Financial Risk Management
Tutorial Class — Session 5

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The bivariate Pareto copula

Exercise

We consider the bivariate Pareto distribution:

$$\begin{aligned}\mathbf{F}(x_1, x_2) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + x_2}{\theta_2} - 1\right)^{-\alpha}\end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\alpha > 0$.

The bivariate Pareto copula

Question 1

Show that the marginal functions of $\mathbf{F}(x_1, x_2)$ correspond to univariate Pareto distributions.

The bivariate Pareto copula

We have:

$$\begin{aligned}\mathbf{F}_1(x_1) &= \Pr\{X_1 \leq x_1\} \\ &= \Pr\{X_1 \leq x_1, X_2 \leq \infty\} \\ &= \mathbf{F}(x_1, \infty)\end{aligned}$$

We deduce that:

$$\begin{aligned}\mathbf{F}_1(x_1) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + \infty}{\theta_2}\right)^{-\alpha} + \\ & \quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + \infty}{\theta_2} - 1\right)^{-\alpha} \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha}\end{aligned}$$

We conclude that $\mathbf{F}_1$ (and $\mathbf{F}_2$) is a Pareto distribution.

The bivariate Pareto copula

Question 2

Find the copula function associated to the bivariate Pareto distribution.

The bivariate Pareto copula

We have:

$$\mathbf{C}(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)$$

It follows that:

$$\begin{aligned} 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= u_1 \\ \Leftrightarrow \quad \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= 1 - u_1 \\ \Leftrightarrow \quad \frac{\theta_1 + x_1}{\theta_1} &= (1 - u_1)^{-1/\alpha} \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= 1 - (1 - u_1) - (1 - u_2) + \\ & \quad \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \\ &= u_1 + u_2 - 1 + \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \end{aligned}$$

The bivariate Pareto copula

Question 3

Deduce the copula density function.

The bivariate Pareto copula

We have:

$$\begin{aligned}
\frac{\partial\,\mathbf{C}\left(u_1,u_2\right)}{\partial\,u_1} &= 1-\alpha\left(\left(1-u_1\right)^{-1/\alpha}+\left(1-u_2\right)^{-1/\alpha}-1\right)^{-\alpha-1}\times \\
&\quad \left(-\frac{1}{\alpha}\right)\left(1-u_1\right)^{-1/\alpha-1}\times\left(-1\right) \\
&= 1-\left(\left(1-u_1\right)^{-1/\alpha}+\left(1-u_2\right)^{-1/\alpha}-1\right)^{-\alpha-1}\times \\
&\quad \left(1-u_1\right)^{-1/\alpha-1}
\end{aligned}$$

The bivariate Pareto copula

We deduce that the probability density function of the copula is:

$$
\begin{aligned}
c(u_1, u_2) &= \frac{\partial^2 \mathbf{C}(u_1, u_2)}{\partial\, u_1\, \partial\, u_2} \\
&= -(-\alpha - 1)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\
&\quad \left(-\frac{1}{\alpha}\right)(1-u_2)^{-1/\alpha-1} \times (-1) \times (1-u_1)^{-1/\alpha-1} \\
&= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\
&\quad (1 - u_1 - u_2 + u_1 u_2)^{-1/\alpha-1}
\end{aligned}
$$

The bivariate Pareto copula

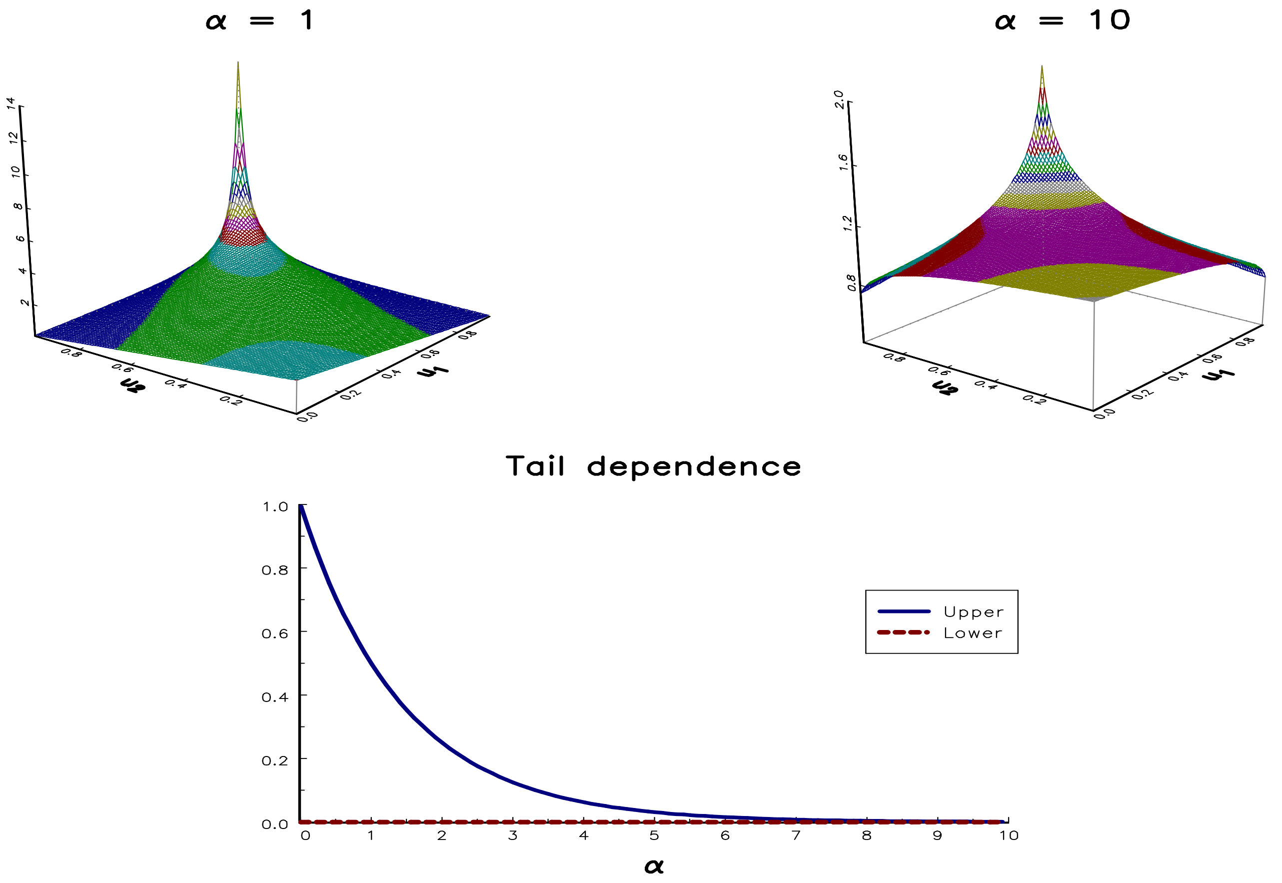
Remark

Another expression of $c(u_1, u_2)$ is:

$$
\begin{aligned}
c(u_1, u_2) &= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)(1-u_2)\right)^{1/\alpha} \times \\
&\quad \left((1-u_1)^{1/\alpha} + (1-u_2)^{1/\alpha} - (1-u_1)^{1/\alpha}(1-u_2)^{1/\alpha}\right)^{-\alpha-2}
\end{aligned}
$$

The bivariate Pareto copula

In this Figure, we have reported the density of the Pareto copula when α is equal to 1 and 10.



The bivariate Pareto copula

Question 4

Show that the bivariate Pareto copula function has no lower tail dependence, but an upper tail dependence.

The bivariate Pareto copula

We have:

$$\begin{aligned}
\lambda^- &= \lim_{u \to 0^+} \frac{\mathbf{C}(u,u)}{u} \\
&= 2 \lim_{u \to 0^+} \frac{\partial\, \mathbf{C}(u,u)}{\partial\, u_1} \\
&= 2 \lim_{u \to 0^+} 1 - \left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1 \right)^{-\alpha-1} (1-u)^{-1/\alpha-1} \\
&= 2 \lim_{u \to 0^+} (1-1) \\
&= 0
\end{aligned}$$

The bivariate Pareto copula

We have:

$$\begin{aligned}
\lambda^{+} &= \lim_{u \to 1^{-}} \frac{1 - 2u + \mathbf{C}(u, u)}{1 - u} \\
&= \lim_{u \to 1^{-}} \frac{\left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1 \right)^{-\alpha}}{1 - u} \\
&= \lim_{u \to 1^{-}} \left(1 + 1 - (1-u)^{1/\alpha} \right)^{-\alpha} \\
&= 2^{-\alpha}
\end{aligned}$$

The tail dependence coefficients λ^{-} and λ^{+} are given with respect to the parameter α in previous Figure. We deduce that the bivariate Pareto copula function has no lower tail dependence ($\lambda^{-} = 0$), but an upper tail dependence ($\lambda^{+} = 2^{-\alpha}$).

The bivariate Pareto copula

Question 5

Do you think that the bivariate Pareto copula family can reach the copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$? Justify your answer.

The bivariate Pareto copula

The bivariate Pareto copula family cannot reach $\mathbf{C}^-$ because λ^- is never equal to 1. We notice that:

$$\lim_{\alpha \to \infty} \lambda^+ = 0$$

and

$$\lim_{\alpha \to 0} \lambda^+ = 1$$

This implies that the bivariate Pareto copula may reach $\mathbf{C}^\perp$ and $\mathbf{C}^+$ for these two limit cases: $\alpha \to \infty$ and $\alpha \to 0$. In fact, $\alpha \to 0$ does not correspond to the copula $\mathbf{C}^+$ because λ^- is always equal to 0.

The bivariate Pareto copula

Question 6

Let X_1 and X_2 be two Pareto-distributed random variables, whose parameters are (α_1, θ_1) and (α_2, θ_2).

The bivariate Pareto copula

Question 6.a

Show that the linear correlation between X_1 and X_2 is equal to 1 if and only if the parameters α_1 and α_2 are equal.

The bivariate Pareto copula

We note $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. X_1 and X_2 are comonotonic if and only if:

$$U_2 = U_1$$

We deduce that:

$$\begin{aligned}
1 - \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad X_2 = \theta_2 &\left(\left(\frac{\theta_1 + X_1}{\theta_1}\right)^{\alpha_1/\alpha_2} - 1\right)
\end{aligned}$$

We know that $\rho\langle X_1, X_2\rangle = 1$ if and only if there is an increasing linear relationship between X_1 and X_2. This implies that:

$$\frac{\alpha_1}{\alpha_2} = 1$$

The bivariate Pareto copula

Question 6.b

Show that the linear correlation between X_1 and X_2 can never reached the lower bound -1.

The bivariate Pareto copula

X_1 and X_2 are countermonotonic if and only if:

$$U_2 = 1 - U_1$$

We deduce that:

$$\begin{aligned} & \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad & \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad & X_2 = \theta_2 \left(\left(1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1}\right)^{1/\alpha_2} - 1\right) \end{aligned}$$

It is not possible to obtain a decreasing linear function between X_1 and X_2. This implies that $\rho \langle X_1, X_2 \rangle > -1$.

The bivariate Pareto copula

Question 6.c

Build a new bivariate Pareto distribution by assuming that the marginal distributions are $\mathcal{P}(\alpha_1, \theta_1)$ and $\mathcal{P}(\alpha_2, \theta_2)$ and the dependence is a bivariate Pareto copula function with parameter α. What is the relevance of this approach for building bivariate Pareto distributions?

The bivariate Pareto copula

We have:

$$\begin{aligned}\mathbf{F}'(x_1, x_2) &= \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha_1} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha_2} + \\ &\quad \left(\left(\frac{\theta_1 + x_1}{\theta_1}\right)^{\alpha_1/\alpha} + \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{\alpha_2/\alpha} - 1\right)^{-\alpha}\end{aligned}$$

The traditional bivariate Pareto distribution $\mathbf{F}(x_1, x_2)$ is a special case of $\mathbf{F}'(x_1, x_2)$ when:

$$\alpha_1 = \alpha_2 = \alpha$$

Using $\mathbf{F}'$ instead of $\mathbf{F}$, we can control the tail dependence, but also the univariate tail index of the two margins.

Calculation of correlation bounds

Question 1

Give the mathematical definition of the copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$. What is the probabilistic interpretation of these copulas?

Calculation of correlation bounds

We have:

$$\begin{aligned}
\mathbf{C}^{-}(u_1,u_2) &= \max(u_1+u_2-1,0) \\
\mathbf{C}^{\perp}(u_1,u_2) &= u_1 u_2 \\
\mathbf{C}^{+}(u_1,u_2) &= \min(u_1,u_2)
\end{aligned}$$

Let X_1 and X_2 be two random variables. We have:

(i) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{-}$ if and only if there exists a non-increasing function f such that we have $X_2 = f(X_1)$;

(ii) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{\perp}$ if and only if X_1 and X_2 are independent;

(iii) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{+}$ if and only if there exists a non-decreasing function f such that we have $X_2 = f(X_1)$.

Calculation of correlation bounds

Question 2

We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and $\mathrm{LGD} \sim \mathcal{U}_{[0,1]}$.

Calculation of correlation bounds

We note $U_1 = 1 - \exp(-\lambda \boldsymbol{\tau})$ and $U_2 = \mathrm{LGD}$.

Calculation of correlation bounds

Question 2.a

Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:

$$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

Calculation of correlation bounds

The dependence between $\boldsymbol{\tau}$ and LGD is maximum when we have $\mathbf{C}\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle = \mathbf{C}^{+}$. Since we have $U_1 = U_2$, we conclude that:

$$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

Calculation of correlation bounds

Question 2.b

Show that the linear correlation $\rho\left(\boldsymbol{\tau}, \mathrm{LGD}\right)$ verifies the following inequality:

$$\left|\rho\left\langle\boldsymbol{\tau}, \mathrm{LGD}\right\rangle\right| \leq \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We know that:

$$\rho \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle \in \left[\rho_{\mathsf{min}} \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle, \rho_{\mathsf{max}} \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle \right]$$

where $\rho_{\mathsf{min}} \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle$ (resp. $\rho_{\mathsf{max}} \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle$) is the linear correlation corresponding to the copula $\mathbf{C}^-$ (resp. $\mathbf{C}^+$). It comes that:

$$\mathbb{E}[\boldsymbol{\tau}] = \sigma(\boldsymbol{\tau}) = \frac{1}{\lambda}$$

and:

$$\begin{aligned} \mathbb{E}[\mathrm{LGD}] &= \frac{1}{2} \\ \sigma(\mathrm{LGD}) &= \sqrt{\frac{1}{12}} \end{aligned}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \mathbf{C}^-$, we have $U_1 = 1 - U_2$. It follows that $\mathrm{LGD} = e^{-\lambda\boldsymbol{\tau}}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right] = \mathbb{E}\left[\boldsymbol{\tau} e^{-\lambda\boldsymbol{\tau}}\right] &= \int_0^\infty t e^{-\lambda t}\lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left[-\frac{t e^{-2\lambda t}}{2}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2\lambda t}\,\mathrm{d}t \\
&= 0 + \frac{1}{2}\left[-\frac{e^{-2\lambda t}}{2\lambda}\right]_0^\infty \\
&= \frac{1}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\min}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \left(\frac{1}{4\lambda} - \frac{1}{2\lambda}\right) \Big/ \left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right) = -\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \mathbf{C}^{+}$, we have $\mathrm{LGD} = 1 - e^{-\lambda\tau}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right] = \mathbb{E}\left[\boldsymbol{\tau}\left(1 - e^{-\lambda\boldsymbol{\tau}}\right)\right] &= \int_0^\infty t\left(1 - e^{-\lambda t}\right)\lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-\lambda t}\,\mathrm{d}t - \int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left(\left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\,\mathrm{d}t\right) - \frac{1}{4\lambda} \\
&= 0 + \left[-\frac{e^{-\lambda t}}{\lambda}\right]_0^\infty - \frac{1}{4\lambda} \\
&= \frac{3}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\max}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \left(\frac{3}{4\lambda} - \frac{1}{2\lambda}\right) \Big/ \left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right) = \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We finally obtain the following result:

$$\left|\rho\left\langle\boldsymbol{\tau},\mathrm{LGD}\right\rangle\right|\leq\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

Question 2.c

Comment on these results.

Calculation of correlation bounds

We notice that $|\rho\langle\boldsymbol{\tau},\mathrm{LGD}\rangle|$ is lower than 86.6%, implying that the bounds -1 and $+1$ can not be reached.

Calculation of correlation bounds

Question 3

We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

Calculation of correlation bounds

Question 3.a

We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relation is true:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

Calculation of correlation bounds

If the copula function of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is the Fréchet upper bound copula, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are comonotone. We deduce that:

$$U_1 = U_2 \Longleftrightarrow 1 - e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

Calculation of correlation bounds

Question 3.b

Show that there exists a function f such that $\boldsymbol{\tau}_2 = f(\boldsymbol{\tau}_2)$ when the dependence function is $\mathbf{C}^-$.

Calculation of correlation bounds

We have $U_1 = 1 - U_2$. It follows that $\mathbf{S}_1(\boldsymbol{\tau}_1) = 1 - \mathbf{S}_2(\boldsymbol{\tau}_2)$. We deduce that:

$$e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{-\ln\left(1 - e^{-\lambda_2 \boldsymbol{\tau}_2}\right)}{\lambda_1}$$

There exists then a function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$ with:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

Calculation of correlation bounds

Question 3.c

Show that the lower and upper bounds of the linear correlation satisfy the following relationship:

$$-1 < \rho \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle \leq 1$$

Calculation of correlation bounds

Using Question 2(b), we known that $\rho \in [\rho_{\min}, \rho_{\max}]$ where $\rho_{\min}$ and $\rho_{\max}$ are the correlations of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ when the copula function is respectively $\mathbf{C}^-$ and $\mathbf{C}^+$. We also know that $\rho = 1$ (resp. $\rho = -1$) if there exists a linear and increasing (resp. decreasing) function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$. When the copula is $\mathbf{C}^+$, we have $f(t) = \frac{\lambda_2}{\lambda_1} t$ and $f'(t) = \frac{\lambda_2}{\lambda_1} > 0$. As it is a linear and increasing function, we deduce that $\rho_{\max} = 1$. When the copula is $\mathbf{C}^-$, we have:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

and:

$$f'(t) = -\frac{\lambda_2 e^{-\lambda_2 t} \ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1 \left(1 - e^{-\lambda_2 t}\right)} < 0$$

The function $f(t)$ is decreasing, but it is not linear. We deduce that $\rho_{\min} \neq -1$ and:

$$-1 < \rho \leq 1$$

Calculation of correlation bounds

Question 3.d

In the more general case, show that the linear correlation of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

Calculation of correlation bounds

When the copula is $\mathbf{C}^-$, we know that there exists a decreasing function f such that $X_2 = f(X_1)$. We also know that the linear correlation reaches the lower bound -1 if the function f is linear:

$$X_2 = a + bX_1$$

This implies that $b < 0$. When X_1 takes the value $+\infty$, we obtain:

$$X_2 = a + b \times \infty$$

As the lower bound of X_2 is equal to zero 0, we deduce that $a = +\infty$. This means that the function $f(x) = a + bx$ does not exist. We conclude that the lower bound $\rho = -1$ can not be reached.

Calculation of correlation bounds

Question 4

We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

Calculation of correlation bounds

Question 4.a

Find the probability distribution of $X_1 + X_2$.

Calculation of correlation bounds

$X_1 + X_2$ is a Gaussian random variable because it is a linear combination of the Gaussian random vector (X_1, X_2). We have:

$$\mathbb{E}\left[X_1 + X_2\right] = \mu_1 + \mu_2$$

and:

$$\mathrm{var}\left(X_1 + X_2\right) = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

We deduce that:

$$X_1 + X_2 \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$$

Calculation of correlation bounds

Question 4.b

Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

$$\operatorname{cov}\left(Y_1, Y_2\right) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

Calculation of correlation bounds

We have:

$$\begin{aligned}\operatorname{cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2] \\ &= \mathbb{E}\left[e^{X_1+X_2}\right] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2]\end{aligned}$$

We know that $e^{X_1+X_2}$ is a lognormal random variable. We deduce that:

$$\begin{aligned}\mathbb{E}\left[e^{X_1+X_2}\right] &= \exp\left(\mathbb{E}[X_1+X_2] + \frac{1}{2}\operatorname{var}(X_1+X_2)\right) \\ &= \exp\left(\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2\right)\right) \\ &= e^{\mu_1+\frac{1}{2}\sigma_1^2} e^{\mu_2+\frac{1}{2}\sigma_2^2} e^{\rho\sigma_1\sigma_2}\end{aligned}$$

We finally obtain:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1+\frac{1}{2}\sigma_1^2} e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)$$

Calculation of correlation bounds

Question 4.c

Deduce the correlation between Y_1 and Y_2.

Calculation of correlation bounds

We have:

$$
\begin{aligned}
\rho\left\langle Y_1, Y_2\right\rangle &= \frac{e^{\mu_1+\frac{1}{2}\sigma_1^2} e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)}{\sqrt{e^{2\mu_1+\sigma_1^2}\left(e^{\sigma_1^2}-1\right)}\sqrt{e^{2\mu_2+\sigma_2^2}\left(e^{\sigma_2^2}-1\right)}} \\
&= \frac{e^{\rho\sigma_1\sigma_2}-1}{\sqrt{e^{\sigma_1^2}-1}\sqrt{e^{\sigma_2^2}-1}}
\end{aligned}
$$

Calculation of correlation bounds

Question 4.d

For which values of θ does the equality $\rho \langle Y_1, Y_2 \rangle = +1$ hold? Same question when $\rho \langle Y_1, Y_2 \rangle = -1$.

Calculation of correlation bounds

$\rho\langle Y_1, Y_2\rangle$ is an increasing function with respect to ρ. We deduce that:

$$\rho\langle Y_1, Y_2\rangle = 1 \Longleftrightarrow \rho = 1 \text{ and } \sigma_1 = \sigma_2$$

The lower bound of $\rho\langle Y_1, Y_2\rangle$ is reached if ρ is equal to -1. In this case, we have:

$$\rho\langle Y_1, Y_2\rangle = \frac{e^{-\sigma_1\sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1}\sqrt{e^{\sigma_2^2} - 1}} > -1$$

It follows that $\rho\langle Y_1, Y_2\rangle \neq -1$.

Calculation of correlation bounds

Question 4.e

We consider the bivariate Black-Scholes model:

$$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

with $\mathbb{E}[W_1(t)\,W_2(t)] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t\to\infty} \rho\langle S_1(t), S_2(t)\rangle$.

Calculation of correlation bounds

It is obvious that:

$$\rho \langle S_1(t), S_2(t) \rangle = \frac{e^{\rho\sigma_1\sigma_2 t} - 1}{\sqrt{e^{\sigma_1^2 t} - 1}\sqrt{e^{\sigma_2^2 t} - 1}}$$

In the case $\sigma_1 = \sigma_2$ and $\rho = 1$, we have $\rho \langle S_1(t), S_2(t) \rangle = 1$. Otherwise, we obtain:

$$\lim_{t \to \infty} \rho \langle S_1(t), S_2(t) \rangle = 0$$

Calculation of correlation bounds

Question 4.f

Comment on these results.

Calculation of correlation bounds

In the case of lognormal random variables, the linear correlation does not necessarily range between -1 and $+1$.

Extreme value theory in the bivariate case

Question 1

What is an extreme value (EV) copula $\mathbf{C}$?

Extreme value theory in the bivariate case

An extreme value copula $\mathbf{C}$ satisfies the following relationship:

$$\mathbf{C}\left(u_1^t, u_2^t\right) = \mathbf{C}^t\left(u_1, u_2\right)$$

for all $t > 0$.

Extreme value theory in the bivariate case

Question 2

Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are EV copulas. Why $\mathbf{C}^{-}$ can not be an EV copula?

Extreme value theory in the bivariate case

The product copula $\mathbf{C}^{\perp}$ is an EV copula because we have:

$$\begin{aligned}
\mathbf{C}^{\perp}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t \\
&= \left(u_1 u_2\right)^t \\
&= \left[\mathbf{C}^{\perp}\left(u_1, u_2\right)\right]^t
\end{aligned}$$

Extreme value theory in the bivariate case

For the copula $\mathbf{C}^{+}$, we obtain:

$$\begin{aligned}
\mathbf{C}^{+}\left(u_1^t, u_2^t\right) &= \min\left(u_1^t, u_2^t\right) \\
&= \begin{cases} u_1^t & \text{if } u_1 \leq u_2 \\ u_2^t & \text{otherwise} \end{cases} \\
&= \left(\min\left(u_1, u_2\right)\right)^t \\
&= \left[\mathbf{C}^{+}\left(u_1, u_2\right)\right]^t
\end{aligned}$$

Extreme value theory in the bivariate case

However, the EV property does not hold for the Fréchet lower bound copula $\mathbf{C}^-$:

$$\mathbf{C}^-\left(u_1^t, u_2^t\right) = \max\left(u_1^t + u_2^t - 1, 0\right) \neq \max\left(u_1 + u_2 - 1, 0\right)^t$$

Indeed, we have $\mathbf{C}^-(0.5, 0.8) = \max(0.5 + 0.8 - 1, 0) = 0.3$ and:

$$\begin{array}{rcl} \mathbf{C}^-\left(0.5^2, 0.8^2\right) & = & \max(0.25 + 0.64 - 1, 0) \\ & = & 0 \\ & \neq & 0.3^2 \end{array}$$

Extreme value theory in the bivariate case

Question 3

We define the Gumbel-Hougaard copula as follows:

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right)$$

with $\theta \geq 1$. Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^\theta + \left(-\ln u_2^t\right)^\theta\right]^{1/\theta}\right) \\
&= \exp\left(-\left[\left(-t\ln u_1\right)^\theta + \left(-t\ln u_2\right)^\theta\right]^{1/\theta}\right) \\
&= \exp\left(-t\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}\right) \\
&= \left(e^{-\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}}\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Extreme value theory in the bivariate case

Question 4

What is the definition of the upper tail dependence λ? What is its usefulness in multivariate extreme value theory?

Extreme value theory in the bivariate case

The upper tail dependence λ is defined as follows:

$$\lambda = \lim_{u \to 1^+} \frac{1 - 2u + \mathbf{C}(u_1, u_2)}{1 - u}$$

It measures the probability to have an extreme in one direction knowing that we have already an extreme in the other direction. If λ is equal to 0, extremes are independent and the EV copula is the product copula $\mathbf{C}^{\perp}$. If λ is equal to 1, extremes are comonotonic and the EV copula is the Fréchet upper bound copula $\mathbf{C}^{+}$. Moreover, the upper tail dependence of the copula between the random variables is equal to the upper tail dependence of the copula between the extremes.

Extreme value theory in the bivariate case

Question 5

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$. If $g'(x_0) \neq 0$, L'Hospital's rule states that:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

Deduce that the upper tail dependence λ of the Gumbel-Hougaard copula is $2 - 2^{1/\theta}$. What is the correlation of two extremes when $\theta = 1$?

Extreme value theory in the bivariate case

Using L'Hospital's rule, we have:

$$\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[(-\ln u)^\theta + (-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[2(-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + 2^{1/\theta} u^{2^{1/\theta}-1}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2^{1/\theta} u^{2^{1/\theta}-1} \\
&= 2 - 2^{1/\theta}
\end{aligned}$$

Extreme value theory in the bivariate case

If θ is equal to 1, we obtain $\lambda = 0$. It comes that the EV copula is the product copula. Extremes are then not correlated. This result is not surprising because the Gumbel-Houggard copula is equal to the product copula when $\theta = 1$:

$$e^{-\left[(-\ln u_1)^1+(-\ln u_2)^1\right]^1} = u_1 u_2 = \mathbf{C}^{\perp}\left(u_1, u_2\right)$$

Extreme value theory in the bivariate case

Question 6

We define the Marshall-Olkin copula as follows:

$$\mathbf{C}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)$$

with $\{\theta_1, \theta_2\} \in [0, 1]^2$.

Extreme value theory in the bivariate case

Question 6.a

Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= u_1^{t(1-\theta_1)} u_2^{t(1-\theta_2)} \min\left(u_1^{t\theta_1}, u_2^{t\theta_2}\right) \\
&= \left(u_1^{1-\theta_1}\right)^t \left(u_2^{1-\theta_2}\right)^t \left(\min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \left(u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Extreme value theory in the bivariate case

Question 6.b

Find the upper tail dependence λ of the Marshall-Olkin copula.

Extreme value theory in the bivariate case

If $\theta_1 > \theta_2$, we obtain:

$$
\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} \min\left(u^{\theta_1}, u^{\theta_2}\right)}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} u^{\theta_1}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2-\theta_2}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + (2-\theta_2) u^{1-\theta_2}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2u^{1-\theta_2} + \theta_2 u^{1-\theta_2} \\
&= \theta_2
\end{aligned}
$$

If $\theta_2 > \theta_1$, we have $\lambda = \theta_1$. We deduce that the upper tail dependence of the Marshall-Olkin copula is $\min\left(\theta_1, \theta_2\right)$.

Extreme value theory in the bivariate case

Question 6.c

What is the correlation of two extremes when $\min(\theta_1, \theta_2) = 0$?

Extreme value theory in the bivariate case

If $\theta_1 = 0$ or $\theta_2 = 0$, we obtain $\lambda = 0$. It comes that the copula of the extremes is the product copula. Extremes are then not correlated.

Extreme value theory in the bivariate case

Question 6.d

In which case are two extremes perfectly correlated?

Extreme value theory in the bivariate case

Two extremes are perfectly correlated when we have $\theta_1 = \theta_2 = 1$. In this case, we obtain:

$$\mathbf{C}(u_1, u_2) = \min(u_1, u_2) = \mathbf{C}^+(u_1, u_2)$$

Maximum domain of attraction in the bivariate case

Question 1

We consider the following distributions of probability:

Distribution		$\mathbf{F}(x)$
Exponential	$\mathcal{E}(\lambda)$	$1 - e^{-\lambda x}$
Uniform	$\mathcal{U}_{[0,1]}$	x
Pareto	$\mathcal{P}(\alpha, \theta)$	$1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha}$

Maximum domain of attraction in the bivariate case

Question 1

For each distribution, we give the normalization parameters a_n and b_n of the Fisher-Tippet theorem and the corresponding limit distribution distribution $\mathbf{G}(x)$:

Distribution	a_n	b_n	$\mathbf{G}(x)$
Exponential	λ^{-1}	$\lambda^{-1} \ln n$	$\mathbf{\Lambda}(x) = e^{-e^{-x}}$
Uniform	n^{-1}	$1 - n^{-1}$	$\mathbf{\Psi}_1(x-1) = e^{x-1}$
Pareto	$\theta\alpha^{-1}n^{1/\alpha}$	$\theta n^{1/\alpha} - \theta$	$\mathbf{\Phi}_\alpha\left(1 + \frac{x}{\alpha}\right) = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}$

We note $\mathbf{G}(x_1, x_2)$ the asymptotic distribution of the bivariate random vector $(X_{1,n:n}, X_{2,n:n})$ where $X_{1,i}$ (resp. $X_{2,i}$) are *iid* random variables.

Maximum domain of attraction in the bivariate case

Let $(X_{1,}X_2)$ be a bivariate random variable whose probability distribution is:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}_{\langle X_1, X_2 \rangle}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

We know that the corresponding EV probability distribution is:

$$\mathbf{G}(x_1, x_2) = \mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2))$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are the two univariate EV probability distributions and $\mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}$ is the EV copula associated to $\mathbf{C}_{\langle X_1, X_2 \rangle}$.

Maximum domain of attraction in the bivariate case

Question 1.a

What is the expression of $\mathbf{G}(x_1, x_2)$ when $X_{1,i}$ and $X_{2,i}$ are independent, $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{U}_{[0,1]}$?

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned}
\mathbf{G}(x_1, x_2) &= \mathbf{C}^{\perp}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2)) \\
&= \mathbf{\Lambda}(x_1)\,\mathbf{\Psi}_1(x_2 - 1) \\
&= \exp\left(-e^{-x_1} + x_2 - 1\right)
\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.b

Same question when $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned}
\mathbf{G}(x_1, x_2) &= \boldsymbol{\Lambda}(x_1)\,\boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\
&= \exp\left(-e^{-x_1} - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)
\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.c

Same question when $X_{1,i} \sim \mathcal{U}_{[0,1]}$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned}\mathbf{G}\left(x_1, x_2\right) &= \boldsymbol{\Psi}_1\left(x_1 - 1\right)\boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\ &= \exp\left(x_1 - 1 - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 2

What becomes the previous results when the dependence function between $X_{1,i}$ and $X_{2,i}$ is the Normal copula with parameter $\rho < 1$?

Maximum domain of attraction in the bivariate case

We know that the upper tail dependence is equal to zero for the Normal copula when $\rho < 1$. We deduce that the EV copula is the product copula. We then obtain the same results as previously.

Maximum domain of attraction in the bivariate case

Question 3

Same question when the parameter of the Normal copula is equal to one.

Maximum domain of attraction in the bivariate case

When the parameter ρ is equal to 1, the Normal copula is the Fréchet upper bound copula $\mathbf{C}^{+}$, which is an EV copula. We deduce the following results:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Lambda}(x_1), \mathbf{\Psi}_1(x_2 - 1)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp(x_2 - 1)\right) \qquad \text{(a)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Lambda}(x_1), \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(b)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Psi}_1(x_1 - 1), \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp(x_2 - 1), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(c)}\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 4

Find the expression of $\mathbf{G}(x_1, x_2)$ when the dependence function is the Gumbel-Hougaard copula.

Maximum domain of attraction in the bivariate case

In the previous exercise, we have shown that the Gumbel-Houggard copula is an EV copula.

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^\theta + (-\ln \mathbf{\Psi}_1(x_2 - 1))^\theta\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1} + (1 - x_2)^\theta\right]^{1/\theta}\right) \qquad \text{(a)} \end{aligned}$$

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^\theta + \left(-\ln \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right)^\theta\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1} + \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(b)} \end{aligned}$$

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Psi}_1(x_1 - 1))^\theta + \left(-\ln \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right)^\theta\right]^{1/\theta}} \\ &= \exp\left(-\left[(1 - x_1)^\theta + \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(c)} \end{aligned}$$

Simulation of the bivariate Normal copula

Exercise

Let $X = (X_1, X_2)$ be a standard Gaussian vector with correlation ρ. We note $U_1 = \Phi(X_1)$ and $U_2 = \Phi(X_2)$.

Simulation of the bivariate Normal copula

Question 1

We note Σ the matrix defined as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Calculate the Cholesky decomposition of Σ. Deduce an algorithm to simulate X.

Simulation of the bivariate Normal copula

P is a lower triangular matrix such that we have $\Sigma = PP^\top$. We know that:

$$P = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

We verify that:

$$\begin{aligned} PP^\top &= \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \end{aligned}$$

Simulation of the bivariate Normal copula

We deduce that:

$$\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{array}\right)\left(\begin{array}{c} N_1 \\ N_2 \end{array}\right)$$

where N_1 and N_2 are two independent standardized Gaussian random variables. Let n_1 and n_2 be two independent random variates, whose probability distribution is $\mathcal{N}(0,1)$. Using the Cholesky decomposition, we deduce that can simulate X in the following way:

$$\left\{\begin{array}{l} x_1 \leftarrow n_1 \\ x_2 \leftarrow \rho n_1 + \sqrt{1-\rho^2}\, n_2 \end{array}\right.$$

Simulation of the bivariate Normal copula

Question 2

Show that the copula of (X_1, X_2) is the same that the copula of the random vector (U_1, U_2).

Simulation of the bivariate Normal copula

We have

$$\begin{aligned} \mathbf{C}\left\langle X_1, X_2 \right\rangle &= \mathbf{C}\left\langle \Phi\left(X_1\right), \Phi\left(X_2\right) \right\rangle \\ &= \mathbf{C}\left\langle U_1, U_2 \right\rangle \end{aligned}$$

because the function $\Phi(x)$ is non-decreasing. The copula of $U = (U_1, U_2)$ is then the copula of $X = (X_1, X_2)$.

Simulation of the bivariate Normal copula

Question 3

Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We deduce that we can simulate U with the following algorithm:

$$\left\{ \begin{array}{l} u_1 \leftarrow \Phi\left(x_1\right) = \Phi\left(n_1\right) \\ u_2 \leftarrow \Phi\left(x_2\right) = \Phi\left(\rho n_1 + \sqrt{1-\rho^2} n_2\right) \end{array} \right.$$

Simulation of the bivariate Normal copula

Question 4

Calculate the conditional distribution of X_2 knowing that $X_1 = x$. Then show that:

$$\Phi_2\left(x_1, x_2; \rho\right) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi\left(x\right) \mathrm{d}x$$

Simulation of the bivariate Normal copula

Let X_3 be a Gaussian random variable, which is independent from X_1 and X_2. Using the Cholesky decomposition, we know that:

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

It follows that:

$$\begin{aligned}
\Pr\left\{ X_2 \leq x_2 \mid X_1 = x \right\} &= \Pr\left\{ \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2 \,\middle|\, X_1 = x \right\} \\
&= \Pr\left\{ X_3 \leq \frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right\} \\
&= \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right)
\end{aligned}$$

Simulation of the bivariate Normal copula

Then we deduce that:

$$\begin{aligned}
\Phi_2(x_1, x_2; \rho) &= \Pr\{X_1 \le x_1, X_2 \le x_2\} \\
&= \Pr\left\{X_1 \le x_1, X_3 \le \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}}\right\} \\
&= \mathbb{E}\left[\Pr\left\{X_1 \le x_1, X_3 \le \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}} \,\middle|\, X_1\right\}\right] \\
&= \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\,\mathrm{d}x
\end{aligned}$$

Simulation of the bivariate Normal copula

Question 5

Deduce an expression of the Normal copula.

Simulation of the bivariate Normal copula

Using the relationships $u_1 = \Phi(x_1)$, $u_2 = \Phi(x_2)$ and $\Phi_2(x_1, x_2; \rho) = \mathbf{C}(\Phi(x_1), \Phi(x_2); \rho)$, we obtain:

$$\begin{aligned} \mathbf{C}(u_1, u_2; \rho) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\,\mathrm{d}x \\ &= \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u \end{aligned}$$

Simulation of the bivariate Normal copula

Question 6

Calculate the conditional copula function $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We have:

$$\begin{aligned}\mathbf{C}_{2|1}\left(u_2 \mid u_1\right) &= \partial_{u_1} \mathbf{C}\left(u_1, u_2\right) \\ &= \Phi\left(\frac{\Phi^{-1}\left(u_2\right) - \rho\Phi^{-1}\left(u_1\right)}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

Let v_1 and v_2 be two independent uniform random variates. The simulation algorithm corresponds to the following steps:

$$\left\{\begin{array}{l} u_1 = v_1 \\ \mathbf{C}_{2|1}\left(u_1, u_2\right) = v_2 \end{array}\right.$$

We deduce that:

$$\left\{\begin{array}{l} u_1 \leftarrow v_1 \\ u_2 \leftarrow \Phi\left(\rho\Phi^{-1}\left(v_1\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(v_2\right)\right) \end{array}\right.$$

Simulation of the bivariate Normal copula

Question 7

Show that this algorithm is equivalent to the Cholesky algorithm found in Question 3.

Simulation of the bivariate Normal copula

We obtain the same algorithm, because we have the following correspondence:

$$\left\{ \begin{array}{l} v_1 = \Phi(n_1) \\ v_2 = \Phi(n_2) \end{array} \right.$$

The algorithm described in Question 6 is then a special case of the Cholesky algorithm if we take $n_1 = \Phi^{-1}(v_1)$ and $n_2 = \Phi^{-1}(v_2)$. Whereas n_1 and n_2 are directly simulated in the Cholesky algorithm with a Gaussian random generator, they are simulated using the inverse transform in the conditional distribution method.

Construction of a stress scenario with the GEV distribution

Question 1

We note a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

Construction of a stress scenario with the GEV distribution

We recall that:

$$\begin{aligned}
\Pr\left\{\frac{X_{n:n}-b_n}{a_n}\leq x\right\} &= \Pr\left\{X_{n:n}\leq a_n x+b_n\right\} \\
&= \mathbf{F}^n\left(a_n x+b_n\right)
\end{aligned}$$

and:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\mathbf{F}^n\left(a_n x+b_n\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.a

Find the limit distribution $\mathbf{G}$ when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}\mathbf{F}^n\left(a_n x + b_n\right) &= \left(1 - e^{-\lambda\left(\lambda^{-1}x + \lambda^{-1}\ln n\right)}\right)^n \\ &= \left(1 - \frac{1}{n}e^{-x}\right)^n\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}e^{-x}\right)^n = e^{-e^{-x}} = \mathbf{\Lambda}\left(x\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.b

Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}\mathbf{F}^n\left(a_n x + b_n\right) &= \left(n^{-1}x + 1 - n^{-1}\right)^n \\ &= \left(1 + \frac{1}{n}\left(x-1\right)\right)^n\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\left(x-1\right)\right)^n = e^{x-1} = \mathbf{\Psi}_1\left(x-1\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.c

Same question when X is a Pareto distribution:

$$\mathbf{F}(x) = 1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha},$$

$a_n = \theta\alpha^{-1}n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}
\mathbf{F}^{n}\left(a_n x + b_n\right) &= \left(1 - \left(\frac{\theta}{\theta + \theta\alpha^{-1}n^{1/\alpha}x + \theta n^{1/\alpha} - \theta}\right)^{\alpha}\right)^{n} \\
&= \left(1 - \left(\frac{1}{\alpha^{-1}n^{1/\alpha}x + n^{1/\alpha}}\right)^{\alpha}\right)^{n} \\
&= \left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^{n}
\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^{n} = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}} = \mathbf{\Phi}_{\alpha}\left(1 + \frac{x}{\alpha}\right)$$

Construction of a stress scenario with the GEV distribution

Question 2

We denote by **G** the GEV probability distribution:

$$\mathbf{G}(x) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1, \ldots, x_T\}$.

Construction of a stress scenario with the GEV distribution

The GEV distribution encompasses the three EV probability distributions. This is an interesting property, because we have not to choose between the three EV distributions. We have:

$$g(x) = \frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\left(\frac{1+\xi}{\xi}\right)} \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\}$$

We deduce that:

$$\begin{aligned}\ell &= -\frac{n}{2}\ln\sigma^2 - \left(\frac{1+\xi}{\xi}\right)\sum_{i=1}^{n}\ln\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right) - \\ &\quad \sum_{i=1}^{n}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 3

Show that for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the Gumbel distribution:

$$\boldsymbol{\Lambda}(x) = \exp\left(-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right)$$

Construction of a stress scenario with the GEV distribution

We notice that:

$$\lim_{\xi \to 0} (1 + \xi x)^{-1/\xi} = e^{-x}$$

Then we obtain:

$$\begin{aligned}
\lim_{\xi \to 0} \mathbf{G}(x) &= \lim_{\xi \to 0} \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\
&= \exp \left\{ - \lim_{\xi \to 0} \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\
&= \exp \left(- \exp \left(- \left(\frac{x - \mu}{\sigma} \right) \right) \right)
\end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 4

We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

Construction of a stress scenario with the GEV distribution

Question 4.a

Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

$$r(\mathcal{T}) \simeq -\left(\hat{\mu} + \left(\frac{\mathcal{T}}{n} - 1\right)\hat{\sigma}\right)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of GEV parameters.

Construction of a stress scenario with the GEV distribution

We have:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\xi^{-1}\left[1 - (-\ln\alpha)^{-\xi}\right]$$

When the parameter ξ is equal to 1, we obtain:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\left(1 - (-\ln\alpha)^{-1}\right)$$

By definition, we have $\mathcal{T} = (1-\alpha)^{-1} n$. The return period $\mathcal{T}$ is then associate to the confidence level $\alpha = 1 - n/\mathcal{T}$. We deduce that:

$$\begin{aligned}
R(\mathcal{T}) &\approx -\mathbf{G}^{-1}(1 - n/\mathfrak{t}) \\
&= -\left(\mu - \sigma\left(1 - (-\ln(1 - n/\mathcal{T}))^{-1}\right)\right) \\
&= -\left(\mu + \left(\frac{\mathcal{T}}{n} - 1\right)\sigma\right)
\end{aligned}$$

We then replace μ and σ by their ML estimates $\hat{\mu}$ and $\hat{\sigma}$.

Construction of a stress scenario with the GEV distribution

Question 4.b

We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.

Construction of a stress scenario with the GEV distribution

For Portfolio #1, we obtain:

$$R(1\mathrm{Y}) = -\left(1\% + \left(\frac{252}{21} - 1\right) \times 3\%\right) = -34\%$$

For Portfolio #2, the stress scenario is equal to:

$$R(1\mathrm{Y}) = -\left(10\% + \left(\frac{252}{21} - 1\right) \times 2\%\right) = -32\%$$

We conclude that Portfolio #1 is more risky than Portfolio #2 if we consider a stress scenario analysis.