2 - \left(\frac{\alpha}{\alpha - 1} x_-\right)^2 \\
&= \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)} x_-^2
\end{aligned}$$

Estimation of the loss severity distribution

We can then estimate the parameter α by considering the following empirical moments:

$$\begin{aligned} h_{i,1}(\alpha) &= x_i - \frac{\alpha}{\alpha - 1} x_{-} \\ h_{i,2}(\alpha) &= \left(x_i - \frac{\alpha}{\alpha - 1} x_{-} \right)^2 - \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)} x_{-}^2 \end{aligned}$$

The generalized method of moments can consider either the first moment $h_{i,1}(\alpha)$, the second moment $h_{i,2}(\alpha)$ or the joint moments $(h_{i,1}(\alpha), h_{i,2}(\alpha))$. In the first case, the estimator is:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i - n x_{-}}$$

Estimation of the loss severity distribution

Question 2.b

Find the maximum likelihood estimator $\hat{\alpha}$.

Estimation of the loss severity distribution

The log-likelihood function is:

$$\ell(\alpha) = \sum_{i=1}^{n} \ln f(x_i) = n \ln \alpha - (\alpha + 1) \sum_{i=1}^{n} \ln x_i + n\alpha \ln x_-$$

The first-order condition is:

$$\partial_\alpha \, \ell(\alpha) = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln x_- = 0$$

We deduce that:

$$n = \alpha \sum_{i=1}^{n} \ln \frac{x_i}{x_-}$$

The ML estimator is then:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} (\ln x_i - \ln x_-)}$$

Estimation of the loss severity distribution

Question 3

We consider the case *(iii)*. Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$. Deduce the first-order conditions of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$.

(iii) X follows a gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \, \mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

Estimation of the loss severity distribution

The probability density function of *(iii)* is:

$$f(x) = \frac{\partial \Pr\{X \leq x\}}{\partial x} = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

It follows that the log-likelihood function is:

$$\ell(\alpha,\beta) = \sum_{i=1}^{n} \ln f(x_i) = -n \ln \Gamma(\alpha) + n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} x_i$$

The first-order conditions $\partial_{\alpha}\,\ell(\alpha,\beta) = 0$ and $\partial_{\beta}\,\ell(\alpha,\beta) = 0$ imply that:

$$n\left(\ln \beta - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right) + \sum_{i=1}^{n} \ln x_i = 0$$

and:

$$n\frac{\alpha}{\beta} - \sum_{i=1}^{n} x_i = 0$$

Estimation of the loss severity distribution

Question 4

We consider the case *(iv)*. Show that the probability density function of X is:

$$f(x) = \frac{\beta^{\alpha} (\ln x)^{\alpha - 1}}{\Gamma(\alpha) x^{\beta + 1}}$$

What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$.

(iv) The natural logarithm of the loss X follows a gamma distribution: $\ln X \sim \Gamma(\alpha; \beta)$.

Estimation of the loss severity distribution

Let $Y \sim \Gamma(\alpha, \beta)$ and $X = \exp Y$. We have:

$$f_X(x)\,|\mathrm{d}x| = f_Y(y)\,|\mathrm{d}y|$$

where f_X and f_Y are the probability density functions of X and Y. We deduce that:

$$\begin{aligned} f_X(x) &= \frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} \times \frac{1}{e^{y}} \\ &= \frac{\beta^{\alpha} (\ln x)^{\alpha-1} e^{-\beta \ln x}}{x\Gamma(\alpha)} \\ &= \frac{\beta^{\alpha} (\ln x)^{\alpha-1}}{\Gamma(\alpha)\, x^{\beta+1}} \end{aligned}$$

The support of this probability density function is $[0, +\infty)$.

Estimation of the loss severity distribution

The log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$ is:

$$\begin{aligned}\boldsymbol{\ell}(\alpha, \beta) &= \sum_{i=1}^{n} \ln f(x_i) \\ &= -n \ln \Gamma(\alpha) + n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^{n} \ln(\ln x_i) - (\beta + 1) \sum_{i=1}^{n} \ln x_i\end{aligned}$$

Estimation of the loss severity distribution

Question 5

We now assume that the losses $\{x_1, \ldots, x_n\}$ have been collected beyond a threshold H meaning that $X \geq H$.

Estimation of the loss severity distribution

Question 5.a

What becomes the generalized method of moments in the case *(i)*.

(i) X follows a log-normal distribution $\mathcal{LN}\left(\mu,\sigma^2\right)$.

Estimation of the loss severity distribution

Using Bayes' formula, we have:

$$\begin{aligned}\Pr\{X \leq x \mid X \geq H\} &= \frac{\Pr\{H \leq X \leq x\}}{\Pr\{X \geq H\}} \\ &= \frac{\mathbf{F}(x) - \mathbf{F}(H)}{1 - \mathbf{F}(H)}\end{aligned}$$

where $\mathbf{F}$ is the cdf of X. We deduce that the conditional probability density function is:

$$\begin{aligned}f(x \mid X \geq H) &= \partial_x \Pr\{X \leq x \mid X \geq H\} \\ &= \frac{f(x)}{1 - \mathbf{F}(H)} \times \mathbb{1}\{x \geq H\}\end{aligned}$$

Estimation of the loss severity distribution

For the log-normal probability distribution, we obtain:

$$\begin{aligned} f(x \mid X \geq H) &= \frac{1}{1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} \,\mathrm{d}x \\ &= \varphi \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} \,\mathrm{d}x \end{aligned}$$

Estimation of the loss severity distribution

We note $\mathcal{M}_m(\mu,\sigma)$ the conditional moment $\mathbb{E}\left[X^m \mid X \geq H\right]$. We have:

$$\begin{aligned}
\mathcal{M}_m(\mu,\sigma) &= \varphi \times \int_H^\infty \frac{x^{m-1}}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} \,\mathrm{d}x \\
&= \varphi \times \int_{\ln H}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2+mx} \,\mathrm{d}x \\
&= \varphi \times e^{m\mu+\frac{1}{2}m^2\sigma^2} \times \int_{\ln H}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(x-\left(\mu+m\sigma^2\right)\right)^2}{\sigma^2}} \,\mathrm{d}x \\
&= \frac{1-\Phi\left(\frac{\ln H-\mu-m\sigma^2}{\sigma}\right)}{1-\Phi\left(\frac{\ln H-\mu}{\sigma}\right)} e^{m\mu+\frac{1}{2}m^2\sigma^2}
\end{aligned}$$

Estimation of the loss severity distribution

The first two moments of $X \mid X \geq H$ are then:

$$\mathcal{M}_1(\mu,\sigma) = \mathbb{E}[X \mid X \geq H] = \frac{1-\Phi\left(\frac{\ln H - \mu - \sigma^2}{\sigma}\right)}{1-\Phi\left(\frac{\ln H - \mu}{\sigma}\right)} e^{\mu + \frac{1}{2}\sigma^2}$$

and:

$$\mathcal{M}_2(\mu,\sigma) = \mathbb{E}\left[X^2 \mid X \geq H\right] = \frac{1-\Phi\left(\frac{\ln H - \mu - 2\sigma^2}{\sigma}\right)}{1-\Phi\left(\frac{\ln H - \mu}{\sigma}\right)} e^{2\mu + 2\sigma^2}$$

Estimation of the loss severity distribution

We can therefore estimate μ and σ by considering the following empirical moments:

$$\left\{\begin{array}{l} h_{i,1}\left(\mu,\sigma\right) = x_i - \mathcal{M}_1\left(\mu,\sigma\right) \\ h_{i,2}\left(\mu,\sigma\right) = \left(x_i - \mathcal{M}_1\left(\mu,\sigma\right)\right)^2 - \left(\mathcal{M}_2\left(\mu,\sigma\right) - \mathcal{M}_1^2\left(\mu,\sigma\right)\right) \end{array}\right.$$

Estimation of the loss severity distribution

Question 5.b

Calculate the maximum likelihood estimator $\hat{\alpha}$ in the case *(ii)*.

(ii) X follows a Pareto distribution $\mathcal{P}\left(\alpha, x^{-}\right)$ defined by:

$$\Pr\left\{X \leq x\right\} = 1 - \left(\frac{x}{x_{-}}\right)^{-\alpha}$$

with $x \geq x_{-}$ and $\alpha > 0$.

Estimation of the loss severity distribution

We have:

$$\begin{aligned} f(x \mid X \geq H) &= \frac{f(x)}{1 - \mathbf{F}(H)} \times \mathbb{1}\{x \geq H\} \\ &= \left(\alpha \frac{x^{-(\alpha+1)}}{x_-^{-\alpha}}\right) \Big/ \left(\frac{H^{-\alpha}}{x_-^{-\alpha}}\right) \\ &= \alpha \frac{x^{-(\alpha+1)}}{H^{-\alpha}} \end{aligned}$$

The conditional probability function is then a Pareto distribution with the same parameter α but with a new threshold $x_- = H$. We can then deduce that the ML estimator $\hat{\alpha}$ is:

$$\hat{\alpha} = \frac{n}{\left(\sum_{i=1}^{n} \ln x_i\right) - n \ln H}$$

Estimation of the loss severity distribution

Question 5.c

Write the log-likelihood function in the case *(iii)*.

(iii) X follows a gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \,\mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

Estimation of the loss severity distribution

The conditional probability density function is:

$$
\begin{aligned}
f(x \mid X \geq H) &= \frac{f(x)}{1 - \mathbf{F}(H)} \times \mathbb{1}\{x \geq H\} \\
&= \left(\frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}\right) \Big/ \int_H^{\infty} \frac{\beta^{\alpha} t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \, \mathrm{d}t \\
&= \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\int_H^{\infty} \beta^{\alpha} t^{\alpha-1} e^{-\beta t} \, \mathrm{d}t}
\end{aligned}
$$

We deduce that the log-likelihood function is:

$$
\begin{aligned}
\ell(\alpha, \beta) &= n\alpha \ln \beta - n \ln \left(\int_H^{\infty} \beta^{\alpha} t^{\alpha-1} e^{-\beta t} \, \mathrm{d}t\right) + \\
&\quad (\alpha - 1) \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} x_i
\end{aligned}
$$

Estimation of the loss frequency distribution

Exercise

We consider a dataset of individual losses $\{x_1, \ldots, x_n\}$ corresponding to a sample of T annual loss numbers $\{N_{Y_1}, \ldots, N_{Y_T}\}$. This implies that:

$$\sum_{t=1}^{T} N_{Y_t} = n$$

If we measure the number of losses per quarter $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$, we use the notation:

$$\sum_{t=1}^{4T} N_{Q_t} = n$$

Estimation of the loss frequency distribution

Question 1

We assume that the annual number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Y)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Y$ associated to the sample $\{N_{Y_1}, \ldots, N_{Y_T}\}$.

Estimation of the loss frequency distribution

We have:

$$\Pr\left\{N = n\right\} = e^{-\lambda_Y} \frac{\lambda_Y^n}{n!}$$

We deduce that the expression of the log-likelihood function is:

$$\ell\left(\lambda_Y\right) = \sum_{t=1}^{T} \ln \Pr\left\{N = N_{Y_t}\right\} = -\lambda_Y T + \left(\sum_{t=1}^{T} N_{Y_t}\right) \ln \lambda_Y - \sum_{t=1}^{T} \ln\left(N_{Y_t}!\right)$$

The first-order condition is:

$$\frac{\partial\, \ell\left(\lambda_Y\right)}{\partial\, \lambda_Y} = -T + \frac{1}{\lambda_Y}\left(\sum_{t=1}^{T} N_{Y_t}\right) = 0$$

We deduce that the ML estimator is:

$$\hat{\lambda}_Y = \frac{1}{T}\sum_{t=1}^{T} N_{Y_t} = \frac{n}{T}$$

Estimation of the loss frequency distribution

Question 2

We assume that the quarterly number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Q)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Q$ associated to the sample $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$.

Estimation of the loss frequency distribution

Using the same arguments, we obtain:

$$\hat{\lambda}_Q = \frac{1}{4T} \sum_{t=1}^{4T} N_{Q_t} = \frac{n}{4T} = \frac{\hat{\lambda}_Y}{4}$$

Estimation of the loss frequency distribution

Question 3

What is the impact of considering a quarterly or annual basis on the computation of the capital charge?

Estimation of the loss frequency distribution

Considering a quarterly or annual basis has no impact on the capital charge. Indeed, the capital charge is computed with a one-year time horizon. If we use a quarterly basis, we have to find the distribution of the annual loss number. In this case, the annual loss number is the sum of the four quarterly loss numbers:

$$N_Y = N_{Q_1} + N_{Q_2} + N_{Q_3} + N_{Q_4}$$

We know that each quarterly loss number follows a Poisson distribution $\mathcal{P}\left(\hat{\lambda}_Q\right)$ and that they are independent. Because the Poisson distribution is infinitely divisible, we obtain:

$$N_{Q_1} + N_{Q_2} + N_{Q_3} + N_{Q_4} \sim \mathcal{P}\left(4\hat{\lambda}_Q\right)$$

We deduce that the annual loss number follows a Poisson distribution $\mathcal{P}\left(\hat{\lambda}_Y\right)$ in both cases.

Estimation of the loss frequency distribution

Question 4

What does this result become if we consider a method of moments based on the first moment?

Estimation of the loss frequency distribution

Since we have $\mathbb{E}\left[\mathcal{P}\left(\lambda\right)\right] = \lambda$, the MM estimator in the case of annual loss numbers is:

$$\hat{\lambda}_Y = \frac{1}{T} \sum_{t=1}^{T} N_{Y_t} = \frac{n}{T}$$

The MM estimator is exactly the ML estimator.

Estimation of the loss frequency distribution

Question 5

Same question if we consider a method of moments based on the second moment.

Estimation of the loss frequency distribution

Since we have $\operatorname{var}(\mathcal{P}(\lambda)) = \lambda$, the MM estimator in the case of annual loss numbers is:

$$\hat{\lambda}_Y = \frac{1}{T}\sum_{t=1}^{T} N_{Y_t}^2 - \frac{n^2}{T^2}$$

If we use a quarterly basis, we obtain:

$$\begin{aligned}\hat{\lambda}_Q &= \frac{1}{4}\left(\frac{1}{T}\sum_{t=1}^{4T} N_{Q_t}^2 - \frac{n^2}{4T^2}\right) \\ &\neq \frac{\hat{\lambda}_Y}{4}\end{aligned}$$

There is no reason that $\hat{\lambda}_Y = 4\hat{\lambda}_Q$ meaning that the capital charge will not be the same.

Computation of the amortization functions

Exercise

In what follows, we consider a debt instrument, whose remaining maturity is equal to m. We note t the current date and $T = t + m$ the maturity date.

Computation of the amortization functions

Question 1

We consider a bullet repayment debt. Define its amortization function $\mathbf{S}(t, u)$. Calculate the survival function $\mathbf{S}^{\star}(t, u)$ of the stock. Show that:

$$\mathbf{S}^{\star}(t, u) = \mathbb{1}\{t \leq u < t + m\} \cdot \left(1 - \frac{u - t}{m}\right)$$

in the case where the new production is constant. Comment on this result.

Computation of the amortization functions

By definition, we have:

$$\mathbf{S}(t,u) = \mathbb{1}\{t \leq u < t+m\} = \begin{cases} 1 & \text{if } u \in [t, t+m[\\ 0 & \text{otherwise} \end{cases}$$

This means that the survival function is equal to one when u is between the current date t and the maturity date $T = t + m$. When u reaches T, the outstanding amount is repaid, implying that $\mathbf{S}(t, T)$ is equal to zero. It follows that:

$$\begin{aligned} \mathbf{S}^{\star}(t,u) &= \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, \mathbf{S}(s,u)\, \mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, \mathbf{S}(s,t)\, \mathrm{d}s} \\ &= \frac{\int_{-\infty}^{t} \mathrm{NP}(s) \cdot \mathbb{1}\{s \leq u < s+m\}\, \mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s) \cdot \mathbb{1}\{s \leq t < s+m\}\, \mathrm{d}s} \end{aligned}$$

Computation of the amortization functions

For the numerator, we have:

$$\begin{aligned} \mathbb{1}\{s \leq u < s + m\} = 1 \quad & \Rightarrow \quad u < s + m \\ & \Leftrightarrow \quad s > u - m \end{aligned}$$

and:

$$\int_{-\infty}^{t} \mathrm{NP}(s) \cdot \mathbb{1}\{s \leq u < s + m\} \, \mathrm{d}s = \int_{u-m}^{t} \mathrm{NP}(s) \, \mathrm{d}s$$

Computation of the amortization functions

For the denominator, we have:

$$\begin{aligned} \mathbb{1}\{s \leq t < s+m\} = 1 & \Rightarrow & t < s+m \\ & \Leftrightarrow & s > t-m \end{aligned}$$

and:

$$\int_{-\infty}^{t} \mathrm{NP}(s) \cdot \mathbb{1}\{s \leq t < s+m\} \,\mathrm{d}s = \int_{t-m}^{t} \mathrm{NP}(s) \,\mathrm{d}s$$

We deduce that:

$$\mathbf{S}^{\star}(t, u) = \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\int_{u-m}^{t} \mathrm{NP}(s) \,\mathrm{d}s}{\int_{t-m}^{t} \mathrm{NP}(s) \,\mathrm{d}s}$$

Computation of the amortization functions

In the case where the new production is a constant, we have $\mathrm{NP}(s) = c$ and:

$$
\begin{aligned}
\mathbf{S}^\star(t, u) &= \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\int_{u-m}^{t} \mathrm{d}s}{\int_{t-m}^{t} \mathrm{d}s} \\
&= \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\left[s\right]_{u-m}^{t}}{\left[s\right]_{t-m}^{t}} \\
&= \mathbb{1}\{t \leq u < t+m\} \cdot \left(\frac{t-u+m}{t-t+m}\right) \\
&= \mathbb{1}\{t \leq u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)
\end{aligned}
$$

The survival function $\mathbf{S}^\star(t, u)$ corresponds to the case of a linear amortization.

Computation of the amortization functions

Question 2

Same question if we consider a debt instrument, whose amortization rate is constant.

Computation of the amortization functions

If the amortization is linear, we have:

$$\mathbf{S}(t,u) = \mathbb{1}\{t \leq u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)$$

We deduce that:

$$\mathbf{S}^{\star}(t,u) = \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\int_{u-m}^{t} \mathrm{NP}(s)\left(1 - \frac{u-s}{m}\right) \mathrm{d}s}{\int_{t-m}^{t} \mathrm{NP}(s)\left(1 - \frac{t-s}{m}\right) \mathrm{d}s}$$

In the case where the new production is a constant, we obtain:

$$\mathbf{S}^{\star}(t,u) = \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\int_{u-m}^{t} \left(1 - \frac{u-s}{m}\right) \mathrm{d}s}{\int_{t-m}^{t} \left(1 - \frac{t-s}{m}\right) \mathrm{d}s}$$

Computation of the amortization functions

For the numerator, we have:

$$
\begin{aligned}
\int_{u-m}^{t}\left(1-\frac{u-s}{m}\right)\mathrm{d}s &= \left[s-\frac{su}{m}+\frac{s^2}{2m}\right]_{u-m}^{t} \\
&= \left(t-\frac{tu}{m}+\frac{t^2}{2m}\right)- \\
&\quad \left(u-m-\frac{u^2-mu}{m}+\frac{(u-m)^2}{2m}\right) \\
&= \left(t-\frac{tu}{m}+\frac{t^2}{2m}\right)-\left(u-\frac{m}{2}-\frac{u^2}{2m}\right) \\
&= \frac{m^2+u^2+t^2+2mt-2mu-2tu}{2m} \\
&= \frac{(m-u+t)^2}{2m}
\end{aligned}
$$

Computation of the amortization functions

For the denominator, we use the previous result and we set $u = t$:

$$\begin{aligned}\int_{t-m}^{t}\left(1-\frac{t-s}{m}\right)\mathrm{d}s &= \frac{(m-t+t)^2}{2m} \\ &= \frac{m}{2}\end{aligned}$$

Computation of the amortization functions

We deduce that:

$$\begin{aligned}
\mathbf{S}^{\star}(t,u) &= \mathbb{1}\{t \leq u < t+m\} \cdot \frac{\dfrac{(m-u+t)^2}{2m}}{\dfrac{m}{2}} \\
&= \mathbb{1}\{t \leq u < t+m\} \cdot \frac{(m-u+t)^2}{m^2} \\
&= \mathbb{1}\{t \leq u < t+m\} \cdot \left(1 - \frac{u-t}{m}\right)^2
\end{aligned}$$

The survival function $\mathbf{S}^{\star}(t,u)$ corresponds to the case of a parabolic amortization.

Computation of the amortization functions

Question 3

Same question if we assume[a] that the amortization function is exponential with parameter λ.

[a] By definition of the exponential amortization, we have $m = +\infty$.

Computation of the amortization functions

If the amortization is exponential, we have:

$$\mathbf{S}(t,u) = e^{-\int_t^u \lambda \,\mathrm{d}s} = e^{-\lambda(u-t)}$$

It follows that:

$$\mathbf{S}^{\star}(t,u) = \frac{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\lambda(u-s)}\,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s)\, e^{-\lambda(t-s)}\,\mathrm{d}s}$$

In the case where the new production is a constant, we obtain:

$$\begin{aligned}
\mathbf{S}^{\star}(t,u) &= \frac{\int_{-\infty}^{t} e^{-\lambda(u-s)}\,\mathrm{d}s}{\int_{-\infty}^{t} e^{-\lambda(t-s)}\,\mathrm{d}s} \\
&= \frac{\left[\lambda^{-1} e^{-\lambda(u-s)}\right]_{-\infty}^{t}}{\left[\lambda^{-1} e^{-\lambda(t-s)}\right]_{-\infty}^{t}} \\
&= e^{-\lambda(u-t)} \\
&= \mathbf{S}(t,u)
\end{aligned}$$

The stock amortization function is equal to the flow amortization function.

Computation of the amortization functions

Question 4

Find the expression of $\mathcal{D}^{\star}(t)$ when the new production is constant.

Computation of the amortization functions

We recall that the liquidity duration is equal to:

$$\mathcal{D}(t) = \int_t^{\infty} (u - t) f(t, u) \,\mathrm{d}u$$

where $f(t, u)$ is the density function associated to the survival function $\mathbf{S}(t, u)$. For the stock, we have:

$$\mathcal{D}^{\star}(t) = \int_t^{\infty} (u - t) f^{\star}(t, u) \,\mathrm{d}u$$

where $f^{\star}(t, u)$ is the density function associated to the survival function $\mathbf{S}^{\star}(t, u)$:

$$f^{\star}(t, u) = \frac{\int_{-\infty}^{t} \mathrm{NP}(s) f(s, u) \,\mathrm{d}s}{\int_{-\infty}^{t} \mathrm{NP}(s) \mathbf{S}(s, t) \,\mathrm{d}s}$$

Computation of the amortization functions

In the case where the new production is constant, we obtain:

$$\mathcal{D}^{\star}(t) = \frac{\int_t^{\infty} (u - t) \int_{-\infty}^{t} f(s, u) \,\mathrm{d}s \,\mathrm{d}u}{\int_{-\infty}^{t} \mathbf{S}(s, t) \,\mathrm{d}s}$$

Since we have $\int_{-\infty}^{t} f(s, u) \,\mathrm{d}s = \mathbf{S}(t, u)$, we deduce that:

$$\mathcal{D}^{\star}(t) = \frac{\int_t^{\infty} (u - t) \mathbf{S}(t, u) \,\mathrm{d}u}{\int_{-\infty}^{t} \mathbf{S}(s, t) \,\mathrm{d}s}$$

Computation of the amortization functions

Question 5

Calculate the durations $\mathcal{D}(t)$ and $\mathcal{D}^{\star}(t)$ for the three previous cases.

Computation of the amortization functions

In the case of the bullet repayment debt, we have:

$$\mathcal{D}(t) = m$$

and:

$$\begin{aligned}
\mathcal{D}^{\star}(t) &= \frac{\int_t^{t+m} (u - t)\,\mathrm{d}u}{\int_{t-m}^{t} \mathrm{d}s} \\
&= \frac{\left[\frac{1}{2}(u - t)^2\right]_t^{t+m}}{\left[s\right]_{t-m}^{t}} \\
&= \frac{m}{2}
\end{aligned}$$

Computation of the amortization functions

In the case of the linear amortization, we have:

$$f(t,u) = \mathbb{1}\{t \leq u < t+m\} \cdot \frac{1}{m}$$

and:

$$\begin{aligned} \mathcal{D}(t) &= \int_t^{t+m} \frac{(u-t)}{m} \,\mathrm{d}u \\ &= \frac{1}{m}\left[\frac{1}{2}(u-t)^2\right]_t^{t+m} \\ &= \frac{m}{2} \end{aligned}$$

Computation of the amortization functions

For the stock duration, we deduce that

$$
\begin{aligned}
\mathcal{D}^{\star}(t) &= \frac{\displaystyle\int_t^{t+m} (u-t)\left(1-\frac{u-t}{m}\right)\mathrm{d}u}{\displaystyle\int_{t-m}^{t}\left(1-\frac{t-s}{m}\right)\mathrm{d}s} \\
&= \frac{\displaystyle\int_t^{t+m}\left(u-t-\frac{u^2}{m}+2\frac{tu}{m}-\frac{t^2}{m}\right)\mathrm{d}u}{\displaystyle\int_{t-m}^{t}\left(1-\frac{t}{m}+\frac{s}{m}\right)\mathrm{d}s} \\
&= \frac{\left[\dfrac{u^2}{2}-tu-\dfrac{u^3}{3m}+\dfrac{tu^2}{m}-\dfrac{t^2u}{m}\right]_t^{t+m}}{\left[s-\dfrac{st}{m}+\dfrac{s^2}{2m}\right]_{t-m}^{t}}
\end{aligned}
$$

Computation of the amortization functions

The numerator is equal to:

$$\begin{aligned}
(*) &= \left[\frac{u^2}{2} - tu - \frac{u^3}{3m} + \frac{tu^2}{m} - \frac{t^2u}{m}\right]_t^{t+m} \\
&= \frac{1}{6m}\left[3mu^2 - 6mtu - 2u^3 + 6tu^2 - 6t^2u\right]_t^{t+m} \\
&= \frac{1}{6m}\left(m^3 - 3mt^2 - 2t^3\right) + \frac{1}{6m}\left(3mt^2 + 2t^3\right) \\
&= \frac{m^2}{6}
\end{aligned}$$

Computation of the amortization functions

The denominator is equal to:

$$
\begin{aligned}
(*) &= \left[s - \frac{st}{m} + \frac{s^2}{2m}\right]_{t-m}^{t} \\
&= \frac{1}{2m}\left[s^2 - 2s(t-m)\right]_{t-m}^{t} \\
&= \frac{1}{2m}\left(t^2 - 2t(t-m) - (t-m)^2 + 2(t-m)^2\right) \\
&= \frac{1}{2m}\left(t^2 - 2t^2 + 2mt + t^2 - 2mt + m^2\right) \\
&= \frac{m}{2}
\end{aligned}
$$

Computation of the amortization functions

We deduce that:

$$\mathcal{D}^{\star}(t) = \frac{m}{3}$$

Computation of the amortization functions

For the exponential amortization, we have:

$$f(t,u) = \lambda e^{-\lambda(u-t)}$$

and[1]:

$$\mathcal{D}(t) = \int_t^{\infty} (u-t)\,\lambda e^{-\lambda(u-t)}\,\mathrm{d}u = \int_0^{\infty} v\lambda e^{-\lambda v}\,\mathrm{d}v = \frac{1}{\lambda}$$

For the stock duration, we deduce that:

$$\mathcal{D}^{\star}(t) = \frac{\int_t^{\infty} (u-t)\, e^{-\lambda(u-t)}\,\mathrm{d}u}{\int_{-\infty}^{t} e^{-\lambda(t-s)}\,\mathrm{d}s} = \frac{\int_0^{\infty} v e^{-\lambda v}\,\mathrm{d}v}{\int_0^{\infty} e^{-\lambda v}\,\mathrm{d}v} = \frac{1}{\lambda}$$

We verify that $\mathcal{D}(t) = \mathcal{D}^{\star}(t)$ since we have demonstrated that $\mathbf{S}^{\star}(t,u) = \mathbf{S}(t,u)$.

[1] We use the change of variable $v = u - t$.

Computation of the amortization functions

Question 6

Calculate the corresponding dynamics $\mathrm{d}N(t)$.

Computation of the amortization functions

In the case of the bullet repayment debt, we have:

$$\mathrm{d}N(t) = (\mathrm{NP}(t) - \mathrm{NP}(t-m))\,\mathrm{d}t$$

Computation of the amortization functions

In the case of the linear amortization, we have:

$$f(s,t) = \frac{\mathbf{1}\{s \leq t < s+m\}}{m}$$

It follows that:

$$\begin{aligned}
\int_{-\infty}^{t} \mathrm{NP}(s)\, f(s,t)\, \mathrm{d}s &= \frac{1}{m}\int_{-\infty}^{t} \mathbf{1}\{s \leq t < s+m\} \cdot \mathrm{NP}(s)\, \mathrm{d}s \\
&= \frac{1}{m}\int_{t-m}^{t} \mathrm{NP}(s)\, \mathrm{d}s
\end{aligned}$$

We deduce that:

$$\mathrm{d}N(t) = \left(\mathrm{NP}(t) - \frac{1}{m}\int_{t-m}^{t} \mathrm{NP}(s)\, \mathrm{d}s\right)\mathrm{d}t$$

Computation of the amortization functions

For the exponential amortization, we have:

$$f(s,t) = \lambda e^{-\lambda(t-s)}$$

and:

$$\begin{aligned}
\int_{-\infty}^{t} \mathrm{NP}(s) f(s,t)\,\mathrm{d}s &= \int_{-\infty}^{t} \mathrm{NP}(s)\,\lambda e^{-\lambda(t-s)}\,\mathrm{d}s \\
&= \lambda \int_{-\infty}^{t} \mathrm{NP}(s)\,e^{-\lambda(t-s)}\,\mathrm{d}s \\
&= \lambda N(t)
\end{aligned}$$

We deduce that:

$$\mathrm{d}N(t) = (\mathrm{NP}(t) - \lambda N(t))\,\mathrm{d}t$$

Impact of prepayment

Exercise

We recall that the outstanding balance of a CAM (constant amortization mortgage) at time t is given by:

$$N(t) = \mathbf{1}\{t < m\} \cdot N_0 \cdot \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}$$

where N_0 is the notional, i is the interest rate and m is the maturity.

Impact of prepayment

Question 1

Find the dynamics $\mathrm{d}N(t)$.

Impact of prepayment

We deduce that the dynamics of $N(t)$ is equal to:

$$\begin{aligned}
\mathrm{d}N(t) &= \mathbb{1}\{t<m\} \cdot N_0 \frac{-ie^{-i(m-t)}}{1-e^{-im}}\,\mathrm{d}t \\
&= -ie^{-i(m-t)}\left(\mathbb{1}\{t<m\} \cdot N_0 \frac{1}{1-e^{-im}}\right)\mathrm{d}t \\
&= -\frac{ie^{-i(m-t)}}{1-e^{-i(m-t)}} N(t)\,\mathrm{d}t
\end{aligned}$$

Impact of prepayment

Question 2

We note $\tilde{N}(t)$ the modified outstanding balance that takes into account the prepayment risk. Let $\lambda_p(t)$ be the prepayment rate at time t. Write the dynamics of $\tilde{N}(t)$.

Impact of prepayment

The prepayment rate has a negative impact on $\mathrm{d}N(t)$ because it reduces the outstanding amount $N(t)$:

$$\mathrm{d}\tilde{N}(t) = -\frac{ie^{-i(m-t)}}{1-e^{-i(m-t)}}\tilde{N}(t)\,\mathrm{d}t - \lambda_p(t)\,\tilde{N}(t)\,\mathrm{d}t$$

Impact of prepayment

Question 3

Show that $\tilde{N}(t) = N(t)\,\mathbf{S}_p(t)$ where $\mathbf{S}_p(t)$ is the prepayment-based survival function.

Impact of prepayment

It follows that:

$$\mathrm{d} \ln \tilde{N}(t) = -\left(\frac{ie^{-i(m-t)}}{1 - e^{-i(m-t)}} + \lambda_p(t)\right) \mathrm{d}t$$

and:

$$\begin{aligned}
\ln \tilde{N}(t) - \ln \tilde{N}(0) &= \int_0^t \frac{-ie^{-i(m-s)}}{1 - e^{-i(m-s)}} \,\mathrm{d}s - \int_0^t \lambda_p(s) \,\mathrm{d}s \\
&= \left[\ln\left(1 - e^{-i(m-s)}\right)\right]_0^t - \int_0^t \lambda_p(s) \,\mathrm{d}s \\
&= \ln\left(\frac{1 - e^{-i(m-t)}}{1 - e^{-im}}\right) - \int_0^t \lambda_p(s) \,\mathrm{d}s
\end{aligned}$$

Impact of prepayment

We deduce that:

$$\begin{aligned}\tilde{N}(t) &= \left(N_0 \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}\right) e^{-\int_0^t \lambda_p(s)\,\mathrm{d}s} \\ &= N(t)\,\mathbf{S}_p(t)\end{aligned}$$

where $\mathbf{S}_p(t)$ is the survival function associated to the hazard rate $\lambda_p(t)$.

Impact of prepayment

Question 4

Calculate the liquidity duration $\tilde{\mathcal{D}}(t)$ associated to the outstanding balance $\tilde{N}(t)$ when the hazard rate of prepayments is constant and equal to λ_p.

Impact of prepayment

We have:

$$\tilde{N}(t,u) = \mathbf{1}\{t \leq u < t+m\} \cdot N(t) \frac{1-e^{-i(t+m-u)}}{1-e^{-im}} e^{-\lambda_p(u-t)}$$

this implies that:

$$\tilde{\mathbf{S}}(t,u) = \mathbf{1}\{t \leq u < t+m\} \cdot \frac{e^{-\lambda_p(u-t)} - e^{-im+(i-\lambda_p)(u-t)}}{1-e^{-im}}$$

and:

$$\tilde{f}(t,u) = \mathbf{1}\{t \leq u < t+m\} \cdot \frac{\lambda_p e^{-\lambda_p(u-t)} + (i-\lambda_p) e^{-im+(i-\lambda_p)(u-t)}}{1-e^{-im}}$$

Impact of prepayment

It follows that:

$$\begin{aligned}
\tilde{\mathcal{D}}(t) &= \frac{\lambda_p}{1-e^{-im}}\int_t^{t+m}(u-t)\,e^{-\lambda_p(u-t)}\,\mathrm{d}u + \\
&\quad \frac{(i-\lambda_p)\,e^{-im}}{1-e^{-im}}\int_t^{t+m}(u-t)\,e^{(i-\lambda_p)(u-t)}\,\mathrm{d}u \\
&= \frac{\lambda_p}{1-e^{-im}}\int_0^{m} ve^{-\lambda_p v}\,\mathrm{d}v + \frac{(i-\lambda_p)\,e^{-im}}{1-e^{-im}}\int_0^{m} ve^{(i-\lambda_p)v}\,\mathrm{d}v \\
&= \frac{\lambda_p}{1-e^{-im}}\left(\frac{me^{-\lambda_p m}}{-\lambda_p}-\frac{e^{-\lambda_p m}-1}{\lambda_p^2}\right) + \\
&\quad \frac{(i-\lambda_p)\,e^{-im}}{1-e^{-im}}\left(\frac{me^{(i-\lambda_p)m}}{(i-\lambda_p)}-\frac{e^{(i-\lambda_p)m}-1}{(i-\lambda_p)^2}\right) \\
&= \frac{1}{1-e^{-im}}\left(\frac{e^{-im}-e^{-\lambda_p m}}{i-\lambda_p}+\frac{1-e^{-\lambda_p m}}{\lambda_p}\right)
\end{aligned}$$

Impact of prepayment

because we have:

$$\begin{aligned}
\int_0^m v e^{\alpha v} \,\mathrm{d}v &= \left[\frac{v e^{\alpha v}}{\alpha}\right]_0^m - \int_0^m \frac{e^{\alpha v}}{\alpha} \,\mathrm{d}v \\
&= \left[\frac{v e^{\alpha v}}{\alpha}\right]_0^m - \left[\frac{e^{\alpha v}}{\alpha^2}\right]_0^m \\
&= \frac{m e^{\alpha m}}{\alpha} - \frac{e^{\alpha m} - 1}{\alpha^2}
\end{aligned}$$